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J. Math. Pures Appl. 99 (2013) 330–342

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

Geometric and differentiable rigidity of submanifolds in spheres

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Received 31 March 2012

Available online 12 October 2012

Abstract

In this paper, we investigate rigidity of geometric and differentiable structures of complete submanifolds via an extrinsic geometrical quantity $\tau(x)$ defined by the second fundamental form. We verify a geometric rigidity theorem for complete submanifolds with parallel mean curvature in a unit sphere $\mathbb{S}^{n+p}$. Inspired by the rigidity theorem, we prove a differentiable sphere theorem for complete submanifolds in $\mathbb{S}^{n+p}$. Moreover, we obtain a differentiable pinching theorem for complete submanifolds in a $\delta(>\frac{1}{4})$-pinched Riemannian manifold.

Résumé

Dans cet article, on étudie la rigidité de structures géométriques et différentiable de sous-variétés complètes via une quantité géométrique extrinsèque $\tau(x)$ définie par sa deuxième forme fondamentale. On démonte un théorème de rigidité géométrique pour les sous-variétés complètes à courbure moyenne parallèle dans une sphère unité $\mathbb{S}^{n+p}$. Inspiré par le théorème de rigidité, on établit un théorème de sphère différentiable pour les sous-variétés complètes dans $\mathbb{S}^{n+p}$. On obtient aussi un théorème de pincement différentiable pour les sous-variétés complètes dans une variété riemannienne $\delta(>\frac{1}{4})$-pincée.

MSC: 53C20; 53C24; 53C40

Keywords: Complete submanifolds; Geometric and differentiable structures; Rigidity theorem; Ricci flow; Stable currents

1. Introduction

The study of relations between geometric invariants and the structures of the geometry and topology of a manifold is a longstanding subject in global differential geometry (see [2,5,8,24]). In 1930's, Hopf conjectured that a compact, simply connected Riemannian manifold whose sectional curvatures are close to 1 is homeomorphic to a sphere. This conjecture is known as the curvature pinching problem, which was first taken up by Rauch [23]. In 1960's, making use of the comparison technique, Berger [1] and Klingenberg [16] proved that a compact, simply connected manifold M whose sectional curvatures all lie in the interval $(1/4, 1]$ is homeomorphic to the sphere $\mathbb{S}^n$. Since the complex projective space $\mathbb{CP}^m$ $(m \geqslant 2)$ has sectional curvatures in the interval $[1/4, 1]$, the pinching constant $1/4$ is optimal

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0021-7824/$ – see front matter
http://dx.doi.org/10.1016/j.matpur.2012.09.003

for even dimensional cases. Later, Micallef and Moore [20] obtained a topological sphere theorem for pointwise 1/4-pinched manifolds via the technique of minimal surfaces. Meanwhile, the geometry of compact Kähler manifolds admitting negatively 1/4-pinched Riemannian metric was investigated by Yau and Zheng [34]. In 2008, Böhm and Wilking [3] proved that a compact Riemannian manifold with 2-positive curvature operator is diffeomorphic to a spherical space form. Recently, Brendle and Schoen [6] proved a remarkable differentiable pinching theorem for pointwise 1/4-pinched Riemannian manifolds by developing Hamilton's Ricci flow theory [15]. Furthermore, they gave a classification of weakly pointwise 1/4-pinched Riemannian manifolds [7]. More recently, Brendle [4] obtained the celebrated convergence result for the Ricci flow.

Theorem 1.1. *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geqslant 4$. Assume that*

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0, \tag{1}$$

for all orthonormal four frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)} + \frac{2}{n} r_{g(t)} g(t),$$

exists for all time and converges to a metric with positive constant sectional curvature as $t \to +\infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

The purpose of the present article is to investigate the geometry and topology from the viewpoint of submanifolds. After the pioneering rigidity theorem for closed minimal submanifolds in a unit sphere due to Simons [26], Lawson [17] and Chern, do Carmo and Kobayashi [12] obtained a classification of n-dimensional closed minimal submanifolds in $\mathbb{S}^{n+p}$ whose squared norm of the second fundamental form satisfies $S \leqslant n/(2 - 1/p)$. Further discussions on the rigidity theorems have been carried out by many other authors (see [21,22,33], etc.). In 1990, Xu [29] proved a rigidity theorem for compact submanifolds with parallel mean curvature in a sphere.

Theorem 1.2. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$. Denote by H and S the mean curvature and squared norm of the second fundamental form, respectively. If $S \leqslant C(n, p, H)$, then M is either a totally umbilic sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$, a Clifford hypersurface in an $(n+1)$-sphere, or the Veronese surface in $\mathbb{S}^4(\frac{1}{\sqrt{1+H^2}})$. Here the constant $C(n, p, H)$ is defined by*

$$C(n,p,H) = \begin{cases} \alpha(n,H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\ \frac{n}{2-\frac{1}{p}}, & \text{for } p \geqslant 2 \text{ and } H = 0, \\ \min\{\alpha(n,H), \frac{n+nH^2}{2-\frac{1}{p-1}} + nH^2\}, & \text{for } p \geqslant 3 \text{ and } H \neq 0, \end{cases}$$

$$\alpha(n,H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.$$

Later, the above pinching constant $C(n, p, H)$ was improved, by Li and Li [19] for $H = 0$ and by Xu [30] for $H \neq 0$, to

$$C'(n,p,H) = \begin{cases} \alpha(n,H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\ \min\{\alpha(n,H), \frac{1}{3}(2n + 5nH^2)\}, & \text{otherwise.} \end{cases}$$

The special case of Theorem 1.2 for $p = 1$ was also studied by Cheng and Nakagawa [11] independently.

On the other hand, Lawson and Simons [18] proved a topological sphere theorem for closed submanifolds in a unit sphere by using the non-existence for stable currents on compact submanifolds in a sphere. Shiohama and Xu [25] improved Lawson–Simons' result and proved a topological sphere theorem for complete submanifolds in a simply connected space form with nonnegative constant curvature. Recently, Xu and Zhao [32] and Xu and Gu [31] obtained some differentiable sphere theorems for complete submanifolds.

Let M^n be an n-dimensional Riemannian submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. Set $UM = \bigcup_{x\in M} U_xM$, where $U_xM = \{u \in T_xM\colon \|u\| = 1\}$ and T_xM is the tangent space at $x \in M$. In 1986, Gauchman [13] proved that if M is an n-dimensional closed minimal submanifold in $\mathbb{S}^{n+p}$, and if $\sigma(u) \leqslant \frac{1}{3}$ for any unit vector $u \in UM$, where $\sigma(u) = \|h(u,u)\|^2$ and h is the second fundamental form of M, then either $\sigma(u) \equiv 0$ or $\sigma(u) \equiv \frac{1}{3}$. Xu, Fang and Xiang [28] generalized this rigidity result to the case where M is an n-dimensional closed submanifold with parallel mean curvature in $\mathbb{S}^{n+p}$. In [32], Xu and Zhao proved the following theorem.

Theorem 1.3. *Let M be an n-dimensional complete submanifold in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$. If $\sigma(u) < \frac{1}{3}$ for all $u \in UM$, then M is diffeomorphic to the standard unit n-sphere $\mathbb{S}^n$.*

Putting $\beta(u,v) = \|h(u,u) - h(v,v)\|^2$ for $u, v \in U_xM$, we define

$$\tau(x) = \max_{u,v\in U_xM,\, u\perp v} \beta(u,v).$$

Note that if $\|h(u,u)\|^2 < c$ for any $u \in U_xM$, where c is some positive constant, then $\beta(u,v) < 4c$ for any $u, v \in U_xM$. This implies $\tau(x) < 4c$.

In this paper, we first obtain the following geometric rigidity theorem.

Theorem A. *Let M be an n-dimensional complete submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$.*

(1) *When $p = 1$, or $p = 2$ and $H \neq 0$, if M is compact and $\tau(x) \leqslant 4$ for all $x \in M$, then either $\tau(x) \equiv 0$ and M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$, or $\tau(x) \equiv 4$.*
(2) *When $p \geqslant 3$, or $p = 2$ and $H = 0$, if $\tau(x) \leqslant \frac{4}{3}$ for all $x \in M$, then either $\tau(x) \equiv 0$ and M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$, or $\tau(x) \equiv \frac{4}{3}$.*

Corollary 1.4. *(See [13,28].) Let M be an n-dimensional complete submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$.*

(1) *When $p = 1$, or $p = 2$ and $H \neq 0$, if M is compact and $\sigma(u) \leqslant 1$ for all $u \in UM$, then either $\sigma(u) \equiv 0$ and M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$, or $\max_{u\in UM} \sigma(u) = 1$.*
(2) *When $p \geqslant 3$, or $p = 2$ and $H = 0$, if $\sigma(u) \leqslant \frac{1}{3}$ for all $u \in UM$, then either $\sigma(u) \equiv 0$ and M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$, or $\sigma(u) \equiv \frac{1}{3}$.*

Then we prove the following differentiable rigidity theorem, which is a generalization of Theorem 1.3.

Theorem B. *Let M be an n-dimensional complete submanifold in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$. If $\tau(x) < \frac{4}{3}$ for all $x \in M$, then M is diffeomorphic to the standard unit n-sphere $\mathbb{S}^n$.*

Remark 1.5. Examples 3.5 and 3.6 in Section 3 show that our pinching constants in Theorems A and B are sharp.

If the ambient space is a general Riemannian manifold, we obtained the following differentiable pinching theorem.

Theorem C. *Let M be an n-dimensional complete submanifold in an $(n+p)$-dimensional pointwise $\delta(> 1/4)$-pinched Riemannian manifold N^{n+p}. Denote by $\overline{K}(x,\pi)$ the sectional curvature of N for 2-plane $\pi \subset T_xN$ and point $x \in N$. Set $\overline{K}_{\max}(x) := \max_{\pi\subset T_xN} \overline{K}(x,\pi)$ for $x \in M$. If $\tau(x) < \frac{16}{9}\overline{K}_{\max}(x)(\delta - \frac{1}{4})$ for all $x \in M$, then M is diffeomorphic to a space form. In particular, if M is simply connected, then M is diffeomorphic to the standard unit n-sphere $\mathbb{S}^n$ or the Euclidean space $\mathbb{R}^n$.*

Our paper is organized as follows. In Section 2, we introduce some basic equations in the geometry of submanifolds and give a lower bound for the sectional curvature of the submanifold in terms of the extrinsic quantity $\tau(x)$. We prove two rigidity theorems for submanifolds of spheres in Section 3. Then we give the proof of Theorem A. In Section 4, Theorems B and C are proved by using convergence results for the Ricci flow of Hamilton [15] and Brendle [4], and the non-existence theorem for stable currents due to Lawson and Simons [18]. Moreover, we present a differentiable pinching theorem for even dimensional submanifolds in pinched Riemannian manifolds.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold isometrically immersed in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We shall make use of the following convention on the range of indices:

$$1 \leqslant A, B, C, \ldots \leqslant n+p, \quad 1 \leqslant i, j, k, \ldots \leqslant n, \quad n+1 \leqslant \alpha, \beta, \gamma, \ldots \leqslant n+p.$$

Choose a local orthonormal frame field $\{e_A\}$ on N^{n+p} such that e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame field of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of N^{n+p}. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji},$$

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha,$$

$$\xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii}e_\alpha,$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right), \tag{2}$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i \left(h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}\right), \tag{3}$$

where h, ξ, R_{ijkl}, $R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the Riemannian curvature tensor and the normal curvature tensor of M, and K_{ABCD} is the Riemannian curvature tensor of N, respectively. We set

$$H = \|\xi\|, \qquad H^\alpha = \left(h^\alpha_{ij}\right)_{n\times n}.$$

Denote the first and second covariant derivatives of h^α_{ij} by h^α_{ijk} and h^α_{ijkl}, respectively. Then we have

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ik} + \sum_k h^\alpha_{ik}\omega_{jk} + \sum_\beta h^\beta_{ij}\omega_{\alpha\beta},$$

$$\sum_k h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\omega_{il} + \sum_l h^\alpha_{ilk}\omega_{jl} + \sum_l h^\alpha_{ijl}\omega_{kl} + \sum_\beta h^\beta_{ijk}\omega_{\alpha\beta}.$$

The Laplacian of h is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. If the mean curvature vector of M is parallel in the normal bundle $T^\perp M$, i.e., $\nabla^\perp_X \xi = 0$ for any $X \in \Gamma(TM)$, following [28,30,33], etc. we have

$$\sum_i h^\alpha_{iik} = 0, \qquad \sum_i h^\alpha_{iikl} = 0, \quad \text{for all } k, l, \alpha, \tag{4}$$

$$\Delta h^\alpha_{ij} = \sum_{k,m} h^\alpha_{km}R_{mijk} + \sum_{k,m} h^\alpha_{mi}R_{mkjk} + \sum_{k,\beta} h^\beta_{ki}R_{\beta\alpha jk}, \tag{5}$$

$$\sum_\alpha R_{\alpha\beta kl}\left(\operatorname{tr} H^\alpha\right) = 0. \tag{6}$$

Let UM denote the unit tangent bundle on M and U_xM its fiber over $x \in M$. Then $UM = \bigcup_{x\in M} U_xM$, where $U_xM = \{u \in T_xM\colon \|u\| = 1\}$, T_xM is the tangent space of M at x. Set

$$\beta(u,v)=\big\|h(u,u)-h(v,v)\big\|^2,$$

for $u,v\in U_xM$. We define a new extrinsic geometrical invariant of M by

$$\tau(x)=\max_{u,v\in U_xM,u\perp v}\beta(u,v),\quad x\in M.$$

Proposition 2.1. *Let M be an n-dimensional submanifold in a Riemannian manifold N^{n+p}, and x a fixed point in M. Then $\tau(x)=0$ if and only if x is a totally umbilical point.*

Proof. A point $x\in M$ is totally umbilical if and only if for each $\alpha=n+1,\dots,n+p$, the eigenvalues of the matrix H^α are equal. Suppose $\{e_1,\dots,e_n\}$ is an orthonormal basis of T_xM such that $\langle h(e_i,e_j),e_\alpha\rangle=\kappa_i^\alpha\delta_{ij}$. Then κ_i^α, $i=1,\dots,n$, are n eigenvalues of H^α. If $\tau(x)=0$, then $h(e_i,e_i)=h(e_j,e_j)$. This implies that $\kappa_i^\alpha=\langle h(e_i,e_i),e_\alpha\rangle=\langle h(e_j,e_j),e_\alpha\rangle=\kappa_j^\alpha$ for $i\neq j$. That is to say, H^α has only one eigenvalue with multiplication n. Note that this holds for arbitrary $\alpha=n+1,\dots,n+p$. Hence M is totally umbilical at x. The inverse is obvious since that M is totally umbilical at x is equivalent to $h(X,Y)=\langle X,Y\rangle\xi$ for any $X,Y\in T_xM$. □

The following lemma gives a lower bound for the sectional curvature of a submanifold.

Lemma 2.2. *Let M be an n-dimensional submanifold in a Riemannian manifold N^{n+p}. Denote by $\overline{K}(x,\pi)$ the sectional curvature of N for 2-plane $\pi\subset T_xN$ and point $x\in N$. Set $\overline{K}_{\min}(x):=\min_{\pi\subset T_xN}\overline{K}(x,\pi)$ for $x\in M$. Then the sectional curvature of M at $x\in M$ is no less than $\overline{K}_{\min}(x)-\frac{\tau(x)}{2}$.*

Proof. For $x\in M$, let u, v be two perpendicular unit vectors in U_xM. Then by Gauss equation (2) we have

$$R(u,v,u,v)=K(u,v,u,v)+\big\langle h(u,u),h(v,v)\big\rangle-\big\|h(u,v)\big\|^2. \tag{7}$$

Note that

$$\tau(x)\geqslant\big\|h(u,u)-h(v,v)\big\|^2=\big\|h(u,u)\big\|^2+\big\|h(v,v)\big\|^2-2\big\langle h(u,u),h(v,v)\big\rangle.$$

Therefore,

$$\begin{aligned}\big\langle h(u,u),h(v,v)\big\rangle&\geqslant\frac{1}{4}\big(\big\|h(u,u)\big\|^2+\big\|h(v,v)\big\|^2-\tau(x)\big)+\frac{1}{2}\big\langle h(u,u),h(v,v)\big\rangle\\&\geqslant-\frac{\tau(x)}{4}.\end{aligned}\tag{8}$$

On the other hand, we have

$$h(u,v)=\frac{1}{2}\left(h\left(\frac{u+v}{\sqrt2},\frac{u+v}{\sqrt2}\right)-h\left(\frac{u-v}{\sqrt2},\frac{u-v}{\sqrt2}\right)\right).$$

Since $(u+v)/\sqrt2$ and $(u-v)/\sqrt2$ are two perpendicular unit vectors, we have

$$\big\|h(u,v)\big\|^2=\frac{1}{4}\left\|h\left(\frac{u+v}{\sqrt2},\frac{u+v}{\sqrt2}\right)-h\left(\frac{u-v}{\sqrt2},\frac{u-v}{\sqrt2}\right)\right\|^2\leqslant\frac{\tau(x)}{4}. \tag{9}$$

Combining (7), (8) and (9), we complete the proof of the lemma. □

3. Rigidity of submanifolds with parallel mean curvature

In this section, we investigate the geometric rigidity of submanifolds with parallel mean curvature. Let M^n be an n-dimensional complete submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$. Then $K_{ABCD}=\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}$.

Suppose $\tau(x_0)\neq0$ at $x_0\in M$. Then there exist two perpendicular unit vectors $u_0,v_0\in U_{x_0}M$ such that

$$\max_{u,v\in U_{x_0}M,u\perp v}\big\|h(u,u)-h(v,v)\big\|^2=\big\|h(u_0,u_0)-h(v_0,v_0)\big\|^2.$$

Choose an orthonormal frame $\{e_A\}$ at x_0 such that

$$e_{n+1}=\frac{h(u_0,u_0)-h(v_0,v_0)}{\|h(u_0,u_0)-h(v_0,v_0)\|}, \tag{10}$$

and the matrix (h_{ij}^{n+1}) satisfies

$$h_{11}^{n+1}\geqslant\cdots\geqslant h_{nn}^{n+1},\qquad h_{ij}^{n+1}=0\quad(i\neq j). \tag{11}$$

Let $u_0=\sum_i x^i e_i$, $v_0=\sum_i y^i e_i$. Since

$$\begin{aligned}\tau(x_0)&=\langle h(u_0,u_0)-h(v_0,v_0),e_{n+1}\rangle^2\\&=\Big[\sum_i\big(x_i^2-y_i^2\big)h_{ii}^{n+1}\Big]^2\\&\leqslant\Big[\sum_i x_i^2h_{11}^{n+1}-\sum_i y_i^2h_{nn}^{n+1}\Big]^2\\&=\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2,\end{aligned}$$

by the definition of τ, we have $\tau(x_0)=(h_{11}^{n+1}-h_{nn}^{n+1})^2$. Thus we can take

$$e_1=u_0,\qquad e_n=v_0. \tag{12}$$

From (10), we know that $h(e_1,e_1)-h(e_n,e_n)$ is parallel to e_{n+1}. Hence

$$h_{11}^{\alpha}=h_{nn}^{\alpha},\quad \alpha\neq n+1. \tag{13}$$

On the other hand, for any $x\in M$, $e_i,e_j\in T_xM$, $i\neq j$, let

$$u=\frac{e_i+e_j}{\sqrt2},\qquad v=\frac{e_i-e_j}{\sqrt2}.$$

Since u, v are perpendicular, we have

$$\sum_\alpha\big(h_{ij}^\alpha\big)^2(x)=\frac14\big\|h(u,u)-h(v,v)\big\|^2 \tag{14}$$

$$\leqslant\frac14\tau(x). \tag{15}$$

Define a tensor field $A=(A_{ijkl})$ on M by

$$A_{ijkl}=\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha.$$

Under the frame satisfying (10)–(13), we have

$$\tau(x_0)=A_{1111}+A_{nnnn}-2A_{11nn}. \tag{16}$$

By the definition of Laplacian, we obtain $\Delta\tau(x_0)=\Delta(1,n)=(\Delta A)_{1111}+(\Delta A)_{nnnn}-2(\Delta A)_{11nn}$, and

$$\begin{aligned}\frac12(\Delta A)_{1111}&=\sum_{\alpha,i}\big[h_{11}^\alpha h_{11ii}^\alpha+\big(h_{11i}^\alpha\big)^2\big],\\ \frac12(\Delta A)_{nnnn}&=\sum_{\alpha,i}\big[h_{nn}^\alpha h_{nnii}^\alpha+\big(h_{nni}^\alpha\big)^2\big],\\ (\Delta A)_{11nn}&=\sum_{\alpha,i}\big[h_{11}^\alpha h_{nnii}^\alpha+h_{nn}^\alpha h_{11ii}^\alpha+2h_{11i}^\alpha h_{nni}^\alpha\big].\end{aligned}$$

Using (13), at x_0 we have

$$\begin{aligned}\frac{1}{2}\Delta(1,n) &= \sum_i \big(h_{11}^{n+1}-h_{nn}^{n+1}\big)\big(h_{11ii}^{n+1}-h_{nnii}^{n+1}\big)+\sum_{\alpha,i}\big(h_{11i}^{\alpha}-h_{nni}^{\alpha}\big)^2\\ &\geqslant \sum_i \big(h_{11}^{n+1}-h_{nn}^{n+1}\big)\big(h_{11ii}^{n+1}-h_{nnii}^{n+1}\big).\end{aligned}\tag{17}$$

Substituting (2), (3) and (5) into (17), we have

$$\begin{aligned}\frac{1}{2}\Delta(1,n) \geqslant{}& n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\\ &-2\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)\sum_{i,\alpha}\big\{\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)\big(h_{1i}^{\alpha}\big)^2+\big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)\big(h_{ni}^{\alpha}\big)^2\big\}\\ &+\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)\sum_{i,\alpha}\big\{\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)h_{11}^{\alpha}h_{ii}^{\alpha}+\big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)h_{nn}^{\alpha}h_{ii}^{\alpha}\big\}.\end{aligned}\tag{18}$$

Lemma 3.1. *Let M be an n-dimensional submanifold with parallel mean curvature in a unit sphere $\mathbb{S}^{n+p}$. Suppose $\tau(x)\neq 0$ at $x\in M$. Let $\{e_A\}$ be an adapted frame satisfying (10)–(13).*

(1) *If $p=1$, or $p=2$ and $H\neq 0$, then*

$$\frac{1}{2}\Delta(1,n)\geqslant n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\left(1-\frac{1}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\right);$$

(2) *If $p\geqslant 3$, or $p=2$ and $H=0$, then*

$$\frac{1}{2}\Delta(1,n)\geqslant n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\left(1-\frac{3}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\right).$$

Proof. (1) If $p=1$, by (10) we see that the second term of the right hand side of (18) vanishes. If $p=2$ and $H\neq 0$, by (6) we have

$$\sum_\alpha R_{\alpha\beta kl}\big(\mathrm{tr}H^\alpha\big)=0.$$

Let α, β be $n+1$ and $n+2$, respectively. Then we have

$$\begin{cases} R_{(n+1)(n+2)kl}\big(\mathrm{tr}\,H^{n+1}\big)=0,\\ R_{(n+2)(n+1)kl}\big(\mathrm{tr}\,H^{n+2}\big)=0.\end{cases}$$

Since $H\neq 0$, $\mathrm{tr}\,H^{n+1}$ and $\mathrm{tr}\,H^{n+2}$ do not equal to 0 at the same time. Hence

$$R_{(n+1)(n+2)kl}=0$$

for all $k,l=1,\dots,n$. By (3) and (11) we have

$$\begin{aligned}R_{(n+1)(n+2)kl}&=\sum_{i=1}^n\big(h_{ik}^{n+1}h_{il}^{n+2}-h_{il}^{n+1}h_{ik}^{n+2}\big)\\ &=h_{kk}^{n+1}h_{kl}^{n+2}-h_{ll}^{n+1}h_{lk}^{n+2}\\ &=h_{kl}^{n+2}\big(h_{kk}^{n+1}-h_{ll}^{n+1}\big)\\ &=0.\end{aligned}$$

Particularly,

$$h_{1i}^{n+2}\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)=0,\qquad h_{ni}^{n+2}\big(h_{nn}^{n+1}-h_{ii}^{n+1}\big)=0.$$

So the second term of the right hand side of (18) vanishes either. Hence we have the following estimate from (18):

$$\begin{aligned}
\frac{1}{2}\Delta(1,n) &\geqslant n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2 \\
&\quad + \big(h_{11}^{n+1}-h_{nn}^{n+1}\big)\sum_i\Big\{\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)\sum_\alpha h_{11}^\alpha h_{ii}^\alpha + \big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)\sum_\alpha h_{nn}^\alpha h_{ii}^\alpha\Big\} \\
&\geqslant n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2 + \big(h_{11}^{n+1}-h_{ii}^{n+1}\big) \\
&\quad \times \sum_i\Big\{-\frac{1}{4}\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2 - \frac{1}{4}\big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\Big\} \\
&= n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\Big[1-\frac{1}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\Big].
\end{aligned} \tag{19}$$

In the second inequality we have used the following estimate:

$$\begin{aligned}
-\sum_\alpha h_{11}^\alpha h_{ii}^\alpha &\leqslant \sum_\alpha \frac{1}{4}\big(h_{11}^\alpha - h_{ii}^\alpha\big)^2 \\
&= \frac{1}{4}\big\|h(e_1,e_1)-h(e_i,e_i)\big\|^2 \\
&\leqslant \frac{1}{4}\big\|h(e_1,e_1)-h(e_n,e_n)\big\|^2 \\
&= \frac{1}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2.
\end{aligned} \tag{20}$$

(2) If $p \geqslant 3$, or $p=2$ and $H=0$, (18) can be rewritten as

$$\begin{aligned}
\frac{1}{2}\Delta(1,n) &= n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2 - \big(h_{11}^{n+1} \\
&\quad - h_{nn}^{n+1}\big)\Big\{\sum_{i=2}^n\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)\Big[2\sum_\alpha\big(h_{1i}^\alpha\big)^2 - \sum_\alpha h_{11}^\alpha h_{ii}^\alpha\Big] \\
&\quad + \sum_{i=1}^{n-1}\big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)\Big[2\sum_\alpha\big(h_{ni}^\alpha\big)^2 - \sum_\alpha h_{nn}^\alpha h_{ii}^\alpha\Big]\Big\}.
\end{aligned} \tag{21}$$

For $i \neq 1$, by (15) we have

$$2\sum_\alpha\big(h_{1i}^\alpha\big)^2 \leqslant \frac{1}{2}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2.$$

From this and (20) we obtain

$$\begin{aligned}
&\sum_{i=2}^n\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)\Big(2\sum_\alpha\big(h_{1i}^\alpha\big)^2 - \sum_\alpha h_{11}^\alpha h_{ii}^\alpha\Big) \\
&\quad \leqslant \sum_{i=2}^n\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)\times\frac{3}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2.
\end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned}
&\sum_{i=1}^{n-1}\big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)\Big(2\sum_\alpha\big(h_{ni}^\alpha\big)^2 - \sum_\alpha h_{nn}^\alpha h_{ii}^\alpha\Big) \\
&\quad \leqslant \sum_{i=1}^{n-1}\big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)\times\frac{3}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2.
\end{aligned} \tag{23}$$

Substituting (22) and (23) into (21), we get

$$\begin{aligned}\frac{1}{2}\Delta(1,n) &\geqslant n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2-\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)\\ &\quad\times\frac{3}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\left(\sum_{i=2}^{n}\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)+\sum_{i=1}^{n-1}\big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)\right)\\ &=n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2-\frac{3}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^3\\ &\quad\times\left(\sum_{i=1}^{n}\big(h_{11}^{n+1}-h_{ii}^{n+1}\big)+\sum_{i=1}^{n}\big(h_{ii}^{n+1}-h_{nn}^{n+1}\big)\right)\\ &=n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2-\frac{3}{4}n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^4\\ &=n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\left(1-\frac{3}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\right).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.2. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $\mathbb{S}^{n+p}$. We have*

(1) *If $p=1$, or $p=2$ and $H\neq 0$, and $\tau(x)\leqslant 4$ for any $x\in M$, then $\tau(x)$ is a constant function on M.*
(2) *If $p\geqslant 3$, or $p=2$ and $H=0$, and $\tau(x)\leqslant \frac{4}{3}$ for any $x\in M$, then $\tau(x)$ is a constant function on M.*

Proof. It suffices to show that $\tau(x)$ is a subharmonic function. Fix a point $x\in M$. For the Laplacian of continuous functions, we have the generalized definition

$$(\Delta\tau)(x)=C\lim_{r\to 0}\frac{1}{r^2}\left(\int_{B(x,r)}\tau\Big/\int_{B(x,r)}1-\tau(x)\right),$$

where C is a positive constant and $B(x,r)$ is a geodesic ball centered at x with radius r.

If $\tau(x)=0$, then $\Delta\tau(x)\geqslant 0$ since τ is nonnegative on M.

If $\tau(x)\neq 0$, in a small neighborhood O_x of x within the cut-locus of x, we denote by $u(y), v(y)$ two vectors tangent to M obtained by parallel transport of $u(x)=e_1$, $v(x)=e_n$ along the unique geodesic joining x to y. Define $f_x(y)=\|h(u(y),u(y))-h(v(y),v(y))\|^2$. Then

$$\Delta f_x(y)|_{y=x}=\Delta\big\{A(u,u,u,u)+A(v,v,v,v)-2A(u,u,v,v)\big\}\big|_{y=x}=\Delta(1,n).$$

Since $\tau(x)=f_x(x)$ and $\tau(y)\geqslant f_x(y)$, we have $\Delta\tau(x)\geqslant\Delta f_x(x)$. From Lemma 3.1 we know that $\Delta(1,n)\geqslant 0$. Hence $\Delta\tau(x)\geqslant 0$.

Since x is arbitrary, we see that $\tau(x)$ is subharmonic, which implies that τ is a constant function. This completes the proof of Lemma 3.2. □

Theorem 3.3. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $\mathbb{S}^{n+p}$. If $p=1$, or $p=2$ and $H\neq 0$, and if $\tau(x)\leqslant 4$ for all $x\in M$, then either $\tau(x)\equiv 0$ and M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$, or $\tau(x)\equiv 4$.*

Proof. By Lemma 3.2, τ is a constant function. When $\tau=0$, from Lemma 2.1 we know that M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$. When $\tau\neq 0$, from the definition of $f_x(y)$ in the proof of Lemma 3.2 we see that $f_x(x)$ is the maximal value of f_x. Hence, $\Delta(1,n)=\Delta f_x(x)\leqslant 0$. By Lemma 3.1 we have

$$\frac{1}{2}\Delta(1,n)\geqslant n\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\left(1-\frac{1}{4}\big(h_{11}^{n+1}-h_{nn}^{n+1}\big)^2\right).$$

Since $\tau(x)=(h_{11}^{n+1}-h_{nn}^{n+1})^2\leqslant 4$, we obtain from the above inequality that $\tau(x)=4$. This completes the proof of Theorem 3.3. □

Theorem 3.4. *Let M be an n-dimensional complete submanifold with parallel mean curvature in a unit sphere $\mathbb{S}^{n+p}$. If $p \geqslant 3$, or $p = 2$ and $H = 0$, and if $\tau(x) \leqslant \frac{4}{3}$ for all $x \in M$, then either $\tau(x) \equiv 0$ and M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$, or $\tau(x) \equiv \frac{4}{3}$.*

Proof. From Lemma 2.2, we have $K_M \geqslant \frac{1}{3} > 0$. Hence M is a compact submanifold by Myers' theorem. When $\tau = 0$, M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$. When τ is a nonzero constant, we have $\Delta(1,n) \leqslant 0$. From Lemma 3.1 we have $\frac{1}{2}\Delta(1,n) \geqslant n(h_{11}^{n+1} - h_{nn}^{n+1})^2(1 - \frac{3}{4}(h_{11}^{n+1} - h_{nn}^{n+1})^2)$. Hence if $\tau(x) = (h_{11}^{n+1} - h_{nn}^{n+1})^2 \leqslant \frac{4}{3}$, we obtain that $\tau = \frac{4}{3}$. This completes the proof of Theorem 3.4. □

Now we are in a position to give the proof of Theorem A.

Proof of Theorem A. Combing Theorems 3.3 and 3.4, we obtain the assertion of Theorem A. □

Example 3.5. Let $\mathbb{S}^q(r)$ be the q-dimensional sphere of radius r in $\mathbb{R}^{q+1}$, and let $1 \leqslant k \leqslant n-1$. We embed $\mathbb{S}^k(1/\sqrt{2}) \times \mathbb{S}^{n-k}(1/\sqrt{2})$ in $\mathbb{S}^{n+1}(1)$ as follows. Let $u \in \mathbb{S}^k(1/\sqrt{2})$ and $v \in \mathbb{S}^{n-k}(1/\sqrt{2})$ be vectors of length $1/\sqrt{2}$ in $\mathbb{R}^{k+1}$ and $\mathbb{R}^{n-k+1}$, respectively. We can consider (u, v) as a unit vector in $\mathbb{R}^{n+2} = \mathbb{R}^{k+1} \times \mathbb{R}^{n-k+1}$. It is easy to see that $\mathbb{S}^k(1/\sqrt{2}) \times \mathbb{S}^{n-k}(1/\sqrt{2})$ is a submanifold in $\mathbb{S}^{n+1}(1)$ with constant mean curvature

$$H = \left|\frac{2k-n}{n}\right|.$$

In particular, $\mathbb{S}^k(1/\sqrt{2}) \times \mathbb{S}^k(1/\sqrt{2}) \subset \mathbb{S}^{n+1}(1)$ is minimal if $n = 2k$. Since $\mathbb{S}^{n+1}(1) \subset \mathbb{S}^{n+2}(1)$ is totally geodesic, as a submanifold in $\mathbb{S}^{n+2}(1)$, $\mathbb{S}^k(1/\sqrt{2}) \times \mathbb{S}^{n-k}(1/\sqrt{2}) \subset \mathbb{S}^{n+2}(1)$ is a submanifold with parallel mean curvature vector of norm $H = |\frac{2k-n}{n}|$.

Example 3.6. Denote by $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{QP}^2$ and $\mathbb{C}\mathrm{ay}\mathbb{P}^2$ the projective planes over the real numbers, complex numbers, quaternions and octonions, and by $\psi_1 : \mathbb{RP}^2 \to \mathbb{S}^4(1)$, $\psi_2 : \mathbb{CP}^2 \to \mathbb{S}^7(1)$, $\psi_3 : \mathbb{QP}^2 \to \mathbb{S}^{13}(1)$ and $\psi_4 : \mathbb{C}\mathrm{ay}\mathbb{P}^2 \to \mathbb{S}^{25}(1)$ the corresponding isometric embeddings. Let $\psi_1' : \mathbb{S}^2(\sqrt{3}) \to \mathbb{S}^4(1)$ be the isometric immersion defined by $\psi_1' = \psi_1 \circ \pi$, where $\pi : \mathbb{S}^2(\sqrt{3}) \to \mathbb{RP}^2$ is the canonical projection.

For $n \geqslant 2$, $m \geqslant 0$, let $\mathbb{S}^n(1)$ be the great sphere in $\mathbb{S}^{n+m}(1)$ given by

$$\mathbb{S}^n(1) = \{(x_1, \dots, x_{n+m+1}) \in \mathbb{S}^{n+m}(1):\ x_{n+2} = \dots = x_{n+m+1} = 0\},$$

and $\eta_{n,m} : \mathbb{S}^n(1) \to \mathbb{S}^{n+m}(1)$ the inclusion. We set

$$\begin{aligned}
\phi_{1,p} &= \eta_{4,p-2} \circ \psi_1 : \mathbb{RP}^2 \to \mathbb{S}^{2+p}(1), \quad p \geqslant 2,\\
\phi_{2,p} &= \eta_{7,p-3} \circ \psi_2 : \mathbb{CP}^2 \to \mathbb{S}^{4+p}(1), \quad p \geqslant 3,\\
\phi_{3,p} &= \eta_{13,p-5} \circ \psi_3 : \mathbb{QP}^2 \to \mathbb{S}^{8+p}(1), \quad p \geqslant 5,\\
\phi_{4,p} &= \eta_{25,p-2} \circ \psi_4 : \mathbb{C}\mathrm{ay}\mathbb{P}^2 \to \mathbb{S}^{16+p}(1), \quad p \geqslant 9,\\
\phi_{1,p}' &= \eta_{4,p-2} \circ \psi_1' : \mathbb{S}^2(\sqrt{3}) \to \mathbb{S}^{2+p}(1), \quad p \geqslant 2.
\end{aligned}$$

Then $\phi_{i,p}$ is a minimal isometric embedding and $\phi_{1,p}'$ is a minimal isometric immersion.

From the proof of Main Theorem 1.6 in [28] we see that, for submanifolds described in Example 3.5, there exist $u, v \in U_xM$ at every $x \in M$ which are perpendicular such that $\tau(u, v) = 4$. Hence $\tau = 4$ on M. This implies that the condition $\tau \leqslant 4$ of Theorem A (1) is optimal. Similarly, we have $\tau \equiv \frac{4}{3}$ on $\phi_{i,p}$, $i = 1, 2, 3, 4$, and $\phi_{1,p}'$ described in Example 3.6. So the condition $\tau(x) \leqslant \frac{4}{3}$ in Theorem A (2) and Theorem B is optimal in the sense that there exist submanifolds which satisfy $\tau \equiv \frac{4}{3}$ but not isometric or diffeomorphic to a sphere.

Theorem A also improves the rigidity result due to Chen [10].

4. Differentiable sphere theorems for submanifolds

In this section, we first prove a differentiable sphere theorem for compact submanifolds in a $\delta(>1/4)$-pinched Riemannian manifold by using the convergence result for the Ricci flow due to Brendle [4].

Theorem 4.1. *Let M be an n-dimensional compact submanifold in an $(n+p)$-dimensional pointwise $\delta(>1/4)$-pinched Riemannian manifold N^{n+p}. Denote by $\overline{K}(x,\pi)$ the sectional curvature of N for 2-plane $\pi\subset T_xN$ and point $x\in N$. For $x\in M$, set $\overline{K}_{\max}(x):=\max_{\pi\subset T_xN}\overline{K}(x,\pi)$. If $\tau(x)<\frac{16}{9}\overline{K}_{\max}(x)(\delta-\frac{1}{4})$ for all $x\in M$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to the standard unit n-sphere $\mathbb{S}^n$.*

Proof. Since M is compact and $\overline{K}_{\max}(x)>0$ for all $x\in N$, we see from Lemma 2.2 that the sectional curvature K_M of M satisfies

$$K_M>\frac{\delta+2}{9}\inf_{x\in M}\overline{K}_{\max}(x)>0.$$

When $n=2$, it is easy to see that M is diffeomorphic to $\mathbb{S}^2$ or $\mathbb{RP}^2$. When $n=3$, the Hamilton theorem in [15] says that M is diffeomorphic to a spherical space form. When $n\geqslant 4$, let $\{e_1,e_2,e_3,e_4\}$ be an orthonormal four-frame at $x\in M$ and $\lambda\in\mathbb{R}$. By Lemma 2.2 and Berger's inequality we have

$$\begin{aligned}
&R_{1313}+\lambda^2R_{1414}+R_{2323}+\lambda^2R_{2424}-2\lambda R_{1234}\\
&\quad>2\big(1+\lambda^2\big)\frac{\delta+2}{9}\overline{K}_{\max}(x)-2|\lambda|\left(\frac{2}{3}(1-\delta)\overline{K}_{\max}(x)+\frac{8}{9}\overline{K}_{\max}(x)\left(\delta-\frac{1}{4}\right)\right)\\
&\quad=\frac{2}{9}(\delta+2)\overline{K}_{\max}(x)\big(1-2|\lambda|+\lambda^2\big)\\
&\quad\geqslant 0.
\end{aligned}$$

This together with Theorem 1.1 implies that M admits a metric with positive constant sectional curvature. Hence M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M must be diffeomorphic to the standard unit n-sphere $\mathbb{S}^n$. This completes the proof. □

Lemma 4.2. *(See [18].) Let M be an n-dimensional compact submanifold in a unit sphere $\mathbb{S}^{n+p}$. Let k and l be positive integers with $k+l=n$. If the following inequality*:

$$\sum_{j=k+1}^{n}\sum_{i=1}^{k}\big(2\|h(e_i,e_j)\|^2-\langle h(e_i,e_i),h(e_j,e_j)\rangle\big)<kl,$$

holds for any orthonormal basis $\{e_1,e_2,\dots,e_n\}$ of the tangent space at any point $x\in M$, then there is no stable k-current in M. Moreover,

$$H_k(M,\mathbb{Z})=H_l(M,\mathbb{Z})=0,$$

where $H_i(M,\mathbb{Z})$ is the i-th homology group on M with integer coefficients.

Now we give the proofs of Theorems B and C.

Proof of Theorem B. By Lemma 2.2, we know that $K_M\geqslant\frac{1}{3}>0$. Then M is compact by Myers' theorem. Let $\{e_1,e_2,\dots,e_n\}$ be any orthonormal basis of the tangent space at any point $x\in M$. By (8) and (9) we know that $\langle h(e_i,e_i),h(e_j,e_j)\rangle>-\frac{1}{3}$ and $\|h(e_i,e_j)\|^2<\frac{1}{3}$. Hence for any $1\leqslant k\leqslant n-1$,

$$\sum_{j=k+1}^{n}\sum_{i=1}^{k}\big(2\|h(e_i,e_j)\|^2-\langle h(e_i,e_i),h(e_j,e_j)\rangle\big)<\sum_{j=k+1}^{n}\sum_{i=1}^{k}\left(2\times\frac{1}{3}+\frac{1}{3}\right)=k(n-k).$$

By Lemma 4.2, we know that there does not exist any stable integral current in M.

Suppose that $\pi_1(M) \neq 0$. Since M is compact, it follows from a classical theorem due to Cartan and Hadamard that there exists a minimal closed geodesic in any nontrivial homotopy class in $\pi_1(M)$. Then we get a contradiction. Therefore $\pi_1(M) = 0$ and M is simply connected. This together with Theorem 4.1 implies that M is diffeomorphic to $\mathbb{S}^n$. This completes the proof of Theorem B. □

Proof of Theorem C. By Lemma 2.2 we see that $K_M > \frac{\delta+2}{9} \inf_{x\in M} \overline{K}_{\max}(x) \geqslant 0$. When M is compact, we see from Theorem 4.1 that M is diffeomorphic to a spherical space form. When M is complete noncompact, it follows from Cheeger–Gromoll–Myers's soul theorem [9,14] that M is diffeomorphic to $\mathbb{R}^n$. In particular, if M is simply connected, then M is diffeomorphic to $\mathbb{S}^n$ or $\mathbb{R}^n$. This completes the proof of Theorem C. □

Theorem C improves the sphere theorems due to Xia [27] and Xu and Zhao [32]. Furthermore, we present a differentiable pinching theorem for even dimensional complete submanifolds as follows.

Theorem 4.3. *Let M^n be an even dimensional complete oriented submanifold in an $(n+p)$-dimensional pointwise $\delta(>1/4)$-pinched Riemannian manifold N^{n+p}. If*

$$\tau(x) < \frac{16}{9}\overline{K}_{\max}(x)\left(\delta - \frac{1}{4}\right)$$

for all $x \in M$, then M is diffeomorphic to the standard unit n-sphere $\mathbb{S}^n$ or the Euclidean space $\mathbb{R}^n$.

Proof. It follows from the assumption and Lemma 2.2 that

$$K_M > \frac{\delta+2}{9}\inf_{x\in M}\overline{K}_{\max}(x) \geqslant 0.$$

When M is compact, it is seen from the assumption and Synge's theorem that M is simply connected. From Theorem 4.1, we see that M is diffeomorphic to $\mathbb{S}^n$. When M is complete noncompact, it follows from Cheeger–Gromoll–Myers's soul theorem [9,14] that M is diffeomorphic to $\mathbb{R}^n$. Therefore, we conclude that M is diffeomorphic to $\mathbb{S}^n$ or $\mathbb{R}^n$. □

Finally, motivated by Theorems A and B, we would like to propose the following conjecture on the mean curvature flow in higher codimensions.

Conjecture 4.4. *Let $F_0 : M \to \mathbb{S}^{n+p}$ be an n-dimensional compact submanifold in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$. If $\tau(x) < \frac{4}{3}$ for all $x \in M$, then the mean curvature flow*

$$\begin{cases} \dfrac{\partial}{\partial t}F(x,t) = n\xi(x,t), & x \in M,\ t \geqslant 0, \\ F(\cdot,0) = F_0(\cdot), \end{cases}$$

has a unique solution $F : M \times [0,T) \to \mathbb{S}^{n+d}$, and either

(1) *$T < \infty$ and M_t converges to a round point as $t \to T$; or*
(2) *$T = \infty$ and M_t converges to a totally geodesic sphere in $\mathbb{S}^{n+d}$ as $t \to \infty$.*

In particular, we give the following:

Conjecture 4.5. *Let $F_0 : M \to \mathbb{S}^{n+p}$ be an n-dimensional compact submanifold in an $(n+p)$-dimensional unit sphere $\mathbb{S}^{n+p}$. If $\sigma(u) < \frac{1}{3}$ for any unit vector $u \in UM$, then the mean curvature flow with F_0 as initial value has a unique solution $F : M \times [0,T) \to \mathbb{S}^{n+d}$, and either*

(1) *$T < \infty$ and M_t converges to a round point as $t \to T$; or*
(2) *$T = \infty$ and M_t converges to a totally geodesic sphere in $\mathbb{S}^{n+d}$ as $t \to \infty$.*

Acknowledgements

Research supported by the NSFC, Grant Nos. 11071211, 11201416; the Trans-Century Training Programme Foundation for Talents by the Ministry of Education of China.

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