A NEW VERSION OF HUISKEN'S CONVERGENCE THEOREM FOR MEAN CURVATURE FLOW IN SPHERES

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Abstract. We prove that if the initial hypersurface of the mean curvature flow in spheres satisfies a sharp pinching condition, then the solution of the flow converges to a round point or a totally geodesic sphere. Our result improves the famous convergence theorem due to Huisken [9]. Moreover, we prove a convergence theorem under the weakly pinching condition. In particular, we obtain a classification theorem for weakly pinched hypersurfaces. It should be emphasized that our pinching condition implies that the Ricci curvature of the initial hypersurface is positive, but does not imply positivity of the sectional curvature.

1. Introduction

Let $F_0 : M^n \to N^{n+1}$ be an n-dimensional immersed hypersurface in a Riemannian manifold. The mean curvature flow with initial value F_0 is a smooth family of immersions $F : M \times [0, T) \to N^{n+1}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = \vec{H}(x,t), \\ F(\cdot,0) = F_0, \end{cases} \tag{1.1}$$

where $\vec{H}(x,t)$ is the mean curvature vector of the hypersurface $M_t = F_t(M)$, $F_t = F(\cdot, t)$.

In 1984, Huisken [7] first proved that compact uniformly convex hypersurfaces in Euclidean space will converge to a round point along the mean curvature flow. Afterwards, Huisken[9] verified the following important convergence result for mean curvature flow of pinched hypersurfaces in spheres.

Theorem A. *Let $F_0 : M^n \to \mathbb{F}^{n+1}(c)$ be an n-dimensional ($n \geqslant 3$) closed hypersurface immersed in a spherical space form of sectional curvature c. If F_0 satisfies $|h|^2 < \frac{1}{n-1}H^2 + 2c$, then the mean curvature flow with initial value F_0 converges to a round point in finite time, or converges to a totally geodesic hypersurface as $t \to \infty$.*

For any fixed positive constant ε, Huisken [9] constructed examples to show that the pinching condition above can not be improved to $|h|^2 < \frac{1}{n-1}H^2 + 2c + \varepsilon$. An attractive problem is: Is it possible to improve Huisken's pinching condition? During the past nearly three decades, there has been no progress on this aspect. For more convergence results of mean curvature flow, we refer the readers to [2, 3, 8, 12, 13, 16, 17, 18].

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Let M be an n-dimensional hypersurface in a space form $\mathbb{F}^{n+1}(c)$ with constant curvature c. Set

$$\alpha(n, H, c) := nc + \frac{n}{2(n-1)}H^2 - \frac{n-2}{2(n-1)}\sqrt{H^4 + 4(n-1)cH^2}. \tag{1.2}$$

Based on the pioneering work on closed minimal submanifolds in a sphere due to Simons [22], Lawson [11] and Chern-do Carmo-Kobayashi [5] proved the famous rigidity theorem for n-dimensional closed minimal submanifolds in $\mathbb{S}^{n+q}$ whose squared norm of the second fundamental form satisfies $|h|^2 \leqslant n/(2-1/q)$. After the work due to Okumura [19, 20] and Yau [25], the second author [23] verified the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in $\mathbb{S}^{n+q}$. In particular, Cheng-Nakagawa [4] and Xu [23] got the following rigidity theorem for constant mean curvature hypersurfaces, independently.

Theorem B. *Let M^n be a compact hypersurface with constant mean curvature in $\mathbb{S}^{n+1}(1/\sqrt{c})$. If $|h|^2 \leqslant \alpha(n, H, c)$, then either M is the totally umbilical sphere $\mathbb{S}^n\left(n/\sqrt{H^2+n^2c}\right)$, one of the Clifford torus $\mathbb{S}^k\left(\sqrt{k/(nc)}\right)\times\mathbb{S}^{n-k}\left(\sqrt{(n-k)/(nc)}\right)$, $1 \leqslant k \leqslant n-1$, or the isoparametric hypersurface $\mathbb{S}^{n-1}\left(1/\sqrt{c+\lambda^2}\right)\times\mathbb{S}^1\left(\lambda/\sqrt{c^2+c\lambda^2}\right)$, where $\lambda = \frac{|H|+\sqrt{H^2+4(n-1)c}}{2(n-1)}$.*

For the refined rigidity results in higher codimensions, we refer the readers to [14, 24]. Motivated by Theorem B, the optimal topological sphere theorem due to Shiohama-Xu [21] and Andrews' suggestion on the nonlinear parabolic flow [1], we have the following conjecture (see [16]).

Conjecture C. *Let $F_0 : M^n \to \mathbb{S}^{n+1}(1/\sqrt{c})$ be an n-dimensional closed hypersurface satisfying $|h|^2 < \alpha(n, H, c)$. Then the mean curvature flow with initial value F_0 converges to a round point in finite time or converges to a totally geodesic sphere as $t \to \infty$.*

In particular, noting that $\min_H \alpha(n, H, 1) = 2\sqrt{n-1}$, we have the following.

Conjecture D. *Let $F_0 : M^n \to \mathbb{S}^{n+1}(1)$ be an n-dimensional closed hypersurface satisfying $|h|^2 < 2\sqrt{n-1}$. Then the mean curvature flow with initial value F_0 converges to a round point or a totally geodesic sphere.*

In [15], Li, Xu and Zhao investigated the conjectures above, and proved the following convergence result for the mean curvature flow in a sphere.

Theorem E. *Let $F_0 : M^n \to \mathbb{S}^{n+1}(1)$ be an n-dimensional ($n \geqslant 3$) closed hypersurface immersed in the unit sphere. If there exists a positive constant $\varepsilon < \frac{1}{8}$, such that F_0 satisfies*

$$|h|^2 \leqslant n - 4\varepsilon + \frac{n}{2(n-1+\varepsilon)}H^2 - \frac{n-2}{2(n-1+\varepsilon)}\sqrt{H^4+4(n-1)H^2} \quad \text{and} \quad H \geqslant \frac{n}{\sqrt[4]{\varepsilon}},$$

then the mean curvature flow with initial value F_0 converges to a round point in finite time.

The purpose of the present paper is to prove a sharp convergence theorem for the mean curvature flow of hypersurfaces in spherical space forms, which is a refined version of the famous convergence theorem due to Huisken [9].

Theorem 1.1. *Let $F_0 : M^n \to \mathbb{F}^{n+1}(c)$ be an n-dimensional ($n \geqslant 3$) closed hypersurface immersed in a spherical space form. If F_0 satisfies*

$$|h|^2 < \gamma(n, H, c),$$

then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0, T) \to \mathbb{F}^{n+1}(c)$, and F_t converges to a round point in finite time, or converges to a totally geodesic hypersurface as $t \to \infty$. Here $\gamma(n, H, c)$ is an explicit positive scalar defined by

$$\gamma(n, H, c) = \min\{\alpha(H^2), \beta(H^2)\},$$

where

$$\alpha(x) = nc + \frac{n}{2(n-1)}x - \frac{n-2}{2(n-1)}\sqrt{x^2 + 4(n-1)cx},$$

$$\beta(x) = \alpha(x_0) + \alpha'(x_0)(x - x_0) + \frac{1}{2}\alpha''(x_0)(x - x_0)^2,$$

$$x_0 = y_n c, \quad y_n = 4(1-n) + \frac{2(n^2-4)}{\sqrt{2n-5}}\cos\left(\frac{1}{3}\arctan\frac{n^2-4n+6}{2(n-1)\sqrt{2n-5}}\right).$$

Remark 1. *Notice that $\gamma(n, H, c) > \frac{1}{n-1}H^2 + 2c$ and $\sqrt{8}n^2 > y_n$. Furthermore, a computation shows that $\gamma(n, H, c) > \frac{9}{5}\sqrt{n-1}c$. Therefore, Theorem 1.1 substantially improves Theorem A as well as Theorem E.*

Put $\alpha_1(x) = n + \frac{n}{2(n-1)}x - \frac{n-2}{2(n-1)}\sqrt{x^2 + 4(n-1)x}$. As a consequence of Theorem 1.1, we obtain the following convergence result.

Theorem 1.2. *Let $F_0 : M^n \to \mathbb{F}^{n+1}(c)$ be an n-dimensional ($n \geqslant 3$) closed hypersurface immersed in a spherical space form. If F_0 satisfies*

$$|h|^2 < k_n c,$$

then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0, T) \to \mathbb{F}^{n+1}(c)$, and F_t converges to a round point in finite time, or converges to a totally geodesic hypersurface as $t \to \infty$. Here k_n is an explicit positive constant defined by

$$k_n = \begin{cases} \alpha_1(y_n) - \alpha_1'(y_n)y_n + \frac{1}{2}\alpha_1''(y_n)y_n^2, & n = 3, \\ \alpha_1(y_n) - \frac{\alpha_1'(y_n)^2}{2\alpha_1''(y_n)}, & n \geqslant 4. \end{cases}$$

Remark 2. *By a computation, we have $k_n > \frac{9}{5}\sqrt{n-1}$. In particular, if $5 \leqslant n \leqslant 10$, then $k_n > 1.999\sqrt{n-1}$. In fact, $k_{10} = 6$. This shows that the pinching constant $k_n c$ in Theorem 1.2 is sharp.*

Corollary 1.3. *Let $F_0 : M^n \to \mathbb{F}^{n+1}(c)$ be an n-dimensional ($n \geqslant 3$) closed hypersurface immersed in a spherical space form. If F_0 satisfies*

$$|h|^2 \leqslant \frac{9}{5}\sqrt{n-1}c,$$

then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0, T) \to \mathbb{F}^{n+1}(c)$, and F_t converges to a round point in finite time, or converges to a totally geodesic hypersurface as $t \to \infty$.

If the ambient space is a sphere, we have the sharp differentiable sphere theorem.

Theorem 1.4. *Let M_0 be an n-dimensional ($n \geqslant 3$) closed hypersurface immersed in $\mathbb{S}^{n+1}(1/\sqrt{c})$ which satisfies $|h|^2 < \gamma(n,H,c)$, then M_0 is diffeomorphic to the standard n-sphere $\mathbb{S}^n$. In particular, if M_0 satisfies $|h|^2 < k_n c$, then M_0 is diffeomorphic to $\mathbb{S}^n$.*

Furthermore, for compact hypersurfaces in spheres, we have the following convergence theorem for the mean curvature flow under the weakly pinching condition.

Theorem 1.5. *Let $F_0 : M^n \to \mathbb{S}^{n+1}(1/\sqrt{c})$ be an n-dimensional ($n \geqslant 3$) closed hypersurface immersed in a sphere. If F_0 satisfies*

$$|h|^2 \leqslant \gamma(n,H,c),$$

then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0,T) \to \mathbb{S}^{n+1}(1/\sqrt{c})$, and either

(*i*) *T is finite, and F_t converges to a round point as $t \to T$,*
(*ii*) *$T = \infty$, and F_t converges to a totally geodesic sphere as $t \to \infty$, or*
(*iii*) *T is finite, M_t is congruent to $\mathbb{S}^{n-1}(r_1(t)) \times \mathbb{S}^1(r_2(t))$, where $r_1(t)^2 + r_2(t)^2 = 1/c$, $r_1(t)^2 = \frac{n-1}{nc}(1 - \mathrm{e}^{2nc(t-T)})$, and F_t converges to a great circle as $t \to T$.*

Theorem 1.5 (iii) shows that our pinching conditions in Theorem 1.5 is sharp. As a consequence of Theorem 1.5, we have the following classification theorem.

Corollary 1.6. *Let M_0 be an n-dimensional ($n \geqslant 3$) closed hypersurface immersed in $\mathbb{S}^{n+1}(1/\sqrt{c})$ which satisfies $|h|^2 \leqslant \gamma(n,H,c)$. Then M_0 is either diffeomorphic to the standard n-sphere $\mathbb{S}^n$, or congruent to $\mathbb{S}^{n-1}(r_1) \times \mathbb{S}^1(r_2)$, where $r_1^2 + r_2^2 = 1/c$ and $r_1^2 < \frac{n-1}{nc}$.*

From the fact that $k_{10} = 6$, we get the following optimal convergence result.

Corollary 1.7. *Let $F_0 : M \to \mathbb{S}^{11}(1/\sqrt{c})$ be a 10-dimensional closed hypersurface which satisfies $|h|^2 \leqslant 6c$. Then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0,T) \to \mathbb{S}^{11}(1/\sqrt{c})$, and either*

(*i*) *T is finite, and F_t converges to a round point as $t \to T$,*
(*ii*) *$T = \infty$, and F_t converges to a totally geodesic sphere as $t \to \infty$, or*
(*iii*) *T is finite, M_t is congruent to $\mathbb{S}^9(r_1(t)) \times \mathbb{S}^1(r_2(t))$, where $r_1(t)^2 + r_2(t)^2 = 1/c$, $r_1(t)^2 = \frac{9}{10c}(1 - \frac{1}{6}\mathrm{e}^{20ct})$, and F_t converges to a great circle as $t \to T$.*

In particular, M_0 is diffeomorphic to $\mathbb{S}^{10}$, or congruent to $\mathbb{S}^9\left(\sqrt{\frac{3}{4c}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{1}{4c}}\right)$.

2. Preservation of curvature pinching

Let $F : M^n \times [0,T) \to \mathbb{F}^{n+1}(c)$ be a mean curvature flow in a spherical space form. Let $(x^1, \cdots, x^n)$ be the local coordinates of an open neighborhood in M. We consider the hypersurface M_t at time t. In the local coordinates, the first fundamental form of M_t can be written as symmetric matrix (g_{ij}). Denote by (g^{ij}) the inverse matrix of (g_{ij}). With the suitable choice of the unit normal vector field, the second fundamental form of M_t can be written as symmetric matrix (h_{ij}). We adopt the Einstein summation convention. Let $h_i^j = g^{jk}h_{ik}$ and $h^{ij} = g^{ik}g^{jl}h_{kl}$. Then the mean curvature and the squared norm of the second fundamental form of M_t can be written as $H = h_i^i$ and $|h|^2 = h^{ij}h_{ij}$, respectively. Let $\mathring{h}_{ij} = h_{ij} - \frac{H}{n}g_{ij}$ be the traceless second fundamental form, whose squared norm satisfies $|\mathring{h}|^2 = |h|^2 - \frac{H^2}{n}$.

Similar to Lemma 1.2 of [9], the gradient and Laplacian of the second fundamental form have the following properties.

Lemma 2.1. *For any hypersurface of $\mathbb{F}^{n+1}(c)$, we have*

(i) $|\nabla h|^2 \geqslant \frac{3}{n+2}|\nabla H|^2$,

(ii) $\Delta|h|^2 = 2\langle h, \nabla^2 H\rangle + 2|\nabla h|^2 + 2W$, *where* $W = Hh_i^j h_j^k h_k^i - |h|^4 + nc|\mathring{h}|^2$.

The evolution equations take the same form as in [9].

Lemma 2.2. *For the mean curvature flow* $F : M \times [0,T) \to \mathbb{F}^{n+1}(c)$, *we have*

(i) $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}$,

(ii) $\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2Hh_{ik}h_j^k + |h|^2 h_{ij} + 2cHg_{ij} - nch_{ij}$,

(iii) $\frac{\partial}{\partial t} H = \Delta H + H(|h|^2 + nc)$,

(iv) $\frac{\partial}{\partial t}|h|^2 = \Delta|h|^2 - 2|\nabla h|^2 + 4cH^2 + 2|h|^4 - 2nc|h|^2$,

(v) $\frac{\partial}{\partial t}|\mathring{h}|^2 = \Delta|\mathring{h}|^2 - 2|\nabla h|^2 + \frac{2}{n}|\nabla H|^2 + 2|\mathring{h}|^2(|h|^2 - nc)$.

We define

$$\alpha(x) = nc + \frac{n}{2(n-1)}x - \frac{n-2}{2(n-1)}\sqrt{x^2 + 4(n-1)cx}, \quad x \geqslant 0,$$

and

$$\beta(x) = \alpha(x_0) + \alpha'(x_0)(x - x_0) + \frac{1}{2}\alpha''(x_0)(x - x_0)^2, \quad x \geqslant 0,$$

where

$$x_0 = y_n c, \quad y_n = 4(1-n) + \frac{2(n^2-4)}{\sqrt{2n-5}}\cos\left(\frac{1}{3}\arctan\frac{n^2-4n+6}{2(n-1)\sqrt{2n-5}}\right).$$

Then we set

$$\gamma(x) = \begin{cases} \alpha(x), & x \geqslant x_0, \\ \beta(x), & 0 \leqslant x < x_0. \end{cases} \tag{2.1}$$

It is easy to see that γ is a C^2-function on $[0, +\infty)$. Moreover, we obtain the following.

Lemma 2.3. *For* $n \geqslant 3$, $c > 0$ *and* $x \geqslant 0$, *the* C^2*-function* $\gamma(x)$ *satisfies*

(i) $2x\gamma''(x) + \gamma'(x) \leqslant \frac{3}{n+2}$, *and the equality holds if and only if* $x = x_0$,

(ii) $(\gamma(x) + nc)x\gamma'(x) \geqslant 2cx + \gamma(x)^2 - nc\gamma(x)$, *and the equality holds if and only if* $x \geqslant x_0$,

(iii) $\gamma(x) > x\gamma'(x)$,

(iv) $\gamma(x) = \min\{\alpha(x), \beta(x)\}$,

(v) $\frac{x}{n-1} + 2c < \gamma(x) < \frac{x}{n-1} + nc$,

(vi) $\frac{n-2}{\sqrt{n(n-1)}}\sqrt{x\left(\gamma(x) - \frac{1}{n}x\right)} + \gamma(x) \leqslant \frac{2}{n}x + nc$.

Proof. (i) By direct computations, for $x > 0$, we get

$$\alpha'(x) = \frac{n}{2(n-1)} - \frac{n-2}{2(n-1)}\frac{x + 2(n-1)c}{\sqrt{x^2 + 4(n-1)cx}},$$

$$\alpha''(x) = \frac{2(n-2)(n-1)c^2}{(x^2 + 4(n-1)cx)^{3/2}},$$

$$\alpha'''(x) = -\frac{6(n-2)(n-1)(x + 2(n-1)c)c^2}{(x^2 + 4(n-1)cx)^{5/2}}.$$

Notice that y_n is the only positive root of equation

$$y_n(y_n + 6(n-1))^2 = \left(\frac{n^2-4n+6}{n^2-4}\right)^2 (y_n + 4(n-1))^3. \tag{2.2}$$

Let $\varphi_1(x) = 2x\alpha''(x) + \alpha'(x) = \frac{n}{2(n-1)} - \frac{(n-2)x^2(x+6(n-1)c)}{2(n-1)(x^2+4(n-1)cx)^{3/2}}$. From (2.2) we get $\varphi_1(x_0) = \frac{3}{n+2}$. We have $\lim_{x\to\infty} \varphi_1(x) = \frac{1}{n-1}$. Since $\varphi_1'(x) = -\frac{6(n-2)(n-1)c^2x^2}{(x^2+4(n-1)cx)^{5/2}} < 0$, we get $\frac{1}{n-1} < \varphi_1(x) < \frac{3}{n+2}$ if $x > x_0$.

Let $\varphi_2(x) = 2x\beta''(x) + \beta'(x) = \alpha'(x_0) + \alpha''(x_0)(3x - x_0)$. Then we get $\varphi_2'(x) = 3\alpha''(x_0) > 0$. Thus we have $\varphi_2(x) < \varphi_2(x_0) = \frac{3}{n+2}$ if $0 \leqslant x < x_0$. Hence assertion (i) follows.

(ii) It's easy to check that α and α' satisfy

$$(\alpha(x) + nc)x\alpha'(x) = 2cx + \alpha(x)^2 - nc\alpha(x). \tag{2.3}$$

Let $\psi(x) = (\beta(x) + nc)x\beta'(x) - 2cx - \beta(x)^2 + nc\beta(x)$. From the C^2-continuity of γ and (2.3), we get $\psi(x_0) = \psi'(x_0) = 0$. Making calculation, we get

$$\psi''(x) = 3\alpha''(x_0)[\alpha''(x_0)x^2 + (\alpha'(x_0) - x_0\alpha''(x_0))x + nc].$$

Let $x_1 = \left(n\sqrt{n-1} - 2n + 2\right)c$. We have $\alpha'(x_1) = 0$, $\alpha''(x_1) = \frac{2}{(n-2)^2\sqrt{n-1}c}$ and $\varphi_1(x_1) = \frac{4}{2\sqrt{n-1}+n}$. Since $\varphi_1'(x) < 0$ and $\varphi_1(x_0) \leqslant \varphi_1(x_1)$, we get $x_0 \geqslant x_1$. Since $\alpha''(x) > 0$, we have $\alpha'(x_0) \geqslant \alpha'(x_1) = 0$. By the definition of y_n, we have $y_n < 4(1-n) + \frac{2(n^2-4)}{\sqrt{2n-5}} \leqslant \frac{2}{15}n(n+2) = \frac{2n}{5\varphi_1(x_0)}$. This yields

$$2x_0\alpha''(x_0) + \alpha'(x_0) < \frac{2nc}{5x_0}.$$

Thus we have $\psi''(x) \geqslant 3\alpha''(x_0)nc\left(1 - \frac{x}{5x_0}\right)$. If $0 \leqslant x < x_0$, then $\psi''(x) > 0$. Therefore, we have $\psi'(x) < 0 < \psi(x)$ if $0 < x < x_0$. This proves (ii).

(iii) We have

$$\alpha(x) - x\alpha'(x) = nc - \frac{(n-2)cx}{\sqrt{x^2 + 4(n-1)cx}} > 0.$$

Since $(\beta(x) - x\beta'(x))' = -\alpha''(x_0)x \leqslant 0$, we get $\beta(x) - x\beta'(x) \geqslant \alpha(x_0) - x_0\alpha'(x_0) > 0$ if $0 \leqslant x < x_0$. Hence assertion (iii) is proved.

(iv) We have $\alpha'''(x) < 0 = \beta'''(x)$. From $\alpha(x_0) = \beta(x_0)$, $\alpha'(x_0) = \beta'(x_0)$ and $\alpha''(x_0) = \beta''(H_0)$, we obtain $\alpha(x) > \beta(x)$ for $0 \leqslant x < x_0$, and $\alpha(x) < \beta(x)$ for $x > x_0$.

(v) It's easy to verify that $2c < \alpha(x) - \frac{x}{n-1} \leqslant nc$. Since $\alpha''(x) > 0$ and $\beta''(x) = \alpha''(x_0) > 0$, we get $\gamma''(x) > 0$. So, $\lim_{x\to\infty} \gamma'(x) = \frac{1}{n-1}$ implies $\gamma'(x) < \frac{1}{n-1}$. Then we have $\gamma(x) - \frac{x}{n-1} > \lim_{x\to\infty}(\alpha(x) - \frac{x}{n-1}) = 2c$. This proves (v).

(vi) Note that α satisfies the following identity

$$\frac{n-2}{\sqrt{n(n-1)}}\sqrt{x\left(\alpha(x) - \frac{1}{n}x\right)} + \alpha(x) = \frac{2}{n}x + nc. \tag{2.4}$$

Combing (2.4) and $\gamma(x) \leqslant \alpha(x)$, we prove (vi). $\square$

The following lemma will be used in the proofs of Corollaries 1.3 and 1.7.

Lemma 2.4. *We have $\gamma(x) > \frac{9}{5}\sqrt{n-1}c$, where $\gamma(x)$ is defined by (2.1). In particular, $k_n > 1.999\sqrt{n-1}$ for $5 \leqslant n \leqslant 9$, and $k_{10} = 6$.*

Proof. Let $x_1 = \left(n\sqrt{n-1} - 2n + 2\right)c$. Since $\alpha'(x_1) = 0$ and $a''(x) > 0$, we get $\alpha(x) \geqslant \alpha(x_1) = 2\sqrt{n-1}c$.

If $n = 3$, we have $\beta(x) > 0.027\frac{x^2}{c} + 0.304x + 2.661c > 1.8\sqrt{2}c$.

If $n \geqslant 4$, by the definition of β, we have $\beta(x) \geqslant \alpha(x_0) - \frac{\alpha'(x_0)^2}{2\alpha''(x_0)} = k_n c$.

Making a calculation, we get $k_4 > 3.443 > 1.8\sqrt{3}$ and $k_5 > 3.998 = 1.999 \times 2$.

If $n \geqslant 6$, putting $x_2 = \sqrt{2(n-1)}\left(\sqrt{n-1} - 1/\sqrt{2}\right)^2 c$, we have $x_0 < [4(1-n) + \frac{2(n^2-4)}{\sqrt{2n-5}}]c < x_2$. Then we get $\alpha'(x_0) < \alpha'(x_2) = \frac{1}{2n-3}$ and $\alpha''(x_0) > \alpha''(x_2) = \frac{4\sqrt{2}(n-2)}{\sqrt{n-1}(2n-3)^3 c}$. Thus we obtain $\beta(x) > 2\sqrt{n-1}c - \frac{\sqrt{n-1}(2n-3)c}{8\sqrt{2}(n-2)} > \frac{9}{5}\sqrt{n-1}c$.

In fact, by more computations, we get $k_n > 1.999\sqrt{n-1}$ if $5 \leqslant n \leqslant 9$, and $k_{10} = 6$. $\square$

Let $\omega \in C^2[0, +\infty)$ be a positive function which takes the following form

$$\omega(x) = \frac{x^2}{\sqrt{x^2 + 4(n-1)cx}}\left[\left(1 + \frac{n^2c}{x}\right)\frac{\frac{n}{n-2} - (1+4(n-1)c/x)^{-1/2}}{\frac{n}{n-2} + (1+4(n-1)c/x)^{-1/2}}\right]^2 \quad \text{for} \quad x \geqslant x_0.$$

Then ω has the following properties.

Lemma 2.5. *For $n \geqslant 3$, $c > 0$ and $x \geqslant 0$, ω satisfies*

(i) $(\gamma(x) + nc)x\frac{\omega'(x)}{\omega(x)} = 2\gamma(x) - x\gamma'(x) - 3nc$ *if* $x \geqslant x_0$,

(ii) $2x_0\omega''(x_0) + \omega'(x_0) > 0$, $\lim_{x\to\infty}(2x\omega''(x) + \omega'(x)) > 0$,

(iii) $\omega(x) - x\omega'(x)$ *is bounded.*

Proof. (i) When $x \geqslant x_0$, we see that ω satisfies the following identity

$$\frac{\omega'(x)}{\omega(x)} = \frac{2\alpha(x) - x\alpha'(x) - 3nc}{x(\alpha(x) + nc)}, \tag{2.5}$$

which implies (i).

(ii) By a computation, we get that $\lim_{x\to\infty}(2x\omega''(x) + \omega'(x)) = \frac{1}{(n-1)^2}$.

If $n = 3$, we have $2x_0\omega''(x_0) + \omega'(x_0) \approx 11.4 > 0$.

If $n \geqslant 4$, we have $y_n < 4(1-n) + \frac{2(n^2-4)}{\sqrt{2n-5}} < (n-2)^2$. So, we get $\alpha(x_0) < \alpha((n-2)^2c) = nc$. From (2.5), for $x \geqslant x_0$, we have

$$\begin{aligned}
\frac{2x\omega''(x) + \omega'(x)}{\omega(x)} &\geqslant 2x\left(\frac{\omega'(x)}{\omega(x)}\right)' + \frac{\omega'(x)}{\omega(x)} \\
&= \frac{2\alpha'(x)(x\alpha'(x) + 5nc)}{(\alpha(x) + nc)^2} + \frac{5nc - 2x^2\alpha''(x) - x\alpha'(x)}{x(\alpha(x) + nc)} - \frac{2}{x}.
\end{aligned}$$

Combing the above inequality with $\alpha(x_0) < nc$, $\alpha'(x_0) \geqslant 0$ and $2x_0\alpha''(x_0) + \alpha'(x_0) < \frac{2nc}{5x_0}$, we obtain $2x_0\omega''(x_0) + \omega'(x_0) > 0$.

(iii) The assertion follows from $\lim_{x\to\infty}(\omega(x) - x\omega'(x)) = \frac{2(2n-1)c}{n-1}$. $\square$

For convenience, we denote $\gamma(H^2)$, $\gamma'(H^2)$, $\gamma''(H^2)$, $\omega(H^2)$, $\omega'(H^2)$ and $\omega''(H^2)$ by γ, γ', γ'', ω, ω' and ω'', respectively.

Suppose that M_0 is a compact hypersurface satisfying $|h|^2 < \gamma$. Then there exists a sufficiently small positive number ε, such that

$$|h|^2 < \gamma - \varepsilon\omega.$$

Now we show that this pinching condition is preserved.

Theorem 2.6. *If M_0 satisfies $|h|^2 < \gamma - \varepsilon\omega$, then this condition holds for all time $t \in [0, T)$.*

Proof. Let $U = |h|^2 - \gamma + \varepsilon\omega$. From the evolution equations we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) U &= -2|\nabla h|^2 + 2(2H^2\gamma'' + \gamma' - \varepsilon(2H^2\omega'' + \omega'))|\nabla H|^2 \\ &\quad +4cH^2 + 2|h|^4 - 2nc|h|^2 - 2(\gamma' - \varepsilon\omega') \cdot H^2(|h|^2 + nc).\end{aligned}$$

By Lemma 2.3 (i) and Lemma 2.5 (ii), the coefficient of $|\nabla H|^2$ in the above formula is less than $\frac{6}{n+2}$. Then Lemma 2.1 (i) yields $-2|\nabla h|^2 + \frac{6}{n+2}|\nabla H|^2 \leqslant 0$. Then replacing $|h|^2$ by $U + \gamma - \varepsilon\omega$, we get

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) U &\leqslant 2U^2 + 2U[2\gamma - H^2\gamma' - nc + \varepsilon(H^2\omega' - 2\omega)] \\ &\quad +4cH^2 + 2\gamma^2 - 2nc\gamma - 2(\gamma + nc)H^2\gamma' \\ &\quad +2\varepsilon\omega\left[-2\gamma + H^2\gamma' + nc + (\gamma + nc)H^2\frac{\omega'}{\omega}\right] + 2\varepsilon^2\omega \cdot (\omega - H^2\omega').\end{aligned} \tag{2.6}$$

By Lemma 2.3 (ii), Lemma 2.5 (i) and (iii), when ε is small enough, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) U < 2U^2 + \upsilon \cdot U.$$

Therefore, the conclusion follows from the maximum principle. □

3. An estimate for traceless second fundamental form

In this section, we derive an estimate for the traceless second fundamental form, which shows that the principal curvatures will approach each other along the mean curvature flow.

Theorem 3.1. *If M_0 satisfies $|h|^2 < \gamma - \varepsilon\omega$, then there exist constants $0 < \sigma < 1$ and $C_0 > 0$ depending only on M_0, such that for all $t \in [0, T)$, we have*

$$|\mathring{h}|^2 \leqslant C_0(H^2 + c)^{1-\sigma}\mathrm{e}^{-2\sigma ct}.$$

Let $\mathring{\gamma} = \gamma - \frac{1}{n}H^2$. Theorem 2.6 says that $|\mathring{h}|^2 < \mathring{\gamma} - \varepsilon\omega$ holds for all time. We denote by $\mathring{\gamma}' = \gamma' - \frac{1}{n}$, $\mathring{\gamma}'' = \gamma''$ the first and second derivatives of $\mathring{\gamma}$ with respect to H^2. For $0 < \sigma < 1$, we set

$$f_\sigma = \frac{|\mathring{h}|^2}{\mathring{\gamma}^{1-\sigma}}.$$

To prove Theorem 3.1, we need to show that f_σ decays exponentially. First, we make an estimate for the time derivative of f_σ.

Lemma 3.2. *There exists a positive constants C_1 depending only on n, c, such that*

$$\frac{\partial}{\partial t}f_\sigma \leqslant \Delta f_\sigma + \frac{2C_1}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| - \frac{2\varepsilon f_\sigma}{C_1|\mathring{h}|^2}|\nabla H|^2 + 2\sigma(|h|^2 - nc)f_\sigma.$$

Proof. By a straightforward calculation, we have

$$\frac{\partial}{\partial t}f_\sigma = f_\sigma\left(\frac{\frac{\partial}{\partial t}|\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{\frac{\partial}{\partial t}\mathring{\gamma}}{\mathring{\gamma}}\right).$$

The gradient of f_σ can be written as

$$\nabla f_\sigma = f_\sigma \left(\frac{\nabla |\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{\nabla \mathring{\gamma}}{\mathring{\gamma}} \right).$$

The Laplacian of f_σ is given by

$$\Delta f_\sigma = f_\sigma \left(\frac{\Delta |\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma)\frac{\Delta \mathring{\gamma}}{\mathring{\gamma}} \right) - 2(1-\sigma)\frac{\langle \nabla f_\sigma, \nabla \mathring{\gamma}\rangle}{\mathring{\gamma}} + \sigma(1-\sigma) f_\sigma \frac{|\nabla \mathring{\gamma}|^2}{|\mathring{\gamma}|^2}. \tag{3.1}$$

By the evolution equations, we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) f_\sigma &= 2(1-\sigma)\frac{\langle \nabla f_\sigma, \nabla \mathring{\gamma}\rangle}{\mathring{\gamma}} - \sigma(1-\sigma) f_\sigma \frac{|\nabla \mathring{\gamma}|^2}{|\mathring{\gamma}|^2} \\
&\quad + \frac{2 f_\sigma}{|\mathring{h}|^2}\left(\frac{|\nabla H|^2}{n} - |\nabla h|^2\right) + 2(1-\sigma) f_\sigma \frac{2H^2\mathring{\gamma}'' + \mathring{\gamma}'}{\mathring{\gamma}}|\nabla H|^2 \\
&\quad + 2 f_\sigma \left[(|h|^2 - nc) - (1-\sigma)(|h|^2+nc)\frac{H^2\mathring{\gamma}'}{\mathring{\gamma}} \right] \\
&\leqslant \frac{2}{\mathring{\gamma}} |\nabla f_\sigma|\, |\nabla \mathring{\gamma}| \qquad (3.2)\\
&\quad + 2 f_\sigma \left[\frac{1}{|\mathring{h}|^2}\left(\frac{|\nabla H|^2}{n} - |\nabla h|^2\right) + (1-\sigma)\frac{2H^2\mathring{\gamma}'' + \mathring{\gamma}'}{\mathring{\gamma}}|\nabla H|^2 \right] \\
&\quad + 2 f_\sigma \left[\sigma(|h|^2+nc) + (1-\sigma)(|h|^2+nc)\left(1 - \frac{H^2\mathring{\gamma}'}{\mathring{\gamma}}\right) - 2nc \right].
\end{aligned}$$

From the definitions of $\mathring{\gamma}$, ω, there exist a positive constant C_1 depending only on n, c, such that

$$\frac{2|H|\,|\mathring{\gamma}'|}{\sqrt{\mathring{\gamma}}} < C_1 \quad \text{and} \quad \frac{n(n+2)\mathring{\gamma}}{2(n-1)\omega} < C_1.$$

This together with $|\mathring{h}|^2 < \mathring{\gamma}$ implies

$$\frac{|\nabla \mathring{\gamma}|}{\mathring{\gamma}} = \frac{2\,|\mathring{\gamma}'|\,|H||\nabla H|}{\mathring{\gamma}} \leqslant C_1 \frac{|\nabla H|}{|\mathring{h}|}. \tag{3.3}$$

Next we estimate the expression in the first square bracket of the right hand side of (3.2). Lemma 2.3 (i) implies $2H^2\mathring{\gamma}'' + \mathring{\gamma}' \leqslant \frac{2(n-1)}{n(n+2)}$. From Lemma 2.1 (i), we have

$$\begin{aligned}
&\frac{1}{|\mathring{h}|^2}\left(\frac{|\nabla H|^2}{n} - |\nabla h|^2\right) + (1-\sigma)\frac{2H^2\mathring{\gamma}'' + \mathring{\gamma}'}{\mathring{\gamma}}|\nabla H|^2 \\
\leqslant\ & \left(\frac{2(1-n)}{n(n+2)}\frac{1}{|\mathring{h}|^2} + \frac{(1-\sigma)}{\mathring{\gamma}}\frac{2(n-1)}{n(n+2)} \right)|\nabla H|^2 \\
\leqslant\ & \frac{2(n-1)}{n(n+2)}\left(\frac{1}{\mathring{\gamma}} - \frac{1}{|\mathring{h}|^2}\right)|\nabla H|^2 \\
\leqslant\ & -\frac{2(n-1)}{n(n+2)}\frac{\varepsilon\omega}{\mathring{\gamma}|\mathring{h}|^2}|\nabla H|^2 \\
\leqslant\ & -\frac{\varepsilon}{C_1|\mathring{h}|^2}|\nabla H|^2.
\end{aligned}$$

Then we estimate the expression in the second square bracket of the right hand side of (3.2). From Lemma 2.3 (iii), we get $(|h|^2+nc)\left(1-\frac{H^2\mathring{\gamma}'}{\mathring{\gamma}}\right)\leqslant\frac{1}{\mathring{\gamma}}(\gamma+nc)(\gamma-H^2\gamma')$. Lemma 2.3 (ii) yields $(\gamma+nc)(\gamma-H^2\gamma')\leqslant 2nc\mathring{\gamma}$. Thus we obtain

$$\sigma(|h|^2+nc)+(1-\sigma)(|h|^2+nc)\left(1-\frac{H^2\mathring{\gamma}'}{\mathring{\gamma}}\right)-2nc\leqslant\sigma(|h|^2-nc).$$

This completes the proof of the Lemma 3.2. □

To estimate the term $2\sigma|h|^2f_\sigma$ in Lemma 3.2, we need the following.

Lemma 3.3. *If M is an n-dimensional $(n\geqslant 3)$ hypersurface in $\mathbb{F}^{n+1}(c)$ $(c>0)$ which satisfies $|\mathring{h}|^2<\mathring{\gamma}-\varepsilon\omega$, then there exists a positive constant C_2 depending only on n and c, such that*

$$\Delta|\mathring{h}|^2\geqslant 2\left\langle\mathring{h},\nabla^2H\right\rangle+2\varepsilon C_2|h|^2|\mathring{h}|^2.$$

Proof. From Lemma 2.1, we have

$$\Delta|\mathring{h}|^2=2\left\langle\mathring{h},\nabla^2H\right\rangle+2|\nabla h|^2-\frac{2}{n}|\nabla H|^2+2W\geqslant 2\left\langle\mathring{h},\nabla^2H\right\rangle+2W.$$

Let λ_i $(1\leqslant i\leqslant n)$ be the principal curvatures of M. Put $\mathring{\lambda}_i=\lambda_i-\frac{H}{n}$. Then $\sum\mathring{\lambda}_i=0$, $\sum\mathring{\lambda}_i^2=|\mathring{h}|^2$. Thus we have

$$h_i^jh_j^kh_k^i=\sum_i\lambda_i^3=\sum_i\mathring{\lambda}_i^3+\frac{3}{n}H|\mathring{h}|^2+\frac{1}{n^2}H^3.$$

Using the inequality $\left|\sum_i\mathring{\lambda}_i^3\right|\leqslant\frac{n-2}{\sqrt{n(n-1)}}|\mathring{h}|^3$ (see [20], Lemma 2.1), we have

$$\begin{aligned}W&=Hh_i^jh_j^kh_k^i-|h|^4+nc|\mathring{h}|^2\\&\geqslant-\frac{n-2}{\sqrt{n(n-1)}}|H||\mathring{h}|^3+\frac{1}{n}H^2|\mathring{h}|^2-|\mathring{h}|^4+nc|\mathring{h}|^2\\&\geqslant|\mathring{h}|^2\left(-\frac{n-2}{\sqrt{n(n-1)}}|H|\sqrt{\mathring{\gamma}}+\frac{1}{n}H^2-(\mathring{\gamma}-\varepsilon\omega)+nc\right).\end{aligned}\tag{3.4}$$

Let $C_2=\inf(\omega/\gamma)$. It follows from Lemma 2.3 (vi) that $W\geqslant\varepsilon\omega|\mathring{h}|^2\geqslant\varepsilon C_2|h|^2|\mathring{h}|^2$. □

From (3.1), (3.3) and Lemma 3.3, we have

$$\begin{aligned}\Delta f_\sigma&\geqslant f_\sigma\frac{\Delta|\mathring{h}|^2}{|\mathring{h}|^2}-(1-\sigma)f_\sigma\frac{\Delta\mathring{\gamma}}{\mathring{\gamma}}-2(1-\sigma)\frac{\langle\nabla f_\sigma,\nabla\mathring{\gamma}\rangle}{\mathring{\gamma}}\\&\geqslant\frac{2f_\sigma}{|\mathring{h}|^2}\left\langle\mathring{h},\nabla^2H\right\rangle+2\varepsilon C_2|h|^2f_\sigma-(1-\sigma)\frac{f_\sigma}{\mathring{\gamma}}\Delta\mathring{\gamma}-\frac{2C_1}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|.\end{aligned}$$

This is equivalent to

$$2\varepsilon C_2|h|^2f_\sigma\leqslant\Delta f_\sigma-\frac{2f_\sigma}{|\mathring{h}|^2}\left\langle\mathring{h},\nabla^2H\right\rangle+\frac{2C_1}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|+(1-\sigma)\frac{f_\sigma}{\mathring{\gamma}}\Delta\mathring{\gamma}.\tag{3.5}$$

Multiplying both sides of the above inequality by f_σ^{p-1}, then integrating them over M_t, applying the divergence theorem and the relation $\nabla_i \mathring{h}^i_j = \frac{n-1}{n}\nabla_j H$, we get

$$\int_{M_t} f_\sigma^{p-1}\Delta f_\sigma \mathrm{d}\mu_t = -(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 \mathrm{d}\mu_t \leqslant 0, \tag{3.6}$$

$$\begin{aligned}
-\int_{M_t}\frac{f_\sigma^p}{|\mathring{h}|^2}\left\langle \mathring{h}, \nabla^2 H\right\rangle \mathrm{d}\mu_t &= -\int_{M_t}\frac{f_\sigma^{p-1}}{\mathring{\gamma}^{1-\sigma}}\mathring{h}^{ij}\nabla^2_{i,j}H\mathrm{d}\mu_t\\
&= \int_{M_t}\nabla_i\left(\frac{f_\sigma^{p-1}}{\mathring{\gamma}^{1-\sigma}}\mathring{h}^{ij}\right)\nabla_j H\mathrm{d}\mu_t\\
&= \int_{M_t}\Bigg[(p-1)\frac{f_\sigma^{p-2}}{\mathring{\gamma}^{1-\sigma}}\mathring{h}^{ij}\nabla_i f_\sigma \nabla_j H\\
&\quad -(1-\sigma)\frac{f_\sigma^{p-1}}{\mathring{\gamma}^{2-\sigma}}\mathring{h}^{ij}\nabla_i\mathring{\gamma}\nabla_j H + \frac{n-1}{n}\frac{f_\sigma^{p-1}}{\mathring{\gamma}^{1-\sigma}}|\nabla H|^2\Bigg]\mathrm{d}\mu_t\\
&\leqslant \int_{M_t}\Bigg[(p-1)\frac{f_\sigma^{p-2}}{\mathring{\gamma}^{1-\sigma}}|\mathring{h}||\nabla f_\sigma||\nabla H|\\
&\quad +\frac{f_\sigma^{p-1}}{\mathring{\gamma}^{2-\sigma}}|\mathring{h}|\,|\nabla\mathring{\gamma}|\,|\nabla H| + \frac{f_\sigma^{p-1}}{\mathring{\gamma}^{1-\sigma}}|\nabla H|^2\Bigg]\mathrm{d}\mu_t\\
&\leqslant \int_{M_t}\left[(p-1)\frac{f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| + (C_1+1)\frac{f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2\right]\mathrm{d}\mu_t,
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
\int_{M_t}\frac{f_\sigma^p}{\mathring{\gamma}}\Delta\mathring{\gamma}\mathrm{d}\mu_t &= -\int_{M_t}\left\langle\nabla\left(\frac{f_\sigma^p}{\mathring{\gamma}}\right), \nabla\mathring{\gamma}\right\rangle\mathrm{d}\mu_t\\
&= \int_{M_t}\left[-p\frac{f_\sigma^{p-1}}{\mathring{\gamma}}\langle\nabla f_\sigma, \nabla\mathring{\gamma}\rangle + f_\sigma^p\left(\frac{|\nabla\mathring{\gamma}|}{\mathring{\gamma}}\right)^2\right]\mathrm{d}\mu_t\\
&\leqslant \int_{M_t}\left[pC_1\frac{f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| + C_1^2\frac{f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2\right]\mathrm{d}\mu_t.
\end{aligned}\tag{3.8}$$

Putting (3.5)-(3.8) together, we obtain

$$2\varepsilon\int_{M_t}|h|^2 f_\sigma^p \mathrm{d}\mu_t \leqslant C_3\int_{M_t}\left[\frac{pf_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| + \frac{f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2\right]\mathrm{d}\mu_t, \tag{3.9}$$

where C_3 is a positive constant depending on n and c.

Using (3.9) and Lemma 3.2, we make an estimate for the time derivative of the integral of f_σ^p.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f_\sigma^p \mathrm{d}\mu_t &= p\int_{M_t} f_\sigma^{p-1}\frac{\partial f_\sigma}{\partial t}\mathrm{d}\mu_t - \int_{M_t} f_\sigma^p H^2 \mathrm{d}\mu_t \\
&\leqslant p\int_{M_t}\left[f_\sigma^{p-1}\Delta f_\sigma + \frac{2C_1 f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|\right. \\
&\quad \left. -\frac{2\varepsilon f_\sigma^p}{C_1|\mathring{h}|^2}|\nabla H|^2 + 2\sigma(|h|^2 - nc)f_\sigma^p\right]\mathrm{d}\mu_t \\
&\leqslant p\int_{M_t}\left[-(p-1)f_\sigma^{p-2}|\nabla f_\sigma|^2 + \frac{2C_1 f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| - \frac{2\varepsilon f_\sigma^p}{C_1|\mathring{h}|^2}|\nabla H|^2\right. \\
&\quad \left. +\frac{C_3\sigma p}{\varepsilon}\frac{f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma||\nabla H| + \frac{C_3\sigma}{\varepsilon}\frac{f_\sigma^p}{|\mathring{h}|^2}|\nabla H|^2 - 2\sigma nc f_\sigma^p\right]\mathrm{d}\mu_t \qquad (3.10)\\
&= p\int_{M_t} f_\sigma^{p-2}\left[-(p-1)|\nabla f_\sigma|^2 + \left(2C_1 + \frac{C_3\sigma p}{\varepsilon}\right)\frac{f_\sigma}{|\mathring{h}|}|\nabla f_\sigma||\nabla H|\right. \\
&\quad \left. -\left(\frac{2\varepsilon}{C_1} - \frac{C_3\sigma}{\varepsilon}\right)\frac{f_\sigma^2}{|\mathring{h}|^2}|\nabla H|^2\right]\mathrm{d}\mu_t - 2p\sigma nc\int_{M_t} f_\sigma^p \mathrm{d}\mu_t.
\end{aligned}$$

In the following we show that the L^p-norm of f_σ decays exponentially.

Lemma 3.4. *There exists a constant C_4 depending on M_0 such that for all $p \geqslant 8C_1^3\varepsilon^{-1}$ and $\sigma \leqslant \varepsilon^2 p^{-1/2}$, we have*

$$\left(\int_{M_t} f_\sigma^p \mathrm{d}\mu_t\right)^{\frac{1}{p}} < C_4 \mathrm{e}^{-3\sigma ct}.$$

Proof. The expression in the square bracket of the right hand side of (3.10) is a quadratic polynomial. With ε small enough, its discriminant satisfies

$$\begin{aligned}
&\left(2C_1 + \frac{C_3\sigma p}{\varepsilon}\right)^2 - 4(p-1)\left(\frac{2\varepsilon}{C_1} - \frac{C_3\sigma}{\varepsilon}\right) \\
<\ & 8C_1^2 + 2C_3^2\varepsilon^2 p - 2p\varepsilon/C_1 \\
=\ & (8C_1^2 - p\varepsilon/C_1) + (2C_3^2\varepsilon^2 p - p\varepsilon/C_1) \\
<\ & 0.
\end{aligned}$$

Therefore, this quadratic polynomial is non-positive. So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f_\sigma^p \mathrm{d}\mu_t \leqslant -3p\sigma c\int_{M_t} f_\sigma^p \mathrm{d}\mu_t.$$

This implies $\int_{M_t} f_\sigma^p \mathrm{d}\mu_t \leqslant \mathrm{e}^{-3p\sigma ct}\int_{M_0} f_\sigma^p \mathrm{d}\mu_0$. □

Letting $g_\sigma = f_\sigma \mathrm{e}^{2\sigma ct}$, we get

Corollary 3.5. *There exist a constant C_5 depending on M_0 such that for all $r \geqslant 0$, $p \geqslant \max\{4r^2\varepsilon^{-4}, 8C_1^3\varepsilon^{-1}\}$ and $\sigma \leqslant \frac{1}{2}\varepsilon^2 p^{-1/2}$, we have*

$$\left(\int_{M_t} |h|^{2r} g_\sigma^p \mathrm{d}\mu_t\right)^{\frac{1}{p}} < C_5 \mathrm{e}^{-\sigma ct}.$$

Proof. Putting $C_6=\sup(\gamma/\mathring{\gamma})$, we obtain

$$\left(\int_{M_t}|h|^{2r}g_\sigma^p\mathrm{d}\mu_t\right)^{\frac{1}{p}}\leqslant\left(\int_{M_t}\gamma^r g_\sigma^p\mathrm{d}\mu_t\right)^{\frac{1}{p}}\leqslant\left(\int_{M_t}(C_6\mathring{\gamma})^r\,\mathrm{e}^{2p\sigma ct}f_\sigma^p\mathrm{d}\mu_t\right)^{\frac{1}{p}}$$
$$\leqslant C_6^{r/p}\mathrm{e}^{2\sigma ct}\left(\int_{M_t}f^p_{\sigma+\frac{r}{p}}\mathrm{d}\mu_t\right)^{\frac{1}{p}}.$$

With $r/p\leqslant\frac{1}{2}\varepsilon^2/\sqrt{p}$ and $\sigma+r/p<\varepsilon^2/\sqrt{p}$, the conclusion follows from Lemma 3.4. □

From Lemma 3.2, we have an estimate for the time derivative of g_σ.

$$\frac{\partial}{\partial t}g_\sigma\leqslant\Delta g_\sigma+\frac{2C_1}{|\mathring{h}|}|\nabla g_\sigma||\nabla H|-\frac{2\varepsilon g_\sigma}{C_1|\mathring{h}|^2}|\nabla H|^2+2\sigma|h|^2g_\sigma.$$

For $k>0$, define $g_{\sigma,k}=\max\{g_\sigma-k,0\}$, $A(k,t)=\operatorname{supp}g_{\sigma,k}=\overline{\{x\in M_t|g_\sigma(x)>k\}}$. Letting $p\geqslant 8C_1^3/\varepsilon$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t}g^p_{\sigma,k}\mathrm{d}\mu_t &\leqslant p\int_{M_t}g^{p-1}_{\sigma,k}\left[\Delta g_\sigma+\tfrac{2C_1}{|\mathring{h}|}|\nabla g_\sigma||\nabla H|-\tfrac{2\varepsilon g_\sigma}{C_1|\mathring{h}|^2}|\nabla H|^2+2\sigma|h|^2g_\sigma\right]\mathrm{d}\mu_t\\ &\leqslant p\int_{M_t}\left[-(p-1)g^{p-2}_{\sigma,k}|\nabla g_\sigma|^2+\tfrac{2C_1g^{p-1}_{\sigma,k}}{|\mathring{h}|}|\nabla g_\sigma||\nabla H|-\tfrac{2\varepsilon g^p_{\sigma,k}}{C_1|\mathring{h}|^2}|\nabla H|^2\right]\mathrm{d}\mu_t\\ &\quad+2p\sigma\int_{A(k,t)}|h|^2g_\sigma^p\mathrm{d}\mu_t\\ &\leqslant -\frac{1}{2}p(p-1)\int_{M_t}g^{p-2}_{\sigma,k}|\nabla g_\sigma|^2\mathrm{d}\mu_t+2p\sigma\int_{A(k,t)}|h|^2g^p_\sigma\mathrm{d}\mu_t.\end{aligned}$$

Notice that $\frac{1}{2}p(p-1)g^{p-2}_{\sigma,k}|\nabla g_\sigma|^2\geqslant|\nabla g^{p/2}_{\sigma,k}|^2$. Setting $v=g^{p/2}_{\sigma,k}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t}v^2\mathrm{d}\mu_t+\int_{M_t}|\nabla v|^2\mathrm{d}\mu_t\leqslant 2p\sigma\int_{A(k,t)}|h|^2g^p_\sigma\mathrm{d}\mu_t. \tag{3.11}$$

The volume of $\operatorname{supp}v$ satisfies $\int_{A(k,t)}\mathrm{d}\mu_t\leqslant\int_{A(k,t)}g^p_\sigma k^{-p}\mathrm{d}\mu_t<C_5^pk^{-p}$. It is sufficiently small when k is large. By Theorem 2.1 of [6], for the function $u=v^{2(n-1)/(n-2)}$ we have a Sobolev inequality $\left(\int_{M_t}u^{\frac{n}{n-1}}\mathrm{d}\mu_t\right)^{\frac{n-1}{n}}\leqslant C_7\int_{M_t}(|\nabla u|+u|H|)\mathrm{d}\mu_t$, where C_7 is a positive constant depending only on n. Using Hölder's inequality, we have

$$\left(\int_{M_t}v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\leqslant C_7\int_{M_t}|\nabla v|^2\mathrm{d}\mu_t+C_7\left(\int_{A(k,t)}H^n\mathrm{d}\mu_t\right)^{\frac{2}{n}}\left(\int_{M_t}v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}.$$

When $k\geqslant C_5(2nC_7)^{\frac{n}{2p}}$, $p\geqslant n^2/\varepsilon^4$ and $\sigma\leqslant\frac{1}{2}\varepsilon^2/\sqrt{p}$, it follows from Corollary 3.5 that $\left(\int_{A(k,t)}H^n\mathrm{d}\mu_t\right)^{\frac{2}{n}}\leqslant\left(\int_{A(k,t)}n^{\frac{n}{2}}|h|^ng^p_\sigma k^{-p}\mathrm{d}\mu_t\right)^{\frac{2}{n}}<\frac{1}{2C_7}$. Thus we obtain

$$\frac{1}{2C_7}\left(\int_{M_t}v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\leqslant\int_{M_t}|\nabla v|^2\mathrm{d}\mu_t. \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t}v^2\mathrm{d}\mu_t+\frac{1}{2C_7}\left(\int_{M_t}v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}}\leqslant 2p\sigma\int_{A(k,t)}|h|^2g^p_\sigma\mathrm{d}\mu_t. \tag{3.13}$$

Let $k \geqslant \sup_{M_0} g_\sigma$. Then $v|_{t=0} = 0$. For a fixed time s, let $s_1 \in [0,s]$ be the time when $\int_{M_t} v^2 \mathrm{d}\mu_t$ achieves its maximum in $[0,s]$. Integrating (3.13) over $[0,s_1]$ and $[0,s]$ respectively, adding them, we obtain

$$\begin{aligned} \sup_{t\in[0,s]} \int_{M_t} v^2 \mathrm{d}\mu_t \quad &+ \quad \frac{1}{2C_7}\int_0^s \left(\int_{M_t} v^{\frac{2n}{n-2}}\mathrm{d}\mu_t\right)^{\frac{n-2}{n}} \mathrm{d}t \\ &\leqslant \quad 4p\sigma \int_0^s \int_{A(k,t)} |h|^2 g_\sigma^p \mathrm{d}\mu_t \mathrm{d}t. \end{aligned} \tag{3.14}$$

Let $\|A(k)\|_s = \int_0^s \int_{A(k,t)} \mathrm{d}\mu_t \mathrm{d}t$. For a positive number $r > \frac{n+2}{2}$, let $p \geqslant 4r/\varepsilon^4$, $\sigma \leqslant \frac{1}{2}\varepsilon^2/\sqrt{pr}$. Using Hölder's inequality and Corollary 3.5 , we have

$$\begin{aligned} \int_0^s \int_{A(k,t)} |h|^2 g_\sigma^p \mathrm{d}\mu_t \mathrm{d}t \quad &\leqslant \quad \left(\int_0^s C_5^{pr}\mathrm{e}^{-pr\sigma ct}\mathrm{d}t\right)^{\frac{1}{r}} \|A(k)\|_s^{1-\frac{1}{r}} \\ &\leqslant \quad \frac{C_5^p}{(pr\sigma c)^{1/r}} \|A(k)\|_s^{1-\frac{1}{r}}. \end{aligned} \tag{3.15}$$

For $h > k$, we have $f_{\sigma,k} > h-k$ on $A(h,t)$. So

$$(h-k)^p \|A(h)\|_s \leqslant \int_0^s \int_{M_t} v^2 \mathrm{d}\mu_t \mathrm{d}t \leqslant \left(\int_0^s \int_{M_t} v^{\frac{2(n+2)}{n}} \mathrm{d}\mu_t \mathrm{d}t\right)^{\frac{n}{n+2}} \|A(k)\|_s^{\frac{2}{n+2}}. \tag{3.16}$$

We estimate the right hand side of the above inequality as follows.

$$\begin{aligned} &\left(\int_0^s \int_{M_t} v^{\frac{2(n+2)}{n}} \mathrm{d}\mu_t \mathrm{d}t\right)^{\frac{n}{n+2}} \\ \leqslant \quad &\left[\int_0^s \left(\int_{M_t} v^2 \mathrm{d}\mu_t\right)^{\frac{2}{n}} \left(\int_{M_t} v^{\frac{2n}{n-2}} \mathrm{d}\mu_t\right)^{\frac{n-2}{n}} \mathrm{d}t\right]^{\frac{n}{n+2}} \\ \leqslant \quad &\left(\sup_{t\in[0,s]} \int_{M_t} v^2 \mathrm{d}\mu_t\right)^{\frac{2}{n+2}} \left[\int_0^s \left(\int_{M_t} v^{\frac{2n}{n-2}} \mathrm{d}\mu_t\right)^{\frac{n-2}{n}} \mathrm{d}t\right]^{\frac{n}{n+2}} \\ \leqslant \quad &C_8 \cdot \sup_{t\in[0,s]} \int_{M_t} v^2 \mathrm{d}\mu_t + \frac{C_8}{2C_7}\int_0^s \left(\int_{M_t} v^{\frac{2n}{n-2}} \mathrm{d}\mu_t\right)^{\frac{n-2}{n}} \mathrm{d}t. \end{aligned} \tag{3.17}$$

Putting inequalities (3.14)-(3.17) together, for $h > k$, we have

$$(h-k)^p \|A(h)\|_s \leqslant C_9 \|A(k)\|_s^{1-\frac{1}{r}+\frac{2}{n+2}},$$

where C_9 is a positive constant depending on M_0, p, σ and r.

By a lemma of [10] (Chapter II, Lemma B.1), there exists a finite number k_1, such that $\|A(k_1)\|_s = 0$ for all s. Hence we have $g_\sigma \leqslant k_1$. This completes the proof of Theorem 3.1.

4. A gradient estimate

To compare the mean curvature at different points of M_t, we need an estimate for the gradient of mean curvature.

Theorem 4.1. *For all $\eta \in \left(0, \frac{1}{n}\right)$, there exists a constant $C(\eta)$ depending on η and M_0, such that*

$$|\nabla H|^2 < [(\eta H)^4 + C(\eta)^2]\mathrm{e}^{-\sigma ct}.$$

Firstly, we derive an inequality for the time derivative of $|\nabla H|$.

Lemma 4.2. *There exists a positive constant $B_1 > 1$ depending only on n, such that*

$$\frac{\partial}{\partial t}|\nabla H|^2 \leqslant \Delta|\nabla H|^2 + B_1(H^2+c)|\nabla h|^2.$$

Proof. From the evolution equation of H, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i H &= \nabla_i\left(\frac{\partial}{\partial t}H\right)\\ &= \nabla_i(\Delta H + H(|h|^2+nc))\\ &= \nabla_i\Delta H + \nabla_i H(|h|^2+nc) + H\nabla_i|h|^2.\end{aligned}$$

Lemma 2.2 (i) implies $\frac{\partial}{\partial t}g^{ij} = 2Hh^{ij}$. Thus we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2 &= \frac{\partial}{\partial t}(g^{ij}\nabla_i H\nabla_j H)\\ &= 2\langle\nabla\Delta H, \nabla H\rangle + 2|\nabla H|^2(|h|^2+nc)\\ &\quad +2H\langle\nabla|h|^2,\nabla H\rangle + 2Hh^{ij}\nabla_i H\nabla_j H.\end{aligned} \tag{4.1}$$

The Laplacian of $|\nabla H|^2$ is given by

$$\Delta|\nabla H|^2 = 2\langle\Delta\nabla H,\nabla H\rangle + 2|\nabla^2 H|^2. \tag{4.2}$$

From the Gauss equation, we get

$$\nabla\Delta H - \Delta\nabla H = (1-n)c\nabla H + h^{jk}h_{ij}\nabla_k H \mathrm{d}x^i - Hh_i^k\nabla_k H\mathrm{d}x^i. \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we obtain the evolution equation of $|\nabla H|^2$.

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2 &= \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2|\nabla H|^2(|h|^2+c)\\ &\quad +2H\langle\nabla|h|^2,\nabla H\rangle + 2h^{ij}h_i^k\nabla_j H\nabla_k H.\end{aligned} \tag{4.4}$$

Using the Cauchy-Schwarz inequality, we have $H\langle\nabla|h|^2,\nabla H\rangle \leqslant 2|H||h||\nabla h||\nabla H|$ and $h^{ij}h_i^k\nabla_j H\nabla_k H \leqslant |h|^2|\nabla H|^2$. Using $|h|^2 < \gamma$ and $|\nabla H|^2 \leqslant \frac{n+2}{3}|\nabla h|^2$, we obtain the conclusion. ∎

Secondly, we need the following estimates.

Lemma 4.3. *Along the mean curvature flow, we have*

(i) $\frac{\partial}{\partial t}H^4 \geqslant \Delta H^4 - 7nH^2|\nabla h|^2 + \frac{4}{n}H^6$,

(ii) $\frac{\partial}{\partial t}|\mathring{h}|^2 \leqslant \Delta|\mathring{h}|^2 - \frac{8}{9}|\nabla h|^2 + H^2|\mathring{h}|^2$,

(iii) $\frac{\partial}{\partial t}\left(H^2|\mathring{h}|^2\right) \leqslant \Delta\left(H^2|\mathring{h}|^2\right) - \frac{7}{9}H^2|\nabla h|^2 + B_2|\nabla h|^2 + 4nH^2(H^2+c)|\mathring{h}|^2$, *where $B_2 > 2c$ is a positive constant depending on M_0.*

Proof. (i) We derive that

$$\frac{\partial}{\partial t}H^4 = \Delta H^4 - 12H^2|\nabla H|^2 + 4H^4(|h|^2+nc).$$

This together with $|h|^2 \geqslant \frac{1}{n}H^2$ and $|\nabla H|^2 \leqslant \frac{n+2}{3}|\nabla h|^2$ implies inequality (i).

(ii) The evolution equation of $|\mathring{h}|^2$ is given by

$$\frac{\partial}{\partial t}|\mathring{h}|^2 = \Delta|\mathring{h}|^2 - 2|\nabla h|^2 + \frac{2}{n}|\nabla H|^2 + 2|\mathring{h}|^2(|h|^2-nc).$$

Using $\frac{1}{n}|\nabla H|^2 \leqslant \frac{n+2}{3n}|\nabla h|^2 \leqslant \frac{5}{9}|\nabla h|^2$ and $|h|^2 - nc < \gamma - nc \leqslant \frac{1}{2}H^2$, we get inequality (ii).

(iii) It follows from the evolution equations that

$$\begin{aligned}\frac{\partial}{\partial t}\left(H^2|\mathring{h}|^2\right) &= \Delta\left(H^2|\mathring{h}|^2\right) + 4H^2|h|^2|\mathring{h}|^2 - 2H^2\left(|\nabla h|^2 - \frac{1}{n}|\nabla H|^2\right)\\ &\quad -2|\mathring{h}|^2|\nabla H|^2 - 4H\left\langle \nabla H, \nabla|\mathring{h}|^2\right\rangle.\end{aligned}$$

We use $\frac{1}{n}|\nabla H|^2 \leqslant \frac{5}{9}|\nabla h|^2$ again. From $|h|^2 < \gamma$, we have $4H^2|h|^2|\mathring{h}|^2 \leqslant 4nH^2(H^2+c)|\mathring{h}|^2$. From the formula $\nabla_i|\mathring{h}|^2 = 2\mathring{h}^{jk}\nabla_i h_{jk}$ and Young's inequality, we get

$$\begin{aligned}-4H\left\langle \nabla H, \nabla|\mathring{h}|^2\right\rangle &\leqslant 8|H||\nabla H||\mathring{h}||\nabla h|\\ &\leqslant 8\sqrt{\tfrac{n+2}{3}C_0}|H|(H^2+c)^{\frac{1-\sigma}{2}}|\nabla h|^2\\ &\leqslant \left(B_2 + \frac{1}{9}H^2\right)|\nabla h|^2.\end{aligned}$$

This proves inequality (iii). □

Proof of Theorem 4.1. Define the following scalar

$$f = \left(|\nabla H|^2 + 9B_1B_2|\mathring{h}|^2 + 7B_1H^2|\mathring{h}|^2\right)\mathrm{e}^{\sigma ct} - (\eta H)^4, \quad \eta \in \left(0, \frac{1}{n}\right).$$

From Lemma 4.2 and 4.3, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)f &\leqslant \Bigg[B_1(H^2+c)|\nabla h|^2 + 9B_1B_2\left(-\frac{8}{9}|\nabla h|^2 + H^2|\mathring{h}|^2\right)\\ &\quad +7B_1\left(-\frac{7}{9}H^2|\nabla h|^2 + B_2|\nabla h|^2 + 4nH^2(H^2+c)|\mathring{h}|^2\right)\\ &\quad +\sigma c\left(|\nabla H|^2 + 9B_1B_2|\mathring{h}|^2 + 7B_1H^2|\mathring{h}|^2\right)\Bigg]\mathrm{e}^{\sigma ct}\\ &\quad -\eta^4\left(-7nH^2|\nabla h|^2 + \frac{4}{n}H^6\right)\\ &\leqslant \left(-\frac{40}{9}B_1\mathrm{e}^{\sigma ct} + 7n\eta^4\right)H^2|\nabla h|^2 + [(B_1c - B_1B_2)|\nabla h|^2 + \sigma c|\nabla H|^2]\mathrm{e}^{\sigma ct}\\ &\quad +B_3(H^2+c)^2|\mathring{h}|^2\mathrm{e}^{\sigma ct} - \frac{4\eta^4}{n}H^6.\end{aligned}$$

By Theorem 3.1, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)f \leqslant \left[C_0B_3(H^2+c)^{3-\sigma} - \frac{4\eta^4}{n}H^6\right]\mathrm{e}^{-\sigma ct}.$$

We consider the expression in the bracket of the right hand side of the above inequality, which it is a function of H. Since the coefficient of the highest-degree term is negative, the supremum $C_2(\eta)$ of this function is finite. Then we have $\frac{\partial}{\partial t}f < \Delta f + C_2(\eta)\mathrm{e}^{-\sigma ct}$. It follows from the maximum principle that f is bounded. This completes the proof of Theorem 4.1. □

5. Convergence Under Sharp Pinching Condition

In order to estimate the diameter of M_t, we need a lower bound for the Ricci curvature.

Lemma 5.1. *Suppose that M is an n-dimensional ($n \geqslant 3$) hypersurface in $\mathbb{F}^{n+1}(c)$ satisfying $|h|^2 < \gamma - \varepsilon\omega$. Then there exists a positive constant B_4 depending only on n, such that for any unit vector X in the tangent space, the Ricci curvature of M satisfies*

$$\mathrm{Ric}(X) \geqslant B_4\varepsilon(H^2 + c).$$

Proof. Using Proposition 2 of [21] and Lemma 2.3 (vi), we have

$$\begin{aligned} \mathrm{Ric}(X) &\geqslant \frac{n-1}{n}\left(nc + \frac{2}{n}H^2 - |h|^2 - \frac{n-2}{\sqrt{n(n-1)}}|H||\mathring{h}|\right) \\ &> \frac{n-1}{n}\left(nc + \frac{2}{n}H^2 - (\gamma - \varepsilon\omega) - \frac{n-2}{\sqrt{n(n-1)}}|H|\sqrt{\mathring{\gamma}}\right) \\ &\geqslant \frac{n-1}{n}\varepsilon\omega \\ &> B_4\varepsilon(H^2 + c). \end{aligned}$$

□

We also need the well-known Myers theorem.

Theorem 5.2 (**Myers Theorem**). *Let Γ be a geodesic of length at least $\pi/\sqrt{k}$ in M. If the Ricci curvature satisfies $\mathrm{Ric}(X) \geqslant (n-1)k$, for each unit vector $X \in T_xM$, at any point $x \in \Gamma$, then Γ has conjugate points.*

Now we show that M_t converges to a round point or a totally geodesic sphere.

Theorem 5.3. *If T is finite, then F_t converges to a round point as $t \to T$.*

Proof. Let $|H|_{\min} = \min_{M_t}|H|$, $|H|_{\max} = \max_{M_t}|H|$. By Theorem 4.1, for any $\eta \in \left(0, \frac{1}{n}\right)$, there exists $C(\eta) > 1$ such that $|\nabla H| < (\eta H)^2 + C(\eta)$. From Theorem 7.1 of [8], $|H|_{\max}$ becomes unbounded as $t \to T$. So, there exists a time τ depending on η, such that $|H|^2_{\max} > C(\eta)/\eta^2$ on M_τ. Then we have $|\nabla H| < 2\eta^2|H|^2_{\max}$ on M_τ.

Let x be a point in M_τ where $|H|$ achieves its maximum. Then along all geodesics of length $l = (2\eta|H|_{\max})^{-1}$ starting from x, we have $|H| > |H|_{\max} - 2\eta^2|H|^2_{\max}\cdot l = (1-\eta)|H|_{\max}$. With η small enough, Lemma 5.1 implies $\mathrm{Ric} > B_4\varepsilon(1-\eta)^2|H|^2_{\max} > (n-1)\pi^2/l^2$ on these geodesics. Then by Myers' theorem, these geodesics can reach any point of M_τ.

Then we have $|H| > (1-\eta)|H|_{\max} > \frac{1}{2\eta}$ on M_τ. Thus we can assume $H > \frac{1}{2\eta}$ on M_τ without loss of generality. Let η be sufficiently small. From Theorem 3.1, at any point in M_τ, the principal curvatures $\lambda_i \geqslant \frac{H}{n} - |\mathring{h}| > 0$, $1 \leqslant i \leqslant n$. Hence M_τ is a convex hypersurface. By Theorem 1.1 of [8], F_t will converge to a round point (see [26] Chapter 11 for the details). □

Theorem 5.4. *If $T = \infty$, then F_t converges to a totally geodesic hypersurface as $t \to \infty$.*

Proof. Firstly we prove that $|H|_{\max}$ must remains bounded for all $t \in [0, \infty)$. If not, we see that F_t will converge to a round point in finite time from the proof of Theorem 5.3.

Next we prove $|H|_{\min} = 0$ for all $t \in [0, \infty)$. Suppose $|H|_{\min} > 0$ on M_θ. From the evolution equation of H, we have $\frac{\partial}{\partial t}|H| > \Delta|H| + \frac{1}{n}|H|^3$ for $t \geqslant \theta$. By the maximum principle, we get that $|H|$ will tend to infinity in finite time. This leads to a contradiction.

By Lemma 5.1 and Myers' theorem, $\operatorname{diam} M_t$ is uniformly bounded. Applying Theorem 4.1, we have $|H| < C\mathrm{e}^{-\frac{\sigma ct}{2}}$. Therefore, from Theorem 3.1 we obtain $|h|^2 = |\mathring{h}|^2 + \frac{H^2}{n} \leqslant C^2\mathrm{e}^{-\sigma ct}$.

By Lemma 7.2 of [8], $|\nabla^m h|$ is bounded for all $m \in \mathbb{N}$ and $t \in [0, +\infty)$. Then M_t converges to a smooth limit hypersurface M_∞. Since $|h| \to 0$ as $t \to \infty$, M_∞ is totally geodesic. □

Proof of Theorem 1.1. Combining Theorem 5.3 and 5.4, we complete the proof of Theorem 1.1. □

6. Convergence Under Weakly Pinching Condition

Now we are in the position to prove Theorem 1.5. Assume that M_0 is a closed hypersurface immersed in $\mathbb{S}^{n+1}(1/\sqrt{c})$, and M_0 satisfies $|h|^2 \leqslant \gamma$.

Proof of Theorem 1.5. Recall the proof of Theorem 2.6. Letting $\varepsilon = 0$, we get

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(|h|^2 - \gamma) \leqslant\ & \left[-\frac{6}{n+2} + 2(2H^2\gamma'' + \gamma')\right]|\nabla H|^2 \\ & + 2(|h|^2 - \gamma)^2 + 2(|h|^2 - \gamma)(2\gamma - H^2\gamma' - nc) \\ & + 4cH^2 + 2\gamma^2 - 2nc\gamma - 2(\gamma + nc)H^2\gamma'.\end{aligned} \tag{6.1}$$

By Lemma 2.3 (i) and (ii), we get $\left(\frac{\partial}{\partial t} - \Delta\right)(|h|^2 - \gamma) \leqslant 2(|h|^2 - \gamma)(|h|^2 + \gamma - H^2\gamma' - nc)$. Then using the strong maximum principle, we have either $|h|^2 < \gamma$ at some time $t_0 \in (0, T)$, or $|h|^2 \equiv \gamma$ for all $t \in [0, T)$.

If $|h|^2 < \gamma$ at some t_0, it reduces to the case of Theorem 1.1.

If $|h|^2 \equiv \gamma$ for $t \in [0, T)$, we have $\left[-\frac{6}{n+2} + 2(2H^2\gamma'' + \gamma')\right]|\nabla H|^2 \equiv 0$ and $4cH^2 + 2\gamma^2 - 2nc\gamma - 2(\gamma + nc)H^2\gamma' \equiv 0$ for all t. Thus we obtain $\nabla H = 0$ and $H^2 \geqslant x_0$. By Theorem B, M_t is the isoparametric hypersurface

$$\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{c + \lambda^2}}\right) \times \mathbb{S}^1\left(\frac{\lambda}{\sqrt{c^2 + c\lambda^2}}\right),$$

where $\lambda = \frac{|H| + \sqrt{H^2 + 4(n-1)c}}{2(n-1)} > \sqrt{\frac{c}{n-1}}$.

We see that λ is the $(n-1)$-multiple principal curvature and $-\frac{c}{\lambda}$ is the other one. Thus we have $|H| = (n-1)\lambda - \frac{c}{\lambda}$ and $|h|^2 = (n-1)\lambda^2 + \frac{c^2}{\lambda^2}$. Substituting these equalities into the evolution equation of H, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((n-1)\lambda - \frac{c}{\lambda}\right) = \left((n-1)\lambda - \frac{c}{\lambda}\right)\left((n-1)\lambda^2 + \frac{c^2}{\lambda^2} + nc\right).$$

Let $r_1 = \frac{1}{\sqrt{c+\lambda^2}}$, $r_2 = \frac{\lambda}{\sqrt{c^2+c\lambda^2}}$. The above equation implies

$$\frac{\mathrm{d}}{\mathrm{d}t}r_1^2 = 2 - 2n + 2ncr_1^2.$$

Solving the ODE above, we obtain

$$r_1^2 = \frac{n-1}{nc}(1 - d\cdot \mathrm{e}^{2nct}),$$

where $d \in (0,1)$ is a constant of integration.

It's seen from the solution of r_1 that the maximal existence time $T = -\frac{\log d}{2nc}$. Hence we obtain $r_1^2 = \frac{n-1}{nc}(1-\mathrm{e}^{2nc(t-T)})$. We see that M_t converges to a great circle $\mathbb{S}^1(1/\sqrt{c})$ as $t \to T$. This completes the proof of Theorem 1.5. □

Motivated by Theorem B and Theorem 1.5, we propose the following conjecture for the mean curvature flow in a sphere.

Conjecture F. *Let $F_0 : M^n \to \mathbb{S}^{n+1}(1/\sqrt{c})$ be an n-dimensional closed hypersurface immersed in a sphere. If F_0 satisfies $|h|^2 \leqslant \alpha(n,H,c)$, then the mean curvature flow with initial value F_0 has a unique smooth solution $F : M \times [0,T) \to \mathbb{S}^{n+1}(1/\sqrt{c})$, and either*

- (*i*) *T is finite, and F_t converges to a round point as $t \to T$,*
- (*ii*) *$T = \infty$, and F_t converges to a totally geodesic sphere as $t \to \infty$, or*
- (*iii*) *T is finite, M_t is congruent to $\mathbb{S}^{n-1}(r_1(t)) \times \mathbb{S}^1(r_2(t))$, where $r_1(t)^2 + r_2(t)^2 = 1/c$, $r_1(t)^2 = \frac{n-1}{nc}(1-\mathrm{e}^{2nc(t-T)})$, and F_t converges to a great circle as $t \to T$,*
- (*iv*) *$T = \infty$, and M_t is congruent to one of the minimal hypersurfaces $\mathbb{S}^k\left(\sqrt{\frac{k}{nc}}\right) \times \mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{nc}}\right)$, $k = 1, \cdots, n-1$.*

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