AN OPTIMAL CONVERGENCE THEOREM FOR MEAN CURVATURE FLOW OF ARBITRARY CODIMENSION IN HYPERBOLIC SPACES

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ABSTRACT. In this paper, we prove that if the initial submanifold M_0 of dimension $n(\geq 6)$ satisfies an optimal pinching condition, then the mean curvature flow of arbitrary codimension in hyperbolic spaces converges to a round point in finite time. In particular, we obtain the optimal differentiable sphere theorem for submanifolds in hyperbolic spaces. It should be emphasized that our pinching condition implies that the Ricci curvature of the initial submanifold is positive, but does not imply positivity of the sectional curvature of M_0.

1. INTRODUCTION

The investigation of curvature and topology of manifolds is one of the main stream in global differential geometry. The sphere theorem for compact manifolds was initiated by Rauch in 1951. During the past six decades, there are many important progresses on sphere theorems for Riemannian manifolds and submanifolds [4, 7, 8, 13, 15, 31]. As is known, the theory of curvature flows become more and more important in the geometry and topology of manifolds [2, 5, 6, 7, 8, 9, 14, 15, 18, 35, 40, 42], etc. In [8], Brendle and Schoen proved the remarkable differentiable 1/4-pinching sphere theorem via the Ricci flow, which had been open for half a century. Since the dimension of a complex projective space is always even, Brendle and Schoen's differentiable sphere theorem is optimal for even dimensional cases. In [9], Brendle and Schoen obtained a differentiable rigidity theorem for compact manifolds with weakly 1/4-pinched curvatures in the pointwise sense.

Let M^n be an $n(\geq 2)$-dimensional submanifold in an $(n+q)$-dimensional simply connected space form $\mathbb{F}^{n+q}(c)$ with constant curvature c. Denote by H and h the mean curvature vector and the second fundamental form of M, respectively. Set

$$\alpha(n,|H|,c) = nc + \frac{n}{2(n-1)}|H|^2 - \frac{n-2}{2(n-1)}\sqrt{|H|^4 + 4(n-1)c|H|^2}. \tag{1.1}$$

After the pioneering rigidity theorem for closed minimal submanifolds in a sphere due to Simons [34], Lawson [22] and Chern-do Carmo-Kobayashi [10] obtained a classification of n-dimensional closed minimal submanifolds in $\mathbb{S}^{n+q}$ whose squared norm of the second fundamental form satisfies $|h|^2 \leq n/(2-1/q)$. Later Li-Li [25] improved Simons' pinching constant for n-dimensional closed minimal submanifolds in $\mathbb{S}^{n+q}$ to $\max\{\frac{n}{2-1/q}, \frac{2}{3}n\}$. Putting $\alpha_1(n,|H|) = \alpha(n,|H|,1)$, the second author

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[36, 37] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in $\mathbb{S}^{n+q}$ whose squared norm of the second fundamental form satisfies $|h|^2 \le C(n, q, |H|)$. Here

$$C(n,q,|H|) = \begin{cases} \alpha_1(n,|H|), & \text{for } q = 1, \text{ or } q = 2 \text{ and } H \neq 0, \\ \min\left\{\alpha_1(n,|H|), \frac{2n}{3} + \frac{5|H|^2}{3n}\right\}, & \text{otherwise.} \end{cases}$$

In the case where $c < 0$, the second author [36, 38] proved the following optimal rigidity theorem for submanifolds with parallel mean curvature in hyperbolic spaces.

Theorem 1.1. *Let M be an n-dimensional ($n \ge 3$) complete submanifold with parallel mean curvature in the hyperbolic space $\mathbb{H}^{n+q}(c)$. If $\sup_M(|h|^2 - \alpha(n,|H|,c)) < 0$, where $|H|^2 + n^2 c > 0$, then M is the totally umbilical sphere $\mathbb{S}^n(n/\sqrt{|H|^2 + n^2 c})$.*

Using nonexistence for stable currents on compact submanifolds of a sphere and the generalized Poincaré conjecture in dimension $n(\ge 5)$ verified by Smale, Lawson and Simons [23] proved that if $M^n (n \ge 5)$ is an oriented compact submanifold in $\mathbb{S}^{n+p}$, and if $|h|^2 < 2\sqrt{n-1}$, then M is homeomorphic to a sphere. Notice that $\min_{|H|} \alpha(n, |H|, 1) = 2\sqrt{n-1}$. Shiohama and Xu [32] improved Lawson-Simons' result and proved the optimal sphere theorem.

Theorem 1.2. *Let M be an n-dimensional ($n \ge 4$) oriented complete submanifold in $\mathbb{F}^{n+q}(c)$ with $c \ge 0$. Suppose that $\sup_M(|h|^2 - \alpha(n,|H|,c)) < 0$. Then M is homeomorphic to a sphere.*

The following problem is very attractive: *Is it possible to generalize Theorem 1.2 to the case of submanifolds in hyperbolic spaces?* By investigating nonexistence for stable currents on compact submanifolds, Fu and Xu [12] obtained partial solution to this problem.

Let $F_0 : M^n \to N^{n+q}$ be an n-dimensional submanifold smoothly immersed in a Riemannian manifold. The mean curvature flow with initial value F_0 is a smooth family of immersions $F : M \times [0, T) \to N^{n+q}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = H(x,t), \\ F(\cdot, 0) = F_0, \end{cases} \tag{1.2}$$

where $H(x,t)$ is the mean curvature vector of the submanifold $M_t = F_t(M)$, $F_t = F(\cdot, t)$.

In 1984, Huisken [17] first proved that uniformly convex hypersurfaces in Euclidean space will converge to a round point along the mean curvature flow. Further discussions on convergence results for the mean curvature flow of hypersurfaces in certain Riemannian manifolds have been carried out by many other authors [19, 20, 24], etc. After the important work on convergence results for the mean curvature flow of arbitrary codimension in Euclidean spaces and spheres due to Andrews and Baker [2, 3], Liu, Xu, Ye and Zhao [27] proved the following convergence result for pinched submanifolds in hyperbolic spaces.

Theorem 1.3. *Let $F_0 : M^n \to \mathbb{H}^{n+q}(c)$ be an n-dimensional ($n \ge 4$) compact submanifold immersed in the hyperbolic space. If F_0 satisfies $|h|^2 \le \frac{1}{n-1}|H|^2 + 2c$, then the mean curvature flow with initial value F_0 has a unique smooth solution on a finite maximal time interval $[0, T)$, and the solution F_t converges to a round point as $t \to T$.*

When $n \geq 4$, we have the unified convergence theorem [2, 3, 27] of the mean curvature flow in space forms under the pinching condition $|h|^2 \leq \frac{1}{n-1}|H|^2 + 2c$, where $|H|^2 + n^2c > 0$. The initial submanifold satisfying the above pinching condition possesses non-negative sectional curvature. Meanwhile, Gu and Xu [14, 39] obtained a convergence theorem for the Ricci flow of submanifolds in space forms under the same pinching condition. For any fixed positive constant ε, there are examples [2, 3, 20, 27] which show that the pinching condition can not be improved to $|h|^2 < \frac{1}{n-1}|H|^2+2c+\varepsilon$. Motivated by the rigidity, sphere and convergence theorems above, i.e., Theorems 1.1-1.3, Liu, Xu and Zhao [29] proposed the following.

Conjecture A. *Let M_0 be an n-dimensional ($n \geq 3$) complete submanifold immersed in the hyperbolic space $\mathbb{H}^{n+q}(c)$. If M_0 satisfies $\sup_{M_0}(|h|^2 - \alpha(n, |H|, c)) < 0$ and $|H|^2 + n^2c > 0$, then the mean curvature flow with initial value M_0 has a unique smooth solution on a finite maximal time interval $[0, T)$, and the solution M_t converges to a round point as $t \to T$. In particular, M_0 is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.*

The purpose of the present article is to prove Conjecture A in dimension $n(\geq 6)$. We will establish the optimal convergence theorem for the mean curvature flow of arbitrary codimension in hyperbolic spaces, which implies the optimal differentiable sphere theorem.

Main Theorem. *Let $F_0 : M^n \to \mathbb{H}^{n+q}(c)$ be an n-dimensional ($n \geq 6$) complete submanifold immersed in the hyperbolic space with constant curvature c. If F_0 satisfies*

$$\sup_{F_0}(|h|^2 - \alpha(n, |H|, c)) < 0 \quad \text{and} \quad |H|^2 + n^2c > 0,$$

then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0, T) \to \mathbb{H}^{n+q}(c)$ on a finite maximal time interval, and F_t converges to a round point as $t \to T$. In particular, M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

Since $\alpha(n, |H|, c) > \frac{1}{n-1}|H|^2 + 2c$, our main theorem improves Theorem 1.3 for $n \geq 6$. Note that every initial submanifold in the convergence results [1, 2, 3, 14, 17, 19, 20, 27, 28, 39] possesses quasi-positive curvature. The pinching condition in Main Theorem implies that the Ricci curvature of the initial submanifold is positive [32], but does not imply positivity of the sectional curvature. The following example shows the pinching condition in Main Theorem is optimal for arbitrary $n(\geq 6)$.

Example. *Let λ, μ be positive constants satisfying $\lambda\mu = -c$ and $\lambda > \sqrt{-c}$, where $c < 0$. For $n \geq 3$, we consider the submanifold $M = \mathbb{F}^{n-1}(c + \lambda^2) \times \mathbb{F}^1(c + \mu^2) \subset \mathbb{H}^{n+q}(c)$. Then M is a complete submanifold with parallel mean curvature, which satisfies $|H| \equiv (n-1)\lambda + \mu > n\sqrt{-c}$ and $|h|^2 \equiv (n-1)\lambda^2 + \mu^2 = \alpha(n, |H|, c)$.*

The key ingredient of the proof of Main Theorem is to establish the elaborate estimates for the pinching quantity $\mathring{\alpha} = \alpha(n, |H|, c) - \frac{1}{n}|H|^2$, because our pinching condition is sharper than that in Theorem 1.3. Using the properties of $\mathring{\alpha}$ and the evolution equations, we first derive that $|\mathring{h}|^2 < \mathring{\alpha}$ is preserved along the mean curvature flow. Applying a new auxiliary function $f_\sigma = |\mathring{h}|^2/\mathring{\alpha}^{1-\sigma}$, we deduce that $|\mathring{h}|^2 \leq C_0|H|^{2(1-\sigma)}$ via the De Giorgi iteration. We then obtain an estimate for $|\nabla H|$. Finally, using estimates for $|\nabla H|$ and the Ricci curvature, we show that

$\operatorname{diam} M_t \to 0$ and $|H|_{\min}/|H|_{\max} \to 1$ as $t \to T$. This implies the flow shrinks to a round point.

2. Notations and formulas

Let (M^n, g) be a Riemannian submanifold immersed in a space form $\mathbb{F}^{n+q}(c)$ with constant curvature c. We denote by $\bar{\nabla}$ the Levi-Civita connection of the ambient space $\mathbb{F}^{n+q}(c)$. We use the same symbol ∇ to represent the connections of the tangent bundle TM and the normal bundle NM. Denote by $(\cdot)^\top$ and $(\cdot)^\perp$ the projections onto TM and NM, respectively. For $u, v \in \Gamma(TM)$, $\xi \in \Gamma(NM)$, the connections ∇ is given by $\nabla_u v = (\bar{\nabla}_u v)^\top$ and $\nabla_u \xi = (\bar{\nabla}_u \xi)^\perp$. The second fundamental form of M is defined as

$$h(u, v) = (\bar{\nabla}_u v)^\perp.$$

Let $\{e_i \,|\, 1 \le i \le n\}$ be a local orthonormal frame for the tangent bundle and $\{\nu_\alpha \,|\, 1 \le \alpha \le q\}$ be a local orthonormal frame for the normal bundle. Let $\{\omega_i\}$ be the dual frame of $\{e_i\}$. With the local frame, the first and second fundamental forms can be written as $g = \sum_i \omega^i \otimes \omega^i$ and $h = \sum_{i,j,\alpha} h^\alpha_{ij} \omega^i \otimes \omega^j \otimes \nu_\alpha$, respectively. The mean curvature vector is given by

$$H = \sum_\alpha H^\alpha \nu_\alpha, \quad H^\alpha = \sum_i h^\alpha_{ii}.$$

We denote by $\nabla^2_{i,j} T = \nabla_i(\nabla_j T) - \nabla_{\nabla_i e_j} T$ the second order covariant derivative of tensor. Then the Laplacian of a tensor is defined by $\Delta T = \sum_i \nabla^2_{i,i} T$.

We have the following estimates for the gradient of second fundamental form.

Lemma 2.1. *For every submanifold in a space form, we have*

(i) $|\nabla h|^2 \ge \frac{3}{n+2} |\nabla H|^2$,

(ii) $|\nabla |H|^2| \le 2|H||\nabla H|$.

The proof of (i) is the same as in [2, 17], and (ii) follows from the Cauchy-Schwarz inequality.

Let $\mathring{h} = h - \frac{1}{n} g \otimes H$ be the traceless second fundamental form of M. Its norm is given by $|\mathring{h}|^2 = |h|^2 - \frac{1}{n}|H|^2$. As in [2, 3], we define the following scalars on M.

$$R_1 = \sum_{\alpha,\beta} \left(\sum_{i,j} h^\alpha_{ij} h^\beta_{ij} \right)^2 + \sum_{i,j,\alpha,\beta} \left(\sum_k (h^\alpha_{ik} h^\beta_{jk} - h^\beta_{ik} h^\alpha_{jk}) \right)^2,$$

$$R_2 = \sum_{i,j} \left(\sum_\alpha H^\alpha h^\alpha_{ij} \right)^2,$$

$$W = nc|\mathring{h}| - R_1 + \sum_{i,j,k,\alpha,\beta} H^\alpha h^\alpha_{ik} h^\beta_{ij} h^\beta_{jk}.$$

By a direct computation, we get the following identity for the Laplacian of $|\mathring{h}|^2$.

$$\frac{1}{2} \Delta |\mathring{h}|^2 = \left\langle \mathring{h}, \nabla^2 H \right\rangle + |\nabla h|^2 - \frac{1}{n} |\nabla H|^2 + W. \tag{2.1}$$

At a fixed point in M, we choose an orthonormal frame $\{\nu_\alpha\}$ for the normal space, such that $H = |H|\nu_1$, and an orthonormal frame $\{e_i\}$ for the tangent space, such that (h^1_{ij}) is diagonal. Then $\left(\mathring{h}^1_{ij}\right)$ is also diagonal, we denote its diagonal elements by $\mathring{\lambda}_i$. Thus $\mathring{\lambda}_i = h^1_{ii} - \frac{1}{n}|H|$ and $\mathring{h}^\alpha_{ij} = h^\alpha_{ij}$ for $\alpha > 1$. We split $|\mathring{h}|^2$ into three parts

$$|\mathring{h}|^2 = P_1 + P_2, \quad P_2 = Q_1 + Q_2, \tag{2.2}$$

where

$$P_1 = \sum_i \mathring{\lambda}_i^2, \quad Q_1 = \sum_{\substack{\alpha > 1 \\ i}} \left(\mathring{h}^\alpha_{ii}\right)^2, \quad Q_2 = \sum_{\substack{\alpha > 1 \\ i \neq j}} \left(\mathring{h}^\alpha_{ij}\right)^2.$$

With the special frame, R_1 becomes

$$\begin{aligned} R_1 \;\; = \;\; & P_1^2 + \frac{2}{n} P_1 |H|^2 + \frac{1}{n^2}|H|^4 \\ & + 2\sum_{\alpha>1}\left(\sum_i \mathring{\lambda}_i \mathring{h}^\alpha_{ii}\right)^2 + \sum_{\alpha,\beta>1}\left(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^\beta_{ij}\right)^2 \\ & + 2\sum_{\substack{\alpha > 1 \\ i \neq j}} \left(\left(\mathring{\lambda}_i - \mathring{\lambda}_j\right)\mathring{h}^\alpha_{ij}\right)^2 + \sum_{\substack{\alpha, \beta > 1 \\ i, j}} \left(\sum_k \left(\mathring{h}^\alpha_{ik}\mathring{h}^\beta_{jk} - \mathring{h}^\alpha_{jk}\mathring{h}^\beta_{ik}\right)\right)^2. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{\alpha>1}\left(\sum_i \mathring{\lambda}_i \mathring{h}^\alpha_{ii}\right)^2 \leq \sum_{\alpha>1}\left(\sum_i \mathring{\lambda}_i^2\right)\left(\sum_i \left(\mathring{h}^\alpha_{ii}\right)^2\right) = P_1 Q_1. \tag{2.3}$$

We also have

$$\sum_{\substack{\alpha > 1 \\ i \neq j}} \left(\left(\mathring{\lambda}_i - \mathring{\lambda}_j\right)\mathring{h}^\alpha_{ij}\right)^2 \leq \sum_{\substack{\alpha > 1 \\ i \neq j}} 2\left(\mathring{\lambda}_i^2 + \mathring{\lambda}_j^2\right)\left(\mathring{h}^\alpha_{ij}\right)^2 \leq 2P_1 Q_2. \tag{2.4}$$

It follows from Theorem 1 of [25] that

$$\sum_{\alpha,\beta>1}\left(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^\beta_{ij}\right)^2 + \sum_{\substack{\alpha, \beta > 1 \\ i, j}} \left(\sum_k \left(\mathring{h}^\alpha_{ik}\mathring{h}^\beta_{jk} - \mathring{h}^\alpha_{jk}\mathring{h}^\beta_{ik}\right)\right)^2 \leq \frac{3}{2}P_2^2. \tag{2.5}$$

Then we obtain

$$R_1 \leq P_1^2 + \frac{2}{n}P_1|H|^2 + \frac{1}{n^2}|H|^4 + 2P_1Q_1 + 4P_1Q_2 + \frac{3}{2}P_2^2. \tag{2.6}$$

We also have

$$R_2 = \sum_{i,j}(|H|h^1_{ij})^2 = |H|^2\left(P_1 + \frac{1}{n}|H|^2\right). \tag{2.7}$$

Combining (2.2), (2.6) and (2.7), we obtain

Lemma 2.2. *For every submanifold in a space form, we have*

(*i*)

$$R_1 - \frac{1}{n}R_2 \leq |\mathring{h}|^4 + \frac{1}{n}|\mathring{h}|^2|H|^2 + 2P_2|\mathring{h}|^2 - \frac{1}{n}P_2|H|^2,$$

(*ii*)

$$R_2 = |\mathring{h}|^2|H|^2 + \frac{1}{n}|H|^4 - P_2|H|^2.$$

The following proposition (see [30, 33]) will be used to estimate a cubic polynomial of h_{ij}^α, $1 \le i, j \le n$, $1 \le \alpha \le q$.

Proposition 2.3. *Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers satisfying $\sum_i a_i = \sum_i b_i = 0$. Then*

$$\left|\sum_i a_i b_i^2\right| \leqslant \frac{n-2}{\sqrt{n(n-1)}}\left(\sum_i a_i^2\right)^{\frac{1}{2}}\left(\sum_i b_i^2\right),$$

where equality holds if and only if $\sum_i a_i^2 = 0$, or $\sum_i b_i^2 = 0$, or at least $n-1$ pairs of numbers of (a_i, b_i) are equal.

In particular, we have

$$\left|\sum_i a_i^3\right| \leqslant \frac{n-2}{\sqrt{n(n-1)}}\left(\sum_i a_i^2\right)^{\frac{3}{2}},$$

where equality holds if and only if at least $n-1$ numbers of a_i are equal.

3. Preservation of curvature pinching

Let $F : M \times [0, T) \to \mathbb{H}^{n+q}(c)$ be a mean curvature flow in the hyperbolic space $\mathbb{H}^{n+q}(c)$. Let $M_t = F(M, t)$. Suppose that M_0 is an n-dimensional ($n \ge 6$) complete submanifold satisfying $\sup(|h|^2 - \alpha(n, |H|, c)) < 0$ and $|H|^2 + n^2c > 0$.

The evolution equations of the mean curvature flow take the same form as in [3, 27].

Lemma 3.1. *For the mean curvature flow $F : M \times [0, T) \to \mathbb{H}^{n+q}(c)$, we have*

(*i*) $\frac{\partial}{\partial t}|h|^2 = \Delta|h|^2 - 2|\nabla h|^2 + 2R_1 + 4c|H|^2 - 2nc|h|^2$,

(*ii*) $\frac{\partial}{\partial t}|H|^2 = \Delta|H|^2 - 2|\nabla H|^2 + 2R_2 + 2nc|H|^2$,

(*iii*) $\frac{\partial}{\partial t}|\mathring{h}|^2 = \Delta|\mathring{h}|^2 - 2|\nabla h|^2 + \frac{2}{n}|\nabla H|^2 + 2R_1 - \frac{2}{n}R_2 - 2nc|\mathring{h}|^2$.

We define a function $\mathring{\alpha} : (-n^2c, +\infty) \to \mathbb{R}$ by

$$\mathring{\alpha}(y) = nc + \frac{n^2 - 2n + 2}{2(n-1)n}y - \frac{n-2}{2(n-1)}\sqrt{y^2 + 4(n-1)cy}. \tag{3.1}$$

It's obvious that $\mathring{\alpha}(|H|^2) = \alpha(n, |H|, c) - \frac{1}{n}|H|^2$. Moreover, we have the following lemma.

Lemma 3.2. *For $n \ge 6$, $c < 0$ and $y > -n^2c$, $\mathring{\alpha}$ has the following properties.*

(*i*) $y\mathring{\alpha}'(y) \cdot \big(\mathring{\alpha}(y) + \frac{1}{n}y + nc\big) \equiv \mathring{\alpha}(y) \cdot \big(\mathring{\alpha}(y) + \frac{1}{n}y - nc\big)$,

(*ii*) $\frac{n-2}{\sqrt{n(n-1)}}\sqrt{y\mathring{\alpha}(y)} \equiv \frac{1}{n}y - \mathring{\alpha}(y) + nc$,

(*iii*) $0 < \mathring{\alpha}(y) < \frac{y+n^2c}{n(n-1)}$, $0 < \mathring{\alpha}'(y) < \frac{1}{n(n-1)}$, $\mathring{\alpha}''(y) > 0$,

(*iv*) $2\sqrt{y}\mathring{\alpha}'(y) < \sqrt{\mathring{\alpha}(y)}$,

(*v*) $y\mathring{\alpha}'(y) > \mathring{\alpha}(y)$,

(*vi*) $2y\mathring{\alpha}''(y) + \mathring{\alpha}'(y) < \frac{2(n-1)}{n(n+2)}$.

Proof. By direct computations, we get

$$\mathring{\alpha}'(y) = \frac{n^2-2n+2}{2(n-1)n} - \frac{n-2}{2(n-1)}\frac{y+2(n-1)c}{\sqrt{y^2+4(n-1)cy}}, \quad \mathring{\alpha}''(y) = \frac{2(n-1)(n-2)c^2}{(y^2+4(n-1)cy)^{3/2}}.$$

We use a variable substitution $\xi = \frac{y}{\sqrt{y^2+4(n-1)cy}}$ to simplify formulas. Then $y > -n^2c$ implies $1 < \xi < \frac{n}{n-2}$. Hence $y = \frac{4(n-1)c}{\xi^{-2}-1}$ and $\sqrt{y^2+4(n-1)cy} = \frac{4(n-1)c}{\xi^{-1}-\xi}$.

With this variable substitution, one can verify (i) and (ii) easily.

For the rest, we have

$$\frac{y+n^2c}{\mathring{\alpha}(y)} = \frac{2n^2}{n-(n-2)\xi} - n > n(n-1),$$

$$\frac{2\sqrt{y}\mathring{\alpha}'(y)}{\sqrt{\mathring{\alpha}(y)}} = \frac{n(\xi-1)+2}{\sqrt{n(n-1)}} < \frac{4}{n-2}\sqrt{\frac{n-1}{n}} < 1,$$

$$y\mathring{\alpha}'(y) - \mathring{\alpha}(y) = [(n-2)\xi - n]c > 0,$$

and

$$2y\mathring{\alpha}''(y) + \mathring{\alpha}'(y) + \frac{1}{n} = \frac{n}{2(n-1)} + \frac{n-2}{4(n-1)}(\xi^3 - 3\xi) < \frac{n}{(n-2)^2} \leqslant \frac{3}{n+2}.$$

□

For convenience, we denote $\mathring{\alpha}(|H|^2)$, $\mathring{\alpha}'(|H|^2)$ and $\mathring{\alpha}''(|H|^2)$ by $\mathring{\alpha}$, $\mathring{\alpha}'$ and $\mathring{\alpha}''$, respectively. Then we get the evolution equation of $\mathring{\alpha}$.

$$\frac{\partial}{\partial t}\mathring{\alpha} = \Delta\mathring{\alpha} + 2\mathring{\alpha}' \cdot (-|\nabla H|^2 + R_2 + nc|H|^2) - \mathring{\alpha}'' \cdot |\nabla|H|^2|^2. \tag{3.2}$$

It's seen from Theorem 1 of [32] that M_0 is compact. Hence there exists a small positive number ε, such that M_0 satisfies

$$|\mathring{h}|^2 < \mathring{\alpha} - \varepsilon\omega, \qquad \text{where} \quad \omega = |H|^2 + 4(n-1)c. \tag{3.3}$$

In the following we prove that the pinching condition above is preserved along the flow.

Theorem 3.3. *If M_0 satisfies $|\mathring{h}|^2 < \mathring{\alpha} - \varepsilon\omega$ and $|H|^2 + n^2c > 0$, then this condition holds for all time $t \in [0, T)$.*

Proof. Suppose $|\mathring{h}|^2 < \mathring{\alpha} - \varepsilon\omega$ remains true for $t \in [0, \tau)$. Note that $\mathring{\alpha} \to 0$ as $|H|^2 \to -n^2c$. This implies that $|H|^2 + n^2c > 0$ also remains true for $t \in [0, \tau)$. On the time interval $[0, \tau)$, we have the following evolution equation for $U = |\mathring{h}|^2 - \mathring{\alpha} + \varepsilon\omega$.

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) U \;=\;& -2|\nabla h|^2 + \frac{2}{n}|\nabla H|^2 + 2\left(\mathring{\alpha}' - \varepsilon\right)|\nabla H|^2 + \mathring{\alpha}'' \cdot |\nabla|H|^2|^2 \\ & + 2R_1 - \frac{2}{n}R_2 - 2nc|\mathring{h}|^2 - 2\left(\mathring{\alpha}' - \varepsilon\right)(R_2 + nc|H|^2).\end{aligned} \tag{3.4}$$

By Lemma 2.1 and Lemma 3.2 (vi), the first line of the right hand side of (3.4) is not greater than

$$\left[-\frac{2(n-1)}{n(n+2)} + \mathring{\alpha}' + 2|H|^2\mathring{\alpha}''\right]|\nabla H|^2 \leq 0.$$

By Lemma 2.2, the second line of the right hand side of (3.4) is not greater than

$$\begin{aligned}&2|\mathring{h}|^2\left(|\mathring{h}|^2+\frac{1}{n}|H|^2-nc\right)+2P_2\left(2|\mathring{h}|^2-\frac{1}{n}|H|^2\right)\\&-2\left(\mathring{\alpha}'-\varepsilon\right)|H|^2\left(|\mathring{h}|^2+\frac{1}{n}|H|^2+nc\right)+2\left(\mathring{\alpha}'-\varepsilon\right)|H|^2P_2.\end{aligned}$$

Replacing $|\mathring{h}|^2$ by $U+\mathring{\alpha}-\varepsilon\omega$, the right hand side of the inequality above becomes

$$\begin{aligned}&2U\left[2\mathring{\alpha}+\frac{1}{n}|H|^2-nc-\mathring{\alpha}'|H|^2+2P_2+\varepsilon(|H|^2-2\omega)\right]+2U^2\\&+2\left[\mathring{\alpha}\cdot\left(\mathring{\alpha}+\frac{1}{n}|H|^2-nc\right)-\mathring{\alpha}'|H|^2\cdot\left(\mathring{\alpha}+\frac{1}{n}|H|^2+nc\right)\right]\\&+2P_2\left[2\mathring{\alpha}-\frac{1}{n}|H|^2+|H|^2\mathring{\alpha}'-\varepsilon(|H|^2+2\omega)\right]\\&+2\varepsilon\omega\left[-\left(2\mathring{\alpha}+\frac{1}{n}|H|^2-nc-\mathring{\alpha}'|H|^2\right)+\frac{|H|^2}{\omega}\left(\mathring{\alpha}+\frac{1}{n}|H|^2+nc\right)\right]\\&-2\varepsilon^2\omega(|H|^2-\omega).\end{aligned}\tag{3.5}$$

By Lemma 3.2 (i) and (iii), the expression in the second square bracket of the RHS of (3.5) equals zero, and the expression in the third square bracket of the RHS of (3.5) is negative. By a direct computation, the expression in the last square bracket of the RHS of (3.5) equals

$$\left(\frac{3(n-2)|H|^2}{\sqrt{|H|^4+4(n-1)c|H|^2}}-n\right)c,$$

which is negative. So, we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)U<2U\left[2\mathring{\alpha}+\frac{1}{n}|H|^2-nc-\mathring{\alpha}'|H|^2+2P_2+\varepsilon(|H|^2-2\omega)\right]+2U^2.$$

By the maximum principle, the assertion follows. $\square$

From the preservation of $\mathring{\alpha}>\varepsilon\omega$, we have

Corollary 3.4. *There exists a positive constant δ depending on ε, such that $|H|^2+n^2c>\delta$ holds for all time $t\in[0,T)$.*

For completeness, we present a new proof of the following proposition [27], which states that the maximal existence time is finite.

Proposition 3.5. *Let $F: M\times[0,T)\to\mathbb{H}^{n+q}(c)$ be a mean curvature flow. If the initial value M_0 is a closed submanifold, then the maximal existence time T is finite.*

Proof. We use the Minkowski model for hyperbolic spaces. Let $\mathbb{R}^{1,m}$ be the $m+1$ dimensional Minkowski space. For $X,Y\in\mathbb{R}^{1,m}$, where $X=(x_0,\cdots,x_m)$, $Y=(y_0,\cdots,y_m)$, the inner product in $\mathbb{R}^{1,m}$ is defined by

$$\langle X,Y\rangle=-x_0y_0+x_1y_1+\cdots+x_my_m.$$

For $c<0$, we consider the following spacelike hypersurface in $\mathbb{R}^{1,m}$

$$-x_0^2+x_1^2+\cdots+x_m^2=1/c,\quad x_0\geq1/\sqrt{-c}.$$

It has constant sectional curvature c. We identify $\mathbb{H}^m(c)$ with this hypersurface.

Let $X : M^n \to \mathbb{H}^m(c) \subset \mathbb{R}^{1,m}$ be a submanifold immersed in the hyperbolic space $\mathbb{H}^m(c)$. We denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections of M and $\mathbb{H}^m(c)$, respectively. Let u, v be tangent vector fields over M. Since $\langle X, X\rangle = 1/c$, we have $\langle vX, X\rangle = 0$ and $\langle uvX, X\rangle = -\langle uX, vX\rangle = -\langle u, v\rangle$. Thus $uvX = \bar{\nabla}_u v - c\langle u, v\rangle X$. Then we have

$$\begin{aligned} \nabla^2_{u,v}X &= uvX - (\nabla_u v)X \\ &= \bar{\nabla}_u v - c\langle u, v\rangle X - \nabla_u v \\ &= h(u,v) - c\langle u, v\rangle X. \end{aligned} \tag{3.6}$$

Taking the trace of the both sides of (3.6), we obtain $\Delta X = H - ncX$.

Let $X : M \times [0,T) \to \mathbb{H}^{n+q}(c) \subset \mathbb{R}^{1,n+q}$ be a mean curvature flow. Then the equation of mean curvature flow becomes $\frac{\partial}{\partial t}X = \Delta X + ncX$. Particularly, $\frac{\partial}{\partial t}x_0 = \Delta x_0 + ncx_0$. By the maximum principle, we have $x_0 \le \sup_{t=0}(x_0)\cdot \mathrm{e}^{nct}$. Therefore, T is finite. $\square$

Since the maximal existence time is finite, we have $\max_{M_t}|h|^2 \to \infty$ as $t \to T$. This can be shown by using analogous argument in the proof of the corresponding theorem in [2]. Once $|h|^2$ is uniformly bounded, then all higher derivatives $|\nabla^m h|^2$ are uniformly bounded. Hence the solution M_t converge to a limit M_T in C^∞–topology as $t \to T$. Thus, the flow can be extended over time T. This contradicts the maximality of T.

4. An estimate for traceless second fundamental form

In this section, we derive an estimate for the traceless second fundamental form, which shows $|\mathring{h}|$ grows slower than $|H|$ along the mean curvature flow.

Theorem 4.1. *If M_0 satisfies $|\mathring{h}|^2 < \mathring{\alpha} - \varepsilon\omega$ and $|H|^2 + n^2c > 0$, then there exist constants $0 < \sigma < 1$ and $C_0 > 0$ depending only on M_0, such that for all $t \in [0,T)$ we have*

$$|\mathring{h}|^2 \le C_0|H|^{2(1-\sigma)}.$$

To prove Theorem 4.1, we need to study the auxiliary function:

$$f_\sigma = \frac{|\mathring{h}|^2}{\mathring{\alpha}^{1-\sigma}}, \quad 0 < \sigma < 1.$$

First, we derive the evolution equation of f_σ.

Lemma 4.2. *Along the mean curvature flow, we have*

$$\frac{\partial}{\partial t}f_\sigma \le \Delta f_\sigma + \frac{2}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| - 12\varepsilon\frac{f_\sigma}{|\mathring{h}|^2}|\nabla H|^2 - 4cf_\sigma + \sigma|H|^2 f_\sigma.$$

Proof. By a direct computation, we have

$$\frac{\partial}{\partial t}f_\sigma = f_\sigma\left(\frac{\frac{\partial}{\partial t}|\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{\frac{\partial}{\partial t}\mathring{\alpha}}{\mathring{\alpha}}\right). \tag{4.1}$$

The gradient of f_σ can be written as

$$\nabla f_\sigma = f_\sigma\left(\frac{\nabla|\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{\nabla\mathring{\alpha}}{\mathring{\alpha}}\right).$$

The Laplacian of f_σ is given by

$$\Delta f_\sigma = f_\sigma\left(\frac{\Delta|\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{\Delta\mathring{\alpha}}{\mathring{\alpha}}\right) - 2(1-\sigma)\frac{\langle\nabla f_\sigma, \nabla\mathring{\alpha}\rangle}{\mathring{\alpha}} + \sigma(1-\sigma)f_\sigma\frac{|\nabla\mathring{\alpha}|^2}{|\mathring{\alpha}|^2}. \tag{4.2}$$

By (4.1), (4.2) and the evolution equations, we get

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) f_\sigma &= 2(1-\sigma)\frac{\langle\nabla f_\sigma, \nabla\mathring{\alpha}\rangle}{\mathring{\alpha}} - \sigma(1-\sigma)f_\sigma\frac{|\nabla\mathring{\alpha}|^2}{|\mathring{\alpha}|^2} \\
&\quad + \frac{2f_\sigma}{|\mathring{h}|^2}\left(\frac{|\nabla H|^2}{n} - |\nabla h|^2\right) + (1-\sigma)\frac{f_\sigma}{\mathring{\alpha}}\left(2\mathring{\alpha}'|\nabla H|^2 + \mathring{\alpha}''|\nabla|H|^2|^2\right) \\
&\quad + 2f_\sigma\left[\frac{1}{|\mathring{h}|^2}\left(R_1 - \frac{1}{n}R_2\right) - nc - (1-\sigma)\frac{\mathring{\alpha}'}{\mathring{\alpha}}(R_2 + nc|H|^2)\right] \\
&\le \frac{2}{\mathring{\alpha}}|\nabla f_\sigma|\,|\nabla\mathring{\alpha}| \\
&\quad + f_\sigma\left[\frac{2}{|\mathring{h}|^2}\left(\frac{|\nabla H|^2}{n} - |\nabla h|^2\right) + \frac{2\mathring{\alpha}'}{\mathring{\alpha}}|\nabla H|^2 + \frac{\mathring{\alpha}''}{\mathring{\alpha}}|\nabla|H|^2|^2\right] \\
&\quad + 2f_\sigma\left[\frac{1}{|\mathring{h}|^2}\left(R_1 - \frac{1}{n}R_2\right) - nc - (1-\sigma)\frac{\mathring{\alpha}'}{\mathring{\alpha}}(R_2 + nc|H|^2)\right].
\end{aligned} \tag{4.3}$$

By Lemma 2.1 (ii) and Lemma 3.2 (iv), we have

$$\frac{1}{\mathring{\alpha}}\,|\nabla\mathring{\alpha}| = \frac{\mathring{\alpha}'}{\mathring{\alpha}}|\nabla|H|^2| \le \frac{1}{|\mathring{h}|}|\nabla H|. \tag{4.4}$$

Now we estimate the expression in the first square bracket of the right hand side of (4.3). From Lemma 2.1 and Lemma 3.2, we have

$$\begin{aligned}
&\frac{2}{|\mathring{h}|^2}\left(\frac{|\nabla H|^2}{n} - |\nabla h|^2\right) + \frac{2\mathring{\alpha}'}{\mathring{\alpha}}|\nabla H|^2 + \frac{\mathring{\alpha}''}{\mathring{\alpha}}|\nabla|H|^2|^2 \\
\le\ &\left(\frac{4(1-n)}{n(n+2)}\frac{1}{|\mathring{h}|^2} + \frac{2\mathring{\alpha}'}{\mathring{\alpha}} + 4|H|^2\frac{\mathring{\alpha}''}{\mathring{\alpha}}\right)|\nabla H|^2 \\
=\ &\left[\frac{4(1-n)}{n(n+2)}\left(\frac{1}{\mathring{\alpha}} + \frac{\mathring{\alpha} - |\mathring{h}|^2}{\mathring{\alpha}|\mathring{h}|^2}\right) + \frac{2\mathring{\alpha}'}{\mathring{\alpha}} + 4|H|^2\frac{\mathring{\alpha}''}{\mathring{\alpha}}\right]|\nabla H|^2 \\
\le\ &\left[-\frac{4(n-1)}{n(n+2)}\frac{\varepsilon\omega}{\mathring{\alpha}|\mathring{h}|^2} + \frac{1}{\mathring{\alpha}}\left(\frac{4(1-n)}{n(n+2)} + 2\mathring{\alpha}' + 4|H|^2\mathring{\alpha}''\right)\right]|\nabla H|^2 \\
\le\ &-\frac{4(n-1)}{n(n+2)}\frac{\omega}{\mathring{\alpha}}\frac{\varepsilon}{|\mathring{h}|^2}|\nabla H|^2 \\
\le\ &-\frac{12\varepsilon}{|\mathring{h}|^2}|\nabla H|^2.
\end{aligned}$$

Next we estimate the expression in the second square bracket of the right hand side of (4.3). By Lemma 2.2, we have

$$
\begin{aligned}
& \frac{1}{|\mathring{h}|^2}\left(R_1-\frac{1}{n}R_2\right)-nc-(1-\sigma)\frac{\mathring{\alpha}'}{\mathring{\alpha}}(R_2+nc|H|^2) \\
\leq\ & |h|^2-nc-(1-\sigma)\frac{\mathring{\alpha}'}{\mathring{\alpha}}|H|^2(|h|^2+nc) \\
& +P_2\left[\frac{1}{|\mathring{h}|^2}\left(2|\mathring{h}|^2-\frac{1}{n}|H|^2\right)+(1-\sigma)\frac{\mathring{\alpha}'}{\mathring{\alpha}}|H|^2\right] \\
\leq\ & \sigma(|h|^2+nc)+(1-\sigma)(|h|^2+nc)\left(1-\frac{\mathring{\alpha}'}{\mathring{\alpha}}|H|^2\right)-2nc \\
& +P_2\left[2+\left(-\frac{1}{n}+\mathring{\alpha}'\right)\frac{|H|^2}{\mathring{\alpha}}\right].
\end{aligned}
\tag{4.5}
$$

From Lemma 3.2 (iii), we get $2+\left(-\frac{1}{n}+\mathring{\alpha}'\right)\frac{|H|^2}{\mathring{\alpha}}<0$ and $1-\frac{\mathring{\alpha}'}{\mathring{\alpha}}|H|^2<0$. Then we have

$$
\begin{aligned}
(|h|^2+nc)\left(1-\frac{\mathring{\alpha}'}{\mathring{\alpha}}|H|^2\right) &\leq \left(\frac{1}{n}|H|^2+nc\right)\left(1-\frac{\mathring{\alpha}'}{\mathring{\alpha}}|H|^2\right) \\
&= \left(\frac{(n-2)|H|^2}{\sqrt{|H|^4+4(n-1)c|H|^2}}+n\right)c \\
&< (2n-2)c.
\end{aligned}
$$

Therefore, the right hand side of (4.5) is less than

$$
\begin{aligned}
\sigma(|h|^2+nc)+(1-\sigma)(2n-2)c-2nc &= \sigma(|h|^2+(2-n)c)-2c \\
&< \sigma\left(\mathring{\alpha}+\frac{1}{n}|H|^2+(2-n)c\right)-2c \\
&< \frac{\sigma}{2}|H|^2-2c.
\end{aligned}
$$

This proves Lemma 4.2. □

To estimate the term $\sigma|H|^2 f_\sigma$ in Lemma 4.3, we need the following.

Lemma 4.3. *If a submanifold in $\mathbb{H}^{n+q}(c)$ satisfies $|\mathring{h}|^2<\mathring{\alpha}-\varepsilon\omega$ and $|H|^2+n^2c>0$, then we have*

$$\Delta|\mathring{h}|^2\geq 2\left\langle\mathring{h},\nabla^2H\right\rangle+\frac{\varepsilon}{2}|H|^2|\mathring{h}|^2.$$

Proof. From (2.1) and Lemma 2.1 (i), we only need to prove $W\geq\frac{\varepsilon}{4}H^2|\mathring{h}|^2$.

We work with the special local orthonormal frame, such that $\nu_1 = H/|H|$ and $\mathring{h}^1 = \mathrm{diag}\left(\mathring{\lambda}_1, \cdots, \mathring{\lambda}_n\right)$. Then we expand W to get

$$\begin{aligned} W &= nc|\mathring{h}|^2 - P_1^2 + \frac{1}{n}|\mathring{h}|^2|H|^2 \\ &\quad -2\sum_{\alpha>1}\left(\sum_i \mathring{\lambda}_i \mathring{h}^\alpha_{ii}\right)^2 - 2\sum_{\substack{\alpha>1\\ i\neq j}}\left(\left(\mathring{\lambda}_i - \mathring{\lambda}_j\right)\mathring{h}^\alpha_{ij}\right)^2 \\ &\quad -\sum_{\alpha,\beta>1}\left(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^\beta_{ij}\right)^2 - \sum_{\substack{\alpha,\beta>1\\ i,j}}\left(\sum_k\left(\mathring{h}^\alpha_{ik}\mathring{h}^\beta_{jk} - \mathring{h}^\alpha_{jk}\mathring{h}^\beta_{ik}\right)\right)^2 \\ &\quad +|H|\sum_{\alpha,i}\mathring{\lambda}_i\left(\mathring{h}^\alpha_{ii}\right)^2 + |H|\sum_{\substack{\alpha>1\\ i\neq j}}\mathring{\lambda}_i\left(\mathring{h}^\alpha_{ij}\right)^2. \end{aligned}$$

Using (2.3) and (2.5) again, we have

$$\sum_{\alpha>1}\left(\sum_i \mathring{\lambda}_i\mathring{h}^\alpha_{ii}\right)^2 \leqslant P_1 Q_1,$$

$$\sum_{\alpha,\beta>1}\left(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^\beta_{ij}\right)^2 + \sum_{\substack{\alpha,\beta>1\\ i,j}}\left(\sum_k\left(\mathring{h}^\alpha_{ik}\mathring{h}^\beta_{jk} - \mathring{h}^\alpha_{jk}\mathring{h}^\beta_{ik}\right)\right)^2 \leqslant \frac{3}{2}P_2^2.$$

By Proposition 2.3, we have

$$\begin{aligned} |H|\sum_{\alpha,i}\mathring{\lambda}_i\left(\mathring{h}^\alpha_{ii}\right)^2 &\geqslant -\tfrac{n-2}{\sqrt{n(n-1)}}|H|\sqrt{P_1}\left(|\mathring{h}|^2 - Q_2\right) \\ &\geqslant -\tfrac{n-2}{\sqrt{n(n-1)}}|H|\left(\frac{1}{2}\left(P_1 + |\mathring{h}|^2\right)|\mathring{h}| - \sqrt{P_1}Q_2\right) \\ &= -\tfrac{n-2}{\sqrt{n(n-1)}}|H|\left(|\mathring{h}|^3 - \frac{1}{2}|\mathring{h}|P_2 - \sqrt{P_1}Q_2\right). \end{aligned}$$

We estimate the rest terms as following.

$$\begin{aligned} &|H|\sum_{\substack{\alpha>1\\ i\neq j}}\mathring{\lambda}_i\left(\mathring{h}^\alpha_{ij}\right)^2 - 2\sum_{\substack{\alpha>1\\ i\neq j}}\left(\left(\mathring{\lambda}_i - \mathring{\lambda}_j\right)\mathring{h}^\alpha_{ij}\right)^2 \\ = &\sum_{\substack{\alpha>1\\ i\neq j}}\left[\frac{1}{2}|H|\left(\mathring{\lambda}_i + \mathring{\lambda}_j\right) + 2\left(\mathring{\lambda}_i + \mathring{\lambda}_j\right)^2 - 4\left(\mathring{\lambda}_i^2 + \mathring{\lambda}_j^2\right)\right]\left(\mathring{h}^\alpha_{ij}\right)^2 \\ \geqslant &\sum_{\substack{\alpha>1\\ i\neq j}}\left[\frac{1}{2}|H|\left(\mathring{\lambda}_i + \mathring{\lambda}_j\right) + \frac{3}{4}\left(\mathring{\lambda}_i + \mathring{\lambda}_j\right)^2 - 4P_1\right]\left(\mathring{h}^\alpha_{ij}\right)^2. \end{aligned}$$

Letting $y = \mathring{\lambda}_i + \mathring{\lambda}_j$, we have $y \geqslant -\sqrt{2(\mathring{\lambda}_i^2 + \mathring{\lambda}_j^2)} \geqslant -\sqrt{2P_1}$. By Lemma 3.2 (iii), we get $\sqrt{P_1} < \sqrt{\mathring{\alpha}} < \frac{|H|}{\sqrt{n(n-1)}}$. Then the function $\frac{3}{4}y^2 + \frac{1}{2}|H|y$ is increasing for $y \geqslant -\sqrt{2P_1}$. Thus $\frac{3}{4}y^2 + \frac{1}{2}|H|y \geqslant \frac{3}{2}P_1 - \frac{\sqrt{2}}{2}|H|\sqrt{P_1}$. We obtain

$$|H| \sum_{\substack{\alpha > 1 \\ i \neq j}} \mathring{\lambda}_i (\mathring{h}_{ij}^\alpha)^2 - 2 \sum_{\substack{\alpha > 1 \\ i \neq j}} ((\mathring{\lambda}_i - \mathring{\lambda}_j)\mathring{h}_{ij}^\alpha)^2 \geqslant -\left(\tfrac{\sqrt{2}}{2}|H|\sqrt{P_1} + \tfrac{5}{2}P_1\right) Q_2.$$

Applying these estimates, we get

$$\begin{aligned}
W &\geqslant nc|\mathring{h}|^2 - P_1^2 + \frac{1}{n}|\mathring{h}|^2|H|^2 - 2P_1Q_1 - \frac{3}{2}P_2^2 - \frac{5}{2}P_1Q_2 - \frac{\sqrt{2}}{2}|H|\sqrt{P_1}Q_2 \\
&\quad - \tfrac{n-2}{\sqrt{n(n-1)}}|H|\left(|\mathring{h}|^3 - \frac{1}{2}|\mathring{h}|P_2 - \sqrt{P_1}Q_2\right) \\
&= nc|\mathring{h}|^2 - |\mathring{h}|^4 + \frac{1}{n}|\mathring{h}|^2|H|^2 - \tfrac{n-2}{\sqrt{n(n-1)}}|H||\mathring{h}|^3 \\
&\quad + \frac{1}{2}\left(\tfrac{n-2}{\sqrt{n(n-1)}}|H||\mathring{h}|P_2 - P_2^2 - P_1Q_2\right) + \left(\tfrac{n-2}{\sqrt{n(n-1)}} - \frac{\sqrt{2}}{2}\right)|H|\sqrt{P_1}Q_2.
\end{aligned}$$

Since $P_2^2 + P_1Q_2 \leqslant |\mathring{h}|^2 P_2 \leqslant \frac{1}{\sqrt{n(n-1)}}|H||\mathring{h}|P_2$, we have

$$\begin{aligned}
W &\geqslant |\mathring{h}|^2\left(nc - |\mathring{h}|^2 + \frac{1}{n}|H|^2 - \tfrac{n-2}{\sqrt{n(n-1)}}|H||\mathring{h}|\right) \\
&\geqslant |\mathring{h}|^2\left(nc - (\mathring{\alpha} - \varepsilon\omega) + \frac{1}{n}|H|^2 - \tfrac{n-2}{\sqrt{n(n-1)}}|H|\sqrt{\mathring{\alpha}}\right) \\
&= |\mathring{h}|^2\varepsilon\omega \\
&\geqslant \frac{\varepsilon}{4}|H|^2|\mathring{h}|^2.
\end{aligned}$$

□

From (4.2), (4.4) and Lemma 4.3, we have

$$\begin{aligned}
\Delta f_\sigma &\geq \frac{f_\sigma \Delta|\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{f_\sigma}{\mathring{\alpha}}\Delta\mathring{\alpha} - 2(1-\sigma)\frac{\langle \nabla f_\sigma, \nabla\mathring{\alpha}\rangle}{\mathring{\alpha}} \\
&\geq \frac{2f_\sigma}{|\mathring{h}|^2}\left\langle \mathring{h}, \nabla^2 H\right\rangle + \frac{\varepsilon}{2}|H|^2 f_\sigma - (1-\sigma)\frac{f_\sigma}{\mathring{\alpha}}\Delta\mathring{\alpha} - \frac{2}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|.
\end{aligned}$$

This is equivalent to

$$\frac{\varepsilon}{2}|H|^2 f_\sigma \leq \Delta f_\sigma - \frac{2f_\sigma}{|\mathring{h}|^2}\left\langle \mathring{h}, \nabla^2 H\right\rangle + (1-\sigma)\frac{f_\sigma}{\mathring{\alpha}}\Delta\mathring{\alpha} + \frac{2}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|. \tag{4.6}$$

We multiply both sides of this inequality by f_σ^{p-1}, then integrate them over M_t. From the divergence theorem and the relation $\nabla_i \mathring{h}_{ij} = \frac{n-1}{n}\nabla_j H$, we have

$$\int_{M_t} f_\sigma^{p-1}\Delta f_\sigma \mathrm{d}\mu_t = -(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 \mathrm{d}\mu_t \leq 0, \tag{4.7}$$

$$
\begin{aligned}
-\int_{M_t}\frac{f_\sigma^p}{|\mathring{h}|^2}\left\langle\mathring{h},\nabla^2H\right\rangle\mathrm{d}\mu_t &= -\int_{M_t}\frac{f_\sigma^{p-1}}{\mathring{\alpha}^{1-\sigma}}\mathring{h}^\alpha_{ij}\nabla^2_{i,j}H^\alpha\mathrm{d}\mu_t\\
&= \int_{M_t}\nabla_i\left(\frac{f_\sigma^{p-1}}{\mathring{\alpha}^{1-\sigma}}\mathring{h}^\alpha_{ij}\right)\nabla_jH^\alpha\mathrm{d}\mu_t\\
&= \int_{M_t}\left[(p-1)\frac{f_\sigma^{p-2}}{\mathring{\alpha}^{1-\sigma}}\mathring{h}^\alpha_{ij}\nabla_if_\sigma\nabla_jH^\alpha\right.\\
&\quad\left.-(1-\sigma)\frac{f_\sigma^{p-1}}{\mathring{\alpha}^{2-\sigma}}\mathring{h}^\alpha_{ij}\nabla_i\mathring{\alpha}\nabla_jH^\alpha+\frac{n-1}{n}\frac{f_\sigma^{p-1}}{\mathring{\alpha}^{1-\sigma}}|\nabla H|^2\right]\mathrm{d}\mu_t\\
&\le \int_{M_t}\left[(p-1)\frac{f_\sigma^{p-2}}{\mathring{\alpha}^{1-\sigma}}|\mathring{h}||\nabla f_\sigma||\nabla H|\right.\\
&\quad\left.+\frac{f_\sigma^{p-1}}{\mathring{\alpha}^{2-\sigma}}|\mathring{h}|\,|\nabla\mathring{\alpha}|\,|\nabla H|+\frac{f_\sigma^{p-1}}{\mathring{\alpha}^{1-\sigma}}|\nabla H|^2\right]\mathrm{d}\mu_t\\
&\le \int_{M_t}\left[(p-1)\frac{f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|+\frac{2f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2\right]\mathrm{d}\mu_t
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
\int_{M_t}\frac{f_\sigma^p}{\mathring{\alpha}}\Delta\mathring{\alpha}\mathrm{d}\mu_t &= -\int_{M_t}\left\langle\nabla\left(\frac{f_\sigma^p}{\mathring{\alpha}}\right),\nabla\mathring{\alpha}\right\rangle\mathrm{d}\mu_t\\
&= \int_{M_t}\left(-\frac{pf_\sigma^{p-1}}{\mathring{\alpha}}\left\langle\nabla f_\sigma,\nabla\mathring{\alpha}\right\rangle+\frac{f_\sigma^p}{\mathring{\alpha}^2}\left|\nabla\mathring{\alpha}\right|^2\right)\mathrm{d}\mu_t\\
&\le \int_{M_t}\left(\frac{pf_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|+\frac{f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2\right)\mathrm{d}\mu_t.
\end{aligned}
\tag{4.9}
$$

Putting (4.6)-(4.9) together, we obtain

$$
\frac{\varepsilon}{2}\int_{M_t}|H|^2f_\sigma^p\mathrm{d}\mu_t\le\int_{M_t}\left(\frac{3pf_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|+\frac{5f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2\right)\mathrm{d}\mu_t.
\tag{4.10}
$$

From (4.10) and Lemma 4.2, we get an estimate for the time derivative of the integral of f_σ^p.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f_\sigma^p \mathrm{d}\mu_t &= p\int_{M_t} f_\sigma^{p-1}\frac{\partial f_\sigma}{\partial t}\mathrm{d}\mu_t - \int_{M_t} f_\sigma^p |H|^2 \mathrm{d}\mu_t \\
&\le p\int_{M_t}\Bigg[f_\sigma^{p-1}\Delta f_\sigma + \frac{2f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| \\
&\qquad - \frac{12\varepsilon f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2 - 4c f_\sigma^p + \sigma |H|^2 f_\sigma^p\Bigg]\mathrm{d}\mu_t \\
&\le p\int_{M_t}\Bigg[-(p-1)f_\sigma^{p-2}|\nabla f_\sigma|^2 + \frac{2f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| - \frac{12\varepsilon f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2 \\
&\qquad + \frac{6\sigma p}{\varepsilon}\frac{f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| + \frac{10\sigma}{\varepsilon}\frac{f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2 - 4cf_\sigma^p\Bigg]\mathrm{d}\mu_t \qquad (4.11) \\
&= p\int_{M_t} f_\sigma^{p-2}\Bigg[-(p-1)|\nabla f_\sigma|^2 + \left(2+\frac{6\sigma p}{\varepsilon}\right)\frac{f_\sigma}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| \\
&\qquad - \left(12\varepsilon - \frac{10\sigma}{\varepsilon}\right)\frac{f_\sigma^2}{|\mathring{h}|^2}|\nabla H|^2\Bigg]\mathrm{d}\mu_t - 4cp\int_{M_t} f_\sigma^p \mathrm{d}\mu_t.
\end{aligned}
$$

Now we show that the L^p-norm of f_σ is bounded.

Lemma 4.4. *There exists a constant C_1 depending only on M_0 such that for all $p \ge 1/\varepsilon$ and $\sigma \le \varepsilon^2/\sqrt{p}$, we have*

$$\left(\int_{M_t} f_\sigma^p \mathrm{d}\mu_t\right)^{\frac{1}{p}} < C_1.$$

Proof. The expression in the square bracket of the right hand side of (4.11) is a quadratic form. With ε small enough, its discriminant satisfies

$$
\begin{aligned}
&\left(2+\frac{6\sigma p}{\varepsilon}\right)^2 - 4(p-1)\left(12\varepsilon - \frac{10\sigma}{\varepsilon}\right) \\
< \;& (2+6\sqrt{p}\varepsilon)^2 - 24p\varepsilon \\
\le \;& 8 + 72p\varepsilon^2 - 24p\varepsilon \\
< \;& 0.
\end{aligned}
$$

Therefore, this quadratic form is non-positive. Now we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f_\sigma^p \mathrm{d}\mu_t \le -4cp\int_{M_t} f_\sigma^p \mathrm{d}\mu_t.$$

This implies $\int_{M_t} f_\sigma^p \mathrm{d}\mu_t \le e^{-4cpt}\int_{M_0} f_\sigma^p \mathrm{d}\mu_0$. Thus, the assertion follows from the finiteness of T. □

Corollary 4.5. *There exists a constant C_2 depending only on M_0 such that for all $r \ge 1$, $p \ge 4r^2/\varepsilon^4$ and $\sigma \le \frac{1}{2}\varepsilon^2/\sqrt{p}$, we have*

$$\left(\int_{M_t} |H|^{2r} f_\sigma^p \mathrm{d}\mu_t\right)^{\frac{1}{p}} < C_2.$$

Proof. Since the constant $C(\delta,n)=\sup\left\{\frac{|H|^2}{\mathring{\alpha}}\,\middle|\,|H|^2>-n^2c+\delta\right\}$ is finite, we have

$$\left(\int_{M_t}|H|^{2r}f_\sigma^p\mathrm{d}\mu_t\right)^{\frac{1}{p}}\le\left(\int_{M_t}(C(\delta,n)\mathring{\alpha})^r f_\sigma^p\mathrm{d}\mu_t\right)^{\frac{1}{p}}\le C(\delta,n)^{\frac{r}{p}}\left(\int_{M_t}f^p_{\sigma+\frac{r}{p}}\mathrm{d}\mu_t\right)^{\frac{1}{p}}.$$

With $r/p\le\frac{1}{2}\varepsilon^2/\sqrt{p}$ and $\sigma+r/p\le\varepsilon^2/\sqrt{p}$, the conclusion follows from Lemma 4.4. □

We are now in a position to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. For all $k>0$, we define $f_{\sigma,k}=\max\{f_\sigma-k,0\}$, $A(k)=\{x\in M_t\,|\,f_\sigma(x)>k\}$. From Lemma 4.2 and $p\ge1/\varepsilon$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t}f^p_{\sigma,k}\mathrm{d}\mu_t&\le p\int_{M_t}f^{p-1}_{\sigma,k}\left(\Delta f_\sigma+\tfrac{2|\nabla f_\sigma||\nabla H|}{|\mathring{h}|}-\tfrac{12\varepsilon f_\sigma|\nabla H|^2}{|\mathring{h}|^2}-4cf_\sigma+\sigma|H|^2f_\sigma\right)\mathrm{d}\mu_t\\&\le p\int_{M_t}\left[-(p-1)f^{p-2}_{\sigma,k}|\nabla f_\sigma|^2+\frac{2f^{p-1}_{\sigma,k}|\nabla f_\sigma||\nabla H|}{|\mathring{h}|}-\frac{12\varepsilon f^p_{\sigma,k}|\nabla H|^2}{|\mathring{h}|^2}\right]\mathrm{d}\mu_t\\&\quad+p\int_{A(k)}(-4c+\sigma|H|^2)f^p_\sigma\mathrm{d}\mu_t\\&\le-\frac{1}{2}p(p-1)\int_{M_t}f^{p-2}_{\sigma,k}|\nabla f_\sigma|^2\mathrm{d}\mu_t+p\int_{A(k)}(\sigma|H|^2-4c)f^p_\sigma\mathrm{d}\mu_t.\end{aligned}$$

We have $\frac{1}{2}p(p-1)f^{p-2}_{\sigma,k}|\nabla f_\sigma|^2\ge|\nabla f^{p/2}_{\sigma,k}|^2$. Putting $v=f^{p/2}_{\sigma,k}$, we get

$$\frac{\partial}{\partial t}\int_{M_t}v^2\mathrm{d}\mu_t+\int_{M_t}|\nabla v|^2\mathrm{d}\mu_t\le p\int_{A(k)}(\sigma|H|^2-4c)f^p_\sigma\mathrm{d}\mu_t.\tag{4.12}$$

From Theorem 2.1 of [16], we see that if u is a non-negative C^1-function on M_t, then the following Sobolev inequality holds: $(\int_{M_t}u^{\frac{n}{n-1}}\mathrm{d}\mu_t)^{\frac{n-1}{n}}\le C_3\int_{M_t}(|\nabla u|+u|H|)\mathrm{d}\mu_t$, where C_3 is a positive constant depending only on n. Replacing u by $v^{2(n-1)/(n-2)}$ and using Hölder's inequality, we obtain

$$\left(\int_{M_t}v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\le C_3\int_{M_t}|\nabla v|^2\mathrm{d}\mu_t+C_3\left(\int_{A(k)}|H|^n\mathrm{d}\mu_t\right)^{\frac{2}{n}}\left(\int_{M_t}v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}.$$

When $k\ge C_2(2C_3)^{\frac{n}{2p}}$, $p\ge n^2/\varepsilon^4$ and $\sigma\le\frac{1}{2}\varepsilon^2/\sqrt{p}$, it follows from Corollary 4.5 that $(\int_{A(k)}|H|^n\mathrm{d}\mu_t)^{\frac{2}{n}}\le(\int_{A(k)}|H|^nf^p_\sigma k^{-p}\mathrm{d}\mu_t)^{\frac{2}{n}}<\frac{1}{2C_3}$. Thus, we have

$$\frac{1}{2C_3}\left(\int_{M_t}v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\le\int_{M_t}|\nabla v|^2\mathrm{d}\mu_t.\tag{4.13}$$

It follows from (4.12) and (4.13) that

$$\frac{\partial}{\partial t}\int_{M_t}v^2\mathrm{d}\mu_t+\frac{1}{2C_3}\left(\int_{M_t}v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\le p\int_{A(k)}(\sigma|H|^2-4c)f^p_\sigma\mathrm{d}\mu_t.$$

Letting $k \geq \sup_{M_0} f_\sigma$, we have $v = 0$ at $t = 0$. Integrating the both sides of the above inequality, we get

$$\begin{aligned}\sup_{[0,T]}\int_{M_t} v^2 \mathrm{d}\mu_t \quad &+ \quad \frac{1}{2C_3}\int_0^T \left(\int_{M_t} v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\mathrm{d}t \\ &\leq \quad 2p\int_0^T\int_{A(k)}(\sigma|H|^2 - 4c)f_\sigma^p \mathrm{d}\mu_t \mathrm{d}t. \end{aligned}\tag{4.14}$$

Let $\|A(k)\| = \int_0^T \int_{A(k)} \mathrm{d}\mu_t \mathrm{d}t$. For a positive number $r > \frac{n+2}{2}$, let $p \geq 4r/\varepsilon^4$, $\sigma \leq \frac{1}{2}\varepsilon^2/\sqrt{pr}$. Applying Hölder's inequality and Lemma 4.4, we have

$$\int_0^T\int_{A(k)} f_\sigma^p \mathrm{d}\mu_t \mathrm{d}t \leq C_1^p T^{\frac{1}{r}}\|A(k)\|^{1-\frac{1}{r}}.\tag{4.15}$$

By Hölder's inequality and Corollary 4.5, we have

$$\int_0^T\int_{A(k)} |H|^2 f_\sigma^p \mathrm{d}\mu_t \mathrm{d}t \leq C_2^p T^{\frac{1}{r}}\|A(k)\|^{1-\frac{1}{r}}.\tag{4.16}$$

For $h > k$, we have $f_{\sigma,k} > h - k$ on $A(h)$. Thus

$$(h-k)^p\|A(h)\| \leq \int_0^T\int_{M_t} v^2 \mathrm{d}\mu_t \mathrm{d}t \leq \left(\int_0^T\int_{M_t} v^{\frac{2(n+2)}{n}}\mathrm{d}\mu_t \mathrm{d}t\right)^{\frac{n}{n+2}}\|A(k)\|^{\frac{2}{n+2}}.\tag{4.17}$$

We estimate the right hand side of the above inequality as follows.

$$\begin{aligned}&\left(\int_0^T\int_{M_t} v^{\frac{2(n+2)}{n}}\mathrm{d}\mu_t \mathrm{d}t\right)^{\frac{n}{n+2}} \\ \leq\quad & \left[\int_0^T\left(\int_{M_t} v^2\mathrm{d}\mu_t\right)^{\frac{2}{n}}\left(\int_{M_t} v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\mathrm{d}t\right]^{\frac{n}{n+2}} \\ \leq\quad & \left(\sup_{[0,T]}\int_{M_t} v^2\mathrm{d}\mu_t\right)^{\frac{2}{n+2}}\left[\int_0^T\left(\int_{M_t} v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\mathrm{d}t\right]^{\frac{n}{n+2}} \\ \leq\quad & \frac{2}{n+2}\sup_{[0,T]}\int_{M_t} v^2\mathrm{d}\mu_t + \frac{n}{n+2}\int_0^T\left(\int_{M_t} v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\mathrm{d}t \\ \leq\quad & C_4\left[\sup_{[0,T]}\int_{M_t} v^2\mathrm{d}\mu_t + \frac{1}{2C_3}\int_0^T\left(\int_{M_t} v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\mathrm{d}t\right].\end{aligned}\tag{4.18}$$

Putting inequalities (4.14)-(4.18) together, for all $h > k \geq \max\left\{C_2(2C_3)^{\frac{n}{2p}}, \sup_{M_0} f_\sigma\right\}$, we have

$$(h-k)^p\|A(h)\| \leq C_4\|A(k)\|^{1-\frac{1}{r}+\frac{2}{n+2}},$$

where C_4 is a positive constant depending on M_0, p and r.

By a lemma of [21] (Chapter II, Lemma B.1), there exists a finite number k_1, such that $\|A(k_1)\| = 0$. Therefore, the assertion follows from the definition of $A(k)$ and Lemma 3.2 (iii). This completes the proof of Theorem 4.1. □

5. A gradient estimate

To compare the mean curvature at different points of M_t, we need derive an estimate for the gradient of mean curvature.

Theorem 5.1. *For all $\eta \in \left(0, \frac{1}{n}\right)$, there exists a constant $C(\eta)$ independent of t, such that*

$$|\nabla H| < \eta^2 |H|^2 + C(\eta).$$

Firstly, we make an estimate for the time derivative of the gradient of mean curvature.

Lemma 5.2. *There exists a constant $B_1 > 1$ depending only on n, such that*

$$\frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta |\nabla H|^2 + B_1 |H|^2 |\nabla h|^2.$$

Proof. The evolution equation of H is

$$\nabla_{\partial_t} H = \Delta H + ncH + H^{\alpha} h^{\alpha}_{ij} h_{ij}. \tag{5.1}$$

Let u be a tangent vector field on M_t satisfying $[\partial_t, u] = 0$. Then we have $\nabla_{\partial_t} u = (\bar{\nabla}_{\partial_t} u)^{\top} = (\bar{\nabla}_u \partial_t)^{\top} = -\langle H, h(u, e_i)\rangle e_i$.

Using the timelike Ricci equation (see (16) of [2]), we obtain

$$\nabla_{\partial_t}(\nabla_u H) = \nabla_u(\nabla_{\partial_t} H) + \langle H, h(u, e_i)\rangle \nabla_i H - \langle H, \nabla_i H\rangle h(u, e_i).$$

Therefore, we have

$$\begin{aligned} \nabla_{\partial_t} \nabla H(u) &= \nabla_{\partial_t}(\nabla_u H) - \nabla H(\nabla_{\partial_t} u) \\ &= \nabla_u(\Delta H + ncH + H^{\alpha} h^{\alpha}_{ij} h_{ij}) \\ &\quad + 2\langle H, h(u, e_i)\rangle \nabla_i H - \langle H, \nabla_i H\rangle h(u, e_i). \end{aligned}$$

We use Hamilton's $*$ notation. For tensors T and S, $T * S$ means any linear combination of contractions of T and S with the metric. Then the above formula can be written as

$$\nabla_{\partial_t} \nabla H = \nabla \Delta H + nc \nabla H + h * h * \nabla h.$$

It follows from Ricci equation and Gauss equation that

$$\nabla \Delta H = \Delta \nabla H + (1 - n)c \nabla H + h * h * \nabla h.$$

Now we obtain the evolution equation of ∇H

$$\nabla_{\partial_t} \nabla H = \Delta \nabla H + c \nabla H + h * h * \nabla h. \tag{5.2}$$

Then

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla H|^2 &= 2\langle \nabla_{\partial_t} \nabla H, \nabla H\rangle \\ &= 2\langle \Delta \nabla H, \nabla H\rangle + 2c|\nabla H|^2 + h * h * \nabla h * \nabla h \\ &= \Delta |\nabla H|^2 - 2|\nabla^2 H|^2 + 2c|\nabla H|^2 + h * h * \nabla h * \nabla h. \end{aligned} \tag{5.3}$$

From the Cauchy-Schwarz inequality and the pinching condition, we have $|h * h * \nabla h * \nabla h| \leq B_1 |H|^2 |\nabla h|^2$. ∎

Secondly, we need the following estimates.

Lemma 5.3. *Along the mean curvature flow, we have*

(*i*) $\frac{\partial}{\partial t}|H|^4 \geq \Delta |H|^4 - 8n|H|^2|\nabla h|^2 + \frac{4}{n}|H|^6 + 4nc|H|^4$, *(ii)* $\frac{\partial}{\partial t}|\mathring{h}|^2 \le \Delta|\mathring{h}|^2 - |\nabla h|^2 + |H|^4$,

(iii) $\frac{\partial}{\partial t}\left(|H|^2|\mathring{h}|^2\right) \le \Delta\left(|H|^2|\mathring{h}|^2\right) - \frac{1}{2}|H|^2|\nabla h|^2 + B_2|\nabla h|^2 + C_0|H|^{6-2\sigma}$, *where* B_2 *is a positive constant.*

Proof. (i) From the evolution equation we derive that

$$\frac{\partial}{\partial t}|H|^4 = \Delta|H|^4 - 4|H|^2|\nabla H|^2 - 2|\nabla|H|^2|^2 + 4|H|^2R_2 + 4nc|H|^4.$$

Then from Lemma 2.1 and 2.2, we have $R_2 > \frac{1}{n}|H|^4$ and $4|H|^2|\nabla H|^2 + 2|\nabla|H|^2|^2 \le 8n|H|^2|\nabla h|^2$.

(ii) The evolution equation of $|\mathring{h}|^2$ is

$$\frac{\partial}{\partial t}|\mathring{h}|^2 = \Delta|\mathring{h}|^2 - 2|\nabla h|^2 + \frac{2}{n}|\nabla H|^2 + 2R_1 - \frac{2}{n}R_2 - 2nc|\mathring{h}|^2.$$

Then from $\frac{2}{n}|\nabla H|^2 \le |\nabla h|^2$ and $R_1 - \frac{1}{n}R_2 - nc|\mathring{h}|^2 \le |\mathring{h}|^2(|h|^2 - nc) < \frac{1}{2}|H|^4$ we get the conclusion.

(iii) It follows from the evolution equations that

$$\begin{aligned}\frac{\partial}{\partial t}\left(|H|^2|\mathring{h}|^2\right) &= \Delta\left(|H|^2|\mathring{h}|^2\right) + 2|H|^2\left(R_1 - \frac{1}{n}R_2\right) + 2|\mathring{h}|^2R_2\\ &\quad -2|H|^2\left(|\nabla h|^2 - \frac{1}{n}|\nabla H|^2\right) - 2|\mathring{h}|^2|\nabla H|^2 - 2\left\langle\nabla|H|^2, \nabla|\mathring{h}|^2\right\rangle.\end{aligned}$$

We use $\frac{2}{n}|\nabla H|^2 \le |\nabla h|^2$ again. Then by Theorem 4.1 we have

$$2|H|^2\left(R_1 - \frac{1}{n}R_2\right) + 2|\mathring{h}|^2R_2 \le 4|H|^2|h|^2|\mathring{h}|^2 < C_0|H|^{6-2\sigma}.$$

From the formula $\nabla_i|\mathring{h}|^2 = 2\mathring{h}^\alpha_{jk}\nabla_i h^\alpha_{jk}$ and Young's inequality, we get

$$\begin{aligned}-2\left\langle\nabla|H|^2, \nabla|\mathring{h}|^2\right\rangle &\le 8|H||\nabla H||\mathring{h}||\nabla h|\\ &\le 8\sqrt{\tfrac{n+2}{3}C_0}|H|^{2-\sigma}|\nabla h|^2\\ &\le \left(B_2 + \frac{1}{2}|H|^2\right)|\nabla h|^2.\end{aligned}$$

This proves Lemma 5.3. □

Now we can prove Theorem 5.1.

Proof of Theorem 5.1. We define a function on M.

$$f = |\nabla H|^2 - \eta^4|H|^4 + 4B_1|H|^2|\mathring{h}|^2 + 4B_1B_2|\mathring{h}|^2, \quad 0 < \eta < \frac{1}{n}.$$

From Lemmas 5.2 and 5.3, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)f &\le B_1|H|^2|\nabla h|^2 - \eta^4\left(-8n|H|^2|\nabla h|^2 + \frac{4}{n}|H|^6 + 4nc|H|^4\right)\\ &\quad +4B_1\left(-\frac{1}{2}|H|^2|\nabla h|^2 + B_2|\nabla h|^2 + C_0|H|^{6-2\sigma}\right)\\ &\quad +4B_1B_2(-|\nabla h|^2 + |H|^4)\\ &\le -\eta^4\left(\frac{4}{n}|H|^6 + 4nc|H|^4\right) + 4B_1C_0|H|^{6-2\sigma} + 4B_1B_2|H|^4.\end{aligned}$$

We consider the last line of the above inequality, which is a function of $|H|$. Since the coefficient of the highest-degree term is negative, the supremum $C_2(\eta)$ of this function is finite. Then we have

$$\frac{\partial}{\partial t}f < \Delta f + C_2(\eta).$$

It's seen from the maximum principle that f is bounded. This completes the proof of Theorem 5.1. $\square$

6. Convergence

To show that M_t converges to a point, we derive a lower bound for the Ricci curvature.

Lemma 6.1. *If M is a submanifold in $\mathbb{H}^{n+q}(c)$ satisfying $|\mathring{h}|^2 < \mathring{\alpha} - \varepsilon\omega$ and $|H|^2 + n^2c > 0$, then for all unit vector X in the tangent space, the Ricci curvature satisfies*

$$\mathrm{Ric}(X) \geq \frac{n-1}{4n}\varepsilon|H|^2.$$

Proof. Using Proposition 2 in [32] and Lemma 3.2 (ii), we have

$$\begin{aligned}
\mathrm{Ric}(X) &\geq \frac{n-1}{n}\left(nc + \frac{2}{n}|H|^2 - |h|^2 - \frac{n-2}{\sqrt{n(n-1)}}|H||\mathring{h}|\right)\\
&> \frac{n-1}{n}\left(nc + \frac{1}{n}|H|^2 - (\mathring{\alpha} - \varepsilon\omega) - \frac{n-2}{\sqrt{n(n-1)}}|H|\sqrt{\mathring{\alpha}}\right)\\
&= \frac{n-1}{n}\varepsilon\omega\\
&> \frac{n-1}{4n}\varepsilon|H|^2.
\end{aligned}$$

$\square$

To estimate the diameter of M_t, we need the well-known Myers theorem.

Theorem 6.2 (Myers). *Let γ be a geodesic of length at least $\pi/\sqrt{k}$ on M. If the Ricci curvature satisfies $\mathrm{Ric}(X) \geq (n-1)k$ for all $x \in \gamma$, all unit vector $X \in T_xM$, then γ has conjugate points.*

Let $|H|_{\min} = \min_{M_t}|H|$, $|H|_{\max} = \max_{M_t}|H|$. To show the flow converges to a round point, we need the following lemma.

Lemma 6.3. *As $t \to T$, we have $\operatorname{diam} M_t \to 0$ and $|H|_{\min}/|H|_{\max} \to 1$.*

Proof. Theorem 5.1 asserts $|\nabla H| < \eta^2|H|^2 + C(\eta)$ for $0 < \eta < \frac{1}{n}$. Since $|H|_{\max} \to \infty$ as $t \to T$, there exists a time $\tau(\eta)$, such that for $t > \tau(\eta)$, $|H|^2_{\max} > C(\eta)/\eta^2$. Then we have $|\nabla H| < 2\eta^2|H|^2_{\max}$.

At a time $t > \tau(\eta)$, let x be a point on M_t where $|H|$ achieves its maximum. Then along all geodesics of length $l = (2\eta|H|_{\max})^{-1}$ starting from x, we have $|H| > |H|_{\max} - |\nabla H|\cdot l > (1-\eta)|H|_{\max}$. With η small enough, Lemma 6.1 implies $\mathrm{Ric} > \frac{n-1}{4n}\varepsilon(1-\eta)^2|H|^2_{\max} > (n-1)\pi^2/l^2$ on these geodesics. Then from Myers' theorem, these geodesics can reach any point of M_t.

Thus we have $|H|_{\min} > (1-\eta)|H|_{\max}$ and $\operatorname{diam} M_t \leq (2\eta|H|_{\max})^{-1}$ for $t \in (\tau(\eta), T)$. This proves the lemma. $\square$

Now we are in a position to complete the proof of Main Theorem.

Proof of Main Theorem. To prove the flow converges to a round point, we magnify the metric of the ambient space such that the submanifold maintains its volume along the flow. Using the same argument as in [28], we can prove that the rescaled mean curvature flow converges to a totally umbilical sphere as the reparameterized time tends to infinity. This completes the proof of Main Theorem. □

7. Final Remarks

Since $\mathring{\alpha} \to 0$ as $|H|^2 \to -n^2c$, we see that if a connected submanifold satisfies $\sup\left(|\mathring{h}|^2 - \mathring{\alpha}\right) < 0$, then either $|H|^2 > -n^2c$ or $|H|^2 < -n^2c$. Therefore, the condition in Main Theorem can be relaxed.

Theorem 7.1. *Let $F_0 : M^n \to \mathbb{H}^{n+q}(c)$ be an n-dimensional ($n \geq 6$) complete submanifold immersed in the hyperbolic space with constant curvature c. Suppose*

$$\sup_{F_0}(|h|^2 - \alpha(n, |H|, c)) < 0$$

and there exists a point x on F_0 such that $|H(x)|^2 > -n^2c$. Then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0, T) \to \mathbb{H}^{n+q}(c)$ on a finite maximal time interval, and F_t converges to a round point as $t \to T$. In particular, M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

For the compact submanifolds, we have the following convergence theorem under the weakly pinching condition.

Theorem 7.2. *Let $F_0 : M^n \to \mathbb{H}^{n+q}(c)$ be an n-dimensional ($n \geq 6$) closed submanifold immersed in the hyperbolic space with constant curvature c. If F_0 satisfies*

$$|h|^2 \leq \alpha(n, |H|, c) \quad \text{and} \quad |H|^2 + n^2c > 0,$$

then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0, T) \to \mathbb{H}^{n+q}(c)$ on a finite maximal time interval, and F_t converges to a round point as $t \to T$. In particular, M is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.

Proof. By the continuity, there exists $t_0 > 0$ such that $|H|^2 + n^2c > 0$ remains true for $t \in [0, t_0]$. Similar to the proof of Theorem 3.3, we obtain the following estimate on the time interval $[0, t_0]$.

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\left(|\mathring{h}|^2 - \mathring{\alpha}\right) \\
\leq\ & \frac{2(n+2)}{3}\left[-\frac{2(n-1)}{n(n+2)} + \mathring{\alpha}' + 2|H|^2\mathring{\alpha}''\right]|\nabla h|^2 \\
&+2P_2\left[2\mathring{\alpha} - \frac{1}{n}|H|^2 + |H|^2\mathring{\alpha}'\right] \\
&+2\left(|\mathring{h}|^2 - \mathring{\alpha}\right)\left(2\mathring{\alpha} + \frac{1}{n}|H|^2 - nc - \mathring{\alpha}'|H|^2 + 2P_2\right) + 2\left(|\mathring{h}|^2 - \mathring{\alpha}\right)^2.
\end{aligned} \tag{7.1}$$

The expressions in the two square brackets of (7.1) are negative. Thus the weak maximum principle implies $|\mathring{h}|^2 \leq \mathring{\alpha}$ is preserved for $t \in [0, t_0]$.

Then we use the strong maximum principle. If $\sup_{M_{t_0}}\left(|\mathring{h}|^2-\mathring{\alpha}\right)=0$, then $|\mathring{h}|^2\equiv\mathring{\alpha}$ holds for all $t\in[0,t_0]$. So, we have $|\nabla h|^2\equiv P_2\equiv 0$ for $t\in[0,t_0]$. By a theorem due to Erbacher [11], M_0 lies in an $(n+1)$-dimensional totally geodesic submanifold of $\mathbb{H}^{n+q}(c)$. It follows from Theorem 4 of [22] and compactness of M_0 that M_0 is a totally umbilical sphere. This contradicts $|\mathring{h}|^2\equiv\mathring{\alpha}>0$.

Therefore, we have $\sup_{M_{t_0}}\left(|\mathring{h}|^2-\mathring{\alpha}\right)<0$. The assertion follows from Main Theorem. □

Finally we discuss the convergence and sphere theorems in low dimensions. In the case where $n\le 5$, $\mathring{\alpha}$ does not possesses all the properties in Lemma 3.2, which prevents us from getting the same convergence result as Main Theorem.

When $n=5$, we obtain a convergence result for the mean curvature flow, which improves the corresponding result in the convergence theorem due to Liu-Xu-Ye-Zhao [27].

Set

$$\beta(x)=\tfrac{5}{11}c+\tfrac{15}{88}x+\tfrac{\sqrt{7}}{88}\sqrt{7x^2+272cx+4000c^2}.$$

Then we obtain the following convergence result.

Proposition 7.3. *Let $F_0:M^5\to\mathbb{H}^{5+q}(c)$ be a complete submanifold immersed in the hyperbolic space. Suppose F_0 satisfies $\sup_{F_0}(|h|^2-\beta(|H|^2))<0$ and $|H|^2>-25c$. Then the mean curvature flow with initial value F_0 has a unique smooth solution on a finite maximal time interval, and the solution F_t converges to a round point.*

Let $\mathring{\beta}(x)=\beta(x)-\frac{x}{5}$. By direct computations, for $x>-25c$, we have:

(i) $\max\left\{\frac{x}{20}+2c,0\right\}<\mathring{\beta}(x)<\mathring{\alpha}(x)$,

(ii) $\mathring{\beta}(x)\cdot\left(\mathring{\beta}(x)+\frac{x}{5}-5c\right)<|H|^2\mathring{\beta}'(x)\cdot\left(\mathring{\beta}(x)+\frac{x}{5}+5c\right)$,

(iii) $2\sqrt{x}\mathring{\beta}'(x)<\sqrt{\mathring{\beta}(x)}$,

(iv) $2x\mathring{\beta}''(x)+\mathring{\beta}'(x)<\frac{8}{35}$.

With these properties, we can prove Proposition 7.3 as the same as the previous parts of this paper. Notice that $\frac{1}{4}|H|^2+2c<\beta(|H|^2)$. Hence Proposition 7.3 improves Theorem A for $n=5$.

When $n=4$, let M be a 4-dimensional oriented, simply connected and complete submanifold immersed in $\mathbb{H}^{4+q}(c)$. Suppose that M satisfies $\sup_M(|h|^2-\alpha(4,|H|,c))<0$ and $|H|^2>-16c$. This pinching condition implies M is a closed submanifold satisfying $|h|^2<4c+\frac{1}{2}|H|^2$. From Theorem 4.1 of [41], we get that M has positive isotropic curvature and is diffeomorphic to the standard 4-sphere.

When $n=3$, let M be a 3-dimensional oriented complete submanifold immersed in $\mathbb{H}^{3+q}(c)$. Suppose M satisfies $|h|^2<9c+\frac{3}{4}|H|^2-\frac{1}{4}\sqrt{|H|^4+24c|H|^2}$ and $|H|^2>-27c$. Then M is diffeomorphic to a spherical space form. This proposition is the 3-dimensional case of Theorem 1.2 of [12].

For the case $c>0$, the following conjecture proposed by Liu-Xu-Ye-Zhao [26] is still open up to now.

Conjecture B. *Let M_0 be a complete submanifold immersed in a sphere $\mathbb{S}^{n+q}(1/\sqrt{c})$. Suppose that $\sup_{M_0}(|h|^2-\alpha(n,|H|,c))<0$. Then the mean curvature flow with* *initial value M_0 converges to a round point in finite time, or converges to a totally geodesic sphere as $t \to \infty$. In particular, M_0 is diffeomorphic to the standard n-sphere $\mathbb{S}^n$.*

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