MEAN CURVATURE FLOW OF HIGHER CODIMENSION IN RIEMANNIAN MANIFOLDS

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ABSTRACT. We investigate the convergence of the mean curvature flow of arbitrary codimension in Riemannian manifolds with bounded geometry. We prove that if the initial submanifold satisfies a pinching condition, then along the mean curvature flow the submanifold contracts smoothly to a round point in finite time. As a consequence we obtain a differentiable sphere theorem for submanifolds in a Riemannian manifold.

1. INTRODUCTION

Let $F_0 : M^n \to N^{n+d}$ be a smooth immersion from an n-dimensional Riemannian manifold without boundary to an $(n+d)$-dimensional Riemannian manifold. Consider a one-parameter family of smooth immersions $F : M \times [0,T) \to N$ satisfying

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}F(x,t) = H(x,t), \\ F(x,0) = F_0(x), \end{array} \right. \tag{1.1}$$

where $H(x,t)$ is the mean curvature vector of $F_t(M)$ and $F_t(x) = F(x,t)$. We call $F : M \times [0,T) \to N$ the mean curvature flow with initial value $F_0 : M \to N$.

The mean curvature flow was proposed by Mullins [16] to describe the formation of grain boundaries in annealing metals. In [3], Brakke introduced the motion of a submanifold by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, many works on hypersurfaces have been done. Huisken [9] showed that if the initial hypersurface in the Euclidean space is compact and uniformly convex, then the mean curvature flow converges to a round point in a finite time. Later, he generalized this convergence theorem to the mean curvature flow of hypersurfaces in a Riemannian manifold in [10]. He also studied in [11] the mean curvature flow of hypersurfaces satisfying a pinching condition in a sphere.

For the mean curvature flow of submanifolds with higher codimensions, fruitful results were obtained for submanifolds with low dimension or admitting some special structures, see [19, 20, 21, 22, 23, 24, 25, 26] etc. for example. Recently, Andrews and Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying a suitable pinching condition in the Euclidean space. In [13, 15], the authors of the present paper and Ye investigated the integral

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curvature pinching conditions that assure the convergence of the mean curvature flow of submanifolds in an Euclidean space or a sphere. More recently, Baker [2] and Liu-Xu-Ye-Zhao [14] generalized Andrews-Baker's convergence theorem [1] for the mean curvature flow of submanifolds in the Euclidean space to the case of the mean curvature flow of arbitrary codimension in space forms.

In this paper, we study the convergence of the mean curvature flow of submanifolds in a general Riemannian manifold. Let $F: M \to N$ be a smooth submanifold. Suppose the sectional curvature K_N, the first covariant derivative $\bar{\nabla}\bar{R}$ of the Riemannian curvature tensor, and the injectivity radius $\mathrm{inj}(N)$ of the ambient space N satisfy

$$-K_1 \leq K_N \leq K_2, \tag{1.2}$$
$$|\bar{\nabla}\bar{R}| \leq L, \tag{1.3}$$
$$\mathrm{inj}(N) \geq i_N, \tag{1.4}$$

for nonnegative constants K_1, K_2, L and positive constant i_N. Our main result is the following:

Theorem 1.1. *Let $F: M^n \to N^{n+d}$ be an n-dimensional smooth closed and connected submanifold in an $(n+d)$-dimensional smooth complete Riemannian manifold satisfying (1.2)-(1.4). There is an explicitly computable nonnegative constant b_0 depending on n, d, K_1, K_2 and L such that if F satisfies*

$$|A|^2 < \begin{cases} \frac{4}{3n}|H|^2 - b_0, & n = 2, 3, \\ \frac{1}{n-1}|H|^2 - b_0, & n \geq 4, \end{cases} \tag{1.5}$$

then the mean curvature flow with F as initial value contracts to a round point in finite time.

Theorem 1.1 can be considered as an extension of the convergence result of Huisken in [10] to higher codimension case under a curvature pinching condition rather than the convexity of the initial hypersurface. On the other hand, if $N = \mathbb{R}^{n+d}$, then $b_0 = 0$. From Proposition 7 of [1] we see that, under their initial curvature pinching condition (1.5) is satisfied after a short time interval. Hence our Theorem 1.1 may also be considered as a generalization of the convergence theorem in [1].

By the Nash imbedding theorem, every compact Riemannian manifold can be isometrically embedded into an Euclidean space or a higher dimensional Riemannian manifold as a submanifold, in general of higher codimension. By using mean curvature flow techniques developed in this paper we can study certain important problems in Riemannian geometry which will be the content of our forthcoming works.

As a consequence of Theorem 1.1, we obtain the following differentiable sphere theorem for submanifolds in a Riemannian manifold.

Corollary 1.2. *Under the assumption of Theorem 1.1, M is diffeomorphic to the standard unit n-sphere $\mathbb{S}^n$.*

Remark 1.3. In [5, 27, 29], some differentiable sphere theorems for simply connected submanifolds in certain Riemannian manifolds were obtained by using convergence results for the Ricci flow.

The paper is organized as follows. In Section 2, we introduce some basic equations in submanifold theory, and recall some evolution equations along the mean curvature flow. In Section 3, we show that the pinching condition (1.5) for a suitable b_0 is preserved along the mean curvature flow. A pinching estimate for the tracefree second fundamental form is obtained in Section 4, which implies that the submanifold becomes spherical as t tends to the maximal existence time. We also show that, under the initial pinching condition, the maximal existence time is finite. We give an estimate of the gradient of the mean curvature in Section 5, which is used to compare the mean curvature at different points. In Section 6, we show that the submanifold shrinks to a single point in finite time. After the dilation of the ambient space and a reparameterization of time, the ambient space will converges to the Euclidian space and the submanifold will converges to a totally umbilical sphere with the same volume as the initial submanifold.

2. Preliminaries

Let $F: M^n \to N^{n+d}$ be a smooth immersion from an n-dimensional Riemannian manifold M^n without boundary to an $(n+d)$-dimensional Riemannian manifold N^{n+d}. We shall make use of the following convention on the range of indices:

$$1 \le i, j, k, \cdots \le n, \quad 1 \le A, B, C, \cdots \le n+d, \quad \textit{and} \quad n+1 \le \alpha, \beta, \gamma, \cdots \le n+d.$$

Choose a local orthonormal frame field $\{e_A\}$ in N such that e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame field of $\{e_A\}$. The metric g and the volume form $d\mu$ of M are $g = \sum_i \omega_i \otimes \omega_i$ and $d\mu = \omega_1 \wedge \cdots \wedge \omega_n$.

For any $x \in M$, denoted by N_xM the normal space of M at point x, which is the orthogonal complement of T_xM in $F^*T_{F(x)}N$. Here we identify T_xM with its image under the map F_*. Denote by $\bar\nabla$ the Levi-Civita connection on N. The Riemannian curvature tensor $\bar R$ of N is defined by

$$\bar R(U,V)W = -\bar\nabla_U \bar\nabla_V W + \bar\nabla_V \bar\nabla_U W + \bar\nabla_{[U,V]} W$$

for vector fields U, V and W tangent to N. The induced connection ∇ on M is defined by

$$\nabla_X Y = (\bar\nabla_X Y)^\top,$$

for X, Y tangent to M, where $(\ )^\top$ denotes tangential component. Let R be the Riemannian curvature tensor of M.

Given a normal vector field ξ along M, the induced connection $\nabla^\perp$ on the normal bundle is defined by

$$\nabla^\perp_X \xi = (\bar\nabla_X \xi)^\perp,$$

where $(\ )^\perp$ denotes the normal component. Let $R^\perp$ denote the normal curvature tensor.

The second fundamental form is defined to be

$$A(X,Y) = (\bar\nabla_X Y)^\perp$$

as a section of the tensor bundle $T^*M \otimes T^*M \otimes NM$, where T^*M and NM are the cotangential bundle and the normal bundle on M. The mean curvature vector H is the trace of the second fundamental form defined by $H = \mathrm{tr}_g A$.

The first covariant derivative of A is defined as

$$(\widetilde\nabla_X A)(Y,Z) = \nabla^\perp_X A(Y,Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

where $\widetilde{\widetilde{\nabla}}$ is the connection on $T^*M \otimes T^*M \otimes NM$. Similarly, we can define the second covariant derivative of A.

Under the local orthonormal frame field, the components of second fundamental form and its first and second covariant derivatives of A are defined by

$$\begin{aligned} h_{ij}^{\alpha} =& \langle A(e_i, e_j), e_\alpha \rangle, \\ \nabla_k h_{ij}^{\alpha} =& \langle (\widetilde{\nabla}_{e_k} A)(e_i, e_j), e_\alpha \rangle, \\ \nabla_l \nabla_k h_{ij}^{\alpha} =& \langle (\widetilde{\nabla}_{e_l} \widetilde{\nabla}_{e_k} A)(e_i, e_j), e_\alpha \rangle. \end{aligned}$$

The Laplacian of A is defined by $\Delta h_{ij}^{\alpha} = \sum_k \nabla_k \nabla_k h_{ij}^{\alpha}$.

We define the tracefree second fundamental form by $\mathring{A} = A - \frac{1}{n} g \otimes H$, whose components are $\mathring{h}_{ij}^{\alpha} = h_{ij}^{\alpha} - \frac{1}{n} H^\alpha \delta_{ij}$, where $H^\alpha = \sum_k h_{kk}^{\alpha}$. Obviously, we have $\sum_i \mathring{h}_{ii}^{\alpha} = 0$.

Let

$$\begin{aligned} R_{ijkl} &= g(R(e_i, e_j)e_k, e_l), \\ \bar{R}_{ABCD} &= \langle \bar{R}(e_A, e_B)e_C, e_D \rangle, \\ R_{ij\alpha\beta}^{\perp} &= \langle R^{\perp}(e_i, e_j)e_\alpha, e_\beta \rangle. \end{aligned}$$

Then we have the following Gauss, Codazzi and Ricci equations.

$$\begin{aligned} R_{ijkl} &= \bar{R}_{ijkl} + \sum_\alpha \left(h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha} \right), \\ \nabla_k h_{ij}^{\alpha} - \nabla_j h_{ik}^{\alpha} &= -\bar{R}_{\alpha ijk}, \\ R_{ij\alpha\beta}^{\perp} &= \bar{R}_{ij\alpha\beta} + \sum_k \left(h_{ik}^{\alpha} h_{jk}^{\beta} - h_{jk}^{\alpha} h_{ik}^{\beta} \right). \end{aligned}$$

It is standard to show the short-time existence of the mean curvature flow (1.1) with closed initial value. Since the mean curvature flow is a (degenerate) quasilinear parabolic evolution equation, one can obtain the short-time existence by using the Nash-Moser implicit function theorem as in [6]. One can also use the De Turck trick to modify the mean curvature flow equation to a strongly parabolic equation, and the short-time existence follows from the standard parabolic theory.

Let $F : M^n \times [0, T) \to N^{n+d}$ be a mean curvature flow solution. We have the following evolution equations.

Lemma 2.1. *Along the mean curvature flow we have*

$$\frac{\partial}{\partial t} d\mu_t = -|H|^2 d\mu_t, \tag{2.1}$$

(2.2)
$$
\begin{aligned}
\frac{\partial}{\partial t}|A|^2 =&\Delta|A|^2 - 2|\nabla A|^2 + 2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big[\sum_p \Big(h_{ip}^{\alpha}h_{jp}^{\beta} - h_{jp}^{\alpha}h_{ip}^{\beta}\Big)\Big]^2\\
&+ 4\sum_{i,j,p,q}\bar{R}_{ipjq}\Big(\sum_{\alpha} h_{pq}^{\alpha}h_{ij}^{\alpha}\Big) - 4\sum_{j,k,p}\bar{R}_{kjkp}\Big(\sum_{i,\alpha} h_{pi}^{\alpha}h_{ij}^{\alpha}\Big)\\
&+ 2\sum_{k,\alpha,\beta}\bar{R}_{k\alpha k\beta}\Big(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}\Big) - 8\sum_{j,p,\alpha,\beta}\bar{R}_{jp\alpha\beta}\Big(\sum_{i} h_{ip}^{\alpha}h_{ij}^{\beta}\Big)\\
&+ 2\sum_{i,j,k,\beta}\bar{\nabla}_k\bar{R}_{kij\beta}h_{ij}^{\beta} - 2\sum_{i,j,k,\beta}\bar{\nabla}_i\bar{R}_{jkk\beta}h_{ij}^{\beta},
\end{aligned}
$$

$$
\frac{\partial}{\partial t}|H|^2 =\Delta|H|^2 - 2|\nabla H|^2 + 2\sum_{i,j}\Big(\sum_{\alpha} H^{\alpha}h_{ij}^{\alpha}\Big)^2 + 2\sum_{k,\alpha,\beta}\bar{R}_{k\alpha k\beta}H^{\alpha}H^{\beta}. \tag{2.3}
$$

Throughout this paper, we assume that the submanifold is connected, and the ambient space N satisfies (1.2)-(1.4) for nonnegative constants K_1, K_2, L and positive constant i_N. By Berger's inequality (see [4] for a proof), we see that the $|\bar{R}_{ACBC}| \le \frac{1}{2}(K_1+K_2)$ for $A \ne B$ and $|\bar{R}_{ABCD}| \le \frac{2}{3}(K_1+K_2)$ for all distinct indices A, B, C, D.

3. A preserved curvature pinching condition

In this section, we prove that the pinching condition (1.5) for a suitable $b_0 > 0$ is preserved along the mean curvature. But first we prove the following lemma.

Lemma 3.1. *For any $\eta > 0$ we have the following inequalities.*

$$
|\nabla A|^2 \ge \left(\frac{3}{n+2} - \eta\right)|\nabla H|^2 - \frac{2}{n+2}\left(\frac{2}{n+2}\eta^{-1} - \frac{n}{n-1}\right)|w|^2, \tag{3.1}
$$

$$
\begin{aligned}
|\nabla A|^2 - \frac{1}{n}|\nabla H|^2 \ge& \frac{n-1}{2n+1}|\nabla A|^2 - \frac{2n}{(n-1)(2n+1)}|w|^2\\
\ge& \frac{n-1}{2n+1}|\nabla A|^2 - C(n,d)(K_1+K_2)^2.
\end{aligned} \tag{3.2}
$$

Here $w = \sum_{i,j,\alpha}\bar{R}_{\alpha jij}e_i \otimes \omega_\alpha$ and $C(n,d) = \frac{n^4 d}{2(n-1)(2n+1)}$.

Proof. Inequality (3.2) follows from (3.1) with $\eta = \frac{n-1}{n(n+2)}$. To prove (3.1), we set

$$
\begin{aligned}
E_{ijk} =& \frac{1}{n+2}(\nabla_i H g_{jk} + \nabla_j H g_{ik} + \nabla_k H g_{ij})\\
&- \frac{2}{(n+2)(n-1)}w_i g_{jk} + \frac{n}{(n+2)(n-1)}(w_j g_{ik} + w_k g_{ij}).
\end{aligned}
$$

Let $F_{ijk} = \nabla_i h_{jk} - E_{ijk}$. By the Codazzi equation we have $\langle E_{ijk}, F_{ijk}\rangle = 0$. Hence $|\nabla A|^2 \ge |E|^2$. By a direct computation, we have

$$
|E|^2 = \frac{3}{n+2}|\nabla H|^2 + \frac{2n}{(n+2)(n-1)}|w|^2 + \frac{4}{n+2}\langle\nabla H, w\rangle.
$$

Then (3.1) follows from Schwartz's inequality, Young's inequality and Berger's inequality. $\square$

Theorem 3.2. *There is a positive constant b_1 depending on n, d, K_1, K_2, L and a such that if $|A|^2 \le a|H|^2 - b$ holds for some constant $a \le \frac{4}{3n}$ and $b > b_1$ at $t = 0$, then it remains true for $t > 0$.*

Proof. Set $Q = |A|^2 - a|H|^2 + b$, where $a \le \frac{4}{3n}$, $b > b_1$, and b_1 is a positive constant to be determined. We will compute the evolution of Q along the mean curvature flow, and show that if $Q = 0$ at a point in the space-time, then $(\frac{\partial}{\partial t} - \Delta)Q$ is negative at this point. By the maximum principle, the theorem follows.

By Lemma 2.1, we have

$$\frac{\partial}{\partial t}Q = \Delta Q - 2(|\nabla A|^2 - a|\nabla H|^2) + 2R_1 - 2aR_2 + \mathrm{P}_a, \tag{3.3}$$

where

$$R_1 = \sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta\Big)^2 + \sum_{i,j,\alpha,\beta}\Big[\sum_p \Big(h_{ip}^\alpha h_{jp}^\beta - h_{jp}^\alpha h_{ip}^\beta\Big)\Big]^2,$$

$$R_2 = \sum_{i,j}\Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)^2,$$

and $\mathrm{P}_a = I + II + III + IV$ with

$$I = 4\sum_{i,j,p,q} \bar{R}_{ipjq}\Big(\sum_\alpha h_{pq}^\alpha h_{ij}^\alpha\Big) - 4\sum_{j,k,p}\bar{R}_{kjkp}\Big(\sum_{i,\alpha} h_{pi}^\alpha h_{ij}^\alpha\Big),$$

$$II = 2\sum_{k,\alpha,\beta}\bar{R}_{k\alpha k\beta}\Big(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta\Big) - 2a\sum_{k,\alpha,\beta}\bar{R}_{k\alpha k\beta}H^\alpha H^\beta,$$

$$III = -8\sum_{j,p,\alpha,\beta}\bar{R}_{jp\alpha\beta}\Big(\sum_i h_{ip}^\alpha h_{ij}^\beta\Big),$$

$$IV = 2\sum_{i,j,k,\beta}\bar{\nabla}_k\bar{R}_{kij\beta}h_{ij}^\beta - 2\sum_{i,j,k,\beta}\bar{\nabla}_i\bar{R}_{jkk\beta}h_{ij}^\beta.$$

At the point where $Q = 0$, the mean curvature vector is not zero. Hence we choose $e_{n+1} = \frac{H}{|H|}$. The second fundamental form can be written as $A = \sum_\alpha h^\alpha e_\alpha$, where h^α, $n+1 \le \alpha \le n+d$, are symmetric 2-tensors. By the choice of e_{n+1}, we see that $H^{n+1} = \mathrm{tr}h^{n+1} = |H|$ and $H^\alpha = \mathrm{tr}h^\alpha = 0$ for $\alpha \ge n+2$. The tracefree second fundamental form may be rewritten as $\mathring{A} = \sum_\alpha \mathring{h}^\alpha e_\alpha$, where $\mathring{h}^{n+1} = h^{n+1} - \frac{|H|}{n}\mathrm{Id}$ and $\mathring{h}^\alpha = h^\alpha$ for $\alpha \ge n+2$. We set

$$|A|_H^2 = |h^{n+1}|^2, \quad |A|_I^2 = \sum_{\alpha \ge n+2}|h^\alpha|^2 = |A|^2 - |A|_H^2,$$

$$|\mathring{A}|_H^2 = |\mathring{h}^{n+1}|^2, \quad |\mathring{A}|_I^2 = \sum_{\alpha \ge n+2}|\mathring{h}^\alpha|^2 = |\mathring{A}|^2 - |\mathring{A}|_H^2.$$

Notice that $|A|_H^2 = |\mathring{A}|_H^2 + \frac{|H|^2}{n}$ and $|A|_I^2 = |\mathring{A}|_I^2$.

Since $Q = 0$ at this point, we have $|H|^2 = \frac{|\mathring{A}|^2 + b}{a - \frac{1}{n}}$. By the computation in [1] we have

$$\begin{aligned} 2R_1 - 2aR_2 \le & \Big(6 - \frac{2}{n(a - \frac{1}{n})}\Big)|\mathring{A}|_H^2|\mathring{A}|_I^2 + \Big(3 - \frac{2}{n(a-\frac{1}{n})}\Big)|\mathring{A}|_I^4 \\ & - \frac{2nab}{n(a-\frac{1}{n})}|\mathring{A}|_H^2 - \frac{4b}{n(a-\frac{1}{n})}|\mathring{A}|_I^2 - \frac{2b^2}{n(a-\frac{1}{n})}. \end{aligned} \tag{3.4}$$

To estimate I, we fix α and choose e_i's such that $h^\alpha_{ij}=\lambda^\alpha_i\delta_{ij}$. Then

$$\begin{aligned}
&4\sum_{i,j,p,q}\bar R_{ipjq}h^\alpha_{pq}h^\alpha_{ij}-4\sum_{j,k,p}\bar R_{kjkp}\Big(\sum_i h^\alpha_{pi}h^\alpha_{ij}\Big)\\
&=4\sum_{i,p}\bar R_{ipip}\Big(\lambda^\alpha_i\lambda^\alpha_p-(\lambda^\alpha_i)^2\Big)\\
&=-2\sum_{i,p}\bar R_{ipip}\Big(\lambda^\alpha_i-\lambda^\alpha_p\Big)^2\\
&\le 4nK_1|\mathring{h}^\alpha|^2.
\end{aligned}$$

Hence we get

$$I\le 4nK_1(|\mathring{A}|^2_H+|\mathring{A}|^2_I). \tag{3.5}$$

By the choice of e_{n+1}, we have

$$II=II_1+II_2+II_3,$$

where

$$II_1=2\sum_{i,j,k}\bar R_{kn+1kn+1}(h^{n+1}_{ij})^2-2a\sum_k\bar R_{kn+1kn+1}(H^{n+1})^2,$$

$$II_2=4\sum_{k,\alpha\ge n+2}\bar R_{k\alpha kn+1}\Big(\sum_{i,j}h^\alpha_{ij}h^{n+1}_{ij}\Big)-4a\sum_{k,\alpha\ge n+2}\bar R_{k\alpha kn+1}H^{n+1}H^\alpha,$$

$$II_3=2\sum_{k,\alpha,\beta\ge n+2}\bar R_{k\alpha k\beta}\Big(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\Big)-2a\sum_{k,\alpha,\beta\ge n+2}\bar R_{k\alpha k\beta}H^\alpha H^\beta.$$

Since $|H|^2=\frac{|\mathring{A}|^2+b}{a-\frac1n}$ at that point, we have

$$\begin{aligned}
II_1\le&2nK_2|A|^2_H+2naK_1|H|^2\\
=&2nK_2|\mathring{A}|^2_H+2(naK_1+K_2)\cdot\frac{|\mathring{A}|^2+b}{a-\frac1n}\\
=&\Big(2nK_2+\frac{2(naK_1+K_2)}{a-\frac1n}\Big)|\mathring{A}|^2_H+\frac{2(naK_1+K_2)}{a-\frac1n}|\mathring{A}|^2_I+\frac{2(naK_1+K_2)b}{a-\frac1n}.
\end{aligned}$$

Since $H^\alpha=0$ for $\alpha\ge n+2$, we have the following estimates for II_2 and II_3.

$$\begin{aligned}
II_2=&4\sum_{k,\alpha\ge n+2}\bar R_{k\alpha kn+1}\Big(\sum_{i,j}h^\alpha_{ij}h^{n+1}_{ij}\Big)\\
=&4\sum_{k,\alpha\ge n+2}\bar R_{k\alpha kn+1}\Big(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^{n+1}_{ij}\Big)\\
\le&(K_1+K_2)\sum_{k,\alpha\ge n+2}\Big(\frac1\varrho\sum_{i,j}(\mathring{h}^\alpha_{ij})^2+\varrho\sum_{i,j}(\mathring{h}^{n+1}_{ij})^2\Big)\\
=&\varrho n(d-1)(K_1+K_2)|\mathring{A}|^2_H+\frac n\varrho(K_1+K_2)|\mathring{A}|^2_I,
\end{aligned}$$

for any positive constant ϱ.

$$\begin{aligned}
II_3 =&2\sum_{k,\alpha,\beta\geq n+2}\bar{R}_{k\alpha k\beta}\Big(\sum_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta}\Big)\\
=&2\sum_{k,\alpha\geq n+2}\bar{R}_{k\alpha k\alpha}\Big(\sum_{i,j}h_{ij}^{\alpha})^2\Big)+2\sum_{k,\alpha,\beta\geq n+2,\alpha\neq\beta}\bar{R}_{k\alpha k\beta}\Big(\sum_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta}\Big)\\
\leq&2nK_2|\mathring{A}|_I^2+2\sum_{k,\alpha,\beta\geq n+2,\alpha\neq\beta}\bar{R}_{k\alpha k\beta}\Big(\sum_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta}\Big)\\
\leq&2nK_2|\mathring{A}|_I^2+\sum_{k,\alpha,\beta\geq n+2,\alpha\neq\beta}|\bar{R}_{k\alpha k\beta}|\Big(\sum_{i,j}(h_{ij}^{\alpha})^2+(h_{ij}^{\beta})^2\Big)\\
\leq&2nK_2|\mathring{A}|_I^2+(K_1+K_2)\sum_{i,j,k,\alpha,\beta\geq n+2,\alpha\neq\beta}(h_{ij}^{\alpha})^2\\
=&2nK_2|\mathring{A}|_I^2+n(d-2)(K_1+K_2)|\mathring{A}|_H^2.
\end{aligned}$$

Hence we get the following estimate for II.

$$\begin{aligned}
II\leq&\bigg(2nK_2+\frac{2(naK_1+K_2)}{a-\frac{1}{n}}+[\varrho n(d-1)+n(d-2)](K_1+K_2)\bigg)|\mathring{A}|_H^2\\
&+\bigg(\frac{2(naK_1+K_2)}{a-\frac{1}{n}}+\frac{n}{\varrho}(K_1+K_2)+2nK_2\bigg)|\mathring{A}|_I^2\\
&+\frac{2(naK_1+K_2)b}{a-\frac{1}{n}}.
\end{aligned}\tag{3.6}$$

For III, we have

$$III=III_1+III_2,$$

where

$$III_1=-16\sum_{j,p,\alpha\geq n+2}\bar{R}_{jp\alpha n+1}\Big(\sum_i h_{ip}^{\alpha}h_{ij}^{n+1}\Big),$$

$$III_2=-8\sum_{j,p,\alpha,\beta\geq n+2,\alpha\neq\beta}\bar{R}_{jp\alpha\beta}\Big(\sum_i h_{ip}^{\alpha}h_{ij}^{\beta}\Big).$$

We have the following estimates for arbitrary positive constant ρ.

$$\begin{aligned}
III_1=&-16\sum_{j,p,\alpha\geq n+2}\bar{R}_{jp\alpha n+1}\Big(\sum_i\mathring{h}_{ip}^{\alpha}\Big(\mathring{h}_{ij}^{n+1}+\frac{|H|}{n}\delta_{ij}\Big)\Big)\\
=&-16\sum_{j\neq p,\alpha\geq n+2}\bar{R}_{jp\alpha n+1}\Big(\sum_i\mathring{h}_{ip}^{\alpha}\mathring{h}_{ij}^{n+1}\Big)\\
\leq&\frac{16}{3}(K_1+K_2)\sum_{j\neq p,i,\alpha\geq n+2}\Big(\frac{1}{\rho}(\mathring{h}_{ip}^{\alpha})^2+\rho(\mathring{h}_{ij}^{n+1})^2\Big)\\
=&\frac{16}{3}\rho(n-1)(d-1)(K_1+K_2)|\mathring{A}|_H^2+\frac{16}{3\rho}(n-1)(K_1+K_2)|\mathring{A}|_I^2.
\end{aligned}$$

Here for the second equality, we use the fact that $\sum_{j,p}\bar{R}_{jp\alpha n+1}\mathring{h}_{jp}^{\alpha}=0$ since $\bar{R}_{jp\alpha n+1}$ is anti-symmetric for j,p and $\mathring{h}_{jp}^{\alpha}$ is symmetric for j,p.

For any fixed $\beta \geq n+2$, we choose e_i's such that $\mathring{h}^{\beta}_{ij} = \mathring{\lambda}^{\beta}_i \delta_{ij}$. Then

$$\begin{aligned}
III_2 =& -8 \sum_{\beta \geq n+2} \sum_{j \neq p, \alpha \geq n+2, \alpha \neq \beta} \bar{R}_{jp\alpha\beta} \mathring{h}^{\alpha}_{jp} \mathring{\lambda}^{\beta}_j \\
\leq& \frac{8}{3}(K_1 + K_2) \sum_{\beta \geq n+2} \Bigg((n-1)^{\frac{1}{2}} \sum_{j \neq p, \alpha \geq n+2, \alpha \neq \beta} (\mathring{h}^{\alpha}_{jp})^2 + \frac{1}{(n-1)^{\frac{1}{2}}} \sum_{j \neq p, \alpha \geq n+2, \alpha \neq \beta} (\mathring{\lambda}^{\beta}_j)^2 \Bigg) \\
\leq& \frac{8}{3}(K_1 + K_2) \Bigg((n-1)^{\frac{1}{2}} (d-2) |\mathring{A}|^2_I + \sum_{\beta \geq n+2} (n-1)^{\frac{1}{2}} (d-2) |\mathring{h}^{\beta}|^2 \Bigg) \\
=& \frac{8}{3}(n-1)^{\frac{1}{2}} (d-2)(K_1 + K_2) |\mathring{A}|^2_I.
\end{aligned}$$

Hence we have

$$\begin{aligned}
III \leq& \frac{16}{3} \rho (n-1)(d-1)(K_1 + K_2) |\mathring{A}|^2_H \\
&+ \Big(\frac{16}{3\rho}(n-1) + \frac{8}{3}(n-1)^{\frac{1}{2}} (d-2) \Big) (K_1 + K_2) |\mathring{A}|^2_I.
\end{aligned} \tag{3.7}$$

For IV, we choose e_i's such that $h^{n+1}_{ij} = \lambda_i \delta_{ij}$. If $K_1 + K_2 \neq 0$, we have

$$\begin{aligned}
IV =& 2 \sum_{i,k} \bar{\nabla}_k \bar{R}_{kiin+1} (\lambda_i - \lambda_k) - 2 \sum_{i,j,k,\beta \geq n+2} (\bar{\nabla}_k \bar{R}_{kij\beta} - \bar{\nabla}_i \bar{R}_{jkk\beta}) \mathring{h}^{\beta}_{ij} \\
\leq& \sum_{i,k} \Big(\frac{1}{\theta} (\bar{\nabla}_k \bar{R}_{kiin+1})^2 + \theta (\lambda_i - \lambda_k)^2 \Big) \\
&+ \sum_{i,j,k,\beta \geq n+2} \Big(\frac{2}{\vartheta} [(\bar{\nabla}_k \bar{R}_{kij\beta})^2 + (\bar{\nabla}_i \bar{R}_{jkk\beta})^2] + \vartheta (\mathring{h}^{\beta}_{ij})^2 \Big) \\
\leq& \frac{L^2}{\theta} + \theta |\mathring{A}|^2_H + \frac{4L^2}{\vartheta} + n\vartheta |\mathring{A}|^2_I,
\end{aligned} \tag{3.8}$$

for positive constants θ, ϑ. If $K_1 + K_2 = 0$, then $L = 0$, and we may choose $\theta, \vartheta = 0$.
Combining (3.5)-(3.8), we have

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta \Big) Q \leq& \bigg(6 - \frac{2}{n(a - \frac{1}{n})} \bigg) |\mathring{A}|^2_H |\mathring{A}|^2_I + \bigg(3 - \frac{2}{n(a - \frac{1}{n})} \bigg) |\mathring{A}|^4_I \\
&- \frac{2nab}{n(a - \frac{1}{n})} |\mathring{A}|^2_H - \frac{4b}{n(a - \frac{1}{n})} |\mathring{A}|^2_I - \frac{2b^2}{n(a - \frac{1}{n})} \\
&+ C_1 |\mathring{A}|^2_H + C_2 |\mathring{A}|^2_I + C_3 b + C_4.
\end{aligned} \tag{3.9}$$

Here

$$\begin{aligned}
C_1 =&4nK_1+2nK_2+\frac{2(naK_1+K_2)}{a-\frac{1}{n}}\\
&+\Big[\varrho n(d-1)+n(d-2)+\frac{16}{3}\rho(n-1)(d-1)\Big](K_1+K_2)+\theta,\\
C_2 =&4nK_1+2nK_2+\frac{2(naK_1+K_2)}{a-\frac{1}{n}}\\
&+\Big(\frac{n}{\varrho}+\frac{16}{3\rho}(n-1)+\frac{8}{3}(n-1)^{\frac{1}{2}}(d-2)\Big)(K_1+K_2)+n\vartheta,\\
C_3 =&\frac{2(naK_1+K_2)}{a-\frac{1}{n}},\\
C_4 =&\frac{L^2}{\theta}+\frac{4L^2}{\vartheta}\ \text{for } K_1+K_2\neq 0\ \ \text{and } C_4=0 \text{ for } K_1+K_2=0.
\end{aligned}$$

If $K_1+K_2\neq 0$, set $b_1=\max\Big\{\frac{C_1}{2a}(a-\frac{1}{n}),\frac{C_2}{4}n(a-\frac{1}{n}),\frac{1}{4}n(a-\frac{1}{n})\Big(C_3+\sqrt{C_3^2+\frac{8C_4}{n(a-\frac{1}{n})}}\Big)\Big\}$ with $\varrho=\rho=\theta=\vartheta=1$. If $K_1+K_2=0$, set $b_1=0$. So if $b>b_1$, we have $\Big(\frac{\partial}{\partial t}-\Delta\Big)Q<0$. Then by the maximum principle, $|A|^2\leq a|H|^2-b$ is preserved along the mean curvature flow. □

Remark 3.3. When $K_1+K_2\neq 0$, we may get a better b_1 by choosing suitable positive constants $\varrho,\ \rho,\ \theta$ and ϑ.

Now we pick the constant b_0 in (1.5) such that $b_0\geq b_1$. Since the submanifold is compact, if (1.5) is satisfied, then there are positive constants $a_\varepsilon<a$ and $b_\varepsilon>b_1$, where a denotes the cofficient of $|H|^2$, such that $|A|^2\leq a_\varepsilon|H|^2-b_\varepsilon$ holds at $t=0$, and it is preserved along the mean curvature flow by Theorem 3.2. Hence in the remained part of the paper, we always assume that $a_\varepsilon<a$, $b_\varepsilon>b_1$ and omit the index ε.

4. A pinching estimate for the tracefree second fundamental form

In this section, we assume that at the initial time the submanifold satisfies the pinching condition $|A|^2\leq a|H|^2-b$ for positive constants $a,\ b$ such that $a<\frac{4}{3n}$ when $n=2,3$ and $a<\frac{1}{n-1}$ when $n\geq 4$, and $b>b_1$, where b_1 is as in Theorem 3.2. From the last paragraph of Section 3 we see that the positive constants $a,\ b$ do exist under the condition (1.5) and the pinching condition is preserved along the mean curvature. We prove a pinching estimate for the tracefree second fundamental form, which guarantees that M_t becomes totally umbilical along the mean curvature flow.

Theorem 4.1. *There are constants $C_0<\infty$ and $\delta>0$ depending only on M_0 such that along the mean curvature flow there holds*

$$|\mathring{A}|^2\leq C_0|H|^{2-\delta}.$$

To prove Theorem 4.1, we define a function $f_\sigma=\frac{|\mathring{A}|^2}{|H|^{2(1-\sigma)}}$ and wish to find an upper bound of f_σ for sufficiently small σ. We first derive the evolution equation of f_σ.

Proposition 4.2. *There is a constant C depending only on n, d, K_1, K_2 and L such that along the mean curvature flow the following evolution inequality holds.*

$$\frac{\partial}{\partial t}f_\sigma \le \Delta f_\sigma + \frac{4(1-\sigma)}{|H|}\langle\nabla|H|,\nabla f_\sigma\rangle - \frac{2\epsilon_\nabla}{|H|^{2(1-\sigma)}}|\nabla H|^2 + 2\sigma|A|^2 f_\sigma + \frac{C}{|H|^{2(1-\sigma)}} + Cf_\sigma. \tag{4.1}$$

Proof. By the definition of f_σ, we have

$$\frac{\partial}{\partial t}f_\sigma = \frac{\frac{\partial}{\partial t}|\mathring{A}|^2}{|H|^{2(1-\sigma)}} - \frac{(1-\sigma)|\mathring{A}|^2\frac{\partial}{\partial t}|H|^2}{|H|^{2(2-\sigma)}}. \tag{4.2}$$

If we put $a=\frac{1}{n}$ and $b=0$ in (3.3), then get the evolution equation of $|\mathring{A}|^2$. From this and (2.3) we have

$$\begin{aligned}\frac{\partial}{\partial t}f_\sigma =& \frac{1}{|H|^{2(1-\sigma)}}\Big(\Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + 2R_1 - \frac{2}{n}R_2 + \mathrm{P}_{\frac{1}{n}}\Big)\\ &- \frac{(1-\sigma)|\mathring{A}|^2}{|H|^{2(1-\sigma)}}\Big(\Delta|H|^2 - 2|\nabla H|^2 + 2R_2 + \sum_{k,\alpha,\beta}\bar{R}_{k\alpha k\beta}H^\alpha H^\beta\Big).\end{aligned} \tag{4.3}$$

The Laplacian of f_σ can be computed as

$$\begin{aligned}\Delta f_\sigma =& \frac{\Delta|\mathring{A}|^2}{|H|^{2(1-\sigma)}} - \frac{(1-\sigma)|\mathring{A}|^2\Delta|H|^2}{|H|^{2(2-\sigma)}} + \frac{(2-\sigma)(1-\sigma)|\mathring{A}|^2|\nabla|H|^2|^2}{|H|^{2(3-\sigma)}}\\ &- \frac{2(1-\sigma)\langle\nabla|\mathring{A}|^2,\nabla|H|^2\rangle}{|H|^{2(2-\sigma)}}.\end{aligned} \tag{4.4}$$

On the other hand, we have

$$-\frac{2(1-\sigma)\langle\nabla|\mathring{A}|^2,\nabla|H|^2\rangle}{|H|^{2(2-\sigma)}} = -\frac{2(1-\sigma)}{|H|^2}\langle\nabla|H|^2,\nabla f_\sigma\rangle - \frac{8(1-\sigma)^2}{|H|^4}f_\sigma|H|^2|\nabla|H||^2. \tag{4.5}$$

Hence

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)f_\sigma =& \frac{2(1-\sigma)}{|H|^2}\langle\nabla|H|^2,\nabla f_\sigma\rangle - \frac{2}{|H|^{2(1-\sigma)}}\Big(|\nabla A|^2 - \frac{|A|^2}{|H|^2}|\nabla H|^2\Big)\\ &- \frac{2\sigma|\mathring{A}|^2}{|H|^{2(2-\sigma)}}|\nabla H|^2 - \frac{4\sigma(1-\sigma)}{|H|^4}f_\sigma|H|^2|\nabla|H||^2\\ &+ \frac{2\sigma R_2 f_\sigma}{|H|^2} + \frac{2}{|H|^{2(1-\sigma)}}\Big(R_1 - \frac{|A|^2}{|H|^2}R_2\Big)\\ &+ \frac{1}{|H|^{2(1-\sigma)}}\mathrm{P}_{\frac{1}{n}} - \frac{2(1-\sigma)|\mathring{A}|^2}{|H|^{2(2-\sigma)}}\sum_{k,\alpha,\beta}\bar{R}_{k\alpha k\beta}H^\alpha H^\beta.\end{aligned} \tag{4.6}$$

From Lemma 3.1 we have

$$\begin{aligned}|\nabla A|^2 - \frac{|A|^2}{|H|^2}|\nabla H|^2 \ge& \Big(\frac{3}{n+2} - \eta - a\Big)|\nabla H|^2 - C(n,d,K_1,K_2,\eta)\\ =& \epsilon_\nabla|\nabla H|^2 - C(n,d,K_1,K_2,\eta).\end{aligned} \tag{4.7}$$

Here $a<\frac{4}{3n}$ for $n=2$, 3 and $a<\frac{1}{n-1}$ for $n\ge4$, and we choose positive constant η depending only on n such that $\epsilon_\nabla=\frac{3}{n+2}-\eta-a>0$. We also have the following estimates.

$$R_2 \le |A|^2|H|^2,$$

$$R_1 - \frac{|A|^2}{|H|^2} R_2 \leq 0,$$

$$\mathrm{P}_{\frac{1}{n}} \leq C|\mathring{A}|^2 + C,$$

and

$$\sum_{k,\alpha,\beta} \bar{R}_{k\alpha k\beta} H^\alpha H^\beta \leq C|H|^2,$$

where C is a positive constant depending on n, d, K_1, K_2 and L. It follows from these estimates, (4.6) and (4.7) that

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big) f_\sigma \leq & \frac{4(1-\sigma)}{|H|} \langle \nabla |H|, \nabla f_\sigma \rangle - \frac{2\epsilon_\nabla}{|H|^{2(1-\sigma)}} |\nabla H|^2 \\ & + 2\sigma |A|^2 f_\sigma + \frac{C}{|H|^{2(1-\sigma)}} + C f_\sigma.\end{aligned}$$

This completes the proof. □

To handle the reaction term $2\sigma|A|^2 f_\sigma$, we need to compute the Laplacian of $|\mathring{A}|^2$. As in [1], we have

$$\frac{1}{2}\Delta|\mathring{A}|^2 \geq |\nabla \mathring{A}|^2 + \langle \mathring{h}_{ij}, \nabla_i \nabla_j H \rangle + Z - C|H|^2 - C \tag{4.8}$$

for some positive constant C depending on $n,\ d,\ K_1,\ K_2$ and L. Here

(4.9)

$$Z = \sum_{i,j,p,\alpha,\beta} H^\alpha h^\alpha_{ip} h^\beta_{pj} h^\beta_{ij} - \sum_{\alpha,\beta} \Big(\sum_{i,j} h^\alpha_{ij} h^\beta_{ij} \Big)^2 - \sum_{i,j,\alpha,\beta} \Big(\sum_p \Big(h^\alpha_{ip} h^\beta_{jp} - h^\alpha_{jp} h^\beta_{ip} \Big) \Big)^2.$$

Proposition 4.3. *There is a positive constant C depending on $n,\ d,\ K_1,\ K_2,\ L$ and M_0 such that for any $p \geq 2$ and $\eta > 0$, the following inequality holds.*

(4.10)

$$\int_{M_t} |H|^2 f_\sigma^p \leq \frac{2p\eta + C}{\epsilon} \int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} |\nabla H|^2 + \frac{p-1}{\epsilon\eta} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 + C^p.$$

Proof. From (4.4) and (4.8), we have

(4.11)

$$\begin{aligned}\Delta f_\sigma \geq & \frac{2}{|H|^{2(1-\sigma)}} \langle \mathring{h}_{ij}, \nabla_i \nabla_j H \rangle + \frac{2}{|H|^{2(1-\sigma)}} |\nabla \mathring{A}|^2 + \frac{2}{|H|^{2(1-\sigma)}} Z - \frac{2(C|H|^2 + C)}{|H|^{2(1-\sigma)}} \\ & - \frac{2(1-\sigma)}{|H|} f_\sigma \Delta |H| - \frac{2(1-\sigma)}{|H|^2} f_\sigma |\nabla |H||^2 + \frac{4(2-\sigma)(1-\sigma)}{|H|^2} f_\sigma |\nabla |H||^2 \\ & - \frac{4(1-\sigma)}{|H|} \langle \nabla |H|, \nabla f_\sigma \rangle - \frac{8(1-\sigma)^2}{|H|^2} f_\sigma |\nabla |H||^2 \\ = & \frac{2}{|H|^{2(1-\sigma)}} \langle \mathring{h}_{ij}, \nabla_i \nabla_j H \rangle + \frac{2}{|H|^{2(1-\sigma)}} |\nabla \mathring{A}|^2 + \frac{2}{|H|^{2(1-\sigma)}} Z \\ & - \frac{4(1-\sigma)}{|H|} \langle \nabla |H|, \nabla f_\sigma \rangle - \frac{2(1-\sigma)}{|H|} f_\sigma \Delta |H| \\ & - \frac{2(1-\sigma)(1-2\sigma)}{|H|^2} f_\sigma |\nabla |H||^2 - \frac{C}{|H|^{-2\sigma}} - \frac{C}{|H|^{2(1-\sigma)}}.\end{aligned}$$

Multiplying both sides of (4.11) by f_σ^{p-1} and integrating over M_t we obtain

$$
\begin{aligned}
0 \geq & (p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 + 2\int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}\langle \mathring{h}_{ij}, \nabla_i\nabla_j H\rangle + 2\int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla \mathring{A}|^2 \\
& + 2\int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} Z - 4(1-\sigma)\int_{M_t} \frac{f_\sigma^{p-1}}{|H|}\langle \nabla|H|, \nabla f_\sigma\rangle - 2(1-\sigma)\int_{M_t} \frac{f_\sigma^p}{|H|} f_\sigma \Delta|H| \\
& - 2(1-\sigma)(1-2\sigma)\int_{M_t} \frac{f_\sigma^p}{|H|^2}|\nabla|H||^2 - C\int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{-2\sigma}} - C\int_{M_t} \frac{f_\sigma^p}{|H|^{2(1-\sigma)}}.
\end{aligned} \tag{4.12}
$$

The first term on the right hand side of (4.12) is nonnegative. For the second term, we have the following computation.

$$
\begin{aligned}
& 2\int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}\langle \mathring{h}_{ij}, \nabla_i\nabla_j H\rangle \\
= & -2\int_{M_t} \Big\langle \nabla_i\Big(\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}\mathring{h}_{ij}\Big), \nabla_j H\Big\rangle \\
= & -2\int_{M_t} \frac{(p-1)f_\sigma^{p-2}}{|H|^{2(1-\sigma)}}\langle \nabla_i f_\sigma \mathring{h}_{ij}, \nabla_j H\rangle + 4\int_{M_t} \frac{(1-\sigma)f_\sigma^{p-1}}{|H|^{3-2\sigma}}\langle \nabla_i|H|\mathring{h}_{ij}, \nabla_j H\rangle \\
& - \frac{2(n-1)}{n}\int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2 - 2\int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}\langle \sum_{i,j,\alpha} \bar{R}_{jii\alpha} e_\alpha, \nabla_j H\rangle.
\end{aligned} \tag{4.13}
$$

We also have

$$
\begin{aligned}
& -2(1-\sigma)\int_{M_t} \frac{f_\sigma^p}{|H|} f_\sigma \Delta|H| \\
= & 2(1-\sigma)\int_{M_t} \Big\langle \nabla\Big(\frac{f_\sigma^p}{|H|}\Big), \nabla|H|\Big\rangle \\
= & 2(1-\sigma)\int_{M_t} \frac{p f_\sigma^{p-1}}{|H|}\langle \nabla f_\sigma, \nabla|H|\rangle - 2(1-\sigma)\int_{M-t} \frac{f_\sigma^p}{|H|^2}|\nabla|H||^2.
\end{aligned} \tag{4.14}
$$

Combining (4.12), (4.13) and (4.14) implies

$$
\begin{aligned}
2\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}Z \le & 2(p-1)\int_{M_t}\frac{f_\sigma^{p-2}}{|H|^{2(1-\sigma)}}\langle\nabla_i f_\sigma \mathring{h}_{ij},\nabla_j H\rangle \\
& -4(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{3-2\sigma}}\langle\nabla_i|H|\mathring{h}_{ij},\nabla_j H\rangle \\
& +\frac{2(n-1)}{n}\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2 \\
& -2(1-\sigma)(p-2)\int_{M_t}\frac{f_\sigma^{p-1}}{|H|}\langle\nabla|H|,\nabla f_\sigma\rangle \\
& +2(1-\sigma)\int_{M_t}\frac{f\sigma^p}{|H|^2}|\nabla|H||^2 \\
& +C\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{-2\sigma}}+C\int_{M_t}\frac{f_\sigma^p}{|H|^{2(1-\sigma)}} \\
& +C\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|.
\end{aligned}
\tag{4.15}
$$

Here C is a positive constant depending on $n,\ d,\ K_1,\ K_2$ and L.

Notice that $f_\sigma \le C|H|^{2\sigma}$ and $|\mathring{A}|^2 \le f_\sigma |H|^{2(1-\sigma)}$, and we can pick $\sigma\in(0,1)$ sufficiently small. Also notice that the initial pinching condition is preserved and implies that for $t\ge 0$, there holds

$$|H|^2 \ge C > 0$$

for some positive constant C depending on $n,\ d,\ K_1,\ K_2$ and L. This implies that

$$\frac{f_\sigma^{p-1}}{|H|^{-2\sigma}} = \frac{|H|^{\frac{2(p-1)}{p}}f_\sigma^{p-1}}{|H|^{2(1-\frac{1}{p})-2\sigma}} \le C|H|^{\frac{2(p-1)}{p}}f_\sigma^{p-1}, \tag{4.16}$$

$$\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} = \frac{|H|^{\frac{2(p-1)}{p}}f_\sigma^{p-1}}{|H|^{\frac{2(p-1)}{p}+2(1-\sigma)}} \le C|H|^{\frac{2(p-1)}{p}}f_\sigma^{p-1}, \tag{4.17}$$

and

$$\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H| \le \bar\varepsilon\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} + \bar\varepsilon^{-1}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2, \tag{4.18}$$

for any positive constant $\bar\varepsilon$

By using Young's inequality, we have the following

$$|H|^{\frac{2(p-1)}{p}}f_\sigma^{p-1} \le \tilde\varepsilon^{\frac{p}{p-1}}|H|^2 f_\sigma^p + \tilde\varepsilon^{-p} \tag{4.19}$$

for arbitrary positive constant $\tilde\varepsilon$.

From (4.16)-(4.19), the last two line of (4.15) is not bigger than

$$C(1+\bar\varepsilon)\tilde\varepsilon^{\frac{p}{p-1}}\int_{M_t}|H|^2 f_\sigma^p + C\tilde\varepsilon^{-p}\mathrm{Vol}(M_0) + C\bar\varepsilon^{-1}\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} \tag{4.20}$$

for some positive constant C depending on $n,\ d,\ K_1,\ K_2$ and L.

On the other hand, we have the following inequalities as in [1].

$$\begin{aligned}&2(p-1)\int_{M_t}\frac{f_\sigma^{p-2}}{|H|^{2(1-\sigma)}}\langle\nabla_i f_\sigma \mathring{h}_{ij},\nabla_j H\rangle\\&\le\frac{p-1}{\eta}\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2+(p-1)\eta\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2,\end{aligned}\tag{4.21}$$

$$-4(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{3-2\sigma}}\langle\nabla_i|H|\mathring{h}_{ij},\nabla_j H\rangle\le 4\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2,\tag{4.22}$$

$$\begin{aligned}&-2(1-\sigma)(p-2)\int_{M_t}\frac{f_\sigma^{p-1}}{|H|}\langle\nabla|H|,\nabla f_\sigma\rangle\\&\le\frac{p-2}{\mu}\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2+(p-2)\mu\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2,\end{aligned}\tag{4.23}$$

$$2(1-\sigma)\int_{M_t}\frac{f_\sigma^{p}}{|H|^2}|\nabla|H||^2\le 2\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2.\tag{4.24}$$

Combining (4.15), (4.20)-(4.24) implies

$$\begin{aligned}&2\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}Z\\&\le\left(6+\frac{2(n-1)}{n}+(p-1)\eta+(p-2)\mu+C\bar{\varepsilon}^{-1}\right)\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\\&\quad+\left(\frac{p-1}{\eta}+\frac{p-2}{\mu}\right)\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2+C(1+\bar{\varepsilon})\tilde{\varepsilon}^{\frac{p}{p-1}}\int_{M_t}|H|^2f_\sigma^p+C\tilde{\varepsilon}^{-p}\mathrm{Vol}(M_0).\end{aligned}\tag{4.25}$$

From Lemma 4 in [1] we have $Z\ge\epsilon|\mathring{A}|^2|H|^2$ for some positive constant ϵ. Then from (4.25) we have

$$\begin{aligned}&\left(2\epsilon-C(1+\bar{\varepsilon})\tilde{\varepsilon}^{\frac{p}{p-1}}\right)\int_{M_t}|H|^2f_\sigma^p\\&\le\left(6+\frac{2(n-1)}{n}+(p-1)\eta+(p-2)\mu+C\bar{\varepsilon}^{-1}\right)\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2\\&\quad+\left(\frac{p-1}{\eta}+\frac{p-2}{\mu}\right)\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2+C\tilde{\varepsilon}^{-p}\mathrm{Vol}(M_0).\end{aligned}\tag{4.26}$$

Now we put $\bar{\varepsilon}=1$, $\tilde{\varepsilon}=\left(\frac{\epsilon}{2C}\right)^{\frac{p-1}{p}}$, and let $\eta=\mu$. Then (4.26) implies

$$\int_{M_t}|H|^2f_\sigma^p\le\frac{2p\eta+C}{\epsilon}\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2+\frac{p-1}{\epsilon\eta}\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2+C^p.$$

Here C is a positive constant depending on $n,\ d,\ K_1,\ K_2,\ L$ and M_0. □

Proposition 4.4. *For sufficiently small σ and large p, there holds*

$$\frac{d}{dt}\int_{M_t}f_\sigma^p\le Cp\int_{M_t}f_\sigma^p+C^p,\tag{4.27}$$

where C is a positive constant depending on $n,\ d,\ K_1,\ K_2,\ L$ and M_0.

Proof. We first compute the evolution equation of $\int_{M_t} f_\sigma^p$ as follows.

$$
\begin{aligned}
\frac{d}{dt}\int_{M_t} f_\sigma^p \le \int_{M_t} p f_\sigma^{p-1}\frac{\partial f_\sigma}{\partial t}& \\
\le -p(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 + 4(1-\sigma)p\int_{M_t}\frac{f_\sigma^{p-1}}{|H|}|\nabla|H|||\nabla f_\sigma|& \\
-2p\epsilon_\nabla\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2 + 2p\sigma\int_{M_t}|H|^2 f_\sigma^p& \\
+Cp\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} + Cp\int_{M_t} f_\sigma^p.&
\end{aligned}
\tag{4.28}
$$

The second integral on the right hand side may be estimated as

$$
4(1-\sigma)p\int_{M)_t}\frac{f_\sigma^{p-1}}{|H|}|\nabla|H|||\nabla f_\sigma| \le \frac{2p}{\rho}\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 + 2p\rho\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2.
\tag{4.29}
$$

The last second term can be estimated from (4.17) and (4.19) as

$$
Cp\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} \le \bar{C}\tilde{\tilde{\varepsilon}}^{\frac{p}{p-1}}p\int_{M_t}|H|^2 f_\sigma^p + \bar{C}p\tilde{\tilde{\varepsilon}}^{-p}\mathrm{Vol}(M_0),
\tag{4.30}
$$

for a positive constant $\bar{C}$ depending on $n,\ d,\ K_1,\ K_2,\ L$ and M_0. Combining (4.10), (4.28)-(4.30) implies

$$
\begin{aligned}
\frac{d}{dt}\int_{M_t} f_\sigma^p \le \int_{M_t} p f_\sigma^{p-1}\frac{\partial f_\sigma}{\partial t}& \\
\le -p(p-1)\Big(1-\frac{2}{\rho(p-1)}-\frac{2\sigma}{\epsilon\eta}-\frac{\bar{C}\tilde{\tilde{\varepsilon}}^{\frac{p}{p-1}}}{\epsilon\eta}\Big)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2& \\
-2p\epsilon_\nabla\Big(1-\frac{\rho}{\epsilon_\nabla}-\frac{(2p\eta+C)\sigma}{\epsilon\epsilon_\nabla}+\frac{(2p\eta+C)\bar{C}\tilde{\tilde{\varepsilon}}^{\frac{p}{p-1}}}{2\epsilon\epsilon_\nabla}\Big)\int_{M_t}\frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2& \\
+Cp\int_{M_t} f_\sigma^p + \Big(2p\sigma + \bar{C}\tilde{\tilde{\varepsilon}}^{\frac{p}{p-1}}p\Big)C^p + \bar{C}p\tilde{\tilde{\varepsilon}}^{-p}\mathrm{Vol}(M_0).&
\end{aligned}
\tag{4.31}
$$

Now we pick $\rho=\frac{\epsilon_\nabla}{4}$, $\eta=\frac{1}{p}$, $\tilde{\tilde{\varepsilon}}=\Big(\min\{\frac{\epsilon\epsilon_\nabla}{2(2+C)},\frac{\epsilon}{4\bar{C}p}\}\Big)^{\frac{p-1}{p}}$, and let $p\ge\frac{32}{\epsilon_\nabla}+1$ and $\sigma\le\min\{\frac{\epsilon\epsilon_\nabla}{4(2+C)},\frac{\epsilon}{8p}\}$. Then (4.31) reduces to

$$
\frac{d}{dt}\int_{M_t} f_\sigma^p \le Cp\int_{M_t} f_\sigma^p + C^p,
$$

for a positive constant C depending on $n,\ d,\ K_1,\ K_2,\ L$ and M_0. □

To give an upper bound of $\int_{M_t} f_\sigma^p$, we need to show that the maximal existence time of the mean curvature flow under the initial curvature pinching condition is finite.

Lemma 4.5. *Under the initial curvature pinching condition, the mean curvature flow has a smooth solution on a finite maximal time interval* $[0,T)$ *and* $\max_{M_t}|A|\to\infty$ *as* $t\to T$.

Proof. We first show that if the second fundamental form is uniformly bounded by a positive constant K in a finite time interval $[0,T)$, then the mean curvature flow can be extended over the time. For any fixed $x \in M$, any $\tau, \varrho \in [0,T)$ such that $\tau < \varrho$, $F(x,t)$, $t \in [\tau,\varrho]$ is a segment in N connecting $F(x,\tau)$ and $F(x,\varrho)$. We have

$$\operatorname{dist}(F(x,\tau), F(x,\varrho)) \leq \int_\tau^\varrho \Big|\frac{\partial}{\partial t}F(x,t)\Big| dt = \int_\tau^\varrho |H(x,t)| dt \leq CT < \infty. \tag{4.32}$$

Hence $F(x,t)$ converges uniformly to some continuous limit function $F(x,T)$. We want to show that $F(x,T)$ is a smooth immersion. To do this, we only have to establish uniform bounds for all the covariant derivatives of the second fundamental form on M_t, $t \in [0,T)$. Since M_t stays in a compact region on N in view of (4.32), we have $\max_{0\leq l\leq m} |\bar\nabla^m \bar R|\Big|_{M_t} \leq C_m$ for positive constants C_m independent of t.

First, by the evolution of the second fundamental form, as Proposition 7.1 in [21] for example, and induction on m, we have the following evolution equations

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 \leq &\Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 \\ &+ C(n,m)\Big\{\sum_{i+j+k=m} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| \\ &+ \bar C_m \sum_{i\leq m} |\nabla^i A||\nabla^m A| + \bar{\bar C}_m |\nabla^m A|\Big\}.\end{aligned} \tag{4.33}$$

For any $\tau \in [0,T)$, we consider the mean curvature flow on the time interval $[0,\tau]$. Consider the function $G = t|\nabla A|^2 + |A|^2$, which is bounded by K at $t=0$. Differentiating the function we get for some positive constant C independent of t and τ

$$\begin{aligned}\frac{\partial}{\partial t}G \leq &\Delta G + |\nabla A|^2 - 2(t|\nabla^2 A|^2 + |\nabla A|^2) \\ &+ Ct(|A|^2|\nabla A|^2 + |\nabla A|^2 + |A||\nabla A| + |\nabla A|) \\ &+ C(|A|^4 + |A|^2 + |A|).\end{aligned} \tag{4.34}$$

Since we have already assumed that the second fundamental form is uniformly bounded on $[0,T)$ by K, (4.34) may be reduced to

$$\frac{\partial}{\partial t}G \leq \Delta G + (Ct-1)|\nabla A|^2 + C, \tag{4.35}$$

where C is a positive constant independent of t and τ. For $t \in [0, \frac{1}{C}]$, where C is as in (4.35), we obtain from (4.35)

$$\frac{\partial}{\partial t}G \leq \Delta G + C, \tag{4.36}$$

which implies by the maximum principle $G \leq K + Ct$. Hence $|\nabla A|^2 \leq \frac{G}{t} \leq \frac{K}{t} + C$ for $t \in (0, \frac{1}{C}]$. If $t > \frac{1}{C}$, the same argument on time interval $[t - \frac{1}{C}, t]$ yields $|\nabla A|^2 \leq \tilde C$ for some $\tilde C$ independent of t and τ. Since $\tau \in [0,T)$ is arbitrary, we have $|\nabla A|^2 \leq C(1 + \frac{1}{t})$ for $t \in (0,T)$, where C a constant independent of t.

Now we assume that there are positive constants C_k independent of t such that $|\nabla^k A|^2 \leq C_k(1 + \frac{1}{t^k})$ holds for $k = 1, 2, \cdots, m-1$ and $t \in (0,T)$. Using Young's inequality, (4.33) implies

$$
\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k A|^2 \leq& \Delta|\nabla^k A|^2 - 2|\nabla^{k+1}A|^2 + \tilde{C}_k\Big[\Big(1+\frac{1}{t^k}\Big)^{\frac{1}{2}}|\nabla^k A| + |\nabla^k A|^2 + 1\Big] \\
\leq& \Delta|\nabla^k A|^2 - 2|\nabla^{k+1}A|^2 + \tilde{C}_k\Big(|\nabla^k A|^2 + 1 + \frac{1}{t^k}\Big),
\end{aligned} \tag{4.37}
$$

where $\tilde{C}_k$ is a positive constant depending on k and others but independent of t.

Consider $G = t^m|\nabla^m A|^2 + mt^{m-1}|\nabla^{m-1}A|^2$. Differentiating G with respect to t gives

$$
\begin{aligned}
\frac{\partial}{\partial t}G \leq& mt^{m-1}|\nabla^m A|^2 + t^m\Big[\Delta|\nabla^m A|^2 + \tilde{C}_m\Big(|\nabla^m A|^2 + 1 + \frac{1}{t^m}\Big)\Big] \\
&+ m\Big\{(m-1)t^{m-2}|\nabla^{m-1}A|^2 \\
&+ t^{m-1}\Big[\Delta|\nabla^{m-1}A|^2 - 2|\nabla^m A|^2 + \tilde{C}_{m-1}\Big(|\nabla^{m-1}A|^2 + 1 + \frac{1}{t^{m-1}}\Big)\Big]\Big\} \\
\leq& \Delta G + (\tilde{C}_m t - m)t^{m-1}|\nabla^m A|^2 + \hat{C}_m,
\end{aligned} \tag{4.38}
$$

provided $t \leq 1$, for some positive constant $\hat{C}_m$ depending on m and others but independent of t. Hence for $t \in (0, \min\{1, \frac{m}{\tilde{C}_m}\})$, by maximum principle, $|\nabla^m A|^2 \leq \frac{G}{t^m} \leq \frac{\hat{C}_m}{t^{m-1}} \leq C_m(1+\frac{1}{t^m})$ for some positive constant C_m depending on m and others but independent of t. For $t > \min\{1, \frac{m}{\tilde{C}_m}\}$, the same argument gives the bound of $|\nabla^m A|^2$. Hence we have proved that $|\nabla^m A|^2 \leq C_m(1+\frac{1}{t^m})$ for $t \in (0, T)$ for constants C_m independent of t.

Now we prove that, under the initial curvature pinching condition, the second fundamental form of the submanifold will blow up in finite time. Consider the function $Q = |A|^2 - a|H|^2 + b(t)$ where $a = \frac{4}{3n}$ for $n = 2, 3$ and $a = \frac{1}{n-1}$ for $n \geq 4$ and $b(t)$ is a function of t with $b(0)$ such that $Q(0) < 0$. By the initial pinching assumption, we may pick $b(0) = b_\varepsilon > b_1$. We will show that $Q < 0$ is preserved for a suitable $b(t)$. Suppose not, then there is a first time such that $Q = 0$ at a point. Then at this point, we have the following estimate as (3.9)

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)Q \leq& -\frac{2nab}{n(a-\frac{1}{n})}|\mathring{A}|_H^2 - \frac{4b}{n(a-\frac{1}{n})}|\mathring{A}|_I^2 - \frac{2b^2}{n(a-\frac{1}{n})} \\
&+ C_1|\mathring{A}|_H^2 + C_2|\mathring{A}|_I^2 + C_3 b + C_4 + b'(t).
\end{aligned} \tag{4.39}
$$

Now we pick $b(t)$ such that $b'(t) = \frac{2b^2}{n(a-\frac{1}{n})} - C_3 b - C_4$. Then $\Big(\frac{\partial}{\partial t} - \Delta\Big)Q < 0$, which implies that $Q < 0$ is preserved along the mean curvature flow. On the other hand, it is easy to check that $b(t)$ is monotone increasing and becomes unbounded as $t \to t_0$ for some $t_0 < \infty$. Hence $|A|^2$ becomes unbounded as $t \to t_0$. This completes the proof of the lemma.

□

Now from Proposition 4.4 and Lemma 4.5, we see that $\int_{M_t} f_\sigma^p \leq C$ holds for $t \in [0, T)$ with a positive constant C independent of t. We also have a Sobolev inequality for M_t (see [8]). Hence we may apply the same argument as in [9] and [10] to derive a bound for f_σ if σ is small enough. This completes the proof of Theorem 4.1.

5. The gradient estimate of the mean curvature

To compare the mean curvature of the submanifold at different points, we need to give an estimate of the gradient of the mean curvature.

Theorem 5.1. *For any $\eta > 0$, there is a constant $C_\eta < \infty$ independent of t such that*

$$|\nabla H|^2 \leq \eta |H|^4 + C_\eta. \tag{5.1}$$

Proof. First we have the following inequalities by using a similar computation as in [1] and [10]

$$\frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta |\nabla H|^2 - 2|\nabla^2 H|^2 + C|H|^2|\nabla A|^2 + C|\nabla A|^2 + C|H|^2, \tag{5.2}$$

$$\frac{\partial}{\partial t}|H|^4 \geq \Delta |H|^4 - 12|H|^2|\nabla H|^2 + \frac{2}{n}|H|^6 - C, \tag{5.3}$$

where C is a positive constant independent of t.

For any $N_1, N_2 > 0$,

(5.4)
$$\begin{aligned}\frac{\partial}{\partial t}((N_1 + N_2|A|^2)|\mathring{A}|^2) =&\Delta((N_1 + N_2|A|^2)|\mathring{A}|^2) - 2N_2\langle \nabla |A|^2, \nabla |\mathring{A}|^2\rangle \\ &- 2N_2|\mathring{A}|^2|\nabla A|^2 - 2(N_1 + N_2|A|^2)|\nabla \mathring{A}|^2 \\ &+ 2N_2(R_1 + \mathrm{P}_0)|\mathring{A}|^2 + 2(N_1 + N_2|A|^2)(R_1 - \frac{R_2}{n} + \mathrm{P}_{\frac{1}{n}}).\end{aligned}$$

The second term on the right can be estimated as follows.

$$\begin{aligned}-2N_2\langle \nabla |A|^2, \nabla |\mathring{A}|^2\rangle \leq& 8N_2|A||\nabla A||\mathring{A}||\nabla \mathring{A}| \\ \leq& 8N_2C_n|H||\nabla A|^2\sqrt{C_0}|H|^{1-\frac{\delta}{2}} \\ \leq& 8N_2C_n\sqrt{C_0}|\nabla A|^2(\varrho^{\frac{4}{4-\delta}}|H|^2 + \varrho^{-\frac{4}{\delta}}),\end{aligned} \tag{5.5}$$

where $C_n = \frac{2}{\sqrt{3n}}$, C_0 and δ are as in Theorem 4.1, and $\varrho > 0$ is an arbitrary constant. Using Young's inequality and the pinching assumption, we have $R_1 + \mathrm{P}_0 \leq C(|H|^4 + 1)$ and $R_1 - \frac{R_2}{n} + \mathrm{P}_{\frac{1}{n}} \leq C|\mathring{A}|^2(|H|^2 + 1) + C$, where C is independent of

t. These together with (5.4) and (5.5) imply

$$
\begin{aligned}
\frac{\partial}{\partial t}((N_1+N_2|A|^2)|\mathring{A}|^2) \leq & \Delta((N_1+N_2|A|^2)|\mathring{A}|^2) \\
& + 8N_2C_n\sqrt{C_0}|\nabla A|^2(\varrho^{\frac{4}{4-\delta}}|H|^2+\varrho^{-\frac{4}{\delta}}) \\
& - 2N_2|\mathring{A}|^2|\nabla A|^2 - 2(N_1+N_2|A|^2)\Big(\frac{n-1}{2n+1}|\nabla A|^2 - C\Big) \\
& + 2N_2|\mathring{A}|^2(C|H|^4+C) \\
& + 2(N_1+N_2|A|^2)(C|\mathring{A}|^2(|H|^2+1)+C) \\
\leq & \Delta((N_1+N_2|A|^2)|\mathring{A}|^2) \\
& - \Big(\frac{2(n-1)}{n(2n+1)}N_2 - 8N_2C_n\sqrt{C_0}\varrho^{\frac{4}{4-\delta}}\Big)|H|^2|\nabla A|^2 \\
& - \Big(\frac{2(n-1)}{2n+1}N_1 - 8N_2C_n\sqrt{C_0}\varrho^{-\frac{4}{\delta}}\Big)|\nabla A|^2 \\
& + (2N_2C+2N_2CC_n)|\mathring{A}|^2|H|^4 \\
& + C(N_1,N_2)|H|^4 + C(N_1,N_2)|H|^2 + C(N_1,N_2).
\end{aligned} \tag{5.6}
$$

Here C is some positive constant and $C(N_1, N_2)$ is some positive constant depending on N_1, N_2 and others. Choose ϱ such that $\frac{(n-1)}{n(2n+1)} = 8C_n\sqrt{C_0}\varrho^{\frac{4}{4-\delta}}$. Then (5.6) implies

$$
\begin{aligned}
\frac{\partial}{\partial t}((N_1+N_2|A|^2)|\mathring{A}|^2) \leq & \Delta((N_1+N_2|A|^2)|\mathring{A}|^2) \\
& - \frac{(n-1)}{n(2n+1)}N_2|H|^2|\nabla A|^2 \\
& - \Big(\frac{2(n-1)}{2n+1}N_1 - C(N_2)\Big)|\nabla A|^2 \\
& + (2N_2C+2N_2CC_n)|\mathring{A}|^2|H|^4 \\
& + C(N_1,N_2)|H|^4 + C(N_1,N_2)|H|^2 + C(N_1,N_2),
\end{aligned} \tag{5.7}
$$

for some positive constant $C(N_2)$ depending on N_2 and others but independent of N_1.

Now consider the function $f = |\nabla H|^2 + (N_1+N_2|A|^2)|\mathring{A}|^2 - \eta|H|^4$. Then f satisfies

$$
\begin{aligned}
\frac{\partial}{\partial t}f \leq & \Delta f - \Big(\frac{(n-1)}{n(2n+1)}N_2 - C - \frac{12}{n}\eta\Big)|H|^2|\nabla A|^2 \\
& - \Big(\frac{2(n-1)}{2n+1}N_1 - C(N_2) - C\Big)|\nabla A|^2 \\
& + (2N_2C+2N_2CC_n)|\mathring{A}|^2|H|^4 \\
& + C(N_1,N_2)|H|^4 + (C(N_1,N_2)+C)|H|^2 + C(N_1,N_2) \\
& - \frac{2}{n}\eta|H|^6 + C\eta,
\end{aligned} \tag{5.8}
$$

where C is as in (5.2) and (5.3). Now we first choose N_2 large enough such that the gradient term on the first line of (5.8) is nonpositive. Then we choose N_1 large enough such that the gradient term on the second line of (5.8) can be absorbed. The remained terms can be estimated by using Theorem 4.1 and Young's inequality, which gives

$$\frac{\partial}{\partial t} f \leq \Delta f + C(N_1, N_2, \eta), \tag{5.9}$$

where $C(N_1, N_2, \eta)$ is some positive constant depending on N_1, N_2, η and others but independent of t. Since the maximal existence time of the mean curvature flow is finite, we conclude that $f \leq C_\eta$. Then the theorem follows from the definition of f.

□

6. Contraction to a round point

In this section we show that as time tends to T, the submanifold will shrink to a single point. If we dilate the metric of the ambient space by a factor, which is a function of t, then the submanifold will maintain the volume. After a reparameterization of time, the dilated submanifold converges to a totally umbilical sphere in the Euclidean space as the reparameterized time tends to infinity.

We need the following lemma.

Lemma 6.1 ([27])**.** *Let M^n be an n-dimensional submanifold in an $(n+d)$-dimensional Riemannian manifold N^{n+d}, and π a tangent 2-plane on T_xM at point $x \in M$. Choose an orthonormal two-frame $\{e_1, e_2\}$ at x such that $\pi = \mathrm{span}\{e_1, e_2\}$. Then*

$$K(\pi) \geq \frac{1}{2}\Big(2\bar{K}_{\min} + \frac{|H|^2}{n-1} - |A|^2\Big) + \sum_{\alpha=n+1}^{n+d} \sum_{j>i,(i,j)\neq(1,2)} (h_{ij}^\alpha)^2.$$

In Lemma 6.1, $K(\pi)$ is the sectional curvature of M for the 2-plane π, and $\bar{K}_{\min}$ is the minimum of the sectional curvature of N at point x. From our assumption and $a_\varepsilon < \frac{1}{n-1}$, we see that the sectional curvature K_M of the evolving submanifold satisfies $K_M \geq \epsilon^2 |H|^2 > 0$ for some constant ϵ provided we choose $b_0 = \max\{b_1, 2K_1\}$. Note that $b_0 = 0$ if $K_1 + K_2 = 0$. With the same argument as in [1] we have $\frac{|H|_{\max}}{|H|_{\min}} \to 1$ and $\mathrm{diam} M \to 0$ as $t \to T$. Hence the submanifold shrinks to a single point $P \in N$ along the mean curvature flow.

To see that the evolving submanifold becomes spherical, we dilate the metric of the ambient space such that the submanifold with the induced metric by the immersion has fixed volume along the flow. Let ψ be a function of t satisfying

$$\psi^{-1}\frac{d\psi}{dt} = \frac{1}{n}\frac{\int_{M_t} |H|^2 d\mu_t}{\int_{M_t} d\mu_t} := \frac{\hbar}{n}.$$

Let h be the Riemannian metric on N. Now we dilate the metric h such that $(N, \psi(t)^2 h)$, $t \in [0, T)$ is a family of Riemannian manifolds. Let $\tilde{g}(t)$ be the induced metric on the submanifold M from $(N, \psi(t)^2 h)$ by the immersion F_t. We denote by $(\tilde{M}, \tilde{g}(t))$ the dilated submanifold with the isometric immersion $\tilde{F}_t$, where $\tilde{M} = M$ and $\tilde{F}_t = F_t$. We also have the following relations.

Lemma 6.2.

$$\begin{aligned}\tilde{A} &= A,\\ \tilde{H} &= \psi^{-2}H,\\ |\tilde{A}|^2 &= \psi^{-2}|A|^2,\\ |\tilde{H}|^2 &= \psi^{-2}|H|^2,\\ d\tilde{\mu}_{\tilde{g}(t)} &= \psi^n d\mu_{g(t)},\\ \tilde{\nabla} &= \nabla,\\ \tilde{\Delta} &= \psi^{-2}\Delta.\end{aligned}$$

Proof. Let $\{x^i\}$ be a local coordinate system on M. For the induced metric, we have the following

$$\begin{aligned}\tilde{g}_{ij} &= \left\langle \frac{\partial \tilde{F}}{\partial x^i}, \frac{\partial \tilde{F}}{\partial x^j} \right\rangle_{\psi^2 h}\\ &= \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle_{\psi^2 h}\\ &= \psi^2 \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle_{h}\\ &= \psi^2 g_{ij}.\end{aligned}$$

For the second fundamental form, noting that $\tilde{e}_\alpha = \psi^{-1}e_\alpha$, we have

$$\begin{aligned}\tilde{h}^\alpha_{ij} &= -\left\langle \frac{\partial^2 \tilde{F}}{\partial x^i \partial x^j}, \tilde{e}_\alpha \right\rangle_{\psi^2 h}\\ &= -\psi^2 \left\langle \frac{\partial^2 F}{\partial x^i \partial x^j}, \psi^{-1} e_\alpha \right\rangle_{h}\\ &= \psi h^\alpha_{ij}.\end{aligned}$$

Hence

$$\tilde{h}_{ij} = \sum_\alpha \tilde{h}^\alpha_{ij}\tilde{e}_\alpha = \sum_\alpha \psi h^\alpha_{ij}\psi^{-1}e_\alpha = h_{ij},$$

which means $\tilde{A} = A$.

The mean curvature is the trace of the second fundamental form, so

$$\tilde{H} = \tilde{g}^{ij}\tilde{h}_{ij} = \psi^{-2}g^{ij}h_{ij} = \psi^{-2}H.$$

For the squared norm of the second fundamental form and the mean curvature, we have

$$\begin{aligned}|\tilde{A}|^2 &= \tilde{g}^{ik}\tilde{g}^{kl}\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ij}\langle \tilde{e}_\alpha, \tilde{e}_\beta\rangle_{\psi^2 h}\\ &= \psi^{-2}g^{ik}g^{kl}h^\alpha_{ij}h^\beta_{ij}\langle e_\alpha, e_\beta\rangle_h\\ &= \psi^{-2}|A|^2,\end{aligned}$$

$$\begin{aligned}|\tilde{H}|^2 &= \langle \tilde{H}, \tilde{H}\rangle_{\psi^2 h}\\ &= \psi^2\langle \psi^{-2}H, \psi^{-2}H\rangle_h\\ &= \psi^{-2}|H|^2.\end{aligned}$$

The relation of the volume forms follow from $\tilde{g} = \psi^2 g$. The Christoffel symbols are scale invariant and thus the connection is also scale invariant. Finally, from the definition of the Laplacian and the scalar invariance of the connection, we see that $\tilde{\Delta} = \psi^{-2}\Delta$. $\square$

It is easy to check that

$$\frac{d}{dt}\int_{\tilde{M}} d\tilde{\mu}_{\tilde{g}(t)} = 0,$$

which means that the volume of the dilated submanifold is fixed as t tends to T.

Now we define the rescaled time variable $\tilde{t}$ by

$$\tilde{t}(t) = \int_0^t \psi^2(\tau)d\tau.$$

So $\frac{d\tilde{t}}{dt} = \psi^2$.

For $0 \le \tilde{t} < \tilde{T} = \tilde{t}(T)$, we first has the following estimates.

$$|\tilde{A}|^2 \le C_n|\tilde{H}|^2, \tag{6.1}$$

$$\frac{|\tilde{H}|_{\min}}{|\tilde{H}|_{\max}} \to 1 \ as \ \tilde{t} \to \tilde{T}, \tag{6.2}$$

$$\tilde{K}_{\min} \ge \epsilon^2|\tilde{H}|^2. \tag{6.3}$$

For the induced metric we have the following evolution equation.

$$\begin{aligned}
\frac{\partial}{\partial\tilde{t}}\tilde{g}_{ij} &= \psi^{-2}\frac{\partial}{\partial t}\tilde{g}_{ij} \\
&= \psi^{-2}\Big(\frac{\partial}{\partial t}\psi^2 g_{ij} + \psi^2\frac{\partial}{\partial t}g_{ij}\Big) \\
&= \psi^{-2}\Big(2\psi^2\frac{\hbar}{n}g_{ij} - \psi^2 2H\cdot h_{ij}\Big) \\
&= -2H\cdot h_{ij} + \frac{2}{n}\hbar g_{ij} \\
&= -2\tilde{H}\tilde{\cdot}\tilde{h}_{ij} + \frac{2}{n}\tilde{\hbar}\tilde{g}_{ij}.
\end{aligned} \tag{6.4}$$

Here $\tilde{\cdot}$ denotes the inner product with respect to the metric $\psi^2 h$. The volume form $d\tilde{\mu}_{\tilde{t}}$ satisfies the following equation.

$$\frac{\partial}{\partial\tilde{t}}d\tilde{\mu}_{\tilde{t}} = (\tilde{\hbar} - |\tilde{H}|^2)d\tilde{\mu}_{\tilde{t}}. \tag{6.5}$$

Proposition 6.3. $0 < C_{\min} \le |\tilde{H}|_{\min} \le |\tilde{H}|_{\max} \le C_{\max} < \infty$ *holds for* $0 \le \tilde{t} < \tilde{T}$.

Proof. From (6.3), the sectional curvature of $\tilde{M}_{\tilde{t}}$ is nonnegative. The Bishop-Gromov volume comparison theorem implies that $\mathrm{Vol}(\tilde{M}_{\tilde{t}}) \le C\tilde{d}^n$, where $\tilde{d}$ is the diameter of $\tilde{M}_{\tilde{t}}$. From the Bonnet theorem, we also have $\tilde{d} \le \frac{\pi}{\sqrt{|\tilde{H}|_{\min}}}$. On the other hand, since $\mathrm{Vol}(\tilde{M}_{\tilde{t}}) = \mathrm{Vol}(\tilde{M}_0)$, we have $|\tilde{H}|_{\min} \le C$, and then (6.2) implies $|\tilde{H}|_{\max} \le C_{\max}$.

If we can show that $|\tilde{H}|_{\max} \geq C > 0$, then (6.2) implies $|\tilde{H}|_{\min} \geq C_{\min} > 0$ for all $\tilde{t} \in [0, \tilde{T})$. Suppose $|\tilde{H}|_{\max} \to 0$ as $\tilde{t} \to \tilde{T}$, (6.1) implies that $|\tilde{A}|^2_{\max} \to 0$ as $\tilde{t} \to \tilde{T}$. Since $|H|^2$ satisfies

$$\frac{\partial}{\partial t}|H|^2 \leq \Delta |H|^2 + C|H|^2_{\max}|H|^2,$$

we can follow the argument in [6] to show that $\int_0^T |H|^2_{\max} dt = \infty$. On the other hand, since $\frac{|H|_{\max}}{|H|_{\min}} \to 1$ as $t \to T$, then for a $\varsigma > 0$ there is a positive constant $\delta > 0$ such that $\frac{|H|_{\max}}{|H|_{\min}} \leq 1 + \varsigma$ for all $t \in [\delta, T)$. So

$$\begin{aligned}
\infty &= \frac{1}{(1+\varsigma)^2} \int_0^T |H|^2_{\max} dt \\
&= \frac{1}{(1+\varsigma)^2} \int_0^\varsigma |H|^2_{\max} dt + \frac{1}{(1+\varsigma)^2} \int_\varsigma^T |H|^2_{\max} dt \\
&\leq \frac{1}{(1+\varsigma)^2} \int_0^\varsigma |H|^2_{\max} dt + \int_\varsigma^T |H|^2_{\min} dt \\
&\leq \frac{1}{(1+\varsigma)^2} \int_0^\varsigma |H|^2_{\max} dt + \int_0^T |H|^2_{\min} dt.
\end{aligned}$$

This implies $\int_0^T |H|^2_{\min} dt = \infty$ since $\int_0^\varsigma |H|^2_{\max} dt < \infty$. We also have $\frac{d\tilde{t}}{dt} = \psi^2$ and $|\tilde{H}|^2 = \psi^{-2}|H|^2$, hence

$$\int_0^{\tilde{T}} \tilde{\hbar}(\tilde{t}) d\tilde{t} = \int_0^T \hbar(t) dt \geq \int_0^T |H|^2_{\min}(t) dt = \infty.$$

However, we have $\tilde{\hbar} \leq |\tilde{H}|^2_{\max} \leq C^2_{\max}$. Therefore $\tilde{T} = \infty$. By the definition of the rescaling of the time variable, $\tilde{T} = \tilde{t}(T) = \int_0^T \psi^2(\tau) d\tau$. Since $T < \infty$, we have $\psi(t) \to \infty$ as $t \to T$. This implies that $(N, \psi^2 h, P)$ converges to the Euclidean space as $\tilde{t} \to \infty$. Since we have $|\tilde{A}|^2_{\max} \to 0$ as $\tilde{t} \to \tilde{T}$, the family of immersions $\tilde{F} : \tilde{M} \to (N, \psi^2 h)$ for $\tilde{t} \in [0, \infty)$, will converge to the isometric immersion of an n-dimensional Euclidean space into an $(n+d)$-dimensional Euclidean space as $\tilde{t} \to \infty$. This is a contradiction since the volume of $\tilde{M}_{\tilde{t}}$ is unchanged along the flow. This competes the proof of the proposition. □

Now we look at the tracefree second fundamental form $\mathring{\tilde{A}}$ of the immersion $\tilde{F}$. From Theorem 4.1, we have the following inequality

$$|\mathring{\tilde{A}}|^2 \leq C_0 \psi^{-\delta} |\tilde{H}|^{2-\delta},$$

which holds for all $\tilde{t} \in [0, \tilde{T})$. As in the proof of Proposition 6.3, we have $\tilde{T} = \infty$ and $\psi(\tilde{t}) \to \infty$ as $\tilde{t} \to \infty$. Hence there holds

$$|\mathring{\tilde{A}}|^2 \to 0, \quad \frac{|\tilde{A}|^2}{|\tilde{H}|^2} - \frac{1}{n} \to 0 \ \text{ as } \ \tilde{t} \to \infty.$$

Since $(N, \psi^2(\tilde{t})h, P) \to (\mathbb{R}^{n+d}, \delta_{AB}, 0)$ in C^∞_{loc}-topology, we see that $\tilde{F}(\tilde{t})$ tends to a totally umbilical immersion from a standard sphere with the same volume as M_0 into the Euclidean space at least in the C^0-topology.

To see that the convergence is in C^∞-topology, we only need to show that all the covariant derivatives of the second fundamental form are uniformly bounded. The second fundamental form $\tilde{A}$ is uniformly bounded by (6.1) since the mean curvature is uniformly bounded.

Along the mean curvature flow, M_t will stay in a compact region of N, say, $M_t \subset B_h(P,r)$ for $t \in [0,T)$ and a suitable $r>0$. After the dilation, $\tilde{M}_{\tilde{t}} \subset B_h(P,r)$, where $B_h(P,r)$ is in fact equal to $B_{\psi^2(\tilde{t})h}(P,\psi(\tilde{t})r)$ as a set. By the definition of the dilation, the Riemannian curvature and its covariant derivations of the ambient space, when restricted to $\tilde{M}_{\tilde{t}}$, are uniformly bounded by constants independent of $\tilde{t}$.

Lemma 6.4. *Let P and Q be two quantities depending on g and A, and P satisfies the evolution equation $\frac{\partial P}{\partial t} = \Delta P + Q$ along the mean curvature flow. If P has degree α, that is, $\tilde{P} = \psi^\alpha P$, then Q has degree $\alpha - 2$ and after dilation, $\tilde{P}$ satisfies*

$$\frac{\partial \tilde{P}}{\partial \tilde{t}} = \tilde{\Delta}\tilde{P} + \tilde{Q} + \frac{\alpha}{n}\tilde{h}\tilde{P}.$$

Proof. Using the similar computation as in the proof of Lemma 9.1 in [9], we can prove the lemma. □

We also need the following interpolation inequalities.

Lemma 6.5. *Let T be any tensor and m an integer. There is a constant $C(n,m)$ independent of the metric and the connection such that*

$$\int_M |\nabla^i T|^{\frac{2m}{i}} d\mu \le C \cdot \max_M |T|^{\frac{2m}{i}-2} \int_M |\nabla^m T|^2 d\mu$$

holds for any $1 \le i \le m-1$, and

$$\int_M |\nabla^i T|^2 d\mu \le C\Big(\int_M |\nabla^m T|^2 d\mu\Big)^{\frac{i}{m}} \Big(\int_M |T|^2 d\mu\Big)^{1-\frac{i}{m}}$$

holds for any $0 \le i \le m$.

Proof. These interpolation inequalities were proved in [6]. □

From Lemma 6.4, we have the following inequality after dilation.

$$\begin{aligned} \frac{\partial}{\partial \tilde{t}}|\tilde{\nabla}^m \tilde{A}|^2 \le & \tilde{\Delta}|\tilde{\nabla}^m \tilde{A}|^2 - 2|\tilde{\nabla}^{m+1}\tilde{A}|^2 \\ & + C(n,m)\Big\{\sum_{i+j+k=m} |\tilde{\nabla}^i \tilde{A}||\tilde{\nabla}^j \tilde{A}||\tilde{\nabla}^k \tilde{A}||\tilde{\nabla}^m \tilde{A}| \\ & + \bar{C}_m \sum_{i\le m} |\tilde{\nabla}^i \tilde{A}||\tilde{\nabla}^m \tilde{A}| + \bar{\bar{C}}_m |\tilde{\nabla}^m \tilde{A}|\Big\} \\ & + \tilde{C}(n,m)\tilde{h}|\tilde{\nabla}^m \tilde{A}|^2. \end{aligned} \tag{6.6}$$

To prove that for any integer $m \ge 0$, $|\tilde{\nabla}^m \tilde{A}|^2$ is uniformly bounded on $[0,\infty)$, we argue by induction on m. Obviously, for $m=0$, $|\tilde{A}|^2$ is uniformly bounded. Suppose that $|\tilde{\nabla}^i \tilde{A}|^2$ is uniformly bounded on $[0,\infty)$ for $i = 1, 2, \cdots, m-1$. Then from (6.6), we see that there holds

$$\frac{\partial}{\partial \tilde{t}}|\tilde{\nabla}^m \tilde{A}|^2 \le \tilde{\Delta}|\tilde{\nabla}^m \tilde{A}|^2 - 2|\tilde{\nabla}^{m+1}\tilde{A}|^2 + C|\tilde{\nabla}^m \tilde{A}|^2 + C \tag{6.7}$$

for some positive constant C independent of $\tilde{t}$. Here we have used Young's inequality and the boundedness of $\tilde{\hbar}$ which is implied by that the mean curvature is uniformly bounded and the area of the submanifold is fixed. From (6.7) we have

$$\begin{aligned}
\frac{d}{d\tilde{t}}\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}} =& \int_{\tilde{M}_{\tilde{t}}}\frac{\partial}{\partial\tilde{t}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2\frac{\partial}{\partial\tilde{t}}d\tilde{\mu}_{\tilde{t}}\\
\leq& -2\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^{m+1}\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+C\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}\\
&+\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2(\tilde{\hbar}-|\tilde{H}|^2)d\tilde{\mu}_{\tilde{t}}+C\\
\leq& -2\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^{m+1}\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+C\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+C.
\end{aligned}\tag{6.8}$$

By the second interpolation inequality in Lemma 6.5, we have

$$\begin{aligned}
\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}} \leq& \bar{C}\Big(\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^{m+1}\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}\Big)^{\frac{m}{m+1}}\Big(\int_{\tilde{M}_{\tilde{t}}}|\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}\Big)^{\frac{1}{m+1}}\\
\leq& \bar{C}\eta\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^{m+1}\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+\bar{C}\eta^{-m}\int_{\tilde{M}_{\tilde{t}}}|\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}\\
\leq& \bar{C}\eta\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^{m+1}\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+\bar{\bar{C}}\eta^{-m}.
\end{aligned}\tag{6.9}$$

Combining (6.8) and (6.9), we have

$$\begin{aligned}
\frac{d}{d\tilde{t}}\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}} \leq& -2\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^{m+1}\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}-C\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+C\\
&+2C\bar{C}\eta\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^{m+1}\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+2C\bar{\bar{C}}\eta^{-m}\\
=& -C\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}+\tilde{C}.
\end{aligned}\tag{6.10}$$

Here we have chosen $\eta=(C\bar{C})^{-1}$ and $\tilde{C}=C+2C\bar{\bar{C}}(C\bar{C})^m$. Put $f(\tilde{t})=\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}$, then

$$\frac{d}{d\tilde{t}}\Big(f-\frac{\tilde{C}}{C}\Big)=\frac{d}{d\tilde{t}}f\leq -Cf+\tilde{C}=-C\Big(f-\frac{\tilde{C}}{C}\Big).$$

If $f-\frac{\tilde{C}}{C}\leq 0$ at $\tilde{t}=0$, then it holds for any $t\in[0,\infty)$, which implies $\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}\leq C_m$ for some positive constant C_m depending on m but independent of $\tilde{t}$.

If $f-\frac{\tilde{C}}{C}>0$ at $\tilde{t}=0$, then $f-\frac{\tilde{C}}{C}\leq(f(0)-\frac{\tilde{C}}{C})e^{-C\tilde{t}}$. This also implies that $\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^2d\tilde{\mu}_{\tilde{t}}\leq C_m$ for some positive constant C_m depending on m but independent of $\tilde{t}$.

From the first interpolation inequality in Lemma 6.5, we see that

$$\int_{\tilde{M}_{\tilde{t}}}|\tilde{\nabla}^m\tilde{A}|^pd\tilde{\mu}_{\tilde{t}}\leq C_{m,p}$$

holds for large p. Combining this with the Sobolev inequality in [8], we may carry out a Stampacchia iteration process just as in [9] and [10] to get that $|\tilde{\nabla}^m\tilde{A}|^2\leq\tilde{C}_m$ for a constant $\tilde{C}_m<\infty$ depending on m.

Hence we have proved that the convergence is in fact in C^∞-topology.

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