THE EXTENSION AND CONVERGENCE OF MEAN CURVATURE FLOW IN HIGHER CODIMENSION

KEFENG LIU, HONGWEI XU, FEI YE, AND ENTAO ZHAO

Abstract. In this paper, we first investigate the integral curvature condition to extend the mean curvature flow of submanifolds in a Riemannian manifold with codimension $d \geq 1$, which generalizes the extension theorem for the mean curvature flow of hypersurfaces due to Le-Šešum [12] and the authors [25, 26]. Using the extension theorem, we prove two convergence theorems for the mean curvature flow of closed submanifolds in R^{n+d} under suitable integral curvature conditions.

1. Introduction

Let $F_0 : M^n \to N^{n+d}$ be a smooth immersion from an n-dimensional Riemannian manifold without boundary to an $(n+d)$-dimensional Riemannian manifold. Consider a one-parameter family of smooth immersions $F : M \times [0,T) \to N$ satisfying

$$\begin{cases} \left(\frac{\partial}{\partial t}F(x,t)\right)^{\perp} = H(x,t), \\ F(x,0) = F_0(x), \end{cases}$$

where $\left(\frac{\partial}{\partial t}F(x,t)\right)^{\perp}$ is the normal component of $\frac{\partial}{\partial t}F(x,t)$, $H(x,t)$ is the mean curvature vector of $F_t(M)$ and $F_t(x) = F(x,t)$. We call $F : M \times [0,T) \to N$ the mean curvature flow with initial value $F_0 : M \to N$. This is the general form of the mean curvature flow, which is a nonlinear weakly parabolic system and is invariant under reparametrization of M. We can find a family of diffeomorphisms $\phi_t : M \to M$ for $t \in [0,T)$ such that $\bar{F}_t = F_t \circ \phi_t : M \to N$ satisfies $\frac{\partial}{\partial t}\bar{F}(x,t) = \bar{H}(x,t)$. We will study the (reparameterized) mean curvature flow

$$\begin{cases} \frac{\partial}{\partial t}F(x,t) = H(x,t), \\ F(x,0) = F_0(x). \end{cases} \tag{1.1}$$

In [2], Brakke introduced the motion of a submanifold by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, most works have been done on hypersurfaces. Huisken [9, 10] showed that if the second fundamental form is uniformly bounded, then the mean curvature flow can be extended over the time. He then proved that if the initial hypersurface in a complete manifold with bounded

2000 *Mathematics Subject Classification.* 53C44, 53C40.

Key words and phrases. Mean curvature flow, submanifold, maximal existence time, convergence theorem, integral curvature.

Research supported by the National Natural Science Foundation of China, Grant No. 11071211; the Trans-Century Training Programme Foundation for Talents by the Ministry of Education of China, and the China Postdoctoral Science Foundation, Grant No. 20090461379.

geometry is compact and uniformly convex, then the mean curvature flow converges to a round point in finite time. Many other beautiful results have been obtained, and there are various approaches to study the mean curvature flow of hypersurfaces (see [4, 7], etc.). However, relatively little is known about the mean curvature flows of submanifolds in higher codimensions, see [16, 17, 20, 21, 22] etc. for example. Recently, Andrews-Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying suitable pinching condition in the Euclidean space.

On the other hand, Le-Šešum [12] and Xu-Ye-Zhao [25] obtained some integral conditions to extend the mean curvature flow of hypersurfaces in the Euclidean space independently. Later, Xu-Ye-Zhao [26] generalized these extension theorems to the case where the ambient space is a Riemannian manifold with bounded geometry.

For an n-dimensional submanifold M in a Riemannian manifold, we denote by g the induced metric on M. Let A and H be the second fundamental form and the mean curvature vector of M, respectively. In this paper, we first generalize the extension theorems in [12, 25, 26] to the mean curvature flow of submanifolds in a Riemannian manifold with bounded geometry.

Theorem 1.1. *Let $F_t : M^n \to N^{n+d}$ $(n \geq 3)$ be the mean curvature flow solution of closed submanifolds in a finite time interval $[0, T)$, where N has bounded geometry. If*
(i) there exist positive constants a and b such that $|A|^2 \leq a|H|^2 + b$ for $t \in [0, T)$,
(ii) $||H||_{\alpha, M\times[0,T)} = \left(\int_0^T \int_{M_t} |H|^\alpha d\mu_t dt\right)^{\frac{1}{\alpha}} < \infty$ for some $\alpha \geq n+2$,
then this flow can be extended over time T.

Let $\mathring{A}$ be the tracefree second fundamental form, which is defined by $\mathring{A} = A - \frac{1}{n} g \otimes H$. Denote by $||\cdot||_p$ the L^p-norm of a function or a tensor field. We obtain the following convergence theorems for the mean curvature flow of closed submanifolds in the Euclidean space.

Theorem 1.2. *Let $F : M^n \to R^{n+d}$ $(n \geq 3)$ be a smooth closed submanifold. Then for any fixed $p > 1$, there is a positive constant C_1 depending on $n, p, Vol(M)$ and $||A||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_1,$$

then the mean curvature flow with F as initial value has a unique solution $F : M \times [0, T) \to R^{n+d}$ in a finite maximal time interval, and F_t converges uniformly to a point $x \in R^{n+d}$ as $t \to T$. The rescaled maps $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in C^∞ to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit n-sphere in some $(n+1)$-dimensional subspace of R^{n+d}.

Theorem 1.3. *Let $F : M^n \to R^{n+d}$ $(n \geq 3)$ be a smooth closed submanifold. Then for any fixed $p > n$, there is a positive constant C_2 depending on $n, p, Vol(M)$ and $||H||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_2,$$

then the mean curvature flow with F as initial value has a unique solution $F : M \times [0, T) \to R^{n+d}$ in a finite maximal time interval, and F_t converges uniformly to a point $x \in R^{n+d}$ as $t \to T$. The rescaled maps $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in

C^∞ to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit n-sphere in some $(n+1)$-dimensional subspace of R^{n+d}.

As immediate consequences of the convergence theorems, we obtain the following differentiable sphere theorems. First let C_1 be as in Theorem 1.2, we have

Corollary 1.4. *Let $F : M^n \to R^{n+d}$ $(n \geq 3)$ be a smooth closed submanifold. If*

$$||\mathring{A}||_p < C_1,$$

for some $p > 1$, then M is diffeomorphic to the unit n-sphere.

Similarly let C_2 be as Theorem 1.3, we have

Corollary 1.5. *Let $F : M^n \to R^{n+d}$ $(n \geq 3)$ be a smooth closed submanifold. If*

$$||\mathring{A}||_p < C_2,$$

for some $p > n$, then M is diffeomorphic to the unit n-sphere.

We remark that in the above theorems and corollaries, we can replace the volume $Vol(M)$ by a positive lower bound of $|H|$ in which case our method works without change.

The paper is organized as follows. In section 2, we introduce some basic equations in submanifold theory, and recall the evolution equations of the second fundamental form along the mean curvature flow. In section 3, by using the Moser iteration and blow-up method for parabolic equations, we prove Theorem 1.1. Theorems 1.2 and 1.3 are proved in section 4. In section 5, we propose some unsolved problems on convergence of the mean curvature flow in higher codimension.

2. Preliminaries

Let $F : M^n \to N^{n+d}$ be a smooth immersion from an n-dimensional Riemannian manifold M^n without boundary to an $(n+d)$-dimensional Riemannian manifold N^{n+d}. We shall make use of the following convention on the range of indices.

$$1 \leq i, j, k, \cdots \leq n, \quad 1 \leq A, B, C, \cdots \leq n+d, \quad \textit{and} \quad n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+d.$$

The Einstein sum convention is used to sum over the repeated indices.

Suppose $\{x^i\}$ is a local coordinate system on M and $\{y^A\}$ is a local coordinate system on N. The metric $g = \sum g_{ij} dx^i \otimes dx^j$ on M induced from the metric $\langle \ , \ \rangle$ on N by F is

$$g_{ij} = \left\langle F_*\Big(\frac{\partial}{\partial x^i}\Big), F_*\Big(\frac{\partial}{\partial x^j}\Big)\right\rangle.$$

The volume form on M is $d\mu = \sqrt{\det(g_{ij})}dx$.

For any $x \in M$, denoted by N_xM the normal space of M in N at point x, which is the orthogonal complement of T_xM in $F^*T_{F(x)}N$. Denote by $\bar{\nabla}$ the Levi-Civita connection on N. The Riemannian curvature tensor $\bar{R}$ of N is defined by

$$\bar{R}(U,V)W = -\bar{\nabla}_U\bar{\nabla}_V W + \bar{\nabla}_V\bar{\nabla}_U W + \bar{\nabla}_{[U,V]}W$$

for vector fields U, V and W tangent to N. The induced connection ∇ on M is defined by

$$\nabla_X Y = (\bar{\nabla}_X Y)^\top,$$

for X, Y tangent to M, where $(\ )^\top$ denotes tangential component. Let R be the Riemannian curvature tensor of M.

Given a normal vector field ξ along M, the induced connection $\nabla^\perp$ on the normal bundle is defined by

$$\nabla^\perp_X \xi = (\bar{\nabla}_X \xi)^\perp,$$

where $(\ )^\perp$ denotes the normal component. Let $R^\perp$ denote the normal curvature tensor.

The second fundamental form is defined to be

$$A(X,Y) = (\bar{\nabla}_X Y)^\perp$$

as a section of the tensor bundle $T^*M \otimes T^*M \otimes NM$, where T^*M and NM are the cotangential bundle and the normal bundle over M. The mean curvature vector H is the trace of the second fundamental form.

The first covariant derivative of A is defined as

$$(\widetilde{\nabla}_X A)(Y,Z) = \nabla^\perp_X A(Y,Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

where $\widetilde{\nabla}$ is the connection on $T^*M \otimes T^*M \otimes NM$. Similarly, we can define the second covariant derivative of A.

Choosing orthonormal bases $\{e_i\}_{i=1}^n$ for T_xM and $\{e_\alpha\}_{\alpha=n+1}^{n+d}$ for N_xM, the components of the second fundamental form and its first and second covariant derivatives are

$$\begin{aligned} h^\alpha_{ij} =& \langle A(e_i,e_j), e_\alpha\rangle, \\ h^\alpha_{ijk} =& \langle (\widetilde{\nabla}_{e_k} A)(e_i,e_j), e_\alpha\rangle, \\ h^\alpha_{ijkl} =& \langle (\widetilde{\nabla}_{e_l}\widetilde{\nabla}_{e_k} A)(e_i,e_j), e_\alpha\rangle. \end{aligned}$$

The Laplacian of A is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$.

We define the tracefree second fundamental form $\mathring{A}$ by $\mathring{A} = A - \frac{1}{n} g \otimes H$, whose components are $\mathring{A}^\alpha_{ij} = h^\alpha_{ij} - \frac{1}{n} h^\alpha_{kk}\delta_{ij}$. Obviously, we have $\mathring{A}^\alpha_{ii} = 0$.

Let

$$\begin{aligned} R_{ijkl} &= g(R(e_i,e_j)e_k, e_l), \\ \bar{R}_{ABCD} &= \langle \bar{R}(e_A,e_B)e_C, e_D\rangle, \\ R^\perp_{ij\alpha\beta} &= \langle R^\perp(e_i,e_j)e_\alpha, e_\beta\rangle. \end{aligned}$$

Then we have the following Gauss, Codazzi and Ricci equations.

$$\begin{aligned} R_{ijkl} &= \bar{R}_{ijkl} + h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}, \\ h^\alpha_{ijk} - h^\alpha_{ikj} &= -\bar{R}_{\alpha ijk}, \\ R^\perp_{ij\alpha\beta} &= \bar{R}_{ij\alpha\beta} + h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik}. \end{aligned}$$

Suppose $F : M \times [0,T) \to N$ is the mean curvature flow with initial value $F_0 : M \to N$. We have the following evolution equations.

Lemma 2.1 ([20]). *Along the mean curvature flow we have*

$$\frac{\partial}{\partial t} d\mu_t = -|H|^2 d\mu_t, \tag{2.1}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}h^\alpha_{ij} =&\Delta h^\alpha_{ij} + \bar{R}_{\alpha ijk,k} + \bar{R}_{\alpha kik,j}\\
&- 2\bar{R}_{lijk}h^\alpha_{lk} + 2\bar{R}_{\alpha\beta jk}h^\beta_{ik} + 2\bar{R}_{\alpha\beta ik}h^\beta_{jk}\\
&- \bar{R}_{lkik}h^\alpha_{lj} - \bar{R}_{lkjk}h^\alpha_{li} + \bar{R}_{\alpha k\beta k}h^\beta_{ij}\\
&- h^\alpha_{im}(h^\beta_{jm}h^\beta_{ll} - h^\beta_{km}h^\beta_{jk})\\
&- h^\alpha_{km}(h^\beta_{jm}h^\beta_{ik} - h^\beta_{km}h^\beta_{ij})\\
&- h^\beta_{ik}(h^\beta_{jl}h^\alpha_{kl} - h^\beta_{kl}h^\alpha_{jl})\\
&- h^\alpha_{jk}h^\beta_{ij}h^\beta_{ll} + h^\beta\langle e_\alpha, \bar{\nabla}_H e_\beta\rangle,
\end{aligned}
\tag{2.2}
$$

where $\bar{R}_{ABCD,E}$ are the components of the first covariant derivative $\bar{\nabla}\bar{R}$ of $\bar{R}$.

Throughout of the paper, we assume that the submanifold is connected, and the ambient space N has bounded geometry. Recall that a Riemannian manifold is said to have bounded geometry if (i) the sectional curvature is bounded; (ii) the injective radius is bounded from below by a positive constant. We always assume that N is a Riemannian manifold with bounded geometry satisfying $-K_1 \le K_N \le K_2$ for nonnegative constant K_1, K_2, and the injective radius of N is bounded from below by a positive constant i_N.

3. The extension of mean curvature flow

In this section, we prove the extension theorem for the mean curvature flow of submanifolds in arbitrary codimension. The following Sobolev inequality can be found in [8].

Lemma 3.1 ([8])**.** *Let $M^n \subset N^{n+d}$ be an $n(\ge 2)$-dimensional closed submanifold in a Riemannian manifold N^{n+d} with codimension $d \ge 1$. Denote by i_N the positive lower bound of the injective radius of N restricted on M. Assume the sectional curvature K_N of N satisfies $K_N \le b^2$. Let h be a non-negative C^1 function on M. Then*

$$\left(\int_M h^{\frac{n}{n-1}}d\mu\right)^{\frac{n-1}{n}} \le C(n,\alpha)\int_M \Big[|\nabla h| + h|H|\Big]d\mu,$$

provided

$$b^2(1-\alpha)^{-\frac{2}{n}}(\omega_n^{-1}Vol(supp\ h))^{\frac{2}{n}} \le 1 \ and\ 2\rho_0 \le i_N,$$

where

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1} b(1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}Vol(supp\ h))^{\frac{1}{n}} & for\ b\ real,\\ (1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}Vol(supp\ h))^{\frac{1}{n}} & for\ b\ imaginary.\end{cases}$$

Here α is a free parameter, $0<\alpha<1$, and

$$C(n,\alpha) = \frac{1}{2}\pi\cdot 2^{n-2}\alpha^{-1}(1-\alpha)^{-\frac{1}{n}}\frac{n}{n-1}\omega_n^{-\frac{1}{n}}.$$

For b imaginary, we may omit the factor $\frac{1}{2}\pi$ in the definition of $C(n,\alpha)$.

Lemma 3.2. *Let $M^n \subset N^{n+d}$ be an $n(\ge 3)$-dimensional closed submanifold in a Riemannian manifold N^{n+d} with codimension $d \ge 1$. Assume $K_N \le K_2$, where K_2 is a nonnegative constant. Let f be a non-negative C^1 function on M satisfying*

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}Vol(supp\ f))^{\frac{2}{n}} \le 1, \tag{3.1}$$

$$2K_2^{-\frac{1}{2}}\sin^{-1}K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}Vol(supp\ f))^{\frac{1}{n}}\le i_N. \tag{3.2}$$

Then

$$||\nabla f||_2^2\ge\frac{(n-2)^2}{4(n-1)^2(1+s)}\left[\frac{1}{C^2(n)}||f||^2_{\frac{2n}{n-2}}-H_0^2\left(1+\frac{1}{s}\right)||f||_2^2\right],$$

where $H_0=\max_{x\in M}|H|$, $C(n)=C(n,\frac{n}{n+1})$ *and* $s>0$ *is a free parameter.*

Proof. For all $g\in C^1(M)$, $g\ge0$ satisfying (3.1) and (3.2), Lemma 3.1 implies

$$||g||_{\frac{n}{n-1}}\le C(n)\int_M(|\nabla g|+g|H|)d\mu. \tag{3.3}$$

Substituting $g=f^{\frac{2(n-1)}{n-2}}$ into (3.3) gives

$$\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-1}{n}}\le\frac{2(n-1)}{n-2}C(n)\int_M f^{\frac{n}{n-2}}|\nabla f|d\mu+C(n)\int_M Hf^{\frac{2(n-1)}{n-2}}d\mu.$$

By Hölder's inequality, we get

$$\begin{aligned}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-1}{n}}\le&C(n)\Bigg[\frac{2(n-1)}{n-2}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{1}{2}}\left(\int_M|\nabla f|^2d\mu\right)^{\frac{1}{2}}\\&+\left(\int_M H_0^2f^2d\mu\right)^{\frac{1}{2}}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{1}{2}}\Bigg]\end{aligned}$$

Then

$$\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-2}{2n}}\le C(n)\left[\frac{2(n-1)}{n-2}\left(\int_M|\nabla f|^2d\mu\right)^{\frac{1}{2}}+\left(\int_M H_0^2f^2d\mu\right)^{\frac{1}{2}}\right].$$

This implies

$$||f||^2_{\frac{2n}{n-2}}\le C^2(n)\left[\frac{4(n-1)^2(1+s)}{(n-2)^2}||\nabla f||_2^2+H_0^2\left(1+\frac{1}{s}\right)||f||_2^2\right],$$

which is desired. □

Now we establish an inequality involving the maximal value of the squared norm of the mean curvature vector and its L^{n+2}-norm in the space-time.

Proposition 3.3. *Suppose that* $F_t:M^n\to N^{n+d}$ $(n\ge3)$ *is the mean curvature flow solution for* $t\in[0,T_0]$, *where* N *has bounded geometry. Then*

$$\max_{(x,t)\in M\times[\frac{T_0}{2},T_0]}|H|^2(x,t)\le C\left(\int_0^{T_0}\int_{M_t}|H|^{n+2}d\mu_tdt\right)^{\frac{2}{n+2}},$$

where C *is a constant depending only on* n, T_0, $\sup_{(x,t)\in M\times[0,T_0]}|A|$, K_1, K_2 *and* i_N.

Proof. In the following proof, we always denote by C the constant depending on some quantities. We make use of Moser iteration for parabolic equations. Here we follow the computation in [6]. From the evolution equation of the second fundamental form in Lemma 2.1, we have the following differential inequality.

$$\frac{\partial}{\partial t}|H|^2\le\triangle|H|^2+\beta|H|^2, \tag{3.4}$$

where β is a positive constant depending only on n, $\sup_{(x,t)\in M\times[0,T_0]}|A|$, K_1 and K_2. For $0<R<R'<\infty$ and $x\in M$, we set

$$\eta=\begin{cases} 1 & x\in B_{g(0)}(x,R),\\ \eta\in[0,1]\ \mathit{and}\ |\nabla\eta|_{g(0)}\leq\frac{1}{R'-R} & x\in B_{g(0)}(x,R')\setminus B_{g(0)}(x,R),\\ 0 & x\in M\setminus B_{g(0)}(x,R'). \end{cases}$$

Since *supp* $\eta\subseteq B_{g(0)}(x,R')$, we assume that R' is sufficiently small such that η satisfies (3.1) and (3.2) with respect to $g(0)$. On the other hand, the area of some fixed subset in M is non-increasing along the mean curvature flow, hence η satisfies (3.1) and (3.2) with respect to each $g(t)$ for $t\in[0,T_0]$. Putting $f=|H|^2$ and $B(R')=B_{g(0)}(x,R')$, the inequality (3.4) implies that, for any $q\geq 2$,

(3.5)
$$\begin{aligned}\frac{1}{q}\frac{\partial}{\partial t}\int_{B(R')}f^q\eta^2d\mu_t&\leq\int_{B(R')}\left(\eta^2f^{q-1}\triangle fd\mu_t+\beta f^q\eta^2\right)d\mu_t+\int_{B(R')}\frac{1}{q}f^q\eta^2\frac{\partial}{\partial t}d\mu_t\\ &=\int_{B(R')}\left(\eta^2f^{q-1}\triangle fd\mu_t+\beta f^q\eta^2\right)d\mu_t-\int_{B(R')}\frac{1}{q}f^{q+1}\eta^2d\mu_t\\ &\leq\int_{B(R')}\left(\eta^2f^{q-1}\triangle fd\mu_t+\beta f^q\eta^2\right)d\mu_t.\end{aligned}$$

Here we have used the evolution equation of the volume form in Lemma 2.1. Integrating by parts we obtain

(3.6)
$$\begin{aligned}\int_{B(R')}\eta^2f^{q-1}\triangle fd\mu_t=&-\frac{4(q-1)}{q^2}\int_{B(R')}|\nabla(f^{\frac{q}{2}}\eta)|^2d\mu_t+\frac{4}{q^2}\int_{B(R')}|\nabla\eta|^2f^qd\mu_t\\ &+\frac{4(q-2)}{q^2}\int_{B(R')}\langle\nabla(f^{\frac{q}{2}}\eta),f^{\frac{q}{2}}\nabla\eta\rangle d\mu_t\\ \leq&-\frac{2}{q}\int_{B(R')}|\nabla(f^{\frac{q}{2}}\eta)|^2d\mu_t+\frac{2}{q}\int_{B(R')}|\nabla\eta|^2f^qd\mu_t.\end{aligned}$$

Thus by (3.5) and (3.6) we obtain

$$\begin{aligned}\frac{1}{q}\frac{\partial}{\partial t}\int_{B(R')}f^q\eta^2d\mu_t\leq&-\frac{2}{q}\int_{B(R')}|\nabla(f^{\frac{q}{2}}\eta)|^2d\mu_t\\ &+\beta\int_{B(R')}f^q\eta^2d\mu_t+\frac{2}{q}\int_{B(R')}|\nabla\eta|^2f^qd\mu_t.\end{aligned}\tag{3.7}$$

This implies

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{B(R')}f^q\eta^2d\mu_t+\int_{B(R')}|\nabla(f^{\frac{q}{2}}\eta)|^2d\mu_t\\ &\qquad\leq 2\int_{B(R')}|\nabla\eta|^2f^qd\mu_t+\beta q\int_{B(R')}f^q\eta^2d\mu_t.\end{aligned}\tag{3.8}$$

For any $0<\tau<\tau'<T_0$, define a function ψ on $[0,T_0]$ by

$$\psi(t)=\begin{cases} 0 & 0\leq t\leq\tau,\\ \frac{t-\tau}{\tau'-\tau} & \tau\leq t\leq\tau',\\ 1 & \tau'\leq t\leq T_0. \end{cases}$$

Then from (3.8) we get

$$\frac{\partial}{\partial t}\left(\psi\int_{B(R')} f^q\eta^2 d\mu_t\right)d\mu_t+\psi\int_{B(R')}|\nabla(f^{\frac{q}{2}}\eta)|^2 d\mu_t \leq 2\psi\int_{B(R')}|\nabla\eta|^2 f^q d\mu_t+(\beta q\psi+\psi')\int_{B(R')} f^q\eta^2 d\mu_t. \tag{3.9}$$

For any $t\in[\tau',T_0]$, integrating both sides of (3.9) on $[\tau,t]$ implies

$$\int_{B(R')} f^q\eta^2 d\mu_t+\int_{\tau'}^{t}\int_{B(R')}|\nabla(f^{\frac{q}{2}}\eta)|^2 d\mu_t dt \leq 2\int_{\tau}^{T_0}\int_{B(R')}|\nabla\eta|^2 f^q d\mu_t dt+\left(\beta q+\frac{1}{\tau'-\tau}\right)\int_{\tau}^{T_0}\int_{B(R')} f^q\eta^2 d\mu_t dt. \tag{3.10}$$

By the Sobolev inequality in Lemma 3.2, we obtain

$$\begin{aligned}\left(\int_{B(R')} f^{\frac{qn}{n-2}}\eta^{\frac{2n}{n-2}}d\mu_t\right)^{\frac{n-2}{n}} =&||f^{\frac{q}{2}}\eta||^2_{\frac{2n}{n-2}}\\ \leq&\frac{4(n-1)^2(1+s)C^2(n)}{(n-2)^2}||\nabla(f^{\frac{q}{2}}\eta)||_2^2\\ &+CC^2(n)\Big(1+\frac{1}{s}\Big)||f^{\frac{q}{2}}\eta||_2^2,\end{aligned} \tag{3.11}$$

where C depends on n and $\sup_{(x,t)\in M\times[0,T_0]}|A|$. Combining (3.10) and (3.11) implies that

$$\begin{aligned}&\int_{\tau'}^{T_0}\int_{B(R')} f^{q(1+\frac{2}{n})}\eta^{2+\frac{1}{n}}d\mu_t dt\\ \leq&\int_{\tau'}^{T_0}\left(\int_{B(R')} f^q\eta^2 d\mu_t\right)^{\frac{2}{n}}\left(\int_{B(R')} f^{\frac{nq}{n-2}}\eta^{\frac{2n}{n-2}}\mu_t\right)^{\frac{n-2}{n}}dt\\ \leq&\max_{t\in[\tau',T_0]}\left(\int_{B(R')} f^q\eta^2 d\mu_t\right)^{\frac{2}{n}}\times\int_{\tau}^{T_0}\Big[\frac{4(n-1)^2(1+s)C^2(n)}{(n-2)^2}||\nabla(f^{\frac{q}{2}}\eta)||_2^2\\ &+CC^2(n)\Big(1+\frac{1}{s}\Big)||f^{\frac{q}{2}}\eta||_2^2\Big]dt\\ \leq&C\max_{t\in[\tau',T_0]}\left(\int_{B(R')} f^q\eta^2 d\mu_t\right)^{\frac{2}{n}}\times\int_{\tau}^{T_0}\Big[||\nabla(f^{\frac{q}{2}}\eta)||_2^2+||f^{\frac{q}{2}}\eta||_2^2\Big]dt\\ \leq&C\Big[2\int_{\tau}^{T_0}\int_{B(R')}|\nabla\eta|^2 f^q d\mu_t dt\\ &+\Big(\beta q+\frac{1}{\tau'-\tau}\Big)\int_{\tau}^{T_0}\int_{B(R')} f^q\eta^2 d\mu_t dt\Big]^{1+\frac{2}{n}},\end{aligned} \tag{3.12}$$

where we have put $s=1$ and C is a constant depending only on n and $\sup_{(x,t)\in M\times[0,T_0]}|A|$.

Note that $|\nabla\eta|_{g(t)} \leq |\nabla\eta|^2_{g(0)}e^{lt}$, where $l = \max_{0\leq t\leq T_0} ||\frac{\partial g}{\partial t}||_{g(t)}$. Thus

$$\begin{aligned}\int_\tau^{T_0}\int_{B(R')}|\nabla\eta|^2 f^q d\mu_t dt \leq& \int_\tau^{T_0}\int_{B(R')}|\nabla\eta|^2_{g(0)}e^{lt}f^q d\mu_t dt\\ \leq&\frac{e^{CT_0}}{(R'-R)^2}\int_\tau^{T_0}\int_{B(R')}f^q d\mu_t dt,\end{aligned}$$

for some positive constant C depending on n and $\sup_{(x,t)\in M\times[0,T_0]}|A|$. This together with (3.12) implies that

$$\begin{aligned}\int_\tau^{T_0}\int_{B(R)}f^{q(1+\frac{2}{n})}d\mu_t dt \leq& C\left(\beta q+\frac{1}{\tau'-\tau}+\frac{2e^{CT_0}}{(R'-R)^2}\right)^{1+\frac{2}{n}}\\ &\times\left(\int_\tau^{T_0}\int_{B(R')}f^q d\mu_t dt\right)^{1+\frac{2}{n}},\end{aligned} \tag{3.13}$$

where C is a positive constant depending on n and $\sup_{(x,t)\in M\times[0,T_0]}|A|$.

Putting $L(q,t,R) = \int_t^{T_0}\int_{B(R)} f^q d\mu_t dt$, we have from (3.13)

$$L\Big(q\Big(1+\frac{2}{n}\Big),\tau',R\Big) \leq C\left(\beta q+\frac{1}{\tau'-\tau}+\frac{2e^{CT_0}}{(R'-R)^2}\right)^{1+\frac{2}{n}} L(q,\tau,R')^{1+\frac{2}{n}}. \tag{3.14}$$

We set

$$\mu = 1+\frac{2}{n},\ q_k = \frac{n+2}{2}\mu^k,\ \tau_k = \Big(1-\frac{1}{\mu^{k+1}}\Big)t,\ R_k = \frac{R'}{2}(1+\frac{1}{\mu^{k/2}}).$$

Then it follows from (3.14) that

$$\begin{aligned}&L(q_{k+1},\tau_{k+1},R_{k+1})^{\frac{1}{q_{k+1}}}\\ \leq& C^{\frac{1}{q_{k+1}}}\Big[\frac{(n+2)\beta}{2}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}+\frac{4e^{CT_0}}{R'^2}\cdot\frac{\mu}{(\sqrt{\mu}-1)^2}\Big]^{\frac{1}{q_k}}\\ &\times\mu^{\frac{k}{q_k}}L(q_k,\tau_k,R_k)^{\frac{1}{q_k}}.\end{aligned}$$

Hence

$$\begin{aligned}&L(q_{m+1},\tau_{m+1},R_{m+1})^{\frac{1}{q_{m+1}}}\\ \leq& C^{\sum_{k=0}^m\frac{1}{q_{k+1}}}\Big[\frac{(n+2)\beta}{2}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}+\frac{4e^{CT_0}}{R'^2}\cdot\frac{\mu}{(\sqrt{\mu}-1)^2}\Big]^{\sum_{k=0}^m\frac{1}{q_k}}\\ &\times\mu^{\sum_{k=0}^m\frac{k}{q_k}}L(q_0,\tau_0,R_0)^{\frac{1}{q_0}}.\end{aligned} \tag{3.15}$$

As $m\to+\infty$, we conclude from (3.15) that

$$f(x,t)\leq C^{\frac{n}{n+2}}\Big(C+\frac{1}{t}+\frac{e^{CT_0}}{R'^2}\Big)\Big(1+\frac{2}{n}\Big)^{\frac{n}{2}}\left(\int_0^{T_0}\int_{M_t}f^{\frac{n+2}{2}}d\mu_t dt\right)^{\frac{2}{n+2}}, \tag{3.16}$$

for some positive constant C depending on n, $\sup_{M\times[0,T]}$, K_1 and K_2.

Note that we choose R' sufficient small such that

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}Vol_{g(0)}(B(R'))^{\frac{2}{n}}\leq 1, \tag{3.17}$$

and

$$2K_2^{-\frac{1}{2}}\sin^{-1}K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}Vol_{g(0)}(B(R'))^{\frac{1}{n}}\leq i_N. \tag{3.18}$$

For $g(0)$, there is a non-positive constant K depending on n, $\max_{x\in M_0}|A|$, K_1 and K_2 such that the sectional curvature of M_0 is bounded from below by K. By the Bishop-Gromov volume comparison theorem, we have

$$Vol_{g(0)}(B(R')) \le Vol_K(B(R')),$$

where $Vol_K(B(R'))$ is the volume of a ball with radius R' in the n-dimensional complete simply connected space form with constant curvature K. Let R' be the largest number such that

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}Vol_K(B(R'))^{\frac{2}{n}} \le 1,$$

and

$$2K_2^{-\frac{1}{2}}\sin^{-1}K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}Vol_K(B(R'))^{\frac{1}{n}} \le i_N.$$

Then R' depends only on n, K_1, K_2, i_N and $\sup_{(x,t)\in M\times[0,T_0]}|A|$, and $Vol_{g(0)}(B(R'))$ satisfies (3.17) and (3.18). This together with (3.16) implies

$$\max_{(x,t)\in M\times[\frac{T_0}{2},T_0]}|H|^2(x,t) \le C\left(\int_0^{T_0}\int_{M_t}|H|^{n+2}d\mu_t dt\right)^{\frac{2}{n+2}},$$

where C is a constant depending on n, T_0 and $\sup_{(x,t)\in M\times[0,T_0]}|A|$, K_1, K_2 and i_N. □

Now we give a sufficient condition that assures the extension of the mean curvature flow of submanifolds in a Riemannian manifold.

Theorem 3.4. *Let $F_t : M^n \to N^{n+d}$ $(n \ge 3)$ be the mean curvature flow solution of closed submanifolds in a finite time interval $[0,T)$. If*
(i) there exist positive constants a and b such that $|A|^2 \le a|H|^2 + b$ for $t \in [0,T)$,
(ii) $||H||_{\alpha,M\times[0,T)} = \left(\int_0^T\int_{M_t}|H|^\alpha d\mu_t dt\right)^{\frac{1}{\alpha}} < \infty$ for some $\alpha \ge n+2$,
then this flow can be extended over time T.

Proof. By Hölder's inequality, it is sufficient to prove the theorem for $\alpha = n+2$. We will argue by contradiction.

Suppose that the solution of the mean curvature flow can't be extended over T. Then the second fundamental form becomes unbounded as $t \to T$. From assumption (i), $|H|^2$ is unbounded either.

First we choose an increasing time sequence $t^{(i)}$, $i = 1, 2, \cdots$, such that $t^{(i)} \to T$ as $i \to \infty$. Then we take a sequence of points $x^{(i)} \in M$ satisfying

$$|H|^2(x^{(i)}, t^{(i)}) = \max_{(x,t)\in M\times[0,t^{(i)}]}|H|^2(x,t).$$

Put

$$Q^{(i)} = |H|^2(x^{(i)}, t^{(i)}),$$

then $Q^{(i)}, i = 1, 2, \cdots$ is a nondecreasing sequence and $\lim_{i\to\infty} Q^{(i)} = \infty$. This together with $\lim_{i\to\infty} t^{(i)} = T > 0$ implies that there exists a positive integer i_0 such that $Q^{(i)}t^{(i)} \ge 1$ and $Q^{(i)} \ge 1$ for $i \ge i_0$. Let h be the Riemannian metric on N. For $i \ge i_0$ and $t \in [0,1]$, we consider the rescaled mean curvature flows

$$F^{(i)}(t) = F\left(\frac{t-1}{Q^{(i)}} + t^{(i)}\right) : (M, g^{(i)}(t)) \to (N, Q^{(i)}h),$$

where $g^{(i)}(t) = F^{(i)}(t)^*(Q^{(i)}h)$. Let $H_{(i)}$ and $A^{(i)} = h^{(i)}_{jk}$ be the mean curvature vector and the second fundamental form of $F^{(i)}(t)$ respectively. Then we have

$$|H_{(i)}|^2(x,t) \le 1 \quad on \quad M \times [0,1]. \tag{3.19}$$

From assumption (i) again, inequality (3.19) implies $|A^{(i)}| \le C$, where C is a constant independent of i. Since (N,h) has bounded geometry and $Q^{(i)} \ge 1$ for $i \ge i_0$, $(N, Q^{(i)}h)$ also has bounded geometry for $i \ge i_0$ with the same bounding constants as (N,h). It follows from Proposition 3.3 that for $i \ge i_0$

$$\max_{(x,t)\in M^{(i)}\times[\frac{1}{2},1]} |H_{(i)}|^2(x,t) \le C\left(\int_0^1 \int_{M_t} |H_{(i)}|^{n+2} d\mu_{g^{(i)}(t)} dt\right)^{\frac{2}{n+2}},$$

where C is a constant independent of i.

By [5], there is a subsequence of pointed mean curvature flow solutions

$$F^{(i)}(t) : (M, g^{(i)}(t), x^{(i)}) \to (N, Q^{(i)}h),\ t \in [0,1]$$

that converges to a pointed mean curvature flow solution

$$\widetilde{F}(t) : (\widetilde{M}, \widetilde{g}(t), \widetilde{x}) \to R^{n+d},\ t \in [0,1].$$

Denote by $\widetilde{H}$ the mean curvature vector of $\widetilde{F}$, $t \in [0,1]$. Then we have

$$\begin{aligned}\max_{(x,t)\in \widetilde{M}\times[\frac{1}{2},1]} |\widetilde{H}|^2(x,t) &\le \lim_{i\to\infty} C\left(\int_0^1 \int_{M_t} |H_{(i)}|^{n+2} d\mu_{g^{(i)}(t)} dt\right)^{\frac{2}{n+2}} \\ &\le \lim_{i\to\infty} C\left(\int_{t^{(i)}}^{t^{(i)}+(Q^{(i)})^{-1}} \int_{M_t} |H_{(i)}|^{n+2} d\mu dt\right)^{\frac{2}{n+2}} \\ &=0.\end{aligned} \tag{3.20}$$

The equality in (3.20) holds because $\int_0^T \int_M |H|^{n+2} d\mu_t dt < +\infty$ and $(Q^{(i)})^{-1} \to 0$ as $i \to \infty$. On the other hand, according to the choice of the points, we have

$$|\widetilde{H}|^2(\widetilde{x}, 1) = \lim_{i\to\infty} |H_{(i)}|^2(x^{(i)}, 1) = 1.$$

This is a contradiction. □

Remark 3.5. When $d = 1$, Theorem 3.4 generalizes the theorems in [12, 25, 26]. In fact, for $N^{n+1} = R^{n+1}$, we have the following computations.

(i) If $h_{ij} \ge -C$ for $(x,t) \in M \times [0,T)$ with some $C \ge 0$, let λ_i, $i = 1, \cdots, n$ be the principal curvatures. Then $\lambda_i + C \ge 0$, which implies that

$$\sum_i (\lambda_i + C)^2 \le n(\sum_i (\lambda_i + C))^2 \le 2nH^2 + 2n^3C^2.$$

On the other hand,

$$\sum_i (\lambda_i + C)^2 = |A|^2 + 2CH + nC^2 \ge |A|^2 - H^2 + (n-1)C^2.$$

Hence $|A|^2 \le (2n+1)H^2 + (2n^3 - n + 1)C^2$ for $t \in [0,T)$.

(ii) If $H > 0$ at $t = 0$, then there exists a positive constant C such that $|A|^2 \le CH^2$ at $t = 0$. By [9], we know that $H > 0$ for $t > 0$ and

$$\frac{\partial}{\partial t}\left(\frac{|A|^2}{H^2}\right) = \triangle\left(\frac{|A|^2}{H^2}\right) + \frac{2}{H}\left\langle \nabla H, \nabla\left(\frac{|A|^2}{H^2}\right)\right\rangle - \frac{2}{H^4}|H\nabla_i h_{jk} - \nabla_i H \cdot h_{jk}|^2.$$

By the maximum principle we obtain that $|A|^2/H^2$ is uniformly bounded from above by its initial data. Hence $|A|^2 \le CH^2$ for $t \in [0,T)$.

For general N^{n+1} with bounded geometry, we have similar computations. Hence our Theorem 3.4 is a generalization.

At the end of this section, we would like to propose the following

Open Problem 3.1. *Let $F_t : M \to N$ be the mean curvature flow solution of closed submanifolds in a finite time interval $[0,T)$. Suppose $||H||_{\alpha, M\times[0,T)} < \infty$ for some $\alpha \ge n+2$. Is there a positive constant ω such that the solution exists in $[0, T+\omega)$?*

4. The convergence of mean curvature flow

In this section we obtain some convergence theorems for the mean curvature flow. The extension theorem proved in section 3 will be used to give a positive lower bound on the existence time of the mean curvature flow.

We need the following Sobolev inequality for submanifolds in the Euclidean space.

Lemma 4.1. *Let M be an $n(\ge 3)$-dimensional closed submanifold in $\mathbb{R}^{n+d}$. Then for all Lipschitz functions v on M, we have*

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \le C_n \left(\int_M |\nabla v|^2 d\mu + \int_M |H|^{n+2} d\mu \int_M v^2 d\mu\right)$$

where C_n is a positive constant depending only on n.

Proof. The proof of the lemma for $d = 1$ was given in [12]. Using the same method we can prove the lemma for $d > 1$. □

Now we begin to prove the following convergence theorem for the mean curvature flow.

Theorem 4.2. *Let $F_0 : M^n \to R^{n+d}$ $(n \ge 3)$ be a smooth closed submanifold. Then for any fixed $p > 1$, there is a positive constant C_1 depending on $n, p, Vol(M_0)$ and $||A||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_1,$$

then the mean curvature flow with F_0 as initial value has a unique solution $F : M \times [0,T) \to R^{n+d}$ in a finite maximal time interval, and F_t converges uniformly to a point $x \in R^{n+d}$ as $t \to T$. The rescaled maps $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in C^∞ to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit n-sphere in some $(n+1)$-dimensional subspace of R^{n+d}.

Proof. We set $\Lambda = ||A||_{n+2}$. Denote by $T_{\max}$ the maximal existence time of the mean curvature flow with F_0 as initial value. It is easy to show that $T_{\max} < +\infty$ (see [22] for a proof).

We split the proof to several steps.

Step 1. For any fixed positive number ε, we first show that if

$$||\mathring{A}||_p < \varepsilon \tag{4.1}$$

for some $p > 1$, then $T_{\max}$ satisfies $T_{\max} > T_0$ for some positive constant T_0 depending on $n,\ p,\ \Lambda$ and independent of ε, and there hold $||A(t)||_{n+2} < 2\Lambda$, $||\mathring{A}(t)||_p < 2\varepsilon$ for $t \in [0, T_0]$.

Put

$$T=\sup\{t\in[0,T_{\max}):||A(t)||_{n+2}<2\Lambda,||\mathring{A}(t)||_p<2\varepsilon\}.$$

We consider the mean curvature flow on the time interval $[0,T)$.

By the definition of T we have $\int_{M_t}|A|^{n+2}d\mu_t\le(2\Lambda)^{n+2}$ for $t\in[0,T)$. From Lemma 4.1 we have for a Lipschitz function v,

$$\Big(\int_M v^{\frac{2n}{n-2}}d\mu\Big)^{\frac{n-2}{n}}\le C_n\Big(\int_M|\nabla v|^2d\mu+n^{\frac{n+2}{2}}(2\Lambda)^{n+2}\int_M v^2d\mu\Big). \tag{4.2}$$

From (2.2), we have

$$\frac{\partial}{\partial t}|A|^2\le\Delta|A|^2+c_1|A|^4,$$

for some positive constant c_1 depending only on n. Putting $u=|A|^2$, we have

$$\frac{\partial}{\partial t}u\le\Delta u+c_1u^2. \tag{4.3}$$

From (4.3) and (2.1) we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t}u^{\frac{n+2}{2}}d\mu_t=&\int_{M_t}\frac{n+2}{2}u^{\frac{n+2}{2}-1}\frac{\partial}{\partial t}ud\mu_t+\int_{M_t}u^{\frac{n+2}{2}}\frac{\partial}{\partial t}d\mu_t\\ =&\frac{n+2}{2}\int_{M_t}u^{\frac{n+2}{2}-1}(\Delta u+cu^2)d\mu_t-\int_{M_t}H^2u^{\frac{n+2}{2}}d\mu_t\\ \le&-\frac{4n}{n+2}\int_{M_t}|\nabla u^{\frac{n+2}{4}}|^2d\mu_t+\frac{n+2}{2}c_1\int_{M_t}u^{\frac{n+2}{2}+1}d\mu_t.\end{aligned} \tag{4.4}$$

For the second term of the right hand side of (4.4), we have by Hölder's inequality

(4.5)

$$\begin{aligned}\int_{M_t}u^{\frac{n+2}{2}+1}d\mu_t\le&\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^{\frac{2}{n+2}}\cdot\Big(\int_{M_t}(u^{\frac{n+2}{2}})^{\frac{n+2}{n}}d\mu_t\Big)^{\frac{n}{n+2}}\\ \le&\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^{\frac{2}{n+2}}\cdot\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^{\frac{2}{n+2}}\cdot\Big(\int_{M_t}(u^{\frac{n+2}{2}})^{\frac{2n}{n-2}}d\mu_t\Big)^{\frac{n-2}{n+2}}\\ \le&\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^{\frac{2}{n+2}}\cdot\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^{\frac{2}{n+2}}\\ &\times\Big[C_n\Big(\int_{M_t}|\nabla u^{\frac{n+2}{4}}|^2d\mu_t+\int_{M_t}|H|^{n+2}d\mu_t\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)\Big]^{\frac{n}{n+2}}\\ \le&\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^{\frac{4}{n+2}}\cdot\Big[C_n^{\frac{n}{n+2}}\Big(\int_{M_t}|\nabla u^{\frac{n+2}{4}}|^2d\mu_t\Big)^{\frac{n}{n+2}}\\ &+n^{\frac{n}{2}}(2\Lambda)^nC_n^{\frac{n}{n+2}}\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^{\frac{2n}{n+2}}\Big]\\ \le&n^{\frac{n}{2}}(2\Lambda)^nC_n^{\frac{n}{n+2}}\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^2+C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\epsilon^{\frac{n+2}{2}}\Big(\int_{M_t}u^{\frac{n+2}{2}}d\mu_t\Big)^2\\ &+C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}\epsilon^{-\frac{n+2}{n}}\int_{M_t}|\nabla u^{\frac{n+2}{4}}|^2d\mu_t,\end{aligned}$$

for any $\epsilon > 0$. Combining (4.4) and (4.5), we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t} u^{\frac{n+2}{2}} d\mu_t \leq & \frac{n+2}{2} c_1 \Big(n^{\frac{n}{2}} (2\Lambda)^n C_n^{\frac{n}{n+2}} + C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2} \epsilon^{\frac{n+2}{2}} \Big) \Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t \Big)^2 \\ & + \Big(\frac{n}{2} c_1 C_n^{\frac{n}{n+2}} \epsilon^{-\frac{n+2}{n}} - \frac{4n}{n+2} \Big) \int_{M_t} |\nabla u^{\frac{n+2}{4}}|^2 d\mu_t. \end{aligned} \tag{4.6}$$

Picking $\epsilon = \Bigg(\frac{n(n+2)c_1 C_n^{\frac{n}{n+2}}}{8} \Bigg)^{\frac{n}{n+2}}$, inequality (4.6) reduces to

$$\frac{\partial}{\partial t}\int_{M_t} |A|^{n+2} d\mu_t \leq c_2 \Big(\int_{M_t} |A|^{n+2} d\mu_t \Big)^2, \tag{4.7}$$

where $c_2 = \frac{n+2}{2} c_1 \Bigg(n^{\frac{n}{2}} (2\Lambda)^n C_n^{\frac{n}{n+2}} + C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2} \Bigg(\frac{n(n+2)c_1 C_n^{\frac{n}{n+2}}}{8} \Bigg)^{\frac{n}{2}} \Bigg)$.

From (4.7), we see by the maximal principle that, for $t \in [0, \min\{T, T_1\})$, where $T_1 = \frac{1-(\frac{2}{3})^{n+2}}{c_2 \Lambda^{n+2}}$, there holds

$$||A(t)||_{n+2} < \frac{3}{2}\Lambda. \tag{4.8}$$

Now we consider the evolution equation of $|\mathring{A}|^2$. By a simple computation, we have

$$\frac{\partial}{\partial t}|\mathring{A}|^2 \leq \Delta |\mathring{A}|^2 - 2|\nabla \mathring{A}|^2 + c_3 |A|^2 |\mathring{A}|^2, \tag{4.9}$$

where $c_3 \geq c_1$ is a positive constant depending only on n.

Define a tensor $\tilde{\mathring{A}}$ by $\tilde{\mathring{A}}_{ij}^{\alpha} = \mathring{A}_{ij}^{\alpha} + \sigma \eta^{\alpha} \delta_{ij}$, where $\eta^{\alpha} = 1$. Set $h_\sigma = |\tilde{\mathring{A}}| = (|\mathring{A}|^2 + nd\sigma^2)^{\frac{1}{2}}$. Then from (4.9) we have

$$\frac{\partial}{\partial t} h_\sigma \leq \Delta h_\sigma + c_3 |A|^2 h_\sigma. \tag{4.10}$$

For any $r \geq p > 1$, we have

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t = & \int_{M_t} h_\sigma^{r-1} \frac{\partial}{\partial t} h_\sigma d\mu_t + \frac{1}{r}\int_{M_t} h_\sigma^p \frac{\partial}{\partial t} d\mu_t \\ \leq & -\frac{4(r-1)}{r^2}\int_{M_t} |\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t + c_3 \int_{M_t} |A|^2 h_\sigma^r d\mu_t. \end{aligned} \tag{4.11}$$

For the second term of the right hand side of (4.11), we have the following estimate.

$$
\begin{aligned}
\int_{M_t}|A|^2 h_\sigma^r d\mu_t \leq & \left(\int_{M_t}|A|^{n+2} d\mu_t\right)^{\frac{2}{n+2}} \cdot\left(\int_{M_t} h_\sigma^{r \cdot \frac{n+2}{n}} d\mu_t\right)^{\frac{n}{n+2}} \\
\leq & (2\Lambda)^2\left(\int_{M_t} h_\sigma^r d\mu_t\right)^{\frac{2}{n+2}} \cdot\left(\int_{M_t}(h_\sigma^r)^{\frac{n}{n-2}} d\mu_t\right)^{\frac{n-2}{n} \cdot \frac{n}{n+2}} \\
\leq & (2\Lambda)^2\left(\int_{M_t} h_\sigma^r d\mu_t\right)^{\frac{2}{n+2}} \cdot\left[C_n\left(\int_M|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\right.\right. \\
& \left.\left.+n^{\frac{n+2}{2}}(2\Lambda)^{n+2} \int_M h_\sigma^r d\mu_t\right)\right]^{\frac{n}{n+2}} \\
\leq & (2\Lambda)^2\left(\int_{M_t} h_\sigma^r d\mu_t\right)^{\frac{2}{n+2}} \cdot\left[C_n^{\frac{n}{n+2}}\left(\int_M|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\right)^{\frac{n}{n+2}}\right. \\
& \left.+n^{\frac{n}{2}}(2\Lambda)^n C_n^{\frac{n}{n+2}}\left(\int_M h_\sigma^r d\mu_t\right)^{\frac{n}{n+2}}\right] \\
= & n^{\frac{n}{2}}(2\Lambda)^{n+2} C_n^{\frac{n}{n+2}} \int_M h_\sigma^r d\mu_t \\
& +(2\Lambda)^2 C_n^{\frac{n}{n+2}}\left(\int_{M_t} h_\sigma^r d\mu_t\right)^{\frac{2}{n+2}} \cdot\left(\int_M|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\right)^{\frac{n}{n+2}} \\
\leq & n^{\frac{n}{2}}(2\Lambda)^{n+2} C_n^{\frac{n}{n+2}} \int_M h_\sigma^r d\mu_t \\
& +(2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2} \mu^{\frac{n+2}{2}} \int_{M_t} h_\sigma^r d\mu_t \\
& +(2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2} \mu^{-\frac{n+2}{n}} \int_M|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t,
\end{aligned} \tag{4.12}
$$

for any $\mu > 0$. Therefore, combining (4.11) and (4.12) we have

(4.13)
$$
\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t \leq & \left(c_3(2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2}\mu^{-\frac{n+2}{n}} - \frac{4(r-1)}{r^2}\right)\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \\
& + c_3\left(n^{\frac{n}{2}}(2\Lambda)^{n+2}C_n^{\frac{n}{n+2}} + (2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2}\mu^{\frac{n+2}{2}}\right)\int_M h_\sigma^r d\mu_t.
\end{aligned}
$$

Choose $\mu = \left(\frac{c_4 r^2 p}{3rp-4p+r}\right)^{\frac{n}{n+2}}$, where $c_4 = c_3(2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2}$. Then from (4.13), we have

(4.14)
$$
\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t + \left(1-\frac{1}{p}\right)\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \leq \left(c_5 + c_6\left(\frac{r^2 p}{3rp-4p+r}\right)^{\frac{n}{2}}\right)\cdot r \cdot \int_M h_\sigma^r d\mu_t,
$$

where $c_5 = c_3(2\Lambda)^2 C_n^{\frac{n}{n+2}}$ and $c_6 = c_3 n^{\frac{n}{2}}(2\Lambda)^{n+2} C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2} \cdot c_4^{\frac{n}{2}}$.

Let $r = p$. Then (4.17) reduces to

$$
\frac{\partial}{\partial t}\int_{M_t} h_\sigma^p d\mu_t \leq c_7 \int_M h_\sigma^p d\mu_t, \tag{4.15}
$$

where $c_7=\left(c_5+c_6\left(\frac{p^2}{3p-3}\right)^{\frac{n}{2}}\right)\cdot p$.

Letting $\sigma\to 0$, (4.15) becomes

$$\frac{\partial}{\partial t}\int_{M_t}|\mathring{A}|^p d\mu_t\le c_7\int_M|\mathring{A}|^p d\mu_t.$$

This implies by the maximal principle that, for $t\in[0,\min\{T,T_2\})$, where $T_2=\frac{(n+2)\ln\frac{3}{2}}{c_7}$, there holds

$$||\mathring{A}(t)||_p<\frac{3}{2}\varepsilon. \tag{4.16}$$

Set $T_0=\min\{T_1,T_2\}$. We claim that $T>T_0$. We prove this claim by contradiction. Suppose that $T\le T_0$. Then (4.8) and (4.16) hold on $[0,T)$.

If $T<T_{\max}$, from the smoothness of the mean curvature flow we see that there exists a positive constant ϑ such that on $[0,T+\vartheta)$ we have

$$||A(t)||_{n+2}<\frac{5}{3}\Lambda,\quad ||\mathring{A}(t)||_p<\frac{5}{3}\varepsilon.$$

This contradicts to the definition of T.

If $T=T_{\max}$, we will show that the mean curvature flow can be extended over time $T_{\max}$.

From (4.14), we have

$$\frac{\partial}{\partial t}\int_{M_t}h_\sigma^r d\mu_t+\left(1-\frac{1}{p}\right)\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\le c_8 r^{n+1}\cdot\int_M h_\sigma^r d\mu_t, \tag{4.17}$$

where $c_8=\max\{\frac{c_5}{p^n},\frac{c_6}{(3p-3)^{\frac{n}{2}}}\}$.

As in the proof of Proposition 3.3, for any τ,τ' such that $0<\tau<\tau'<T_{\max}-\theta$, and for any $t\in[\tau',T_{\max}-\theta]$, where θ is a small positive constant, we have from (4.17)

$$\int_{M_t}h_\sigma^r d\mu_t+\left(1-\frac{1}{p}\right)\int_{\tau'}^t\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t dt\le\left(c_8 r^{n+1}+\frac{1}{\tau'-\tau}\right)\int_\tau^{T_{\max}-\theta}\int_{M_t}h_\sigma^r d\mu_t dt. \tag{4.18}$$

As in (3.12), we have by (4.2)

$$\begin{aligned}
&\int_{\tau'}^{T_{\max}-\theta}\int_{M_t}h_\sigma^{(1+\frac{2}{n})}d\mu_t dt\\
\le&\int_{\tau'}^{T_{\max}-\theta}\left(\int_{M_t}h_\sigma^r d\mu_t\right)^{\frac{2}{n}}\cdot\left(\int_{M_t}h_\sigma^{\frac{nr}{n-2}}d\mu_t\right)^{\frac{n-2}{n}}dt\\
\le&\max_{t\in[\tau',T_{max}-\theta]}\left(\int_{M_t}h_\sigma^r d\mu_t\right)^{\frac{2}{n}}\cdot\int_{\tau'}^{T_{\max}-\theta}\left(\int_{M_t}h_\sigma^{\frac{nr}{n-2}}d\mu_t\right)^{\frac{n-2}{n}}dt\\
\le&C_n^{\frac{n-2}{n}}\cdot\max_{t\in[\tau',T_{max}-\theta]}\left(\int_{M_t}h_\sigma^r d\mu_t\right)^{\frac{2}{n}}\\
&\times\int_{\tau'}^{T_{\max}-\theta}\left(\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t+n^{\frac{n+2}{2}}(2\Lambda)^{n+2}\int_{M_t}h_\sigma^r d\mu_t\right)dt.
\end{aligned} \tag{4.19}$$

From (4.18) and (4.19), we have

$$\begin{aligned}\int_{\tau'}^{T_{\max}-\theta}\int_{M_t} h_\sigma^{r(1+\frac{2}{n})} d\mu_t dt \leq & c_9\Big(c_8 r^{n+1}+\frac{1}{\tau'-\tau}\Big)^{1+\frac{2}{n}} \\ & \times\Big(\int_{\tau}^{T_{\max}-\theta}\int_{M_t} h_\sigma^r d\mu_t dt\Big)^{1+\frac{2}{n}},\end{aligned} \tag{4.20}$$

where $c_9 = C_n^{\frac{n-2}{n}} \cdot \max\{1, n^{\frac{n+2}{2}}(2\Lambda)^{n+2}T_0 \cdot \frac{p}{p-1}\}$.

We put

$$J(r,t) = \int_t^{T_{\max}-\theta}\int_{M_t} h_\sigma^r d\mu_t dt.$$

Then from (4.20) we have

$$J\Big(r\Big(1+\frac{2}{n}\Big), \tau'\Big) \leq c_9\Big(c_8 r^{n+1}+\frac{1}{\tau'-\tau}\Big)^{1+\frac{2}{n}} J(r,\tau)^{1+\frac{2}{n}}. \tag{4.21}$$

We let

$$\mu = 1+\frac{2}{n}, \quad r_k = p\mu^k, \quad \tau_k = \Big(1-\frac{1}{\mu^{k+1}}\Big)t.$$

Notice that $\mu > 1$. From (4.21) we have

$$J(r_{k+1},\tau_{k+1})^{\frac{1}{r_{k+1}}} \leq c_9^{\frac{1}{r_{k+1}}}\Big(c_8 p^{n+1}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}\Big)^{\frac{1}{r_k}} \mu^{\frac{k}{r_k}\cdot(n+1)} J(r_k,\tau_k)^{\frac{1}{r_k}}.$$

Hence

$$\begin{aligned}J(r_{m+1},\tau_{m+1})^{\frac{1}{r_{m+1}}} \leq & c_9^{\sum_{k=0}^m \frac{1}{r_{k+1}}}\Big(c_8 p^{n+1}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}\Big)^{\sum_{k=0}^m \frac{1}{r_k}} \\ & \cdot \mu^{(n+1)\cdot\sum_{k=0}^m \frac{k}{r_k}} J(p,t)^{\frac{1}{p}}.\end{aligned}$$

As $m \to +\infty$, we conclude that

$$h_\sigma(x,t) \leq \Big(1+\frac{2}{n}\Big)^{\frac{n(n+1)(n+2)}{4p}} c_9^{\frac{n}{2p}}\Big(c_8 p^{n+1}+\frac{(n+2)^2}{2nt}\Big)^{\frac{n+2}{2p}}\Big(\int_0^{T_{\max}-\theta}\int_{M_t} h_\sigma^p d\mu_t dt\Big)^{\frac{1}{p}}. \tag{4.22}$$

Now let $\sigma \to 0$ and $\theta \to 0$. Then we have for $t \in [\frac{T_{\max}}{2}, T_{\max})$,

$$|\mathring{A}|^2(x,t) \leq C(n,p,\Lambda,\varepsilon,T_{\max}) < +\infty.$$

This implies that

$$|A|^2 \leq a|H|^2 + b$$

on $[0,T_{\max})$ for some positive constants a and b independent of t. On the other hand, we also have

$$\int_0^{T_{\max}}\int_{M_t} |H|^{n+2} d\mu_t dt < +\infty,$$

since $T_{\max} < +\infty$. Now we apply Theorem 3.4 to conclude that the mean curvature flow can be extended over time $T_{\max}$. This is a contradiction. This completes the proof of the claim.

By the definition of T, for $t \in [0,T_0]$, we also have

$$||A(t)||_{n+2} < 2\Lambda, \quad ||\mathring{A}(t)||_p < 2\varepsilon. \tag{4.23}$$

This completes Step 1.

Step 2. We denote by $Vol(\Sigma)$ the volume of a Riemannian manifold Σ, and set $V = Vol(M_0)$. In this step we show that if we choose ε sufficiently small, then at some time $T_3 \in [\frac{T_0}{2}, T_0]$, the mean curvature is bounded from bellow by a positive constant depending on n, p, V and Λ.

Since the area of the submanifold is non-increasing along the mean curvature flow, we see that for $t \in [0, T_{\max})$, there holds

$$Vol(M_t) \le V. \tag{4.24}$$

Since M_t is a closed submanifold in the Euclidean space, by the total mean curvature inequality (for the proof see [3]), we have

$$n^n \omega_n \le \int_{M_t} |H|^n d\mu_t \le |H|^n_{\max}(t) Vol(M_t) \le |H|^n_{\max}(t) V.$$

Here $|H|_{\max}(t) = \max_{M_t} |H|(\cdot, t)$. This implies that for $t \in [0, T_{\max})$, there holds

$$|H|^2_{\max}(t) \ge n^n \omega_n V^{-1} := c_{10}. \tag{4.25}$$

On the other hand, by [18], there is a positive constant c_{11} depending only on n such that for $t \in [0, T_{\max})$, we have

$$diam(M_t) \le c_{11} \int_{M_t} |H|^{n-1} d\mu_t,$$

where $diam(M_t)$ denotes the diameter of M_t. This together with the Hölder inequality, (4.23) and (4.24) implies that for $t \in [0, T_{\max})$

$$diam(M_t) \le c_{11} n^{\frac{n-1}{2}} (2\Lambda)^{n-1} V^{\frac{3}{n+2}} := c_{12}. \tag{4.26}$$

Since $T > T_0$, we consider the mean curvature flow on $[\frac{T_0}{2}, T_0]$.

As (4.22), we have for $t \in [\frac{T_0}{2}, T_0]$

$$|\mathring{A}| \le \left(1 + \frac{2}{n}\right)^{\frac{n(n+1)(n+2)}{4p}} c_9^{\frac{n}{2p}} \left(c_8 p^{n+1} + \frac{(n+2)^2}{nT_0}\right)^{\frac{n+2}{2p}} \cdot T_0^{\frac{1}{p}} \cdot 2\varepsilon := c_{13}\varepsilon. \tag{4.27}$$

Here c_{13} depends on n, p, V, Λ and is independent of ε.

For $u = |A|^2$, since $c_1 \le c_3$, we have by (4.3)

$$\frac{\partial}{\partial t} u \le \Delta u + c_3 |A|^2 u. \tag{4.28}$$

Then by a standard Moser iteration process as for h_σ in Step 1, we have for $t \in [\frac{T_0}{2}, T_0]$

$$|A|^2 \le \left(1 + \frac{2}{n}\right)^{\frac{n(n+1)}{2}} c_{15}^{\frac{n}{n+2}} \left(c_{14} \left(\frac{n+2}{2}\right)^{n+1} + \frac{(n+2)^2}{nT_0}\right) \cdot T_0^{\frac{2}{n+2}} \cdot 2\Lambda := c_{16}. \tag{4.29}$$

Here $c_{14} = \max\{\frac{c_5 2^n}{(n+2)^n}, \frac{c_6 2^{\frac{n}{2}}}{(3n)^{\frac{n}{2}}}\}$, and $c_{15} = C_n^{\frac{n-2}{n}} \cdot \max\{1, n^{\frac{n+2}{2}} (2\Lambda)^{n+2} T_0 \cdot \frac{n+2}{n}\}$.

Set

$$G = \left(t - \frac{T_0}{2}\right) |\nabla \mathring{A}|^2 + |\mathring{A}|^2.$$

We consider the evolution inequality of G on $[\frac{T_0}{2}, T_0]$.

As in [1], we have

$$\nabla_t (\nabla \mathring{A}) = \nabla(\nabla_t \mathring{A}) + A * A * \nabla A.$$

Here ∇ is the connection on the spatial vector bundle, which for each t is agree with the Levi-Civita connection of $g(t)$. The evolution equation of $\mathring{A}$ is

$$\nabla_t \mathring{A} = \triangle \mathring{A} + A * A * A.$$

On the other hand, we have

$$\nabla(\triangle \mathring{A}) = \triangle(\nabla \mathring{A}) + A * A * \nabla A.$$

Hence

$$\nabla_t(\nabla \mathring{A}) = \triangle(\nabla \mathring{A}) + A * A * \nabla A.$$

This implies

$$\frac{\partial}{\partial t}|\nabla \mathring{A}|^2 \leq \triangle|\nabla \mathring{A}|^2 + c_{17}|A|^2|\nabla \mathring{A}|^2, \tag{4.30}$$

where c_{17} is a positive constant depending only on n. Here we have used the inequality $|\nabla A|^2 \leq \frac{3n}{2(n-1)}|\nabla \mathring{A}|^2$, which was proved in [1].

Combining (4.9) and (4.30) we have

$$\frac{\partial}{\partial t}G \leq \triangle G + \left(\left(t - \frac{T_0}{2}\right)c_{17}|A|^2 - 1\right)|\nabla \mathring{A}|^2 + c_3|A|^2|\mathring{A}|^2. \tag{4.31}$$

From (4.27), (4.29) and (4.31), we have for $t \in [\frac{T_0}{2}, T_0]$

$$\frac{\partial}{\partial t}G \leq \triangle G + \left(\left(t - \frac{T_0}{2}\right)c_{17}c_{16} - 1\right)|\nabla \mathring{A}|^2 + c_3 c_{16} c_{13}^2 \varepsilon^2. \tag{4.32}$$

Set $T_3 = \min\{T_0, \frac{T_0}{2} + \frac{1}{c_{17}c_{16}}\}$. Then $\frac{T_0}{2} \leq T_3 \leq T_0$. For $t \in [\frac{T_0}{2}, T_3]$, we have from (4.32)

$$\frac{\partial}{\partial t}G \leq \triangle G + c_3 c_{16} c_{13}^2 \varepsilon^2.$$

By the maximal principle, this implies

$$G(t) - G\left(\frac{T_0}{2}\right) \leq c_3 c_{16} c_{13}^2 \left(t - \frac{T_0}{2}\right)\varepsilon^2$$

for $t \in [\frac{T_0}{2}, T_3]$. Hence

$$\begin{aligned}\left(t - \frac{T_0}{2}\right)|\nabla \mathring{A}|^2 \leq & |\mathring{A}|^2\left(\frac{T_0}{2}\right) + c_3 c_{16} c_{13}^2\left(t - \frac{T_0}{2}\right)\varepsilon^2 \\ \leq & c_{13}^2\varepsilon^2 + c_3 c_{16} c_{13}^2 \left(t - \frac{T_0}{2}\right)\varepsilon^2.\end{aligned}$$

Then for $t \in (\frac{T_0}{2}, T_3]$, there holds

$$|\nabla \mathring{A}|^2 \leq \frac{c_{13}^2}{\left(t - \frac{T_0}{2}\right)}\varepsilon^2 + c_3 c_{16} c_{13}^2 \varepsilon^2. \tag{4.33}$$

On the other hand, from [1], we know that $|\nabla H|^2 \leq \frac{3n^2}{2(n-1)}|\nabla \mathring{A}|^2$. Therefore, (4.33) implies that at $t = T_3$, we have

$$|\nabla H|^2 \leq \frac{3n^2}{2(n-1)} \cdot \left(\frac{c_{13}^2}{\left(T_3 - \frac{T_0}{2}\right)} + c_3 c_{16} c_{13}^2\right)\varepsilon^2 := c_{18}^2 \varepsilon^2. \tag{4.34}$$

Now we consider the submanifold M_{T_3} at time T_3. Let $x, y \in M_{T_3}$ be two points such that $|H|(x, T_3) = |H|_{\min}(T_3) := \min_{M_{T_3}}|H|(\cdot, T_3)$ and $|H|(y, T_3) = |H|_{\max}(T_3) :=$

$\max_{M_{T_3}}|H|(\cdot,T_3)$. Let $l:[0,L]\to M_{T_3}$ be the shortest geodesic such that $l(0)=x$ and $l(L)=y$. Define a function $\eta:[0,L]\to R$ by $\eta(s)=|H|^2(l(s),T_3)$ for $s\in[0,L]$. Then $\eta(0)=|H|^2_{\min}(T_3)$ and $\eta(L)=|H|^2_{\max}(T_3)$. By the definition of η, we have

$$\left|\frac{d}{ds}\eta(s)\right|=\left|\frac{d}{ds}|H|^2(l(s),T_3)\right|\le\left|(\nabla|H|^2)(l(s),T_3)\right|\le\left|2(|H||\nabla H|)(l(s),T_3)\right|.$$

This together with (4.29) and (4.34) implies

$$\left|\frac{d}{ds}\eta(s)\right|\le 2n^{\frac{1}{2}}c_{16}c_{18}\varepsilon. \tag{4.35}$$

Then we have

$$\eta(L)-\eta(0)=\int_0^L\frac{d}{ds}\eta ds\le diam(M_{T_3})\cdot 2n^{\frac{1}{2}}c_{16}c_{18}\varepsilon. \tag{4.36}$$

Combining (4.25), (4.26) and (4.36), we obtain

$$|H|^2_{\min}(T_3)\ge c_{10}-c_{19}\varepsilon, \tag{4.37}$$

where $c_{19}=2n^{\frac{1}{2}}c_{16}c_{18}c_{12}$. We put

$$\varepsilon_1=\frac{c_{10}}{2c_{19}}.$$

Then if $\varepsilon\le\varepsilon_1$, (4.37) implies that

$$|H|^2_{\min}(T_3)\ge\frac{c_{10}}{2}. \tag{4.38}$$

Step 3. In this step, we finish the proof of Theorem 4.2.

Consider the submanifold M_{T_3}. Set

$$\varepsilon_2=\frac{c_{10}^{\frac{1}{2}}}{[2n(n-1)]^{\frac{1}{2}}c_{13}}\quad for\quad n\ge4,\quad and\quad \varepsilon_2=\frac{c_{10}^{\frac{1}{2}}}{3\sqrt{2}c_{13}}\quad for\quad n=3.$$

By (4.27) and (4.38), we see that if $\varepsilon\le\min\{\varepsilon_1,\varepsilon_2\}$, then

$$|A|^2(T_3)\le c_{13}^2\varepsilon_2^2+\frac{1}{n}|H|^2(T_3)\le\frac{|H|^2(T_3)}{n-1}\quad for\quad n\ge4,$$

and

$$|A|^2(T_3)\le\frac{4}{9}|H|^2(T_3)\quad for\quad n=3.$$

We pick $C_1=\min\{\varepsilon_1,\varepsilon_2\}$, which depends only on n, p, V and Λ. Then by the uniqueness of the mean curvature flow and the convergence theorem proved in [1], we conclude that the mean curvature flow with initial value F_0 converges to a round point in finite time. This completes the proof of Theorem 4.2.

□

Corollary 4.3. *Let $F_0:M^n\to R^{n+d}$ $(n\ge3)$ be a smooth closed submanifold. Suppose that the mean curvature is nowhere vanishing. Then for any fixed $p>1$, there is a positive constant C_1' depending on n, p, $\min_{M_0}|H|$ and $||A||_{n+2}$, such that if*

$$||\mathring{A}||_p<C_1',$$

then the mean curvature flow with F_0 as initial value has a unique solution $F:M\times[0,T)\to R^{n+d}$ in a finite maximal time interval, and F_t converges uniformly to a point $x\in R^{n+d}$ as $t\to T$. The rescaled maps $\widetilde{F}_t=\frac{F_t-x}{\sqrt{2n(T-t)}}$ converge in

C^∞ to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit n-sphere in some $(n+1)$-dimensional subspace of R^{n+d}.

Proof. It is easily to see that we can choose C_1 in Theorem 4.2 such that $C_1 = C_1(n,p,V,||A||_{n+2})$ depending on $n,\ p,\ ||A||_{n+2}$ and the upper bound V of the volume of M_0. Since

$$||A||_{n+2} \geq n^{\frac{1}{2}}||H||_{n+2} \geq n^{\frac{1}{2}}Vol(M_0)^{\frac{1}{n+2}}\min_{M_0}|H|,$$

we have

$$Vol(M) \leq n^{-\frac{n+2}{2}}(\min_{M_0}|H|)^{-(n+2)}||A||_{n+2}^{n+2} := V'.$$

Then by Theorem 4.2, we can pick $C_1' = C_1(n,p,V',||A||_{n+2})$, which depends on $n,\ p,\ \min_{M_0}|H|$ and $||A||_{n+2}$.

□

Theorem 4.4. *Let $F_0 : M^n \to R^{n+d}$ $(n \geq 3)$ be a smooth closed submanifold. Then for any fixed $p > n$, there is a positive constant C_2 depending on $n, p, Vol(M_0)$ and $||H||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_2,$$

then the mean curvature flow with F_0 as initial value has a unique solution $F : M \times [0,T) \to R^{n+d}$ in a finite maximal time interval, and F_t converges uniformly to a point $x \in R^{n+d}$ as $t \to T$. The rescaled maps $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in C^∞ to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit n-sphere in some $(n+1)$-dimensional subspace of R^{n+d}.

Proof. The idea to prove Theorem 4.4 is similar to the proof of Theorem 4.2. We set $\Lambda = ||H||_{n+2}$. Suppose

$$||\mathring{A}||_p < \varepsilon \tag{4.39}$$

for some fixed $p > n$ and $\varepsilon \in (0, 100]$. Set

$$T' = \sup\{t \in [0, T_{\max}) : ||H||_{n+2} < \Lambda, ||\mathring{A}||_p < 2\varepsilon\}.$$

As in the proof of Theorem 4.2, we consider the mean curvature flow on the time interval $[0, T')$.

For $|H|^2$, we have the following inequality (see [1, 20] for the derivation)

$$\frac{\partial}{\partial t}|H|^2 \leq \Delta|H|^2 - 2|\nabla H|^2 + c_{20}|A|^2|H|^2,$$

for some positive constant c_{20} depending only on n. Set $w = |H|^2$. Then

$$\frac{\partial}{\partial t}w \leq \Delta w + c_{20}|\mathring{A}|^2 w + \frac{c_{20}}{n}w^2. \tag{4.40}$$

From (4.40) we have for $r > 1$

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} w^r d\mu_t \leq & -\frac{4(r-1)}{r^2}\int_{M_t}|\nabla w^{\frac{r}{2}}|^2 d\mu_t \\ & + c_{20}\int_{M_t}|\mathring{A}|^2 w^r d\mu_t + \frac{c_{20}}{n}\int_{M_t} w^{r+1} d\mu_t.\end{aligned} \tag{4.41}$$

Now we let $r = \frac{n+2}{2}$. As in (4.5), we have

(4.42)
$$\begin{aligned}\int_{M_t} w^{\frac{n+2}{2}+1}d\mu_t \leq & C_n^{\frac{n}{n+2}}\left(\int_{M_t} w^{\frac{n+2}{2}}d\mu_t\right)^2 + C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\epsilon^{\frac{n+2}{2}}\left(\int_{M_t} w^{\frac{n+2}{2}}d\mu_t\right)^2 \\ & + C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}\epsilon^{-\frac{n+2}{n}}\int_{M_t}|\nabla w^{\frac{n+2}{4}}|^2 d\mu_t,\end{aligned}$$

for any $\epsilon > 0$.

As in (4.12), we have

$$\begin{aligned}\int_{M_t}|\mathring{A}|^2 w^{\frac{n+2}{2}}d\mu_t \leq & (200)^2 C_n^{\frac{n}{n+2}}\int_M w^{\frac{n+2}{2}}d\mu_t \\ & + (200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\mu^{\frac{n+2}{2}}\int_{M_t} w^{\frac{n+2}{2}}d\mu_t \\ & + (200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}\mu^{-\frac{n+2}{n}}\int_M|\nabla w^{\frac{n+2}{4}}|^2 d\mu_t,\end{aligned} \tag{4.43}$$

for any $\mu > 0$.

Therefore, combining (4.41), (4.42) and (4.43) we have

(4.44)
$$\begin{aligned}\frac{2}{n+2}\cdot\frac{\partial}{\partial t}\int_{M_t} w^{\frac{n+2}{2}}d\mu_t \leq & \Bigg(c_{20}(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}\mu^{-\frac{n+2}{n}} + \frac{c_{20}}{n}\cdot C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}\epsilon^{-\frac{n+2}{n}} \\ & - \frac{8n}{(n+2)^2}\Bigg)\int_{M_t}|\nabla w^{\frac{r}{2}}|^2 d\mu_t \\ & + c_{20}\left((200)^2 C_n^{\frac{n}{n+2}} + (200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\mu^{\frac{n+2}{2}}\right)\int_{M_t} w^{\frac{n+2}{2}}d\mu_t \\ & + \frac{c_{20}}{n}\left(C_n^{\frac{n}{n+2}} + C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\epsilon^{\frac{n+2}{2}}\right)\left(\int_{M_t} w^{\frac{n+2}{2}}d\mu_t\right)^2.\end{aligned}$$

Now we pick

$$\mu = \epsilon = \left(\frac{c_{20}(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2} + \frac{c_{20}}{n}\cdot C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}}{\frac{6n-4}{(n+2)^2}}\right)^{\frac{n}{n+2}}.$$

Then from (4.44), we have

$$\frac{\partial}{\partial t}\int_{M_t} w^{\frac{n+2}{2}}d\mu_t \leq c_{21}\int_{M_t} w^{\frac{n+2}{2}}d\mu_t + c_{22}\left(\int_{M_t} w^{\frac{n+2}{2}}d\mu_t\right)^2, \tag{4.45}$$

where c_{21} and c_{22} are positive constants depending only on n.

Let $\rho(t)$ be the positive solution to the following Bernoulli equation

$$\begin{aligned}\frac{d}{dt}\rho = & c_{21}\rho + c_{22}\rho^2, \\ \rho(0) = & \Lambda^{n+2}.\end{aligned}$$

Then

$$\rho(t)=\frac{e^{c_{21}t}}{\frac{1}{\Lambda^{n+2}}+\frac{c_{22}}{c_{21}}-\frac{c_{22}}{c_{21}}e^{t}},\quad t\in\left[0,\frac{2\ln\left(\frac{c_{21}}{c_{22}\Lambda^{n+2}}+1\right)}{2c_{21}}\right).$$

Let $T_1'>0$ such that $\rho(t)\leq(2\Lambda)^{n+2}$ for $t\in[0,T_1']$. Then by the maximal principle, we see that for $t\in[0,\min\{T',T_1'\})$, there holds

$$\int_{M_t}w^{\frac{n+2}{2}}d\mu_t<\Big(\frac{3}{2}\Lambda\Big)^{n+2},$$

or equivalently,

$$||H(t)||_{n+2}<\frac{3}{2}\Lambda. \tag{4.46}$$

Next, from (4.10) we have

$$\frac{\partial}{\partial t}h_\sigma\leq\Delta h_\sigma+c_3|\mathring{A}|^2h_\sigma+\frac{c_3}{n}|H|^2h_\sigma. \tag{4.47}$$

From (4.47) we have for $r>1$

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t}h_\sigma^r d\mu_t\leq&-\frac{4(r-1)}{r^2}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t\\&+c_3\int_{M_t}|\mathring{A}|^2h_\sigma^r d\mu_t+\frac{c_3}{n}\int_{M_t}|H|^2h_\sigma^r d\mu_t.\end{aligned} \tag{4.48}$$

As in (4.12), we have for $r\geq p>n$, there holds

$$\begin{aligned}\int_{M_t}|\mathring{A}|^2h_\sigma^r d\mu_t\leq&(200)^2C_n^{\frac{n}{p}}\int_M h_\sigma^r d\mu_t\\&+(200)^pC_n^{\frac{n}{p}}\cdot\frac{p-n}{p}\nu^{\frac{p}{p-n}}\int_{M_t}h_\sigma^r d\mu_t\\&+(200)^pC_n^{\frac{n}{p}}\cdot\frac{n}{p}\nu^{-\frac{p}{n}}\int_M|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t,\end{aligned} \tag{4.49}$$

and

$$\begin{aligned}\int_{M_t}|H|^2h_\sigma^r d\mu_t\leq&(200)^2C_n^{\frac{n}{n+2}}\int_M h_\sigma^r d\mu_t\\&+(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\varrho^{\frac{n+2}{2}}\int_{M_t}h_\sigma^r d\mu_t\\&+(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}\varrho^{-\frac{n+2}{n}}\int_M|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t,\end{aligned} \tag{4.50}$$

for any $\nu,\ \varrho>0$.

From (4.48), (4.49) and (4.50), we have

(4.51)
$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t \leq & \Bigg(c_3(200)^p C_n^{\frac{n}{p}}\cdot\frac{n}{p}\nu^{-\frac{p}{n}}+\frac{c_3}{n}(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}\varrho^{-\frac{n+2}{n}}\\ &-\frac{4(r-1)}{r^2}\Bigg)\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\\ &+\Bigg(c_3(200)^2C_n^{\frac{n}{p}}+c_3(200)^pC_n^{\frac{n}{p}}\cdot\frac{p-n}{p}\nu^{\frac{p}{p-n}}\\ &+\frac{c_3}{n}\cdot(200)^2C_n^{\frac{n}{n+2}}+\frac{c_3}{n}\cdot(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\varrho^{\frac{n+2}{2}}\Bigg)\int_{M_t}h_\sigma^r d\mu_t.\end{aligned}$$

Pick

$$\nu^{\frac{p}{n+2}}=\varrho=\left(\frac{c_3(200)^pC_n^{\frac{n}{p}}\cdot\frac{n}{p}+\frac{c_3}{n}(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}}{\frac{3r-4}{r^2}}\right)^{\frac{n}{n+2}}.$$

Since $r\geq p>n$, then

$$\nu^{\frac{p}{n+2}}=\varrho\leq\left(\frac{c_3(200)^pC_n^{\frac{n}{p}}\cdot\frac{n}{p}+\frac{c_3}{n}(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}}{3p-4}\right)^{\frac{n}{n+2}}\cdot r^{\frac{2n}{n+2}}:=c_{23}\cdot r^{\frac{2n}{n+2}}.$$

Then from (4.51), we have

$$\frac{\partial}{\partial t}\int_{M_t}h_\sigma^r d\mu_t+\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t\leq c_{24}r^{\frac{p+n}{p-n}}\int_{M_t}h_\sigma^r d\mu_t, \tag{4.52}$$

where

$$\begin{aligned}c_{24}=\max\Bigg\{&c_3(200)^2C_n^{\frac{n}{p}}+\frac{c_3}{n}\cdot(200)^2C_n^{\frac{n}{n+2}},\quad c_{23}^{\frac{n+2}{p-n}}\cdot c_3(200)^pC_n^{\frac{n}{p}}\cdot\frac{p-n}{p},\\ &c_{23}^{\frac{n+2}{2}}\cdot\frac{c_3}{n}\cdot(200)^{n+2}C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\Bigg\}.\end{aligned}$$

Letting $r=p$, we have from (4.52)

$$\frac{\partial}{\partial t}\int_{M_t}h_\sigma^p d\mu_t\leq c_{24}p^{\frac{p+n}{p-n}}\int_{M_t}h_\sigma^p d\mu_t, \tag{4.53}$$

Now we apply the maximal principle and let $\sigma\to 0$. Then for $t\in[0,\min\{T',T_2'\})$, where $T_2'=c_{24}^{-1}p^{-\frac{2n}{p-n}-1}$, there holds

$$||\mathring{A}(t)||_p<\frac{3}{2}\varepsilon.$$

Set $T_0'=\min\{T_1',T_2'\}$. As in the Step 1 of the proof of Theorem 4.2, we can prove that $T'>T_0'$ by contradiction. In fact, from the smoothness of the mean curvature flow we exclude the case where $T'<T_{\max}$. For the case where $T'=T_{\max}$, since we have (4.52), which has similar form as (4.17), we can apply the standard Moser process to obtain the following estimate for small $\theta>0$.

(4.54)
$$h_\sigma(x,t)\leq\left(1+\frac{2}{n}\right)^{\frac{n(n+2)(p+n)}{4p(p-n)}}c_{25}^{\frac{n}{2p}}\left(c_{24}p^{\frac{p+n}{p-n}}+\frac{(n+2)^2}{2nt}\right)^{\frac{n+2}{2p}}\left(\int_0^{T_{\max}-\theta}\int_{M_t}h_\sigma^p d\mu_t dt\right)^{\frac{1}{p}}.$$

Here $c_{25} = C_n^{\frac{n-2}{n}} \cdot \max\{1, (2\Lambda)^{n+2} T_0'\}$.

Now we let $\sigma \to 0$ and $\theta \to 0$. Then we have for $t \in [\frac{T_{\max}}{2}, T_{\max})$,

$$|\mathring{A}|^2(x,t) \leq C'(n,p,\Lambda,\varepsilon,T_{\max}) < +\infty.$$

This implies that

$$|A|^2 \leq a'|H|^2 + b'$$

on $[0, T_{\max})$ for some positive constants a' and b' independent of t. On the other hand we also have $\int_0^{T_{\max}} \int_{M_t} |H|^{n+2} d\mu_t dt < +\infty$. Applying Theorem 3.4 we conclude that the mean curvature flow can be extended over time $T_{\max}$. This is a contradiction.

We consider the mean curvature flow for $t \in [\frac{T_0'}{2}, T_0']$. As (4.54), we have

$$|\mathring{A}|(x,t) \leq \left(1+\frac{2}{n}\right)^{\frac{n(n+2)(p+n)}{4p(p-n)}} c_{25}^{\frac{n}{2p}} \left(c_{24} p^{\frac{p+n}{p-n}} + \frac{(n+2)^2}{nT_0'}\right)^{\frac{n+2}{2p}} T_0'^{\frac{1}{p}} \cdot 2\varepsilon := c_{27}\varepsilon. \tag{4.55}$$

By (4.40), we have

$$\frac{\partial}{\partial t} w \leq \Delta w + c_{20}|\mathring{A}|^2 w + \frac{c_{20}}{n}|H|^2 w.$$

Then similarly as (4.55), we get for $t \in [\frac{T_0'}{2}, T_0']$

(4.56)

$$|H|^2(x,t) \leq \left(1+\frac{2}{n}\right)^{\frac{n(n+1)}{2}} c_{28}^{\frac{n}{n+2}} \left(c_{27}(n+2)^{n+1} + \frac{(n+2)^2}{nT_0'}\right) T_0'^{\frac{2}{n+2}} \cdot (2\Lambda)^2 := c_{29}.$$

Here

$$c_{27} = \max\Big\{c_{20}(200)^2 C_n^{\frac{n}{n+2}} + \frac{c_{20}}{n} \cdot (200)^2 C_n^{\frac{n}{n+2}},\ \ c_{23}'^{\frac{n+2}{2}} \cdot c_{20}(200)^{n+2} C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2},$$
$$c_{23}'^{\frac{n+2}{2}} \cdot \frac{c_{20}}{n} \cdot (200)^{n+2} C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2}\Big\}.$$

$$c_{28} = C_n^{\frac{n-2}{n}} \cdot \max\{1, (2\Lambda)^{n+2} T_0'\},$$

and

$$c_{23}' = \left(\frac{c_3(200)^{n+2} C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2} + \frac{c_3}{n}(200)^{n+2} C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2}}{3n+2}\right)^{\frac{n}{n+2}}.$$

By (4.55) and (4.56), we have

$$|A|^2(x,t) \leq c_{27}^2 100^2 + \frac{c_{29}}{n} := c_{30}, \tag{4.57}$$

for $t \in [\frac{T_0'}{2}, T_0']$. As in Step 2 of the proof of Theorem 4.2, we have for $t \in [0, T_{\max})$, there hold

$$|H|^2_{\max}(t) \geq n^n \omega_n V^{-1} := c_{31}, \tag{4.58}$$

and

$$diam(M_t) \leq c_{11}(2\Lambda)^{n-1} V^{\frac{3}{n+2}} := c_{32}, \tag{4.59}$$

where $V = Vol(M_0)$.

Using a similar argument, for $t \in [\frac{T_0'}{2}, T_3']$, where $T_3' = \min\{T_0', \frac{T_0'}{2} + \frac{1}{c_{17}c_{30}}\}$, we have

$$|\nabla H|^2 \le \frac{3n^2}{2(n-1)} \cdot \left(\frac{c_{27}^2}{\left(t - \frac{T_0'}{2}\right)} + c_3 c_{31} c_{27}^2 \right) \varepsilon^2 := c_{33}^2 \varepsilon^2. \tag{4.60}$$

Combining (4.58), (4.59) and (4.60), we obtain that, at time T_3', there is

$$\varepsilon_1' = \frac{c_{31}}{2n^{\frac{1}{2}} c_{30} c_{32} c_{33}},$$

such that if $\varepsilon \le \varepsilon_1'$, then

$$|H|^2_{\min}(T_3') \ge \frac{c_{31}}{2}. \tag{4.61}$$

Set

$$\varepsilon_2' = \frac{c_{31}^{\frac{1}{2}}}{[2n(n-1)]^{\frac{1}{2}} c_{27}} \quad for \quad n \ge 4, \quad and \quad \varepsilon_2' = \frac{c_{31}^{\frac{1}{2}}}{3\sqrt{2} c_{27}} \quad for \quad n = 3.$$

By (4.27) and (4.38), we see that if $\varepsilon \le \min\{\varepsilon_1', \varepsilon_2', 100\}$, then

$$|A|^2(T_3') \le c_{27}^2 \varepsilon_2^2 + \frac{1}{n}|H|^2(T_3') \le \frac{|H|^2(T_3')}{n-1} \quad for \quad n \ge 4,$$

and

$$|A|^2(T_3') \le \frac{4}{9}|H|^2(T_3') \quad for \quad n = 3.$$

Then we can pick $C_2 = \min\{\varepsilon_1', \varepsilon_2', 100\}$, which depends only on n, p, V and Λ, and this completes the proof of Theorem 4.4. □

Using a similar argument as in the proof of Corollary 4.3, we have following

Corollary 4.5. *Let $F_0 : M^n \to R^{n+d}$ $(n \ge 3)$ be a smooth closed submanifold. Suppose that the mean curvature is nowhere vanishing. Then for any fixed $p > n$, there is a positive constant C_2' depending on n, p, $\min_{M_0} |H|$ and $||H||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_2',$$

then the mean curvature flow with F_0 as initial value has a unique solution $F : M \times [0, T) \to R^{n+d}$ in a finite maximal time interval, and F_t converges uniformly to a point $x \in R^{n+d}$ as $t \to T$. The rescaled maps $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in C^∞ to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit n-sphere in some $(n+1)$-dimensional subspace of R^{n+d}.

5. Open Problems

In this section, we propose several open problems for the convergence of the mean curvature flow of submanifolds. Denote by $F^{n+d}(c)$ the $(n+d)$-dimensional complete simply connected space form of constant sectional curvature c. Let M be an n-dimensional closed oriented submanifold in $F^{n+d}(c)$ with $c \ge 0$. Shiohama-Xu [15] showed that if $|A|^2 < \alpha(n, H, c)$, then M is homeomorphic to a sphere for $n \ge 4$, or diffeomorphic to a spherical space form for $n = 3$. Here

$$\alpha(n, H, c) = nc + \frac{nH^2}{2(n-1)} - \frac{n-2}{2(n-1)} \sqrt{H^2 + 4(n-1)cH^2}.$$

In [27], Xu-Zhao proved several differentiable sphere theorems for submanifolds satisfying suitable pinching conditions in a Riemannian manifold. Recently, Xu-Gu [24] strengthened Shiohama-Xu's topological sphere theorem for $c = 0$ to be a differentiable sphere theorem. Motivated by these sphere theorems and the convergence theorem for the mean curvature flow due to Andrews and Baker [1], we propose the following

Open Problem 5.1. *Let M be an n-dimensional $(n \geq 2)$ smooth closed submanifold in $F^{n+d}(c)$ with $c > 0$. Let M_t be the solution of the mean curvature flow with M as initial submanifold. Suppose M satisfies*

$$|A|^2 < \alpha(n, H, c).$$

Then one of the following holds.

a) The mean curvature flow has a smooth solution M_t on a finite time interval $0 \leq t < T$ and the M_t's converge uniformly to a round point as $t \to T$.

b) The mean curvature flow has a smooth solution M_t for all $0 \leq t < \infty$ and the M_t's converge in the C^∞-topology to a smooth totally geodesic submanifold M_∞ in $F^{n+d}(c)$.

In particular, M is diffeomorphic to the standard n-sphere.

In [14], Shiohama-Xu obtained a topological sphere theorem for closed submanifolds satisfying $||\mathring{A}||_n < C(n)$ in $F^{n+d}(c)$ with $c \geq 0$ for an explicit positive constant $C(n)$ depending only on n. The following problems arise out of this topological sphere theorem and our convergence theorems.

Open Problem 5.2. *Let M be an n-dimensional $(n \geq 2)$ smooth closed submanifold in R^{n+d} . Let M_t be the solution of the mean curvature flow with M as initial submanifold. Then there exists an positive constant $D(n)$ depending only on n, such that if M satisfies*

$$||\mathring{A}||_n < D(n),$$

then the mean curvature flow has a solution M_t's on a finite time interval $[0, T)$ and M_t converges uniformly to a round point. In particular, M is diffeomorphic to the standard n-sphere.

For any 4-dimensional compact manifold M which is homeomorphic to a sphere, we hope to show that there exists an isometric embedding of the 4-sphere into an Euclidean space such that $||\mathring{A}||_4$ is small enough in the sense of Theorems 1.2 or Open problem 5.2. In fact, Shiohama and the second author [14] proved that for any 4-dimensional compact submanifold M in an Euclidean space, we have $||\mathring{A}||_4 \geq C(\Sigma_{i=1}^3 \beta_i)^{1/4}$, where C is a universal positive constant and β_i is the i-th Betti number of M, $i = 1, 2, 3$. Therefore it's possible to isometrically embed a topological 4-sphere into an Euclidean space with small upper bound for $||\mathring{A}||_4$. If this can be done, then we can deduce that M is diffeomorphic to a sphere. This may open a way to prove the smooth Poincaré conjecture in dimension 4 which is now one of the most challenging problems in geometry and topology.

In general, for a homotopy sphere M, we can try to find its embedding in Euclidean spaces with small integral norm $||\mathring{A}||_n$. Our results on mean curvature flow of arbitrary codimension reduce the problem of proving whether M is diffeomorphic to a sphere to the problem of finding the optimal embeddings of M into Euclidean spaces.

Open Problem 5.3. *Let M be an n-dimensional $(n \geq 2)$ smooth closed submanifold in $F^{n+d}(c)$ with $c > 0$. Let M_t be the solution of the mean curvature flow with M as initial submanifold. Then there exists an positive constant $E(n)$ depending only on n, such that if M satisfies*

$$||\mathring{A}||_n < E(n),$$

then one of the following holds.

a) The mean curvature flow has a smooth solution M_t on a finite time interval $0 \leq t < T$ and the M_t's converge uniformly to a round point as $t \to T$.

b) The mean curvature flow has a smooth solution M_t for all $0 \leq t < \infty$ and the M_t's converge in the C^∞-topology to a smooth totally geodesic submanifold M_∞ in $F^{n+d}(c)$.

In particular, M is diffeomorphic to the standard n-sphere.

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