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Rationally inequivalent points on hypersurfaces in $\mathbb{P}^n$ ☆

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ABSTRACT

We prove a conjecture of Voisin that any two distinct points on a very general hypersurface of degree $2n+2$ in $\mathbb{P}^{n+1}$ are rationally inequivalent.

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Contents

1. Introduction

In this paper, we work exclusively over $\mathbb{C}$. Recall that Claire Voisin proved the following ([18, Theorem 3.1] for hypersurfaces and [19, Theorem 0.6] for complete intersections)

Theorem 1.1 *(C. Voisin). Let X be a very general complete intersection in $\mathbb{P}^{n+k}$ of type $(d_1, d_2, ..., d_k)$.*

- *If $\sum(d_i - 1) \geq 2n+2$, no two distinct points on X are $\mathbb{Q}$-rationally equivalent.*
- *If $(n, k, d_1) = (2, 1, 6)$, there are at most countably many points on X that are $\mathbb{Q}$-rationally equivalent to a fixed point p for all $p \in X$.*

The main purpose of the paper is to generalize this result in two directions. First, we will make a minor improvement by replacing rational equivalence by Roĭtman's Γ-equivalence [14]: fixing a smooth projective curve Γ and two points $0 \neq \infty \in \Gamma$, for every algebraic cycle $\xi \in \mathcal{Z}^k(X \times \Gamma)$ with $\operatorname{supp}(\xi)$ flat over Γ, the fibers ξ_0 and ξ_∞ of ξ over 0 and ∞ are Γ-equivalent, written as $\xi_0 \sim_\Gamma \xi_\infty$.

We will prove

Theorem 1.2. *For a fixed smooth projective curve Γ with two fixed points $0 \neq \infty$, no two distinct points on a very general complete intersection X in $\mathbb{P}^{n+k}$ of type $(d_1, d_2, ..., d_k)$ are Γ-equivalent over $\mathbb{Q}$ if $\sum(d_i - 1) \geq 2n+2$.*

Although the replacement of rational equivalence by Γ-equivalence is only a mild improvement of Voisin's result, it does lead to the following interesting consequence:

Corollary 1.3. *Fixing a quasi-projective variety T, there is no nonconstant rational map from T to a very general complete intersection X in $\mathbb{P}^{n+k}$ of type $(d_1, d_2, ..., d_k)$ if $\sum(d_i - 1) \geq 2n+2$ or $k = 1$ and $d_1 = 2n+2$.*

The second case in the corollary follows from our solution to Voisin's conjecture (see Theorem 1.5 below).

Second, we will try to find the optimal bound for d_i where the result holds. Our most optimistic expectation is

Conjecture 1.4. *For a very general complete intersection $X \subset \mathbb{P}^{n+k}$ of type $(d_1, d_2, ..., d_k)$ and every point $p \in X$,*

$$\dim R_{X,p,\Gamma} \leq 2n - \sum_{i=1}^{k}(d_i - 1) \tag{1.1}$$

where $R_{X,p,\Gamma} = \{q \neq p \in X : N(p-q) \sim_\Gamma 0 \text{ for some } N \in \mathbb{Z}^+\}$ and Γ is a fixed smooth projective curve with two fixed points $0 \neq \infty$.

Note that $R_{X,p,\Gamma}$ is a locally Noetherian scheme.

The case $\sum(d_i - 1) = n+1$ follows from Roĭtman's generalization of Mumford's famous theorem ([13], [14] and [15]). Of course, Voisin proved

$$\dim R_{X,p,\mathbb{P}^1} \leq 2n+1 - \sum_{i=1}^{k}(d_i - 1) \tag{1.2}$$

for $\sum(d_i - 1) \geq 2n+2$ or $(n,k,d_1) = (2,1,6)$. Theorem 1.2 shows that (1.1) holds for $\sum(d_i - 1) \geq 2n+2$.

If our conjecture holds, $R_{X,p,\Gamma} = \emptyset$ when $\sum(d_i - 1) \geq 2n+1$. So the "boundary" case is $\sum(d_i - 1) = 2n+1$. For example, it is expected that $R_{X,p,\Gamma} = \emptyset$ for a very general sextic surface $X \subset \mathbb{P}^3$. Voisin's theorem shows that $\dim R_{X,p,\mathbb{P}^1} = 0$ for such surfaces X. This boundary case is quite challenging, even only for sextic surfaces. We claim the following:

Theorem 1.5. *No two distinct points are Γ-equivalent over $\mathbb{Q}$ on a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $2n+2$ for a fixed smooth projective curve Γ with two fixed points $0 \neq \infty$. That is, (1.1) holds for $k = 1$ and $d_1 = 2n+2$.*

Note that the bound $d \geq 2n+2$ is optimal for hypersurfaces of degree d in $\mathbb{P}^{n+1}$: For a general hypersurface X of degree $d \leq 2n+1$ in $\mathbb{P}^{n+1}$, there exist two lines L_1

and L_2 in $\mathbb{P}^{n+1}$ such that each L_i meets X at a unique point p_i with $p_1 \neq p_2$. Then $dp_1 \sim_{\mathbb{P}^1} L_1.X \sim_{\mathbb{P}^1} L_2.X \sim_{\mathbb{P}^1} dp_2$ and hence p_1 and p_2 are rationally equivalent over $\mathbb{Q}$. Please see Appendix A for details.

Recently, Eric Riedl and David Yang proved Conjecture 1.4 for $k = 1$ and $d_1 \leq 2n+1$ [16]. So together with Theorem 1.5, we know that the conjecture holds for hypersurfaces.

A logarithmic version of Theorem 1.2, with rational equivalence replaced by $\mathbb{A}^1$-equivalence, was given in [3].

We are only interested in the rational equivalence of two points. Voisin's method can be easily extended to the study of the linear dependence of m points under rational equivalence on generic hypersurfaces. This was done in [6], where it was proved that for every $m \in \mathbb{Z}^+$, there exists a number $d(m,n)$ such that m distinct points on a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of $\deg X \geq d(m,n)$ are linearly independent in $\mathrm{CH}_0(X) \otimes \mathbb{Q}$. However, the bound $d(m,n)$ obtained there is certainly not optimal. We are expecting that $d(m,n) = 2n + \lceil (m+1)/2 \rceil$. However, we are not certain whether this is true for $m \geq 3$.

2. Relative cycle map

Voisin's proof consists of two major components. One is relative cycle map. For a relative Chow cycle $Z \in \mathrm{CH}^n_{\mathrm{hom}}(X/B)$ for a smooth projective family $\pi : X \to B$ of relative dimension n, if $\mathrm{AJ}_n(Z_b) = 0$ under the Abel-Jacobi map on each fiber X_b, one can define some topological invariant $\delta Z \in H^n(R^n\pi_*\mathbb{Q})$. This invariant can be defined in a Hodge-theoretical way as in [19]. Please see Appendix B for a comprehensive treatment along this line. Here we take a different approach: we define δZ directly by (2.2) (see below) and then we prove δZ is invariant under rational equivalence. This has the advantage of being elementary: no Hodge theory is involved in the definition of δZ. In addition, we will obtain for free that δZ is invariant under Γ-equivalence. Another advantage of this approach is that δZ is well defined for an arbitrary flat family $\pi : X \to B$ without any extra hypotheses on X/B.

Definition 2.1. Let $\pi : X \to B$ be a flat and surjective morphism of relative dimension n from a smooth variety X onto a smooth variety B of $\dim B = N$. For a multi-section $Z \subset X$, we define

$$\delta Z \in \mathrm{Hom}(\pi_*(\wedge^N \Omega_X), \wedge^N \Omega_B) = \mathrm{Hom}(\pi_* \Omega_X^N, K_B) \tag{2.1}$$

as follows:

$$\delta Z = \mathrm{Tr}_{Z/B} \circ (d\sigma) : \pi_* \Omega_X^N \xrightarrow{d\sigma} (\pi \circ \sigma)_* \Omega_Z^N = (\pi \circ \sigma)_* K_Z \xrightarrow{\mathrm{Tr}_{Z/B}} K_B \tag{2.2}$$

where $\mathrm{Tr}_{Z/B}$ is the trace map and $\sigma : Z \hookrightarrow X$ is the embedding.

We can easily extend δ to the free abelian group $\mathcal{Z}^n(X/B)$ of algebraic cycles Z of pure codimension n in X whose support $\operatorname{supp}(Z)$ is flat over B. For $Z = \sum m_i Z_i$ with Z_i multi-sections of π, we let $\delta Z = \sum m_i \delta Z_i$.

Remark 2.2. The definition (2.2) of δZ might need some further explanation. The differential map $d\sigma$ is usually $d\sigma : \sigma^*\Omega_X^N \to \Omega_Z^N$. In (2.2), it is the composition of $d\sigma$ and $(\pi \circ \sigma)_*$:

$$\begin{array}{ccccc} \pi_*\Omega_X^N & \longrightarrow & \pi_*(\Omega_X^N \otimes \mathcal{O}_Z) & \longrightarrow & (\pi \circ \sigma)_*\Omega_Z^N \\ & & \| & & \\ & & \pi_*(\sigma_*\sigma^*\Omega_X^N). & & \end{array} \tag{2.3}$$

The trace map $\operatorname{Tr}_{Z/B}$ can be defined for $\pi_*(\wedge^m \Omega_Z) \to \wedge^m \Omega_B$ under a generically finite map $\pi : Z \to B$. Obviously, it is well defined outside of the ramification locus of π. Since every meromorphic differential form in $\wedge^m \Omega_B$ is regular if it is regular in codimension 1, it suffices to show that the image of a differential m-form on Z under the trace map can be extended to a regular m-form on B in codimension 1 [8, Proposition 5.77, p. 185]. Moreover, the trace map is well defined for B normal if we follow the convention to define Ω_B to be the sheaf of differential forms regular in codimension 1. However, $\operatorname{Tr}_{Z/B}$ cannot be defined for $\pi_*(\Omega_Z^{\otimes m}) \to \Omega_B^{\otimes m}$ when $m \geq 2$, which is the reason why Mumford's argument cannot be generalized using pluri-canonical forms.

Lemma 2.3. *Let $\pi : X \to B$ be a flat and projective morphism of relative dimension n from X onto a smooth variety B of $\dim B = N$ and let Z be a cycle in $\mathcal{Z}^n(X/B)$. If $\pi_*\Omega_X^N$ is locally free and $Z_b \sim_\Gamma 0$ for all $b \in B$, then $\delta Z = 0$, where Γ is a fixed smooth projective curve with two fixed points $0 \neq \infty$.*

Proof. Since $\pi_*\Omega_X^N$ is locally free, $\delta Z = 0$ if and only if $\delta Z = 0$ at a general point of B. So we may shrink B if necessary.

Using a Hilbert scheme argument, we can find a dominant and generically finite morphism $f : B' \to B$ and a cycle $Y \in \mathcal{Z}^n(X' \times \Gamma)$ such that $\operatorname{supp}(Y)$ is flat over $B' \times \Gamma$ and $Y_0 - Y_\infty = f^*Z$, where $X' = X \times_B B'$, Y_t is the fiber of Y over $t \in \Gamma$ and f^*Z is the pullback of Z under $f : X' \to X$ (we also use f to denote the map $X' \to X$). Note that we can shrink B to guarantee the flatness of Y over $B' \times \Gamma$.

Note that δ commutes with base change. Namely, we have the commutative diagram

$$\begin{array}{ccc} f^*\pi_*\Omega_X^N & \longrightarrow & (\pi')_*\Omega_{X'}^N \\ \big\downarrow {\scriptstyle f^*(\delta Z)} & & \big\downarrow {\scriptstyle \delta(f^*Z)} \\ f^*K_B & \longrightarrow & K_{B'} \end{array} \tag{2.4}$$

where $\pi' : X' \to B'$ is the projection.

Obviously, $\delta Z = 0$ if $\delta f^* Z = 0$ by (2.4) and the fact that $\pi_* \Omega_X^N$ is locally free. So we may simply replace (X, B) by (X', B').

Since Y is flat over $B \times \Gamma$, it defines δY lying

$$\begin{aligned} \delta Y &\in \operatorname{Hom}(\varepsilon_* \Omega_{X\times\Gamma}^{N+1}, K_{B\times\Gamma}) \\ &= \operatorname{Hom}(\eta_1^* \pi_* \Omega_X^{N+1} \oplus \eta_1^* \pi_* \Omega_X^N \otimes \eta_2^* K_\Gamma, \eta_1^* K_B \otimes \eta_2^* K_\Gamma) \\ &= \operatorname{Hom}(\eta_1^* \pi_* \Omega_X^{N+1}, \eta_1^* K_B \otimes \eta_2^* K_\Gamma) \oplus \operatorname{Hom}(\eta_1^* \pi_* \Omega_X^N, \eta_1^* K_B) \end{aligned} \tag{2.5}$$

where ε, η_1 and η_2 are the projections $\varepsilon : X \times \Gamma \to B \times \Gamma$, $\eta_1 : B \times \Gamma \to B$ and $\eta_2 : B \times \Gamma \to \Gamma$, respectively. Let ρ be the projection

$$\begin{array}{ccc} \operatorname{Hom}(\varepsilon_* \Omega_{X\times\Gamma}^{N+1}, K_{B\times\Gamma}) & \xrightarrow{\ \rho\ } & \operatorname{Hom}(\eta_1^* \pi_* \Omega_X^N, \eta_1^* K_B) \\ & & \| \\ & & H^0(\eta_1^*((\pi_* \Omega_X^N)^\vee \otimes K_B)) \end{array} \tag{2.6}$$

given in (2.5).

For every coherent sheaf V on B, we have

$$H^0(\eta_1^* V) = H^0((\eta_1)_* \eta_1^* V) = H^0(V)$$

since Γ is projective. In other words, for every $s \in H^0(\eta_1^* V)$, its restriction s_t to $t \in \Gamma$ is a constant section in $H^0(V)$. Therefore, $\rho(\delta Y)_t$ is constant.

For every $t \in \Gamma$, we clearly have $\delta Y_t = \rho(\delta Y)_t$. Therefore, δY_t is constant. It follows that $\delta Z = \delta Y_0 - \delta Y_\infty = 0$. We are done. □

We say that a closed subscheme $Z \subset X$ or its ideal sheaf $I_Z \subset \mathcal{O}_X$ imposes independent conditions on a coherent sheaf $\mathcal{F}$ or its global sections $H^0(\mathcal{F})$ (resp. a linear series $\mathcal{D} \subset H^0(\mathcal{F})$) on X if $H^0(\mathcal{F}) \to H^0(\mathcal{F} \otimes \mathcal{O}_Z)$ (resp. $\mathcal{D} \to H^0(\mathcal{F} \otimes \mathcal{O}_Z)$) is surjective.

A coherent sheaf $\mathcal{F}$ on a variety X is *globally generated* (resp. *very ample*) if every 0-dimensional subscheme $Z \subset X$ of length $h^0(\mathcal{O}_Z) = 1$ (resp. 2) imposes independent conditions on $\mathcal{F}$. We also say that $\mathcal{F}$ is *weakly very ample* if this holds for all $Z = \{p, q\}$ consisting of two distinct points $p \neq q$.

To show that $\sigma_1(b) \nsim_\Gamma \sigma_2(b)$ over $\mathbb{Q}$ at a general point $b \in B$ for two sections $\sigma_i : B \hookrightarrow X$ of X/B, we only need to find $s \in H^0(U, \pi_* \Omega_X^N)$ satisfying

$$(d\sigma_1)\sigma_1^* s - (d\sigma_2)\sigma_2^* s \neq 0 \tag{2.7}$$

over some open dense subset $U \subset B$. The existence of such s is guaranteed if $\sigma_i(b)$ impose independent conditions on $H^0(X_b, \Omega_X^N)$ for $b \in B$ general. This observation leads to the following:

Proposition 2.4. *Let $\pi : X \to B$ be a smooth and projective morphism from a smooth X onto a smooth variety B of $\dim B = N$. Suppose that $\Omega_X^N \otimes \mathcal{O}_{X_b}$ is weakly very ample on X_b for $b \in B$ general. Then $R_{X_b,p,\Gamma} = \emptyset$ for $b \in B$ very general and all $p \in X_b$, where Γ is a fixed smooth projective curve with two fixed points $0 \neq \infty$. More generally,*

$$R_{X_b,p,\Gamma} \subset \Big\{ q \in X_b : q \neq p \text{ and } \{p,q\} \text{ does not impose independent conditions on } H^0(X_b, \Omega_X^N) \Big\} \tag{2.8}$$

for $b \in B$ very general.

Proof. Suppose that there are a pair of points $p \neq q$ on a general fiber X_b such that $p \sim_\Gamma q$ over $\mathbb{Q}$ and $\{p,q\}$ imposes independent conditions on $H^0(X_b, \Omega_X^N)$. By a base change and shrinking B to an affine variety, we may assume that

- there exists two disjoint sections P and $Q \subset X$ of $\pi : X \to B$ such that $m(P_b - Q_b) \sim_\Gamma 0$ for some $m \in \mathbb{Z}^+$ and all $b \in B$,
- $h^0(X_b, \Omega_X^N)$ is constant for all $b \in B$ and
- $P \sqcup Q$ imposes independent conditions on $H^0(\Omega_X^N)$.

Since $P \sqcup Q$ imposes independent conditions on $H^0(\Omega_X^N)$ and Ω_X^N is locally free, the map

$$H^0(\Omega_X^N \otimes I_P) \xrightarrow{\sigma_Q^*} H^0(\sigma_Q^* \Omega_X^N) \tag{2.9}$$

is a surjection, where σ_P and $\sigma_Q : B \hookrightarrow X$ are the embeddings of P and Q to X, respectively. Combining (2.9) with the pullback map of $\sigma_Q : B \hookrightarrow X$ on differentials, we have a composition of two surjections

$$H^0(\Omega_X^N \otimes I_P) \xrightarrow{\sigma_Q^*} H^0(\sigma_Q^* \Omega_X^N) \xrightarrow{d\sigma_Q} H^0(\Omega_B^N) \tag{2.10}$$

where $d\sigma_P$ and $d\sigma_Q$ are the pullback maps induced by σ_P and σ_Q on the differentials, respectively. Therefore, there exists $s \in H^0(\Omega_X^N)$ such that

$$\sigma_P^* s = 0 \text{ and } (d\sigma_Q)\sigma_Q^* s \neq 0. \tag{2.11}$$

It follows that

$$\langle \delta Z, s \rangle = (d\sigma_P)\sigma_P^* s - (d\sigma_Q)\sigma_Q^* s = -(d\sigma_Q)\sigma_Q^* s \neq 0 \tag{2.12}$$

for $Z = P - Q$. On the other hand, $\delta Z = 0$ by Lemma 2.3. Contradiction.

The above argument shows that no irreducible component of

$$S_{X,\Gamma} = \Big\{(b,p,q) : b \in B \text{ and } p \neq q \in X_b \text{ satisfy that } p \sim_\Gamma q \text{ over } \mathbb{Q} \\ \text{and } \{p,q\} \text{ imposes independent conditions} \\ \text{on } H^0(X_b, \Omega_X^N)\Big\} \tag{2.13}$$

dominates B via the projection $\xi : S_{X,\Gamma} \to B$. Note that $S_{X,\Gamma}$ is a locally Noetherian subscheme of $X \times_B X$. Therefore, for $b \in B \backslash \xi(S_{X,\Gamma})$, (2.8) holds. □

Remark 2.5. Note that the right hand side (RHS) of (2.8) is a subscheme that does not depend on the choice of the triple $(\Gamma, 0, \infty)$.

Proof of Theorem 1.2. Let $X \subset B \times \mathbb{P}^{n+k}$ be the universal family of complete intersections in $\mathbb{P}^{n+k}$ of type $(d_1, d_2, ..., d_k)$. By [18], Ω_X^N is very ample on X_b for $b \in B$ general when $\sum(d_i - 1) \geq 2n+2$. Then it follows from Proposition 2.4 that no two distinct points on X_b are Γ-equivalent over $\mathbb{Q}$ for $b \in B$ very general. □

The rest of the paper will be devoted to the case $k = 1$ and $d_1 = 2n+2$.

3. Positivity of the sheaf of holomorphic N-forms

Throughout this section, unless otherwise stated, we let $\pi : X \to B$ be a smooth projective morphism of relative dimension n from smooth X onto a smooth variety B of $\dim B = N$.

3.1. Ampleness of Ω_X^N

By Proposition 2.4, we can prove that no two distinct points are Γ-equivalent on a very general fiber X_b of X/B if Ω_X^N is weakly very ample when restricted to X_b. We observe that

$$\Omega_X^N \otimes \pi^* K_B^{-1} \cong T_X^n \otimes K_{X/B} \tag{3.1}$$

using the pairing $\Omega_X^N \otimes \Omega_X^n \to K_X$.

When $X \subset \mathbb{P}^{n+k} \times B$ is a family of complete intersections in $\mathbb{P}^{n+k}$ of type $(d_1, d_2, ..., d_k)$, we have

$$\begin{aligned} \Omega_X^N \otimes \pi^* K_B^{-1} &\cong T_X^n \otimes \mathcal{O}_X\left(\sum(d_i - 1) - (n+1)\right) \\ &\cong T_X^n(n) \otimes \mathcal{O}_X\left(\sum(d_i - 1) - (2n+1)\right) \end{aligned} \tag{3.2}$$

where $\mathcal{O}_X(1)$ is the pullback of the hyperplane bundle on $\mathbb{P}^{n+k}$ and

$$T_X^n(n) = \wedge^n T_X \otimes \mathcal{O}_X(n) = \wedge^n(T_X(1)).$$

It is very easy to verify the following simple facts:

(1) The quotients of globally generated/(weakly) very ample coherent sheaves are also globally generated/(weakly) very ample.
(2) If $\mathcal{F}$ is globally generated and $\mathcal{G}$ is globally generated/(weakly) very ample, then $\mathcal{F} \otimes \mathcal{G}$ is also globally generated/(weakly) very ample.
(3) Combining (1) and (2), we see that the wedge and symmetric products of a globally generated/(weakly) very ample coherent sheaf are also globally generated/(weakly) very ample.

When $X \subset \mathbb{P}^{n+k} \times B$ is a versal family of complete intersections, Voisin proved that $T_X(1) = T_X \otimes \mathcal{O}_X(1)$ is globally generated on X_b, generalizing Herbert Clemens' result for hypersurfaces [2] (see also [4] and [5]). Thus, we can conclude that Ω_X^N is very ample on X_b when $\sum(d_i - 1) > 2n + 1$, using the simple facts on global generation and very ampleness as above.

When $\sum(d_i - 1) = 2n + 1$, we naturally ask whether Voisin's result can be improved to show that $\Omega_X^N \otimes \pi^* K_B^{-1} = T_X^n(n)$ is weakly very ample on X_b. Voisin actually had a stronger conjecture that $T_X^2(1) = (\wedge^2 T_X) \otimes \mathcal{O}_X(1)$ is globally generated on X_b [19, Question 2.1].

Unfortunately, $T_X^2(1)$ fails to be globally generated [20]. Even the weaker assertion that $T_X^n(n)$ is weakly very ample is very likely to fail. Thus, in order to tackle the case $\sum(d_i - 1) = 2n + 1$, we cannot rely on the very ampleness of Ω_X^N. Instead, we need to develop some refined criteria to show that $\delta(\sigma_1(B) - \sigma_2(B)) \neq 0$ for two sections σ_i of X/B. These criteria, while still depending on certain positivity of Ω_X^N, do not require it to be weakly very ample.

3.2. Differential map $d\sigma$

A closer examination of the proof of Proposition 2.4 shows that we do not really need Ω_X^N to be weakly very ample on X_b. We only need find $s \in H^0(U, \pi_*\Omega_X^N)$ satisfying (2.7). This is much weaker than the requirement that $p_1 = \sigma_1(b)$ and $p_2 = \sigma_2(b)$ impose independent conditions on $H^0(X_b, \Omega_X^N)$ for b general. For one thing, $(d\sigma_1)\sigma_1^* s - (d\sigma_2)\sigma_2^* s = 0$ imposes only one condition on $\Gamma_b(\Omega_X^N) = H^0(X_b, \Omega_X^N)$.

Let $\Gamma_b(d\sigma_i)$ be the map induced by $d\sigma_i$ on $\Gamma_b(\Omega_X^N)$ as in

$$\begin{array}{c} \Gamma_b(T_X^n \otimes K_X) \\ \Vert \\ \Gamma_b(\Omega_X^N) \xrightarrow{d\sigma_1 \oplus d\sigma_2} \Gamma_b\left(K_{\sigma_1(B)}\right) \oplus \Gamma_b\left(K_{\sigma_2(B)}\right). \end{array} \tag{3.3}$$

Clearly, (2.7) holds for some $s \in H^0(U, \pi_* \Omega_X^N)$ if

$$\ker(\Gamma_b(d\sigma_1)) \neq \ker(\Gamma_b(d\sigma_2)) \tag{3.4}$$

holds at a general point $b \in B$. More precisely, as long as (3.4) holds at a point $b \in B$ such that $h^0(X_t, \Omega_X^N)$ is locally constant for t in an open neighborhood of b, we can find a section $s_b \in \Gamma_b(\Omega_X^N)$ with the property

$$(d\sigma_1)s_b - (d\sigma_2)s_b \neq 0 \tag{3.5}$$

and this s_b can be extended to a section $s \in H^0(U, \pi_* \Omega_X^N)$ over an open neighborhood U of b satisfying (2.7).

Therefore, to show that $\sigma_i(b)$ are not Γ-equivalent over $\mathbb{Q}$ on a general fiber X_b, we just have to prove (3.4). Let us formalize this observation in the following proposition:

Proposition 3.1. *Let X be a smooth projective family of varieties over a smooth variety B of $\dim B = N$ and let $\sigma_i : B \to X$ be two disjoint sections of X/B for $i = 1, 2$. Then $\sigma_1(b)$ and $\sigma_2(b)$ are not Γ-equivalent over $\mathbb{Q}$ on X_b for $b \in B$ general if (3.4) holds at a point b where $h^0(X_t, \Omega_X^N)$ is locally constant in t.*

3.3. Criterion for two fixed sections

In the rest of this section, we assume that $X \subset B \times P$ is a smooth family of subvarieties in some smooth projective variety P. Usually, we take P to be some projective space. But some results apply to arbitrary P. Here we strive for the greatest generality, although we just need to apply these results to hypersurfaces in $\mathbb{P}^n$.

To apply Proposition 3.1, we need an explicit description of the differential maps $d\sigma_i$. They can be made very explicit if $X \subset Y = B \times P$ is a family of varieties in a projective space P passing through two fixed points $p_i \in P$ and $\sigma_i(b) \equiv p_i$ for $i = 1, 2$. On the other hand, for an arbitrary family $X \subset Y$ with two sections σ_i over B, we can always apply an automorphism $\lambda \in B \times \mathrm{Aut}(P)$, after a base change, fiberwise to Y/B such that $\lambda \circ \sigma_i(b) \equiv p_i$ for two fixed points $p_i \in P$; thus, to test (3.4) for a general fiber X_b of X/B, it suffices to test it for a general fiber $\widehat{X}_b$ of $\widehat{X}/B$, $\widehat{X} = \lambda(X)$ and $\widehat{\sigma}_i = \lambda \circ \sigma_i$. Let us first consider families $X \subset B \times P$ with two fixed sections $\sigma_i(b) \equiv p_i$.

To set it up, we let $P = \mathbb{P}^r$ and fix two points $p_1 \neq p_2$ in P. We let $X \subset Y = B \times P$ be a closed subvariety of Y that is flat over B with fibers X_b containing p_1 and p_2 for all $b \in B$. We assume that X and B are smooth of $\dim X = N + n$ and $\dim B = N$, respectively. We have two sections $\sigma_i : B \to X$ sending $\sigma_i(b) = p_i$ for all $b \in B$ and $i = 1, 2$.

To state our next proposition on the differential map $d\sigma$, we need to introduce the filtration $F^\bullet \Omega_X$ associated to the fibration X/B.

For a surjective morphism $f : W \to B$ with B smooth, we have a Leray filtration

$$\begin{gathered}\Omega_W^m = F^0\Omega_W^m \supset F^1\Omega_W^m \supset ... \supset F^{m+1}\Omega_W^m = 0 \\ \text{with } \mathrm{Gr}_F^p\Omega_W^m = \frac{F^p\Omega_W^m}{F^{p+1}\Omega_W^m} = f^*\left(\wedge^p\Omega_B\right) \otimes \wedge^{m-p}\Omega_{W/B}\end{gathered} \tag{3.6}$$

for $\Omega_W^m = \wedge^m\Omega_W$ derived from the short exact sequence

$$0 \longrightarrow f^*\Omega_B \longrightarrow \Omega_W \longrightarrow \Omega_{W/B} \longrightarrow 0. \tag{3.7}$$

Note that F^p is an exact functor.

For $\pi_B : Y \to B$ with $Y = B \times P$, $F^p\Omega_Y^m$ is simply

$$F^p\Omega_Y^m = \bigoplus_{i \geq p} \pi_B^*\Omega_B^i \otimes \pi_P^*\Omega_P^{m-i} \tag{3.8}$$

and we have natural projections $\Omega_Y^m \to F^p\Omega_Y^m$, where π_B and π_P are the projections of Y to B and P, respectively.

We have the so-called adjunction sequence

$$0 \longrightarrow T_X \longrightarrow T_Y \otimes \mathcal{O}_X \xrightarrow{\eta} \mathcal{N}_X \longrightarrow 0 \tag{3.9}$$

associated to $X \subset Y$, where $\mathcal{N}_X$ is the normal bundle of X in Y. From this, we obtain a left exact sequence

$$0 \longrightarrow T_X^m \longrightarrow T_Y^m \xrightarrow{\eta_m} T_Y^{m-1} \otimes \mathcal{N}_X \tag{3.10}$$

on X_b, where η_m is actually the map in the generalized Koszul complex

$$\begin{gathered}\wedge^m T_Y \otimes \mathcal{O}_X \xrightarrow{\eta_m} \wedge^{m-1}T_Y \otimes \mathcal{N}_X \to \wedge^{m-2}T_Y \otimes \mathrm{Sym}^2\mathcal{N}_X \\ \to ... \to T_Y \otimes \mathrm{Sym}^{m-1}\mathcal{N}_X \to \mathrm{Sym}^m\mathcal{N}_X \to 0\end{gathered} \tag{3.11}$$

of $\wedge^{m-\bullet}T_Y \otimes \mathrm{Sym}^\bullet\mathcal{N}_X$ induced by η.

Setting $m = n$ in (3.10), we have

$$\begin{array}{ccccc} T_X^n \otimes K_X & \hookrightarrow & T_Y^n \otimes K_X & \xrightarrow{\eta_n} & T_Y^{n-1} \otimes K_X \otimes \mathcal{N}_X \\ \| & & \| & & \| \\ \Omega_X^N & \hookrightarrow & \Omega_Y^{N+k} \otimes \det(\mathcal{N}_X) & \longrightarrow & \Omega_Y^{N+1+k} \otimes \det(\mathcal{N}_X) \otimes \mathcal{N}_X \end{array} \tag{3.12}$$

where $\det(\mathcal{N}_X) = \wedge^k\mathcal{N}_X$ for $k = \dim Y - \dim X$. Let us first prove:

Proposition 3.2. *Let $X \subset Y = B \times P$ be a smooth projective family of varieties in a smooth projective variety P passing through a fixed point $p \in P$ over a smooth variety B with the section $\sigma : B \to X$ given by $\sigma(b) = p$ for $b \in B$. Then the diagram*

$$\begin{array}{ccccc} \Omega_X^N & \hookrightarrow & \Omega_Y^{N+k} \otimes \det(\mathcal{N}_X) & \longrightarrow & \Omega_Y^{N+k+1} \otimes \det(\mathcal{N}_X) \otimes \mathcal{N}_X \\ \downarrow{\scriptstyle d\sigma} & & \downarrow & & \downarrow \\ \Omega_{\sigma(B)}^N & \hookrightarrow & F^N\Omega_Y^{N+k} \otimes \det(\mathcal{N}_X)\Big|_{\sigma(B)} & \longrightarrow & F^N\Omega_Y^{N+k+1} \otimes \det(\mathcal{N}_X) \otimes \mathcal{N}_X\Big|_{\sigma(B)} \end{array} \tag{3.13}$$

commutes and has left exact rows, where $N = \dim B$, $k = \dim Y - \dim X$, $\det(\mathcal{N}_X) = \wedge^k \mathcal{N}_X$ and the vertical maps in the second and third columns are induced by the projections $\Omega_Y^\bullet \to F^N\Omega_Y^\bullet$ followed by the restrictions to $\sigma(B)$.

Proof. The rows of (3.13) are induced by Koszul complex (3.11) and hence left exact.

We want to point out that the diagram

$$\begin{array}{ccc} \Omega_Y^m & \xrightarrow{\eta} & \Omega_Y^{m+1} \otimes \mathcal{N}_X \\ \downarrow & & \downarrow \\ F^l\Omega_Y^m & \longrightarrow & F^l\Omega_Y^{m+1} \otimes \mathcal{N}_X \end{array} \tag{3.14}$$

does not commute in general. However, it commutes when we restrict the bottom row to $\sigma(B)$. That is, we claim that the diagram

$$\begin{array}{ccc} \Omega_Y^m & \xrightarrow{\eta} & \Omega_Y^{m+1} \otimes \mathcal{N}_X \\ {\scriptstyle \rho_m}\downarrow & & \downarrow{\scriptstyle \rho_{m+1}} \\ F^l\Omega_Y^m\Big|_{\sigma(B)} & \xrightarrow{\eta_\sigma} & F^l\Omega_Y^{m+1} \otimes \mathcal{N}_X\Big|_{\sigma(B)} \end{array} \tag{3.15}$$

commutes. Of course, this implies that the right square of (3.13) commutes.

Let $(x_1, x_2, ..., x_r)$ and $(t_1, t_2, ..., t_N)$ be the local coordinates of P and B, respectively. Let $p = \{x_1 = x_2 = ... = x_r = 0\}$ and

$$X = \{f_1(x,t) = f_2(x,t) = ... = f_k(x,t) = 0\}. \tag{3.16}$$

Then η is given by

$$\eta(\omega) = (\omega \wedge df_1, \omega \wedge df_2, ..., \omega \wedge df_k). \tag{3.17}$$

Since $p \in X_b$ for all $b \in B$, we have $f_i(0,t) \equiv 0$. Hence

$$\left.\frac{\partial f_i}{\partial t_j}\right|_{x=0} = 0 \tag{3.18}$$

for all t, $i = 1, 2, ..., k$ and $j = 1, 2, ..., N$. It follows that

$$\begin{aligned}\rho_{m+1} \circ \eta(\omega_1) &= \rho_{m+1}(\omega_1 \wedge df_1, \omega_1 \wedge df_2, ..., \omega_1 \wedge df_k) \\ &= (\rho_{m+1}(\omega_1 \wedge df_1), \rho_{m+1}(\omega_1 \wedge df_2), ..., \rho_{m+1}(\omega_1 \wedge df_k)) \\ &= 0 = \eta_\sigma \circ \rho_m(\omega_1)\end{aligned} \tag{3.19}$$

for all local sections

$$\omega_1 \in H^0(U, \bigoplus_{i<l} \pi_B^* \Omega_B^i \otimes \pi_P^* \Omega_P^{m-i}) \subset H^0(U, \Omega_Y^m), \tag{3.20}$$

where U is an open subset of Y. Every $\omega \in H^0(U, \Omega_Y^m)$ can be written as

$$\omega = \omega_1 + \omega_2 \tag{3.21}$$

with ω_1 given in (3.20) and $\omega_2 \in H^0(U, F^l \Omega_Y^m)$. It is clear that

$$\rho_{m+1} \circ \eta(\omega_2) = \eta_\sigma \circ \rho_m(\omega_2). \tag{3.22}$$

Combining (3.19) and (3.22), we conclude that

$$\rho_{m+1} \circ \eta(\omega) = \eta_\sigma \circ \rho_m(\omega) \tag{3.23}$$

and hence the diagram (3.15) commutes. It remains to prove that the left square of (3.13) commutes.

Note that Ω_X^N can be identified with the image of the map

$$\Omega_Y^N \otimes \mathcal{O}_X \xrightarrow{\theta} \Omega_Y^{N+k} \otimes \det(\mathcal{N}_X) \tag{3.24}$$

given by

$$\theta(\omega) = \omega \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_k \otimes \frac{\partial}{\partial f_1} \wedge \cdots \wedge \frac{\partial}{\partial f_k}. \tag{3.25}$$

By (3.18) again, we see that the diagram

$$\begin{array}{ccc} \Omega_Y^N \otimes \mathcal{O}_X & \xrightarrow{\quad\theta\quad} & \Omega_Y^{N+k} \otimes \det(\mathcal{N}_X) \\ \downarrow{\scriptstyle d\sigma} & & \downarrow \\ F^N \Omega_Y^N \otimes \mathcal{O}_X\Big|_{\sigma(B)} & \longrightarrow & F^N \Omega_Y^{N+k} \otimes \det(\mathcal{N}_X)\Big|_{\sigma(B)} \end{array} \tag{3.26}$$

commutes. Thus, the diagram

$$\begin{array}{ccccc} \Omega_Y^N \otimes \mathcal{O}_X & \twoheadrightarrow & \Omega_X^N & \hookrightarrow & \Omega_Y^{N+k} \otimes \det(\mathcal{N}_X) \\ \downarrow d\sigma & & \downarrow d\sigma & & \downarrow \\ F^N\Omega_Y^N \otimes \mathcal{O}_X\Big|_{\sigma(B)} & = & \Omega_{\sigma(B)}^N & \hookrightarrow & F^N\Omega_Y^{N+k} \otimes \det(\mathcal{N}_X)\Big|_{\sigma(B)} \end{array} \quad (\text{top: } \theta: \Omega_Y^N \otimes \mathcal{O}_X \to \Omega_Y^{N+k} \otimes \det(\mathcal{N}_X)) \tag{3.27}$$

commutes. □

Note that

$$F^N\Omega_Y^{N+k} = \pi_B^*\Omega_B^N \otimes \pi_P^*\Omega_P^k \tag{3.28}$$

Combining (3.12), (3.13) and (3.28), we obtain commutative diagrams

$$\begin{array}{ccccc} T_X^n \otimes K_X & \hookrightarrow & T_Y^n \otimes K_X & \xrightarrow{\eta_n} & T_Y^{n-1} \otimes K_X \otimes \mathcal{N}_X \\ \downarrow d\sigma_i & & \downarrow \alpha_{n,i} & & \downarrow \alpha_{n-1,i} \\ K_{\sigma_i(B)} & \hookrightarrow & \pi_P^*T_P^n \otimes K_X\Big|_{\sigma_i(B)} & \longrightarrow & \pi_P^*T_P^{n-1} \otimes K_X \otimes \mathcal{N}_X\Big|_{\sigma_i(B)} \end{array} \tag{3.29}$$

with left exact rows for $i = 1, 2$. By the above diagram, we have

$$\ker(\Gamma_b(d\sigma_i)) = \ker(\Gamma_b(\alpha_{n,i})) \cap \ker(\Gamma_b(\eta_n)) \tag{3.30}$$

for $i = 1, 2$. Therefore, (3.4) is equivalent to

$$\boxed{\ker(\Gamma_b(\alpha_{n,1})) \cap \ker(\Gamma_b(\eta_n)) \neq \ker(\Gamma_b(\alpha_{n,2})) \cap \ker(\Gamma_b(\eta_n))}. \tag{3.31}$$

More explicitly, we can write $\Gamma_b(T_Y^n \otimes K_X)$ as

$$\Gamma_b(T_Y^n \otimes K_X) = \Gamma_b(\pi_P^*T_P^n \otimes K_X) \oplus \sum_{j<n} \Gamma_b(\pi_P^*T_P^j \otimes K_X) \otimes T_{B,b}^{n-j}. \tag{3.32}$$

Then the kernel of $\Gamma_b(\alpha_{n,i})$ is

$$\begin{aligned} \ker(\Gamma_b(\alpha_{n,i})) = {} & \Gamma_b(\pi_P^*T_P^n \otimes K_X(-p_i)) \\ & \oplus \sum_{j<n} \Gamma_b(\pi_P^*T_P^j \otimes K_X) \otimes T_{B,b}^{n-j} \end{aligned} \tag{3.33}$$

for $i = 1, 2$, where $K_X(-p_i) = K_X \otimes I_{p_i}$ for I_{p_i} the ideal sheaf of p_i. So (3.31) is equivalent to

$$\boxed{\begin{aligned} &\ker(\Gamma_b(\eta_n)) \cap (\Gamma_b(\pi_P^* T_P^n \otimes K_X(-p_1)) \oplus \sum_{j<n} \Gamma_b(\pi_P^* T_P^j \otimes K_X) \otimes T_{B,b}^{n-j}) \\ \neq &\ker(\Gamma_b(\eta_n)) \cap (\Gamma_b(\pi_P^* T_P^n \otimes K_X(-p_2)) \oplus \sum_{j<n} \Gamma_b(\pi_P^* T_P^j \otimes K_X) \otimes T_{B,b}^{n-j}) \end{aligned}} \tag{3.34}$$

Combining it with Proposition 3.1, we obtain the following criterion:

Proposition 3.3. *Let $X \subset Y = B \times P$ be a smooth projective family of n-dimensional varieties in a projective space P passing through two fixed points $p_1 \neq p_2 \in P$ over a smooth variety B. Then p_1 and p_2 are not Γ-equivalent over $\mathbb{Q}$ on X_b for $b \in B$ general if (3.34) holds at a point b where $h^0(X_t, T_X^n \otimes K_X)$ is locally constant in t.*

Remark 3.4. Since $X \subset B \times P$ is a family of varieties in P passing through p_i, $\eta(\mathbf{v})$ is a section in $H^0(N_{X_b})$ vanishing at p_i for all tangent vectors $\mathbf{v} \in T_{B,b}$ and $i = 1, 2$. It follows that

$$\begin{aligned} &\eta_n \left(\sum_{j=0}^{n-1} \Gamma_b(\pi_P^* T_P^j \otimes K_X) \otimes T_{B,b}^{n-j} \right) \\ &\subset \bigcap_{i=1}^{2} \ker(\Gamma_b(\alpha_{n-1,i})) = \Gamma_b(\pi_P^* T_P^{n-1} \otimes K_X \otimes \mathcal{N}_X(-p_1 - p_2)) \\ &\qquad \oplus \sum_{j=0}^{n-2} \Gamma_b(\pi_P^* T_P^j \otimes K_X \otimes \mathcal{N}_X) \otimes T_{B,b}^{n-1-j}. \end{aligned} \tag{3.35}$$

Let us apply Proposition 3.3 to complete intersections in $P = \mathbb{P}^{n+k}$ of type $(d_1, d_2, ..., d_k)$. When $\sum(d_i - 1) = 2n + 1$, we have $K_X = \mathcal{O}_X(n)$. More general, let us consider a smooth projective family $X \subset Y = B \times P$ of varieties of dimension n in P with $K_X(-n)$ globally generated on each fiber X_b. In this case, we have the following corollary of Proposition 3.3.

Corollary 3.5. *Let $X \subset Y = B \times P$ be a smooth projective family of n-dimensional varieties in a projective space P passing through two fixed points $p_1 \neq p_2 \in P$ over a smooth variety B and let $W_{X,b}$ be the subspace of $\Gamma_b(T_P(1))$ defined by*

$$W_{X,b} = \left\{ \omega \in \Gamma_b(T_P(1)) : \eta(\omega) \in \eta\big(\Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}\big) \right\}, \tag{3.36}$$

where the map η on $\Gamma_b(T_P(1))$ and $\Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}$ are given by the diagram

$$\begin{array}{ccc} \Gamma_b(T_Y(1)) & = & \Gamma_b(T_P(1)) \oplus \Gamma_b(\mathcal{O}(1)) \otimes T_{B,b} \\ \downarrow \eta & \swarrow & \\ \Gamma_b(\mathcal{N}_X(1)) & & \end{array} \tag{3.37}$$

Suppose that there exists a point $b \in B$ such that $h^0(X_t, T_X^n \otimes K_X)$ is constant for t in an open neighborhood of b, each point p_i imposes independent conditions on both $K_{X_b}(-n)$ and $T_X(1) \otimes \mathcal{O}_{X_b}$, i.e., the maps

$$\Gamma_b(K_X(-n)) \twoheadrightarrow K_X(-n) \otimes \mathcal{O}_{p_i} \quad \text{and} \tag{3.38}$$

$$\Gamma_b(T_X(1)) \twoheadrightarrow T_X(1) \otimes \mathcal{O}_{p_i} \tag{3.39}$$

are surjective for $i = 1, 2$ and

$$\{\omega \in W_{X,b} : \omega(p_1) = 0\} \neq \{\omega \in W_{X,b} : \omega(p_2) = 0\}. \tag{3.40}$$

Then p_1 and p_2 are not Γ-equivalent over $\mathbb{Q}$ on X_b for $b \in B$ general.

Proof. By (3.40), there exists $\omega \in W_{X,b}$ such that $\omega(p_i) = 0$ and $\omega(p_{3-i}) \neq 0$ for $i = 1$ or 2. Without loss of generality, let us assume that $\omega_1(p_1) = 0$ and $\omega_1(p_2) \neq 0$ for some $\omega_1 \in W_{X,b}$.

It is easy to see that $W_{X,b}$ is the image of the projection from $\Gamma_b(T_X(1))$ to $\Gamma_b(T_P(1))$ via the diagram

$$\begin{array}{ccccc} \Gamma_b(T_X(1)) & \hookrightarrow & \Gamma_b(T_Y(1)) & \xrightarrow{\eta} & \Gamma_b(\mathcal{N}_X(1)) \\ & \searrow & \downarrow & & \\ & & \Gamma_b(T_P(1)) & & \end{array} \tag{3.41}$$

where $\Gamma_b(T_X(1))$ can be identified with $\ker(\eta)$. In other words, for every $\omega \in W_{X,b}$, there exists $\tau \in \Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}$ such that $\eta(\omega + \tau) = 0$ and hence $\omega + \tau \in \Gamma_b(T_X(1))$.

By (3.39), $\Gamma_b(T_X(1))$ generates the vector space $T_X(1) \otimes \mathcal{O}_{p_2}$. On the other hand, by the diagram

$$\begin{array}{ccccc} T_{X_b}(1) \otimes \mathcal{O}_{p_2} & \hookrightarrow & T_X(1) \otimes \mathcal{O}_{p_2} & \longrightarrow & T_{B,b} \\ \downarrow & & \downarrow & & \| \\ T_{Y_b}(1) \otimes \mathcal{O}_{p_2} & \hookrightarrow & T_Y(1) \otimes \mathcal{O}_{p_2} & \longrightarrow & T_{B,b} \\ & = & \downarrow & & \\ & & T_P(1) \otimes \mathcal{O}_{p_2} & & \end{array} \tag{3.42}$$

we see that the image of the projection $T_X(1) \otimes \mathcal{O}_{p_2} \to T_P(1) \otimes \mathcal{O}_{p_2}$ is the same as the image of the map $T_{X_b}(1) \otimes \mathcal{O}_{p_2} \to T_{Y_b}(1) \otimes \mathcal{O}_{p_2}$ and thus has dimension n. Therefore,

$$\dim\{\omega(p_2) : \omega \in W_{X,b}\} = n. \tag{3.43}$$

And since $\omega_1(p_2) \neq 0$, we can find $\omega_2, ..., \omega_n \in W_{X,b}$ such that $\{\omega_j(p_2)\}$ are linearly independent. On the other hand, $\omega_1(p_1) = 0$ and hence $\{\omega_j(p_1)\}$ are linearly dependent. In other words,

$$\begin{cases} \omega_1(p_1) \wedge \omega_2(p_1) \wedge ... \wedge \omega_n(p_1) = 0 \\ \omega_1(p_2) \wedge \omega_2(p_2) \wedge ... \wedge \omega_n(p_2) \neq 0. \end{cases} \tag{3.44}$$

Let $\eta(\omega_j + \tau_j) = 0$ for some $\tau_j \in \Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}$ and $j = 1, 2, ..., n$. Then

$$\bigwedge_{j=1}^{n} (\omega_j + \tau_j) \otimes s \in \ker(\Gamma_b(\eta_n)) \tag{3.45}$$

for all $s \in \Gamma_b(K_X(-n))$. By (3.44), we have

$$\begin{aligned} &\bigwedge_{j=1}^{n} \omega_j \otimes s \in \Gamma_b(\pi_P^* T_P^n \otimes K_X(-p_1)) \text{ and} \\ &\bigwedge_{j=1}^{n} \omega_j \otimes s \notin \Gamma_b(\pi_P^* T_P^n \otimes K_X(-p_2)) \end{aligned} \tag{3.46}$$

provided that $s(p_2) \neq 0$. The combination of (3.45) and (3.46) yields (3.34). □

Since the validity of (3.40) is determined by the restriction of $W_{X,b}$ to $Z = \{p_1, p_2\}$, we may let $W_{X,b,Z}$ be the subspace of $H^0(Z, T_P(1))$ given by

$$\begin{aligned} W_{X,b,Z} &= W_{X,b} \otimes H^0(\mathcal{O}_Z) \\ &= \Big\{ \omega\big|_Z : \omega \in \Gamma_b(T_P(1)) \text{ and } \eta(\omega) \in \eta\big(\Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}\big) \Big\} \end{aligned} \tag{3.47}$$

and reformulate (3.40) as

$$W_{X,b,Z} \cap H^0(Z, T_P(1) \otimes I_p) \neq 0 \tag{3.48}$$

for some $p \in \operatorname{supp}(Z) = \{p_1, p_2\}$.

3.4. Criterion for two varying sections

So far we have obtained the key criterion, Corollary 3.5, for the Γ-equivalence of two fixed sections of X/B in the ambient space P. To apply it to two arbitrary sections of X/B, we need to use an automorphism $\lambda \in \operatorname{Aut}(Y/B)$ to move these two sections to two fixed points in P, as pointed out before. This line of argument leads to the following:

Proposition 3.6. *Let $X \subset Y = B \times P$ be a smooth projective family of n-dimensional varieties in a projective space P over the N-dimensional polydisk $B = \operatorname{Spec} \mathbb{C}[[t_j]]$ and let $\sigma_i : B \to X$ be two disjoint sections of X/B with $p_i = \sigma_i(b)$ at the origin $b \in B$ for $i = 1, 2$. Let $\lambda \in B \times \operatorname{Aut}(P)$ be an automorphism of Y preserving the base B, satisfying that $\lambda_b = id$ and $\lambda(\sigma_i(t)) \equiv p_i$ for $i = 1, 2$ and all $t \in B$ and given by*

$$\lambda \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_r \end{bmatrix} = \Lambda \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_r \end{bmatrix}, \tag{3.49}$$

where $(x_0, x_1, ..., x_r)$ are the homogeneous coordinates of P and $\Lambda = \Lambda(t)$ is an $(r+1) \times (r+1)$ matrix over $\mathbb{C}[[t_j]]$ satisfying $\Lambda(0) = I$. Let $W_{X,b,Z,\lambda}$ be the subspace of $H^0(Z, T_P(1))$ defined by

$$\begin{gathered} W_{X,b,Z,\lambda} = \Big\{\omega\big|_Z + L_\lambda(\tau) : \omega \in \Gamma_b(T_P(1)), \tau \in \Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}, \\ \eta(\omega + \tau) = 0\Big\} \end{gathered} \tag{3.50}$$

for $Z = \{p_1, p_2\}$, where $L_\lambda : \pi_B^* T_{B,b} \to T_P \otimes \mathcal{O}_Z$ is the map given by

$$L_\lambda\left(\frac{\partial}{\partial t_j}\right) = \begin{bmatrix} x_0 & x_1 & \dots & x_r \end{bmatrix} \frac{\partial \Lambda^T}{\partial t_j}\bigg|_{t=0} \begin{bmatrix} \partial/\partial x_0 \\ \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_r \end{bmatrix}. \tag{3.51}$$

Suppose that

- *$h^0(X_t, T_X^n \otimes K_X)$ is constant over B,*
- *each point p_i imposes independent conditions on*

  $$K_{X_b}(-n) \text{ and } T_X(1) \otimes \mathcal{O}_{X_b}$$

  for $i = 1, 2$,
- *and*

  $$W_{X,b,Z,\lambda} \cap H^0(Z, T_P(1) \otimes I_p) \neq 0 \tag{3.52}$$

  for some $p \in \operatorname{supp}(Z)$.

Then $\sigma_1(t)$ and $\sigma_2(t)$ are not Γ-equivalent over $\mathbb{Q}$ on X_t for $t \in B$ general.

Proof. Note that $W_{X,b,Z,\lambda} = W_{X,b,Z}$ if $L_\lambda = 0$, i.e., $\sigma_i(t) \equiv p_i$.

Let $\widehat{X} = \lambda(X) \subset Y = B \times P$. Obviously, $\widehat{X}$ is a smooth projective family of n-dimensional varieties in P over B passing through the two fixed points $p_1 \neq p_2$.

We define the map $\widehat{\eta} : T_Y \otimes \mathcal{O}_{\widehat{X}} \to \mathcal{N}_{\widehat{X}}$ and the space $W_{\widehat{X},b} \subset \Gamma_b(T_P(1))$ for $\widehat{X} \subset Y = B \times P$ in the same way as η and $W_{X,b}$. Note that since $\lambda_b = \mathrm{id}$, $X_b = \widehat{X}_b$ and we may use $\Gamma_b(\bullet)$ to refer both $H^0(X_b, \bullet)$ and $H^0(\widehat{X}_b, \bullet)$.

Let us consider the commutative diagram:

$$\begin{array}{ccccc} \Gamma_b(T_X(1)) & \hookrightarrow & \Gamma_b(T_Y(1)) & \xrightarrow{\eta} & \Gamma_b(\mathcal{N}_X(1)) \\ \cong\downarrow d\lambda & & \cong\downarrow d\lambda & & \downarrow \\ \Gamma_b(T_{\widehat{X}}(1)) & \hookrightarrow & \Gamma_b(T_Y(1)) & \xrightarrow{\widehat{\eta}} & \Gamma_b(\mathcal{N}_{\widehat{X}}(1)) \\ & \searrow & \downarrow \pi_{P,*} & & \\ & & \Gamma_b(T_P(1)) & & \end{array} \tag{3.53}$$

As pointed out in the proof of Corollary 3.5, $W_{X,b}$ is simply the image of the projection from $\Gamma_b(T_X(1))$ to $\Gamma_b(T_P(1))$ when $\Gamma_b(T_X(1))$ is identified with the kernel of $\eta : \Gamma_b(T_Y(1)) \to \Gamma_b(\mathcal{N}_X(1))$. The same holds for $\widehat{X}$. That is, $W_{\widehat{X},b}$ is simply the image of the projection from $\Gamma_b(T_{\widehat{X}}(1))$ to $\Gamma_b(T_P(1))$ when $\Gamma_b(T_{\widehat{X}}(1))$ is identified with the kernel of $\widehat{\eta} : \Gamma_b(T_Y(1)) \to \Gamma_b(\mathcal{N}_{\widehat{X}}(1))$.

We may regard $W_{\widehat{X},b}$ as the image of $\Gamma_b(T_X(1))$ under the map $\pi_{P,*} \circ d\lambda$ in the above diagram. Note that $\pi_{P,*} \circ d\lambda$ is not the same as the projection $\pi_{P,*} : \Gamma_b(T_Y(1)) \to \Gamma_b(T_P(1))$, i.e.,

$$\pi_{P,*} \circ d\lambda \neq \pi_{P,*}. \tag{3.54}$$

Indeed, we have

$$(d\lambda)(\omega + \tau) = (\omega + \widehat{L}_\lambda(\tau)) + \tau \tag{3.55}$$

for $\omega \in \Gamma_b(T_P(1))$ and $\tau \in \Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}$, where

$$\widehat{L}_\lambda : \ \pi_B^* T_B \longrightarrow \pi_P^* T_P \tag{3.56}$$

is a homomorphism induced by $d\lambda : T_Y \to T_Y$. Thus,

$$\pi_{P,*} \circ (d\lambda)(\omega + \tau) = \omega + \widehat{L}_\lambda(\tau) \neq \omega = \pi_{P,*}(\omega + \tau). \tag{3.57}$$

It follows that

$$\begin{aligned} W_{\widehat{X},b} &= \pi_{P,*} \circ d\lambda(\Gamma_b(T_X(1))) \\ &= \big\{\omega + \widehat{L}_\lambda(\tau) : \omega \in \Gamma_b(T_P(1)), \tau \in \Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}, \\ &\qquad \eta(\omega + \tau) = 0\big\}. \end{aligned} \tag{3.58}$$

We claim that L_λ and $W_{X,b,Z,\lambda}$ are exactly the restrictions of $\widehat{L}_\lambda$ and $W_{\widehat{X},b}$ to Z, respectively. Indeed, the differential map $d\lambda : T_Y \to T_Y$ is given by

$$\begin{aligned}(d\lambda)\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_i}\\ (d\lambda)\left(\frac{\partial}{\partial t_j}\right) &= \frac{\partial}{\partial t_j} + \widehat{L}_\lambda\left(\frac{\partial}{\partial t_j}\right)\\ &= \frac{\partial}{\partial t_j} + \begin{bmatrix} x_0 & x_1 & \dots & x_r\end{bmatrix}\frac{\partial \Lambda^T}{\partial t_j}\begin{bmatrix}\partial/\partial x_0\\ \partial/\partial x_1\\ \vdots\\ \partial/\partial x_r\end{bmatrix}\end{aligned} \tag{3.59}$$

at b. Therefore, L_λ is the restriction of $\widehat{L}_\lambda$ to Z and hence $W_{\widehat{X},b,Z} = W_{X,b,Z,\lambda}$.

In conclusion, the hypothesis (3.52) on $W_{X,b,Z,\lambda}$ translates to

$$\left\{\omega \in W_{\widehat{X},b} : \omega(p_1) = 0\right\} \neq \left\{\omega \in W_{\widehat{X},b} : \omega(p_2) = 0\right\}. \tag{3.60}$$

Then by Corollary 3.5, $\sigma_1(t)$ and $\sigma_2(t)$ are not Γ-equivalent over $\mathbb{Q}$ on a general fiber X_t of X/B. □

Remark 3.7. In the above proof, it is easy to see that

$$\widehat{\eta}\left(\frac{\partial}{\partial x_i}\right) = \eta\left(\frac{\partial}{\partial x_i}\right) \text{ and } \widehat{\eta}\left(\frac{\partial}{\partial t_j}\right) = \eta\left(\frac{\partial}{\partial t_j} - \widehat{L}_\lambda\left(\frac{\partial}{\partial t_j}\right)\right). \tag{3.61}$$

Since $\widehat{X}_t$ passes through p_1 and p_2, $\widehat{\eta}(\tau)$ vanishes at p_i and hence L_λ satisfies

$$\eta(L_\lambda(\tau)) = \eta(\tau)\Big|_Z \text{ for all } \tau \in T_{B,b}. \tag{3.62}$$

There is a more intrinsic way to define L_λ: for every $t \in B$, we consider the line joining the two points $\sigma_i(t)$; we may regard $\sigma_i(t)$ as the image of two fixed points on $\mathbb{P}^1$ mapped to this line and thus interpret L_λ in terms of the deformation of this map $\mathbb{P}^1 \to P$. We can put the above proposition in the following equivalent form.

Proposition 3.8. *Let $X \subset Y = B \times P$ be a smooth projective family of n-dimensional varieties in a projective space P over a smooth variety B and let $v : S = B \times \mathbb{P}^1 \hookrightarrow Y$ be a closed immersion preserving the base B such that $v^*\mathcal{O}_Y(1) = \mathcal{O}_S(1)$ and there are two fixed points $p_1 \neq p_2$ on $\mathbb{P}^1$ with $v_b(p_i) \in X_b$ for all $b \in B$. Let $W_{X,b,Z,\lambda}$ be the subspace of $H^0(Z, v_b^* T_P(1))$ defined by*

$$W_{X,b,Z,\lambda} = \Big\{ v_b^*\omega\big|_Z - L_\lambda(v_b^*\tau) : \omega \in \Gamma_b(T_P(1)), \\ \tau \in \Gamma_b(\mathcal{O}(1)) \otimes T_{B,b}, \\ \eta(\omega+\tau) = 0 \Big\} \tag{3.63}$$

for $Z = \{p_1, p_2\}$, where $L_\lambda : \pi_{S,B}^* T_{B,b} \to v_b^* T_P \otimes \mathcal{O}_Z$ is the map induced by $T_S \to v^* T_Y$ with $\pi_{S,B}$ the projection $S \to B$.

Let b be a general point of B. Suppose that each point $v_b(p_i)$ imposes independent conditions on $K_{X_b}(-n)$ and $T_X(1) \otimes \mathcal{O}_{X_b}$ for $i = 1, 2$ and

$$W_{X,b,Z,\lambda} \cap H^0(Z, v_b^* T_P(1) \otimes I_p) \neq 0 \tag{3.64}$$

for some $p \in Z$. Then $v_b(p_1)$ and $v_b(p_2)$ are not Γ-equivalent over $\mathbb{Q}$ on X_b.

Proof. Note that the hypothesis $v^*\mathcal{O}_Y(1) = \mathcal{O}_S(1)$ simply means that v maps S/B fiberwise to lines in P.

Basically, we want to show that the two spaces $W_{X,b,Z,\lambda}$ defined by (3.50) and (3.63) are identical. In turn, this comes down to showing that the maps L_λ are "essentially" the same up to a sign.

To be more precise, we fix a general point b, replace B by an analytic neighborhood of b and choose $\lambda \in \mathrm{Aut}(Y/B)$ to be an automorphism such that $\lambda(v_t(p_i)) \equiv v_b(p_i)$ for $i = 1, 2$ and all $t \in B$. Then we have a map L_λ defined by (3.51). Let us rename this map to $\overline{L}_\lambda : \pi_B^* T_{B,b} \to T_P \otimes \mathcal{O}_{v_b(Z)}$.

Let $\Delta_i = \pi_{S,\mathbb{P}^1}^{-1}(p_i)$ be the two sections of S/B given by p_i for $i = 1, 2$, where $\pi_{S,\mathbb{P}^1}$ is the projection $S \to \mathbb{P}^1$. Since $\pi_P \circ \lambda \circ v$ is constant on Δ_i, we see that

$$L_{\lambda\circ v} \equiv 0$$

from the commutative diagram

$$\begin{array}{ccccc} \pi_{S,B}^* T_B \otimes \mathcal{O}_{\Delta_i} & = & T_{\Delta_i} & = & (\lambda\circ v)^* T_{v(\Delta_i)} \\ & \searrow & \downarrow & & \downarrow \\ & & T_S \otimes \mathcal{O}_{\Delta_i} & \longrightarrow & (\lambda \circ v)^* T_Y \otimes \mathcal{O}_{\Delta_i} \end{array}$$

where the map $L_{\lambda\circ v} : \pi_{S,B}^* T_{B,b} \to (\lambda \circ v_b)^* T_P \otimes \mathcal{O}_Z$ is defined in an analogous way to L_λ.

Since

$$d(\lambda \circ v) = (d\lambda) \circ (dv)$$

we derive that

$$L_{\lambda\circ v}(v_b^*\tau) = L_\lambda(v_b^*\tau) + v_b^*\left(\overline{L}_\lambda(\tau)\right)$$

using (3.55). Then

$$v_b^* \left(\overline{L}_\lambda(\tau)\right) = -L_\lambda(v_b^* \tau)$$

and the proposition follows. □

Using Proposition 3.6 or 3.8, we obtain the following criterion for the Γ-inequivalence of all pairs of distinct points on X_b.

Corollary 3.9. *Let $X \subset Y = B \times P$ be a smooth projective family of n-dimensional varieties in a projective space P over a smooth variety B and let $W_{X,b,Z,\lambda}$ be the subspace of $H^0(Z, T_P(1))$ defined by (3.50) for a 0-dimensional subscheme $Z \subset X_b$ and $L_\lambda \in \operatorname{Hom}(\pi_B^* T_{B,b}, T_P \otimes \mathcal{O}_Z)$.*

Let b be a very general point of B. Suppose that

- *$K_{X_b}(-n)$ and $T_X(1) \otimes \mathcal{O}_{X_b}$ are globally generated on X_b and*
- *(3.52) holds for all pairs $Z = \{p_1, p_2\}$ of distinct points $p_1 \neq p_2$ on X_b, some $p \in \operatorname{supp}(Z)$ and all $L_\lambda \in \operatorname{Hom}(\pi_B^* T_{B,b}, T_P \otimes \mathcal{O}_Z)$ satisfying (3.62).*

Then no two distinct points on X_b are Γ-equivalent over $\mathbb{Q}$.

We believe that the above corollary will find applications in the future. However, we will not use it to prove our main Theorem 1.5; instead, we will apply Proposition 3.6 directly to families $X \subset B \times \mathbb{P}^{n+1}$ of hypersurfaces of degree $2n + 2$ in $\mathbb{P}^{n+1}$.

4. Hypersurfaces of degree $2n + 2$ in $\mathbb{P}^{n+1}$

In this section, we are going to prove our main Theorem 1.5 using the criteria developed in the previous section. Here is an outline of the proof.

To start, let us choose a versal family $X \subset Y = B \times \mathbb{P}^{n+1}$ of hypersurfaces of degree $2n+2$ in $\mathbb{P}^{n+1}$. Suppose that the theorem fails. Then using a Hilbert scheme argument, we can find two disjoint sections $\sigma_i : U \to X$ in an analytic (or étale) open neighborhood U of a fixed general point $b \in B$ such that $\sigma_1(t)$ and $\sigma_2(t)$ are Γ-equivalent over $\mathbb{Q}$ for all t in U. We shall apply Proposition 3.6 to deduce a contradiction. Shrinking B is necessary, the first item above (3.52) is clearly satisfied. Since $K_{X_b}(-n) = \mathcal{O}_{X_b}$ and $T_X(1)$ is globally generated on X_b for X/B being versal [19], it remains to verify the crucial (3.52). Let $\lambda \in U \times \operatorname{Aut}(P)$ be an automorphism of Y/U such that $\lambda_b = \mathrm{id}$ and $\lambda(\sigma_i(t)) \equiv p_i = \sigma_i(b)$ for $i = 1, 2$. We shall prove that either (3.52) holds or the line joining p_1 and p_2 meets X_b only at these two points. However, using a standard argument, we shall further show that the latter case does not occur. This concludes the proof.

4.1. Versal deformation of the Fermat hypersurface

Let us choose $X \subset Y = B \times P$ to be the family of hypersurfaces of degree d in $P = \mathbb{P}^{n+1}$ given by

$$F(x_0, x_1, ..., x_n, x_{n+1}, t_f) = x_0^d + x_1^d + ... + x_{n+1}^d + \sum_{f \in J_d} t_f f = 0, \tag{4.1}$$

where $(x_0, x_1, ..., x_{n+1})$ are the homogeneous coordinates of $\mathbb{P}^{n+1}$, J_d is the set of monomials in x_i given by

$$\begin{aligned} J_d = \big\{ x_0^{m_0} x_1^{m_1} ... x_{n+1}^{m_{n+1}} :\ & m_0, m_1, ..., m_{n+1} \in \mathbb{N}, \\ & m_0 + m_1 + ... + m_{n+1} = d \text{ and} \\ & m_0, m_1, ..., m_{n+1} \le d - 2 \big\} \end{aligned} \tag{4.2}$$

and (t_f) are the coordinates of the affine space $B = \operatorname{Span}_{\mathbb{C}} J_d \cong \mathbb{A}^N$ for

$$N = h^0(\mathcal{O}_P(d)) - h^0(T_P) - 1 = \binom{d+n+1}{n+1} - (n+2)^2. \tag{4.3}$$

We may regard X/B as a versal deformation of the Fermat hypersurface.

At a general point $b \in B$, X/B is obviously versal, i.e., the Kodaira-Spencer map

$$\begin{array}{ccc} T_{B,b} & \xrightarrow{\sim} & H^0(\mathcal{N}_{X_b})/\eta(H^0(X_b, T_P)) \\ & & \big\downarrow \\ & & H^1(T_{X_b}) \end{array} \tag{4.4}$$

is an isomorphism, where η is the map in

$$0 \longrightarrow T_X \longrightarrow T_Y \otimes \mathcal{O}_X \xrightarrow{\eta} \mathcal{N}_X \longrightarrow 0. \tag{4.5}$$

More explicitly, (4.4) is equivalent to saying

$$\operatorname{Span}\left\{ x_i \frac{\partial F}{\partial x_j} \right\} \oplus \operatorname{Span} J_d = H^0(\mathcal{N}_{X_b}) = H^0(X_b, \mathcal{O}(d)) \tag{4.6}$$

for $b \in B$ general.

Let $\mathcal{E} = \mathcal{O}_P(1)^{\oplus n+2}$ be the Euler bundle on P. Then

$$H^0(T_P) \cong \frac{H^0(\mathcal{E})}{(\alpha)} = \operatorname{Span}\left\{ x_i \frac{\partial}{\partial x_j} \right\} / (\alpha) \tag{4.7}$$

by the Euler sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_P & \longrightarrow & \mathcal{O}_P(1)^{\oplus(n+2)} & \longrightarrow & T_P & \longrightarrow & 0 \\
 & & & & \| & & & & \\
 & & & & \mathcal{E} & & & &
\end{array}
\tag{4.8}
$$

and

$$
\eta\left(x_i \frac{\partial}{\partial x_j}\right) = x_i \frac{\partial F}{\partial x_j} \text{ and } \eta\left(\frac{\partial}{\partial t_f}\right) = \frac{\partial F}{\partial t_f} = f \tag{4.9}
$$

for $j = 0, 1, 2, ..., n+1$ and $f \in J_d$, where

$$
\alpha = \sum_{i=0}^{n+1} x_i \frac{\partial}{\partial x_i}. \tag{4.10}
$$

We are going to show that no two distinct points on a very general fiber X_b of X/B are Γ-equivalent over $\mathbb{Q}$ when $d = 2n + 2 \geq 6$. First a definition.

Definition 4.1. Let Z be a 0-dimensional scheme of length 2 in $P = \mathbb{P}^{n+1}$ with homogeneous coordinates $(x_0, x_1, ..., x_{n+1})$. We call Z *generic* with respect to the homogeneous coordinates (x_i) if

$$
H^0(\mathcal{O}_Z(1)) = \operatorname{Span}\{x_j : j \neq i\} \text{ for every } i = 0, 1, \ldots, n+1. \tag{4.11}
$$

Otherwise, we call Z *special* with respect to (x_i). We call Z *very special* with respect to (x_i) if

$$
\#\{x_i : x_i \in H^0(I_Z(1))\} = n = h^0(\mathcal{O}_P(1)) - 2 \tag{4.12}
$$

where I_Z is the ideal sheaf of Z in P. Geometrically, $Z = \{p \neq q\}$ being special means that Z is projected to one point under the projection sending $(x_0, x_1, ..., x_{n+1})$ to $(x_0, x_1, ..., \widehat{x}_i, ..., x_{n+1})$ for some i and being very special means that Z is contained in a line cut out by n coordinate hyperplanes.

Remark 4.2. Clearly, these notions depend on the choice of homogeneous coordinates of P. More generally, we can define these terms with respect to a basis of $H^0(L)$ for an arbitrary very ample line bundle L on P.

When the choice of homogeneous coordinates is clear, we simply say Z is generic (resp. special/very special).

Obviously, being very special implies being special.

There always exist $i \neq j$ such that x_i and x_j span $H^0(\mathcal{O}_Z(1))$ since $\mathcal{O}_P(1)$ is very ample. Without loss of generality, we usually make the assumption that $(i, j) = (0, 1)$, i.e.,

$$H^0(\mathcal{O}_Z(1)) = \mathrm{Span}\{x_0, x_1\}. \tag{4.13}$$

Under the hypothesis of (4.13) and swapping $0, 1$ if needed, Z is special if and only if

$$\mathrm{Span}\{x_0, x_1\} = H^0(\mathcal{O}_Z(1)) \supsetneq \mathrm{Span}\{x_1, x_2, ..., x_{n+1}\}. \tag{4.14}$$

Furthermore, by re-arranging $x_2, ..., x_{n+1}$, we may assume that there exists $1 \le a \le n+1$ such that

$$x_1, ..., x_a \notin H^0(I_Z(1)) \text{ and } x_{a+1}, ..., x_{n+1} \in H^0(I_Z(1)). \tag{4.15}$$

Of course, Z is very special if and only if $a = 1$.

We are considering two cases: with respect to (x_j), for a very general point $b \in B$,

Generic case: $Z = \{\sigma_1(b), \sigma_2(b)\} = \{p_1, p_2\}$ is generic or
Special case: $Z = \{\sigma_1(b), \sigma_2(b)\}$ is special.

4.2. A basis for $W_{X,b}$

For convenience, we identify the tangent space $T_{B,b}$ with $\mathrm{Span}\, J_d$. Then $\eta(f) = f$ for all $f \in \mathrm{Span}\, J_d$.

We start the verification of (3.52) by studying the space $W_{X,b}$ defined by (3.36). It has a basis given by:

Lemma 4.3. *Let $P = \mathbb{P}^{n+1}$ and $X \subset Y = B \times P$ be the family of hypersurfaces in P given by (4.1) over $B = \mathrm{Span}\, J_d$ for $d \ge 3$. Then*

$$\begin{aligned} \mathcal{W}_{X,b} &= \left\{\omega \in H^0(X_b, \mathcal{E}(1)) : \eta(\omega) \in \mathrm{Span}\, J_{d+1}\right\} \\ &= \mathrm{Span}\left\{\omega_{ijk} : 0 \le i, j, k \le n+1,\ i \le j \text{ and } i, j \ne k\right\} \end{aligned} \tag{4.16}$$

has dimension

$$\dim \mathcal{W}_{X,b} = (n+2)\binom{n+2}{2} \tag{4.17}$$

for $b = (t_f)$ in an open neighborhood of 0, where

$$\begin{aligned} \omega_{ijk} &= x_i x_j \frac{\partial}{\partial x_k} \text{ for } i \ne j \ne k \text{ and} \\ \omega_{iik} &= x_i^2 \frac{\partial}{\partial x_k} - \sum_{j \ne i} c_{ijk} x_i x_j \frac{\partial}{\partial x_i} \text{ for } i \ne k \end{aligned} \tag{4.18}$$

with

$$c_{ijk} = \frac{d-1}{d!}\left(\frac{\partial^d F}{\partial x_i^{d-2}\partial x_j \partial x_k}\right) = \begin{cases} 2d^{-1}t_f & \text{if } i \neq j = k \\ d^{-1}t_f & \text{if } i \neq j \neq k \end{cases} \tag{4.19}$$

for $f = x_i^{d-2}x_jx_k$. *Here we consider* η *as a map* $H^0(\mathcal{E}(1)) \to H^0(\mathcal{O}(d+1))$ *given by* (4.9).

Proof. We have

$$\eta\left(x_ix_j\frac{\partial}{\partial x_k}\right) = x_ix_j\frac{\partial F}{\partial x_k} = dx_ix_jx_k^{d-1} + \sum_{f\in J_d} t_f x_ix_j\frac{\partial f}{\partial x_k}. \tag{4.20}$$

It is easy to check that

$$\eta(\omega_{ijk}) = x_ix_j\frac{\partial F}{\partial x_k} \in \operatorname{Span} J_{d+1} \tag{4.21}$$

for $i \neq j \neq k$ and

$$\eta(\omega_{iik}) = x_i^2\frac{\partial F}{\partial x_k} - \sum_{j\neq i}\frac{d-1}{d!}x_ix_j\left(\frac{\partial^d F}{\partial x_i^{d-2}\partial x_j\partial x_k}\right)\frac{\partial F}{\partial x_i} \in \operatorname{Span} J_{d+1} \tag{4.22}$$

for $i \neq k$. Hence $\omega_{ijk} \in \mathcal{W}_{X,b}$ for all $i, j \neq k$.

To show that $\{\omega_{ijk} : i \leq j \text{ and } i, j \neq k\}$ forms a basis of $\mathcal{W}_{X,b}$ in an open neighborhood of 0, it suffices to verify this for $b = 0$: clearly,

$$\left\{\omega_{ijk}\Big|_{b=0} : i \leq j \text{ and } i,j \neq k\right\} = \left\{x_ix_j\frac{\partial}{\partial x_k} : i \leq j \text{ and } i,j\neq k\right\} \tag{4.23}$$

is a basis of $\mathcal{W}_{X,0}$. Therefore, (4.16) and (4.17) follow. □

Clearly, $W_{X,b}$ is the image of $\mathcal{W}_{X,b}$ under the map

$$H^0(X_b, \mathcal{E}(1)) \twoheadrightarrow H^0(X_b, T_P(1)). \tag{4.24}$$

More precisely, let $\widetilde{W}_{X,b}$ be the lift of $W_{X,b}$ in $H^0(X_b, \mathcal{E}(1))$. Then

$$\widetilde{W}_{X,b} = \mathcal{W}_{X,b} \oplus \alpha \otimes H^0(\mathcal{O}(1)) \tag{4.25}$$

where $\mathcal{W}_{X,b} \cap \alpha \otimes H^0(\mathcal{O}(1)) = 0$ because

$$\operatorname{Span} J_{d+1} \cap \eta\left(\alpha \otimes H^0(\mathcal{O}(1))\right) = \operatorname{Span} J_{d+1} \cap F \otimes H^0(\mathcal{O}(1)) = 0. \tag{4.26}$$

4.3. A key observation on L_λ

We observe the following:

Lemma 4.4. *Let $P = \mathbb{P}^{n+1}$ and $X \subset Y = B \times P$ be the family of hypersurfaces in P given by (4.1) over $B = \operatorname{Span} J_d$. For $b \in B$, a 0-dimensional subscheme $Z \subset X_b$ of length 2 and $L_\lambda \in \operatorname{Hom}(\pi_B^* T_{B,b}, T_P \otimes \mathcal{O}_Z)$, if*

$$L_\lambda(f) \neq 0 \text{ for some } f \in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} \subset \operatorname{Span} J_d, \tag{4.27}$$

then (3.52) holds.

Proof. Obviously, under the hypothesis, (4.27) holds for some $f = lg$ with $l \in H^0(I_Z(1))$ and $g \in J_{d-1}$.

For each point $p \in \operatorname{supp}(Z)$, we choose $l_p \in H^0(\mathcal{O}_P(1))$ such that $l_p(p) = 0$ and $l_p \notin H^0(I_Z(1))$ and let

$$\tau_p = l_p \otimes f - l \otimes l_p g \in H^0(\mathcal{O}_{X_b}(1)) \otimes T_{B,b}. \tag{4.28}$$

Then $\eta(\tau_p) = 0$ so $L_\lambda(\tau_p) \in W_{X,b,Z,\lambda}$. Clearly,

$$L_\lambda(\tau_p) = l_p L_\lambda(f) - l L_\lambda(l_p g) = l_p L_\lambda(f) \tag{4.29}$$

since $l \in H^0(I_Z(1))$. Then by our choice of l_p, $L_\lambda(\tau_p)$ vanishes at p.

If $L_\lambda(\tau_p) \neq 0$ for some $p \in \operatorname{supp}(Z)$, then (3.52) follows. Otherwise,

$$l_p L_\lambda(f) = 0 \text{ for all } p \in \operatorname{supp}(Z). \tag{4.30}$$

Since $l_p \notin H^0(I_Z(1))$, (4.30) implies that $L_\lambda(f)$ vanishes at all $p \in \operatorname{supp}(Z)$.

If Z consists of two distinct points, then we must have

$$L_\lambda(f) = 0, \tag{4.31}$$

which contradicts our hypothesis (4.27). Although we only need the lemma for this case, we will prove it for all Z for the sake of completeness.

If Z is supported at a single point p, then $L_\lambda(f)$ vanishes at p. Applying the same argument to $\tau_q = l_q \otimes f - l \otimes l_q g$ for some $l_q \in H^0(\mathcal{O}_P(1))$ satisfying $l_q(p) \neq 0$, we have

$$L_\lambda(\tau_q) = l_q L_\lambda(f) - l L_\lambda(l_q g) = l_q L_\lambda(f) \in W_{X,b,Z,\lambda} \tag{4.32}$$

vanishing at p. Again, we have either (3.52) or (4.31) since $l_q(p) \neq 0$. □

Let us assume that (4.31) holds for all $f \in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}$. Otherwise, we are done by the above lemma. Then $L_\lambda : T_{B,b} \to H^0(Z, T_P)$ factors through

$$\frac{\operatorname{Span} J_d}{H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}} \tag{4.33}$$

and it can be regarded as a map

$$\frac{\operatorname{Span} J_d}{H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}} \xrightarrow{L_\lambda} H^0(Z, T_P). \tag{4.34}$$

4.4. The space $H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}$

Let us figure out the space (4.33). Obviously,

$$H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} \subset \operatorname{Span} J_d \cap H^0(I_Z(1)) \otimes H^0(\mathcal{O}_P(d-1)). \tag{4.35}$$

Furthermore, since $H^0(I_Z(1)) \otimes H^0(\mathcal{O}_P(d-1))$ is the kernel of the map

$$\begin{array}{ccc} H^0(\mathcal{O}_P(d)) & \xrightarrow{\ \xi\ } & \operatorname{Sym}^d H^0(\mathcal{O}_Z(1)) \\ \| & \nearrow & \\ \operatorname{Sym}^d H^0(\mathcal{O}_P(1)) & & \end{array} \tag{4.36}$$

we may write (4.35) as

$$H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} \subset \operatorname{Span} J_d \cap \ker(\xi). \tag{4.37}$$

Actually, this inclusion is an equality for Z generic:

Lemma 4.5. *Let $P = \mathbb{P}^{n+1}$, J_d be defined in (4.2) and Z be a 0-dimensional subscheme of P of length 2. If $d \geq 5$ and Z is generic with respect to (x_i), then*

$$\begin{aligned} H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} &= \operatorname{Span} J_d \cap H^0(I_Z(1)) \otimes H^0(\mathcal{O}_P(d-1)) \\ &= \operatorname{Span} J_d \cap \ker(\xi). \end{aligned} \tag{4.38}$$

Or equivalently, $H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}$ is the kernel of the map

$$\operatorname{Span} J_d \xrightarrow{\ \xi\ } \operatorname{Sym}^d H^0(\mathcal{O}_Z(1)). \tag{4.39}$$

In addition,

$$\frac{\operatorname{Span} J_d}{H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}} \xrightarrow[\sim]{\xi} \operatorname{Sym}^d H^0(\mathcal{O}_Z(1)) \tag{4.40}$$

is an isomorphism.

Proof. To prove (4.38), it suffices to find a subset $S \subset J_d$ such that

$$\operatorname{Span} J_d = H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}(S) \tag{4.41}$$

and

$$H^0(I_Z(1)) \otimes H^0(\mathcal{O}_P(d-1)) \cap \operatorname{Span}(S) = 0. \tag{4.42}$$

Let us assume (4.13). By (4.11), $H^0(\mathcal{O}_Z(1)) = \operatorname{Span}\{x_1, x_2, ..., x_{n+1}\}$ and hence there exists $i \neq 0, 1$ such that

$$H^0(\mathcal{O}_Z(1)) = \operatorname{Span}\{x_1, x_i\}. \tag{4.43}$$

Similarly, we have $H^0(\mathcal{O}_Z(1)) = \operatorname{Span}\{x_0, x_2, ..., x_{n+1}\}$ and hence there exists $j \neq 0, 1$ such that

$$H^0(\mathcal{O}_Z(1)) = \operatorname{Span}\{x_0, x_j\}. \tag{4.44}$$

Then we let

$$\begin{aligned} S = \Big\{ & x_0^{d-3} x_i^3, x_0^{d-3} x_i^2 x_1, x_0^{d-3} x_i x_1^2, \\ & x_0^{d-3} x_1^3, x_0^{d-4} x_1^4, ..., x_0^3 x_1^{d-3}, \\ & x_0^2 x_1^{d-3} x_j, x_0 x_1^{d-3} x_j^2, x_1^{d-3} x_j^3 \Big\}. \end{aligned} \tag{4.45}$$

By (4.13), (4.43) and (4.44), for every k,

$$\begin{aligned} & x_k \in H^0(I_Z(1)) + \operatorname{Span}\{x_0, x_1\}, \\ & x_k \in H^0(I_Z(1)) + \operatorname{Span}\{x_1, x_i\}, \text{ and} \\ & x_k \in H^0(I_Z(1)) + \operatorname{Span}\{x_0, x_j\}. \end{aligned} \tag{4.46}$$

To see (4.41), it suffices to prove that every monomial in J_d lies in the vector space spanned by $H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}$ and S. For a monomial

$$x_0^{a_0} x_1^{a_1} x_2^{a_2} ... x_{n+1}^{a_{n+1}} \in J_d$$

satisfying $a_0 + a_1 \leq d - 3$, we choose $a_k = \max(a_2, a_3, ..., a_{n+1})$ and write $x_k = l_1 + l_2$ for some $l_1 \in H^0(I_Z(1))$ and $l_2 \in \operatorname{Span}\{x_0, x_1\}$. Then

$$\begin{aligned}
&x_0^{a_0}x_1^{a_1}x_2^{a_2}...x_k^{a_k}...x_{n+1}^{a_{n+1}} = x_0^{a_0}x_1^{a_1}x_2^{a_2}...x_k^{a_k-1}(l_1+l_2)...x_{n+1}^{a_{n+1}} \\
&\in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\Big\{x_0^{a_0+1}x_1^{a_1}x_2^{a_2}...x_k^{a_k-1}...x_{n+1}^{a_{n+1}}, \\
&\qquad x_0^{a_0}x_1^{a_1+1}x_2^{a_2}...x_k^{a_k-1}...x_{n+1}^{a_{n+1}}\Big\}
\end{aligned}$$

Repeating this process, we see that

$$J_d \subset H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\left\{x_0^{a_0}x_1^{a_1}\prod_{k=2}^{n+1} x_k^{a_k} \in J_d : a_0 + a_1 \geq d-2\right\}$$

So it remains to verify that monomials of type

$$x_0^{a_0}x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}} \in J_d$$

with $a_0 + a_1 \geq d-2$ lie in $H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}(S)$:

(1) For a monomial $x_0^{d-3}x_ax_bx_c$ with $b, c \neq 0$, we can write

$$\begin{aligned}
x_a &= l_1 + l_2, \quad \text{for } l_1 \in H^0(I_Z(1)) \text{ and } l_2 \in \operatorname{Span}\{x_1, x_i\} \\
x_b &= l_3 + l_4, \quad \text{for } l_3 \in H^0(I_Z(1)) \text{ and } l_4 \in \operatorname{Span}\{x_1, x_i\} \\
x_c &= l_5 + l_6, \quad \text{for } l_5 \in H^0(I_Z(1)) \text{ and } l_6 \in \operatorname{Span}\{x_1, x_i\}
\end{aligned}$$

Then

$$\begin{aligned}
&x_0^{d-3}x_ax_bx_c = x_0^{d-3}(l_1+l_2)x_bx_c = l_1x_0^{d-3}x_bx_c + l_2x_0^{d-3}x_bx_c \\
&\in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\{x_0^{d-3}x_1x_bx_c, x_0^{d-3}x_ix_bx_c\} \\
&\subset H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\{x_0^{d-3}x_1(l_3+l_4)x_c, x_0^{d-3}x_i(l_3+l_4)x_c\} \\
&\subset H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\{x_0^{d-3}x_1^2x_c, x_0^{d-3}x_1x_ix_c, x_0^{d-3}x_i^2x_c\} \\
&\subset H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\Big\{x_0^{d-3}x_1^2(l_5+l_6), x_0^{d-3}x_1x_i(l_5+l_6), \\
&\qquad x_0^{d-3}x_i^2(l_5+l_6)\Big\} \\
&\subset H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\{x_0^{d-3}x_1^3, x_0^{d-3}x_1^2x_i, x_0^{d-3}x_1x_i^2, x_0^{d-3}x_i^3\} \\
&\subset H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}(S).
\end{aligned}$$

(2) For a monomial $x_1^{d-3}x_ax_bx_c$ with $b, c \neq 1$, we can write

$$\begin{aligned}
x_a &= l_1 + l_2, \quad \text{for } l_1 \in H^0(I_Z(1)) \text{ and } l_2 \in \operatorname{Span}\{x_0, x_j\} \\
x_b &= l_3 + l_4, \quad \text{for } l_3 \in H^0(I_Z(1)) \text{ and } l_4 \in \operatorname{Span}\{x_0, x_j\} \\
x_c &= l_5 + l_6, \quad \text{for } l_5 \in H^0(I_Z(1)) \text{ and } l_6 \in \operatorname{Span}\{x_0, x_j\}
\end{aligned}$$

Then by the same argument as in (1), we obtain

$$\begin{aligned} x_1^{d-3}x_ax_bx_c \in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\Big\{x_1^{d-3}x_0^3, x_1^{d-3}x_0^2x_j,\\ x_1^{d-3}x_0x_j^2, x_1^{d-3}x_j^3\Big\}\\ \subset H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}(S). \end{aligned}$$

(3) For a monomial

$$x_0^{a_0}x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}} \in J_d$$

with $a_0, a_1 \geq 2$, by substituting every x_k for $k \geq 2$ with $x_k = l_1 + l_2$ for some $l_1 \in H^0(I_Z(1))$ and $l_2 \in \operatorname{Span}\{x_0, x_1\}$, we obtain

$$\begin{aligned} x_0^{a_0}x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}} &\in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}\{x_0^r x_1^{d-r} : 2 \leq r \leq d-2\}\\ &\subset H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}(S) \end{aligned}$$

where we have shown that

$$x_0^{d-2}x_1^2, x_0^2x_1^{d-2} \in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} + \operatorname{Span}(S)$$

in (1) and (2).

To see (4.42), we just have to show that $\ker(\xi) \cap \operatorname{Span}(S) = 0$, which is equivalent to

$$\xi\left(\operatorname{Span}(S)\right) = \operatorname{Sym}^d H^0(\mathcal{O}_Z(1)) \tag{4.47}$$

since $|S| = \dim \operatorname{Sym}^d H^0(\mathcal{O}_Z(1)) = d+1$. Again it is easy to see from (4.13), (4.43) and (4.44) that

$$\begin{aligned} \xi\left(\operatorname{Span}(S)\right) &= \xi\left(\operatorname{Span}\{x_0^{d-k}x_1^k : k = 0, 1, \ldots, d\}\right)\\ &= \operatorname{Sym}^d H^0(\mathcal{O}_Z(1)). \end{aligned} \tag{4.48}$$

This also proves that (4.40) is an isomorphism. □

When Z is special, $H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}$ is no longer the kernel of the map (4.39). Instead, we have the following result when Z is special but not very special.

Lemma 4.6. *Let $P = \mathbb{P}^{n+1}$, J_d be defined in (4.2) and Z be a 0-dimensional subscheme of P of length 2. Suppose that $d \geq 4$, Z satisfies (4.14) and $\{x_2, ..., x_{n+1}\} \not\subset H^0(I_Z(1))$. Then*

$$\begin{aligned}\operatorname{Span} J_d \cap \ker(\xi) = H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1} \\ + \operatorname{Span}\Big\{x_0^{d-2} x_i(x_j - c_j x_1) : i \geq 1, j \geq 2 \\ \textit{and } x_j - c_j x_1 \in H^0(I_Z(1))\Big\}.\end{aligned} \tag{4.49}$$

Proof. We leave the verification of (4.49) to the readers. □

4.5. Special case

Let us first prove (3.52) when Z is special for all b. Without loss of generality, let us assume that $Z = \{p_1, p_2\}$ satisfies (4.14) and (4.15) for b general and some a.

We claim that $L_\lambda : \pi_B^* T_{B,b} \to T_P \otimes \mathcal{O}_Z$ factors through a sub-sheaf $\mathcal{G}_Z$ of $T_P \otimes \mathcal{O}_Z$, i.e., $L_\lambda \in \operatorname{Hom}(\pi_B^* T_{B,b}, \mathcal{G}_Z)$ for the sub-sheaf $\mathcal{G}_Z$ of $T_P \otimes \mathcal{O}_Z$ generated by the global sections

$$H^0(\mathcal{G}_Z) = \operatorname{Span}\left\{x_i \frac{\partial}{\partial x_j} : j = 0 \text{ or } 2 \leq i, j \leq a\right\}. \tag{4.50}$$

In addition, if x_0 vanishes at one of p_i for b general, $\mathcal{G}_Z$ is generated by

$$H^0(\mathcal{G}_Z) = \operatorname{Span}\left\{x_i \frac{\partial}{\partial x_j} : i = j = 0 \text{ or } 2 \leq i, j \leq a\right\}. \tag{4.51}$$

To see this, we notice that $(1, 0, ..., 0) \notin X_b$ for all b. So

$$x_1(p_i) \neq 0 \text{ for } i = 1, 2. \tag{4.52}$$

Otherwise, if $x_1 = 0$ at some $p \in Z$, then $x_2 = x_3 = ... = x_{n+1} = 0$ at p by (4.14) and $p = (1, 0, ..., 0)$.

Thus, we may choose λ to be given by

$$\lambda \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} g_1(t) & g_2(t) & & \\ & 1 & & \\ & & A(t) & \\ & & & I_{n-a+1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \tag{4.53}$$

locally at b, for some $g_1(t)$, $g_2(t)$ and $A(t)$ satisfying $g_1(b) = 1$, $g_2(b) = 0$ and $A(b) = I_{a-1}$, where I_m is the $m \times m$ identity matrix. Then by (3.51),

$$L_\lambda(\tau) \in H^0(\mathcal{G}_Z) \tag{4.54}$$

for all $\tau \in T_{B,b}$ with $\mathcal{G}_Z$ generated by (4.50).

When x_0 vanishes at one of p_i for b general, $g_2(t) \equiv 0$ in (4.53) and thus we have (4.51). This proves our claim that L_λ factors through $\mathcal{G}_Z$ given by (4.50) or (4.51).

Let $\Lambda \subset P$ be the line joining p_1 and p_2. Then the map ξ in (4.36) is simply the restriction to Λ as in

$$\begin{array}{ccc} H^0(\mathcal{O}_P(m)) & \xrightarrow{\ \xi\ } & H^0(\mathcal{O}_\Lambda(m)) \\ & \searrow^{\xi} & \| \\ & & \operatorname{Sym}^m H^0(\mathcal{O}_Z(1)) \end{array} \tag{4.55}$$

for $m \in \mathbb{N}$. We will use $\operatorname{Sym}^m H^0(\mathcal{O}_Z(1))$ and $H^0(\mathcal{O}_\Lambda(m))$ interchangeably under this setting. We also use ξ to denote the induced map

$$H^0(\mathcal{O}_{X_b}(m)) \xrightarrow{\ \xi\ } \frac{H^0(\mathcal{O}_\Lambda(m))}{\xi(F) \otimes H^0(\mathcal{O}_\Lambda(m-d))} \tag{4.56}$$

where quotient by $\xi(F)$ is necessary; otherwise it is not well defined as $\xi(F)$ is not zero in $H^0(\mathcal{O}_\Lambda(d))$ unless X_b contains the line Λ.

We further abuse the notation by using ξ for the maps induced by the restriction $H^0(\mathcal{E}(m)) \to H^0(\Lambda, \mathcal{E}(m))$:

$$\begin{array}{ccccc} H^0(\mathcal{E}(m)) & = & H^0(X_b, \mathcal{E}(m)) & \xrightarrow{\ \xi\ } & H^0(\Lambda, \mathcal{E}(m)) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \Big\downarrow \eta & & H^0(X_b, T_P(m)) & \xrightarrow{\ \xi\ } & H^0(\Lambda, T_P(m)) \\ \Big\downarrow & & \Big\downarrow \eta & & \Big\downarrow \eta \\ H^0(\mathcal{O}(m+d)) & \longrightarrow & H^0(\mathcal{O}_{X_b}(m+d)) & \xrightarrow{\ \xi\ } & \dfrac{H^0(\mathcal{O}_\Lambda(m+d))}{\xi(F) \otimes H^0(\mathcal{O}_\Lambda(m))} \end{array} \tag{4.57}$$

for $m \le d-2$, where we also abuse the notation η by using it for three different maps, all defined by (4.9).

Next, let us consider the images of the spaces $\mathcal{W}_{X,b} \subset H^0(X_b, \mathcal{E}(1))$ and $W_{X,b} \subset H^0(X_b, T_P(1))$ under ξ, where $\xi(\mathcal{W}_{X,b})$ and $\xi(W_{X,b})$ are considered as the subspaces of $H^0(\Lambda, \mathcal{E}(1))$ and $H^0(\Lambda, T_P(1))$, respectively.

Lemma 4.7. *Let $P = \mathbb{P}^{n+1}$ and $X \subset Y = B \times P$ be the family of hypersurfaces in P given by (4.1) over $B = \operatorname{Span} J_d$ for $n \ge 2$ and $d \ge 4$. For $b \in B$ general and* ***all*** *0-dimensional subschemes $Z \subset X_b$ of length 2 satisfying (4.14),*

$$\xi(\mathcal{W}_{X,b}) \supset \left\{ x_1^2 \frac{\partial}{\partial x_i} \right\} \cup \left\{ x_0 x_1 \frac{\partial}{\partial x_j} : j \ge 1 \right\} \tag{4.58}$$

and

$$\xi\left(W_{X,b}\right) = H^0(\Lambda, T_P(1)) \tag{4.59}$$

if $\{x_2, ..., x_{n+1}\} \not\subset H^0(I_Z(1))$ *and*

$$\begin{aligned}\xi\left(\mathcal{W}_{X,b}\right) = \operatorname{Span}&\left\{x_0x_1\frac{\partial}{\partial x_k} : k \neq 0,1\right\}\\ &\cup\left\{x_0^2\frac{\partial}{\partial x_k} - c_{01k}x_0x_1\frac{\partial}{\partial x_0} : k\neq 0\right\}\\ &\cup\left\{x_1^2\frac{\partial}{\partial x_k} - c_{10k}x_0x_1\frac{\partial}{\partial x_1} : k\neq 1\right\} \subset H^0(\Lambda,\mathcal{E}(1))\end{aligned} \tag{4.60}$$

if $\{x_2, ..., x_{n+1}\} \subset H^0(I_Z(1))$, *where* $\Lambda \subset P$ *is the line cutting out* Z *on* X_b, ξ *is the map defined in* (4.57) *and* c_{ijk} *are the numbers given by* (4.19).

Proof. Let us first deal with the case that $\{x_2, ..., x_{n+1}\} \not\subset H^0(I_Z(1))$, i.e., Z is special but not very special. Note that under the hypothesis of (4.14), all $x_2, ..., x_{n+1}$ are multiples of x_1 in $H^0(\mathcal{O}_\Lambda(1))$.

We write $u_1 \equiv u_2$ if $\xi(u_1 - u_2) \in \xi(\mathcal{W}_{X,b})$. Of course, $\omega_{ijk} \equiv 0$ for ω_{ijk} given by (4.18). Under this notation, (4.58) is equivalent to

$$\begin{aligned}x_1^2\frac{\partial}{\partial x_0} \equiv x_1^2\frac{\partial}{\partial x_1} \equiv x_1^2\frac{\partial}{\partial x_2} \equiv ... \equiv x_1^2\frac{\partial}{\partial x_{n+1}}&\\ \equiv x_0x_1\frac{\partial}{\partial x_1} \equiv x_0x_1\frac{\partial}{\partial x_2} \equiv ... \equiv x_0x_1\frac{\partial}{\partial x_{n+1}} \equiv 0.&\end{aligned} \tag{4.61}$$

Without loss of generality, let us assume that $x_2 \notin H^0(I_Z(1))$. Then $x_2 = ax_1$ in $H^0(\mathcal{O}_\Lambda(1))$ for some $a \neq 0$. Therefore,

$$\begin{aligned}\omega_{01k} \equiv \omega_{12k} \equiv 0 \Rightarrow x_0x_1\frac{\partial}{\partial x_k} \equiv x_1x_2\frac{\partial}{\partial x_k} \equiv 0&\\ \Rightarrow x_0x_1\frac{\partial}{\partial x_k} \equiv x_1^2\frac{\partial}{\partial x_k} \equiv 0&\end{aligned} \tag{4.62}$$

for $k \geq 3$ and

$$\begin{aligned}\omega_{120} \equiv \omega_{201} \equiv \omega_{012} \equiv 0 \Rightarrow x_1x_2\frac{\partial}{\partial x_0} \equiv x_2x_0\frac{\partial}{\partial x_1} \equiv x_0x_1\frac{\partial}{\partial x_2} \equiv 0&\\ \Rightarrow x_1^2\frac{\partial}{\partial x_0} \equiv x_0x_1\frac{\partial}{\partial x_1} \equiv x_0x_1\frac{\partial}{\partial x_2} \equiv 0.&\end{aligned} \tag{4.63}$$

We claim that (4.62) holds for all $k \geq 1$, i.e.,

$$\begin{aligned}x_0x_1\frac{\partial}{\partial x_k} \equiv x_1^2\frac{\partial}{\partial x_k} \equiv 0 \text{ for all } k \geq 1 \text{ or equivalently}&\\ x_ix_j\frac{\partial}{\partial x_k} \equiv 0 \text{ for all } j,k \geq 1.&\end{aligned} \tag{4.64}$$

If $\{x_3, ..., x_{n+1}\} \not\subset H^0(I_Z(1))$, say $x_3 \notin H^0(I_Z(1))$, then

$$\begin{aligned} \omega_{231} \equiv \omega_{132} \equiv 0 &\Rightarrow x_2 x_3 \frac{\partial}{\partial x_1} \equiv x_1 x_3 \frac{\partial}{\partial x_2} \equiv 0 \\ &\Rightarrow x_1^2 \frac{\partial}{\partial x_1} \equiv x_1^2 \frac{\partial}{\partial x_2} \equiv 0 \end{aligned} \tag{4.65}$$

and together with (4.62) and (4.63), we see that (4.64) follows.

Otherwise, $\{x_3, ..., x_{n+1}\} \subset H^0(I_Z(1))$. Then by

$$\begin{aligned} \omega_{113} \equiv 0 \Rightarrow x_1^2 \frac{\partial}{\partial x_3} &- (c_{103} x_0 + c_{123} x_2) x_1 \frac{\partial}{\partial x_1} \equiv 0 \\ &\Rightarrow x_1^2 \frac{\partial}{\partial x_3} \equiv x_0 x_1 \frac{\partial}{\partial x_1} \equiv 0 \end{aligned} \tag{4.66}$$

we conclude that

$$x_1 x_2 \frac{\partial}{\partial x_1} \equiv 0 \Rightarrow x_1^2 \frac{\partial}{\partial x_1} \equiv 0 \tag{4.67}$$

as long as $c_{123} \neq 0$, which is obvious for $b \in B$ general. Similarly, by considering ω_{223}, we obtain

$$x_1^2 \frac{\partial}{\partial x_2} \equiv 0. \tag{4.68}$$

This concludes the proof of (4.64), which, combined with (4.63), yields (4.61) and hence (4.58).

Next, let us prove (4.59). Note that by (4.25), we have the diagram

$$\begin{array}{ccc} \mathcal{W}_{X,b} & \xrightarrow{\xi} & H^0(\Lambda, \mathcal{E}(1)) \\ \downarrow & & \downarrow \\ W_{X,b} & \xrightarrow{\xi} & H^0(\Lambda, T_P(1)) \end{array} \tag{4.69}$$

and hence

$$\xi(W_{X,b}) = \frac{\xi(\mathcal{W}_{X,b})}{\alpha \otimes H^0(\mathcal{O}_\Lambda(1))} \tag{4.70}$$

for α given by (4.10).

Let us write $u_1 \equiv u_2 \pmod{\alpha}$ if $u_1 - u_2 \in \xi(W_{X,b})$. Then (4.59) is equivalent to

$$x_i x_j \frac{\partial}{\partial x_k} \equiv 0 \pmod{\alpha} \tag{4.71}$$

for all i, j, k. Since $H^0(\mathcal{O}_\Lambda(1)) = \operatorname{Span}\{x_0, x_1\}$, it is enough to prove (4.71) for $0 \le i, j \le 1$.

Obviously,

$$x_i \alpha \equiv 0 \ (\mathrm{mod}\ \alpha) \Rightarrow x_i x_0 \frac{\partial}{\partial x_0} \equiv -x_i \sum_{j=1}^{n+1} x_j \frac{\partial}{\partial x_j} \ (\mathrm{mod}\ \alpha) \tag{4.72}$$

for all i. Combining (4.63), (4.64) and (4.72), we obtain

$$x_0^2 \frac{\partial}{\partial x_0} \equiv x_0 x_1 \frac{\partial}{\partial x_0} \equiv x_1^2 \frac{\partial}{\partial x_0} \equiv 0 \ (\mathrm{mod}\ \alpha) \Rightarrow x_i x_j \frac{\partial}{\partial x_0} \equiv 0 \ (\mathrm{mod}\ \alpha) \tag{4.73}$$

for all i, j.

Finally, by (4.73),

$$\omega_{00k} \equiv 0 \Rightarrow x_0^2 \frac{\partial}{\partial x_k} - \sum_{j=1}^{n+1} c_{0jk} x_0 x_j \frac{\partial}{\partial x_0} \equiv 0 \Rightarrow x_0^2 \frac{\partial}{\partial x_k} \equiv 0 \ (\mathrm{mod}\ \alpha) \tag{4.74}$$

for all $k \ge 1$. Combining (4.64), (4.73) and (4.74), we conclude (4.59).

When $\{x_2, ..., x_{n+1}\} \subset H^0(I_Z(1))$, i.e., Z is very special, (4.60) follows directly from the fact that $\xi(\mathcal{W}_{X,b}) = \operatorname{Span}\{\xi(\omega_{ijk})\}$. $\square$

We want to call attention to the subtle difference and relation between $\xi(\mathcal{W}_{X,b})$ and $\xi(W_{X,b})$ in the above lemma and also Lemma 4.9 below. By (4.69), $\xi(W_{X,b})$ is the image of $\xi(\mathcal{W}_{X,b})$ under $H^0(\Lambda, \mathcal{E}(1)) \twoheadrightarrow H^0(\Lambda, T_P(1))$. However, $\xi(\mathcal{W}_{X,b})$ is not necessarily the lift of $\xi(W_{X,b})$ in $H^0(\Lambda, \mathcal{E}(1))$. In particular, when Z is special but not very special, we have (4.59) but it is easy to check that $\xi(\mathcal{W}_{X,b}) \ne H^0(\Lambda, \mathcal{E}(1))$.

Let us go back to the proof of (3.52) for Z special. Since x_0 and x_1 span $H^0(\mathcal{O}_Z(1))$, we can choose $p \in Z$ such that $x_0 \ne 0$ at p. To prove (3.52), let us consider $\omega \in \mathcal{W}_{X,b}$ such that $\omega(p) = 0$. Note that $\eta(\omega) \in \operatorname{Span} J_{d+1}$ by the definition of $\mathcal{W}_{X,b}$ and $\eta(\omega)$ also vanishes at p. We claim that

$$\eta(\omega) \in H^0(I_p(1)) \otimes \operatorname{Span} J_d. \tag{4.75}$$

This follows from the lemma below.

Lemma 4.8. *Let $P = \mathbb{P}^{n+1}$ and J_d be defined in (4.2) for $d \ge 3$. Then*

$$\operatorname{Span} J_{d+1} \cap H(I_p(d+1)) = H^0(I_p(1)) \otimes \operatorname{Span} J_d \tag{4.76}$$

for every point $p \in P$ satisfying

$$p \notin \{(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1)\}. \tag{4.77}$$

Furthermore, for every 0*-dimensional subscheme* $Z \subset P$ *of length* 2*, a point* $p \in \operatorname{supp}(Z)$ *satisfying* (4.77) *and* $s \in H^0(I_p(1)) \backslash H^0(I_Z(1))$*,*

$$\operatorname{Span} J_{d+1} \cap H^0(I_p(d+1)) = H^0(I_Z(1)) \otimes \operatorname{Span} J_d + s \otimes \operatorname{Span} J_d. \tag{4.78}$$

Proof. By (4.77), there exist $i \neq j$ such that neither x_i nor x_j vanishes at p. Without loss of generality, let us assume that $x_0 \neq 0$ and $x_1 \neq 0$ at p.

It is obvious that

$$\begin{aligned} \operatorname{Span} J_{d+1} \cap H(I_p(d+1)) \supset H^0(I_p(1)) \otimes \operatorname{Span} J_d \text{ and} \\ \dim\left(\operatorname{Span} J_{d+1} \cap H(I_p(d+1))\right) = \dim \operatorname{Span} J_{d+1} - 1. \end{aligned} \tag{4.79}$$

Therefore, to show (4.76), it suffices to show that

$$\operatorname{Span} J_{d+1} = H^0(I_p(1)) \otimes \operatorname{Span} J_d + \operatorname{Span}\left\{x_0^2 x_1^{d-1}\right\} \tag{4.80}$$

which follows from the fact that

$$x_k \in H^0(I_p(1)) + \operatorname{Span}\{x_0\} \text{ and } x_k \in H^0(I_p(1)) + \operatorname{Span}\{x_1\} \tag{4.81}$$

for all k.

To see (4.78), we observe that for all $l \in H^0(I_p(1))$ and $f \in \operatorname{Span} J_d$, lf can be written as

$$lf = (l - cs)f + csf \in H^0(I_Z(1)) \otimes \operatorname{Span} J_d + s \otimes \operatorname{Span} J_d, \tag{4.82}$$

where c is a constant such that $l - cs \in H^0(I_Z(1))$. □

Note that by (4.1), $p \in Z$ always satisfies (4.77).

Suppose that $a = 1$ in (4.15), i.e., Z is very special. By Lemma 4.7,

$$\begin{aligned} &\begin{cases} x_0^2 \dfrac{\partial}{\partial x_2} - c_{012} x_0 x_1 \dfrac{\partial}{\partial x_0} \in \xi(\mathcal{W}_{X,b}) \\ x_0^2 \dfrac{\partial}{\partial x_3} - c_{013} x_0 x_1 \dfrac{\partial}{\partial x_0} \in \xi(\mathcal{W}_{X,b}) \end{cases} \\ \Rightarrow\ & c_{013} x_0^2 \frac{\partial}{\partial x_2} - c_{012} x_0^2 \frac{\partial}{\partial x_3} \in \xi(\mathcal{W}_{X,b}) \end{aligned} \tag{4.83}$$

Similarly,

$$\begin{aligned} &\begin{cases} x_1^2 \dfrac{\partial}{\partial x_2} - c_{102} x_0 x_1 \dfrac{\partial}{\partial x_1} \in \xi(\mathcal{W}_{X,b}) \\ x_1^2 \dfrac{\partial}{\partial x_3} - c_{103} x_0 x_1 \dfrac{\partial}{\partial x_1} \in \xi(\mathcal{W}_{X,b}) \end{cases} \\ \Rightarrow\ & c_{103} x_1^2 \frac{\partial}{\partial x_2} - c_{102} x_1^2 \frac{\partial}{\partial x_3} \in \xi(\mathcal{W}_{X,b}) \end{aligned} \tag{4.84}$$

and hence

$$(c_{013}x_0^2 + c_{103}x_1^2)\frac{\partial}{\partial x_2} - (c_{012}x_0^2 + c_{102}x_1^2)\frac{\partial}{\partial x_3} \in \xi(\mathcal{W}_{X,b}). \tag{4.85}$$

Since $x_2 = ... = x_{n+1} = 0$ at $p \neq (1, 0, ..., 0), (0, 1, 0, ..., 0)$, neither x_0 nor x_1 vanishes at p. Hence there exist numbers r_k such that $c_{01k}x_0^2 + c_{10k}x_1^2 + r_k x_0 x_1$ vanishes at p for $k = 2, 3$. For b general, by (4.19) the numbers c_{ijk} are general with the only relations $c_{ijk} = c_{ikj}$. In particular,

$$\det \begin{bmatrix} c_{012} & c_{102} \\ c_{013} & c_{103} \end{bmatrix} \neq 0. \tag{4.86}$$

Therefore, at least one of $c_{012}x_0^2 + c_{102}x_1^2 + r_2x_0x_1$ and $c_{013}x_0^2 + c_{103}x_1^2 + r_3x_0x_1$ does not vanish on Z. Consequently,

$$\begin{aligned} &(c_{013}x_0^2 + c_{103}x_1^2 + r_3x_0x_1)\frac{\partial}{\partial x_2} \\ &- (c_{012}x_0^2 + c_{102}x_1^2 + r_2x_0x_1)\frac{\partial}{\partial x_3} \in \xi(\mathcal{W}_{X,b}) \end{aligned} \tag{4.87}$$

vanishes at p but not on Z. So we may choose $\omega \in \mathcal{W}_{X,b}$ such that

$$\begin{aligned} \xi(\omega) &= (c_{013}x_0^2 + c_{103}x_1^2 + r_3x_0x_1)\frac{\partial}{\partial x_2} \\ &\quad - (c_{012}x_0^2 + c_{102}x_1^2 + r_2x_0x_1)\frac{\partial}{\partial x_3}, \\ \omega(p) &= 0 \text{ and } \omega\Big|_Z \neq 0. \end{aligned} \tag{4.88}$$

Let us write

$$\xi(\omega) = s_1\left(s_2\frac{\partial}{\partial x_2} + s_3\frac{\partial}{\partial x_3}\right) \tag{4.89}$$

with $s_i \in H^0(\mathcal{O}_P(1))$ satisfying $s_1(p) = 0$ and either $s_1s_2 \neq 0$ or $s_1s_3 \neq 0$ on Z.

Since $\omega(p) = 0$, $\tau = \eta(\omega)$ vanishes at p as well. So by Lemma 4.8, $\tau \in H^0(I_p(1)) \otimes \operatorname{Span} J_d$. When we regard τ as a vector in $H^0(I_p(1)) \otimes T_{B,b}$, we have

$$L_\lambda(\tau) = s_1\gamma \tag{4.90}$$

for some

$$\gamma \in H^0(\mathcal{G}_Z) = \operatorname{Span}\left\{x_0\frac{\partial}{\partial x_0}, x_1\frac{\partial}{\partial x_0}\right\} \tag{4.91}$$

by (4.50) with $a = 1$. Then

$$\omega|_Z - L_\lambda(\tau) = s_1 \left(s_2 \frac{\partial}{\partial x_2} + s_3 \frac{\partial}{\partial x_3} - \gamma \right) \in W_{X,b,Z,\lambda}. \tag{4.92}$$

Obviously, $\omega|_Z - L_\lambda(\tau)$ vanishes at p. But since one of $s_1 s_2$ and $s_1 s_3$ does not vanish on Z and γ lies in the subspace (4.91) of $H^0(Z, T_P)$, it is easy to see that $\omega - L_\lambda(\tau)$ does not vanish in $H^0(Z, T_P(1))$. This finishes the proof for (3.52) when Z is very special.

Suppose that $2 \le a \le n$ in (4.15). Then by (4.59), ξ maps $W_{X,b}$ surjectively onto $H^0(\Lambda, T_P(1))$. So we can choose $\omega \in \mathcal{W}_{X,b}$ such that

$$\xi(\omega) = s x_1 \frac{\partial}{\partial x_{n+1}} \tag{4.93}$$

in $H^0(\Lambda, T_P(1))$ for some $s \in H^0(I_p(1)) \backslash H^0(I_Z(1))$. Note that x_1 does not vanish on either $p_i \in Z$, as explained for (4.52).

By the same argument as before, we have

$$\omega - L_\lambda(\tau) = s \left(x_1 \frac{\partial}{\partial x_{n+1}} - \gamma \right) \in W_{X,b,Z,\lambda} \tag{4.94}$$

for some $\gamma \in H^0(\mathcal{G}_Z)$. Again, $\omega - L_\lambda(\tau)$ vanishes at p and does not vanish in $H^0(Z, T_P(1))$ for $a \le n$ by (4.50). This finishes the proof for (3.52) when $a \le n$.

Suppose that $a \ge 2$ and x_0 vanishes at one of p_i for b general. Then we choose $\omega \in \mathcal{W}_{X,b}$ such that

$$\xi(\omega) = s x_1 \frac{\partial}{\partial x_0} \tag{4.95}$$

in $H^0(\Lambda, T_P(1))$ for some $s \in H^0(I_p(1)) \backslash H^0(I_Z(1))$.

By the same argument as before, we have

$$\omega - L_\lambda(\tau) = s \left(x_1 \frac{\partial}{\partial x_0} - \gamma \right) \in W_{X,b,Z,\lambda} \tag{4.96}$$

for some $\gamma \in H^0(\mathcal{G}_Z)$. Again, $\omega - L_\lambda(\tau)$ vanishes at p. Note that we choose p such that $x_0 \ne 0$ at p. So x_0 must vanish at $Z \backslash \{p\}$. By (4.51),

$$\gamma \in H^0(\mathcal{G}_Z) = \operatorname{Span} \left\{ x_0 \frac{\partial}{\partial x_0}, x_1 \frac{\partial}{\partial x_2}, ..., x_1 \frac{\partial}{\partial x_a} \right\}. \tag{4.97}$$

It follows that $\omega - L_\lambda(\tau) \ne 0$ in $H^0(Z, T_P(1))$. This finishes the proof for (3.52) when $a \ge 2$ and x_0 vanishes at one of p_i.

It remains to verify (3.52) when $a = n + 1$ in (4.15) and $x_0 \ne 0$ at both p_i. In this case,

$$\begin{aligned} H^0(\mathcal{G}_Z) &= \operatorname{Span}\left\{ x_i \frac{\partial}{\partial x_j} : j = 0 \text{ or } 2 \le i, j \le n+1 \right\} \\ &= \operatorname{Span}\left\{ x_1 \frac{\partial}{\partial x_j} : j = 0, 1, ..., n+1 \right\} \end{aligned} \tag{4.98}$$

by (4.50), where the term $x_1(\partial/\partial x_1)$ comes from using the relation $\alpha = 0$ given by the Euler vector field for α defined by (4.10).

Let us choose $s_1 = x_0 - r_1 x_1$ and $s_2 = x_0 - r_2 x_1$ for some constants r_i such that $s_i(p_i) \neq 0$ and $s_i(p_{3-i}) = 0$ for $i = 1, 2$. Clearly, $r_1 \neq r_2 \neq 0$.

Fixing $1 \le k \le n+1$, we let

$$u_k = x_0 \frac{\partial}{\partial x_k} - \sum_{j=1}^{n+1} c_{0jk} x_j \frac{\partial}{\partial x_0}. \tag{4.99}$$

Since $\omega_{00k} = x_0 u_k$, $\xi(x_0 u_k) \in \xi(\mathcal{W}_{X,b})$. And by (4.58), $\xi(x_1 u_k) \in \xi(\mathcal{W}_{X,b})$. Therefore, $\xi(s u_k) \in \xi(\mathcal{W}_{X,b})$ for all $s \in H^0(\mathcal{O}_P(1))$. In particular, there exist $w_{ik} \in \mathcal{W}_{X,b}$ such that

$$w_{ik}\Big|_{\Lambda} = s_i u_k\Big|_{\Lambda} \tag{4.100}$$

in $H^0(\Lambda, \mathcal{E}(1))$ for $i = 1, 2$. Then by Lemma 4.8,

$$\eta(w_{ik}) - s_i \gamma_{ik} \in H^0(I_Z(1)) \otimes \operatorname{Span} J_d \tag{4.101}$$

for some $\gamma_{ik} \in \operatorname{Span} J_d$ and $i = 1, 2$. We may write

$$\eta(w_{ik}) - s_i \gamma_{ik} = \sum l_j \tau_j \tag{4.102}$$

with $l_j \in H^0(\mathcal{O}_P(1))$ and $\tau_j \in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}$. Then

$$w_{ik} - s_i L_\lambda(\gamma_{ik}) - \sum l_j L_\lambda(\tau_j) = s_i \left(u_k - L_\lambda\left(\gamma_{ik}\right)\right) \in W_{X,b,Z,\lambda}, \tag{4.103}$$

when restricted to Z, since L_λ vanishes on $H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}$.

By the same argument as before, we conclude that

$$\left(u_k - L_\lambda\left(\gamma_{ik}\right)\right)\Big|_{p_i} = 0 \tag{4.104}$$

for $i = 1, 2$; otherwise, (3.52) follows. By our choice of s_i and r_i, we see that

$$L_\lambda\left(\gamma_{ik}\right) = r_{3-i} x_1 \frac{\partial}{\partial x_k} - \sum_{j=1}^{n+1} c_{0jk} x_j \frac{\partial}{\partial x_0} \tag{4.105}$$

for $i = 1, 2$. In particular,

$$L_\lambda\left(\gamma_{1k}-\gamma_{2k}\right)=(r_2-r_1)x_1\frac{\partial}{\partial x_k}\neq 0. \tag{4.106}$$

So $x_1(\partial/\partial x_k)$ lies in the image of L_λ for all $k=1,2,...,n+1$.

By (4.101), $\eta(w_{ik})-s_i\gamma_{ik}=0$ in $H^0(\mathcal{O}_\Lambda(d+1))$ and hence

$$\begin{aligned}&\xi(s_i\gamma_{ik})=\xi(\eta(w_{ik}))=\xi(\eta(s_iu_k))=\xi(s_i\eta(u_k))\\&\Rightarrow s_i(\gamma_{ik}-\eta(u_k))\Big|_\Lambda=0\Rightarrow(\gamma_{ik}-\eta(u_k))\Big|_\Lambda=0\\&\Rightarrow\xi(\gamma_{ik})=\xi(\eta(u_k))\end{aligned} \tag{4.107}$$

for $i=1,2$. Therefore, $\xi(\gamma_{1k}-\gamma_{2k})=0$ and hence

$$\gamma_{1k}-\gamma_{2k}\in\operatorname{Span}J_d\cap\ker(\xi). \tag{4.108}$$

Combining (4.106) and (4.108), we conclude that

$$\begin{aligned}&\text{for each } 1\le k\le n+1,\text{ there exists }\gamma_k\in\operatorname{Span}J_d\cap\ker(\xi)\\&\qquad\text{such that } L_\lambda(\gamma_k)=x_1\frac{\partial}{\partial x_k}.\end{aligned} \tag{4.109}$$

On the other hand, we know that

$$\operatorname{Span}J_d\cap\ker(\xi)=H^0(I_Z(1))\otimes\operatorname{Span}J_{d-1}+V \tag{4.110}$$

by (4.49) in Lemma 4.6 for

$$\begin{aligned}V=\operatorname{Span}\Big\{&x_0^{d-2}x_i(x_j-c_jx_1):i\ge 1,j\ge 2\text{ and}\\&x_j-c_jx_1\in H^0(I_Z(1))\Big\}.\end{aligned} \tag{4.111}$$

And since L_λ vanishes on $H^0(I_Z(1))\otimes\operatorname{Span}J_{d-1}$, (4.109) is equivalent to saying that

$$\left\{x_1\frac{\partial}{\partial x_k}:k\ge 1\right\}\subset L_\lambda(V). \tag{4.112}$$

Note that for $x_0^{d-2}x_i(x_j-c_jx_1)\in V$,

$$\eta\left(x_1\otimes x_0^{d-2}x_i(x_j-c_jx_1)-(x_j-c_jx_1)\otimes x_0^{d-2}x_1x_i\right)=0 \tag{4.113}$$

and hence

$$\begin{aligned}&L_\lambda\left(x_1\otimes x_0^{d-2}x_i(x_j-c_jx_1)-(x_j-c_jx_1)\otimes x_0^{d-2}x_1x_i\right)\\&=x_1L_\lambda\left(x_0^{d-2}x_i(x_j-c_jx_1)\right)-(x_j-c_jx_1)L_\lambda\left(x_0^{d-2}x_1x_i\right)\\&=x_1L_\lambda\left(x_0^{d-2}x_i(x_j-c_jx_1)\right)\in W_{X,b,Z,\lambda}.\end{aligned} \tag{4.114}$$

It follows that $x_1 L_\lambda(\gamma) \in W_{X,b,Z,\lambda}$ for all $\gamma \in V$. Consequently,

$$\operatorname{Span}\left\{x_1^2 \frac{\partial}{\partial x_k} : k \geq 1\right\} \subset W_{X,b,Z,\lambda} \tag{4.115}$$

by (4.112).

It remains to find $u \in H^0(\Lambda, \mathcal{E})$ satisfying

$$u \in \operatorname{Span}\left\{x_1 \frac{\partial}{\partial x_k} : k \geq 1\right\},\ u \neq 0 \text{ and } x_0 u \in W_{X,b,Z,\lambda}. \tag{4.116}$$

If such u exists, $u \neq 0$ at both p_i. Then combining (4.115) and (4.116), we see that $(x_0 - r_1 x_1)u \in W_{X,b,Z,\lambda}$ vanishes at p_2 but not p_1.

To construct u satisfying (4.116), let us consider

$$\begin{aligned} \omega &= c_{013}\left(\omega_{012} - \sum_{j=2}^{n+1} c_{02j}\omega_{1j0}\right) - c_{012}\left(\omega_{013} - \sum_{j=2}^{n+1} c_{03j}\omega_{1j0}\right) \\ &= c_{013}\left(x_0 x_1 \frac{\partial}{\partial x_2} - \sum_{j=2}^{n+1} c_{02j} x_1 x_j \frac{\partial}{\partial x_0}\right) \\ &\quad - c_{012}\left(x_0 x_1 \frac{\partial}{\partial x_3} - \sum_{j=2}^{n+1} c_{03j} x_1 x_j \frac{\partial}{\partial x_0}\right) \end{aligned} \tag{4.117}$$

in $\mathcal{W}_{X,b}$. We choose ω in such a way that the expansion of $\eta(\omega)$ does not contain monomials in J_{d+1} of degree $d-1$ in x_0. Thus, we can write

$$\eta(\omega) = \sum_{i=1}^{n+1} x_i \tau_i \tag{4.118}$$

for some $\tau_i \in \operatorname{Span} J_d$. Therefore, by the definition (3.50) of $W_{X,b,Z,\lambda}$,

$$\omega - \sum_{i=1}^{n+1} x_i L_\lambda(\tau_i) \in W_{X,b,Z,\lambda} \tag{4.119}$$

when restricted to Z. Combining it with (4.98) and (4.115), we conclude

$$x_0\left(c_{013} x_1 \frac{\partial}{\partial x_2} - c_{012} x_1 \frac{\partial}{\partial x_3}\right) - \beta_1 x_1^2 \frac{\partial}{\partial x_0} \in W_{X,b,Z,\lambda} \tag{4.120}$$

for some constant β_1. Similarly, we have

$$x_0\left(c_{023} x_2 \frac{\partial}{\partial x_1} - c_{021} x_2 \frac{\partial}{\partial x_3}\right) - \beta_2 x_1^2 \frac{\partial}{\partial x_0} \in W_{X,b,Z,\lambda} \tag{4.121}$$

for some constant β_2 by switching x_1 and x_2. Hence by (4.120) and (4.121),

$$x_0\Big(e_1c_{013}x_1\frac{\partial}{\partial x_2}+e_2c_{023}x_2\frac{\partial}{\partial x_1} - (e_1c_{012}x_1+e_2c_{021}x_2)\frac{\partial}{\partial x_3}\Big)\in W_{X,b,Z,\lambda} \tag{4.122}$$

for constants e_1 and e_2, not all zero, satisfying $e_1\beta_1+e_2\beta_2=0$.

For $b\in B$ general, $c_{013}c_{023}\neq 0$ and hence e_1c_{013} and e_2c_{023} cannot both vanish. Therefore,

$$u=e_1c_{013}x_1\frac{\partial}{\partial x_2}+e_2c_{023}x_2\frac{\partial}{\partial x_1}-(e_1c_{012}x_1+e_2c_{021}x_2)\frac{\partial}{\partial x_3} \tag{4.123}$$

satisfies (4.116).

This finishes the proof of (3.52) for Z special. Thus, if $Z=\{\sigma_1(b),\sigma_2(b)\}$ is special with respect to (x_i) for all $b\in B$, then $\sigma_1(b)$ and $\sigma_2(b)$ are not Γ-equivalent over $\mathbb{Q}$ on X_b for $b\in B$ general.

4.6. Generic case

Next we will finish the proof of our main theorem by proving (3.52) for Z generic. We start with a result on $\xi(\mathcal{W}_{X,b})$ for Z generic, similar to Lemma 4.7.

Lemma 4.9. *Let $P=\mathbb{P}^{n+1}$ and $X\subset Y=B\times P$ be the family of hypersurfaces in P given by (4.1) over $B=\operatorname{Span} J_d$ for $n\geq 2$ and $d\geq 4$. Then ξ is surjective when restricted to $\mathcal{W}_{X,b}$, i.e.,*

$$\xi(\mathcal{W}_{X,b})=H^0(\Lambda,\mathcal{E}(1)) \tag{4.124}$$

for $b\in B$ general and <u>all</u> 0-dimensional subschemes $Z\subset X_b$ of length 2 that are generic with respect to (x_i), where $\Lambda\subset P$ is the line cutting out Z on X_b and ξ is the restriction $H^0(\mathcal{E}(1))\to H^0(\Lambda,\mathcal{E}(1))$.

To keep our argument in sight, we postpone its proof till the end of the subsection.

By the isomorphism (4.40), L_λ actually induces a map

$$\begin{array}{ccc}\dfrac{\operatorname{Span} J_d}{H^0(I_Z(1))\otimes\operatorname{Span} J_{d-1}} & \xrightarrow{L_\lambda} & H^0(Z,T_P)\\ \xi\big\downarrow\cong & & \big\uparrow L_\lambda\\ \operatorname{Sym}^d H^0(\mathcal{O}_Z(1)) & = & H^0(\mathcal{O}_\Lambda(d))\end{array} \tag{4.125}$$

As before, we choose $s_i\in H^0(\mathcal{O}_P(1))$ such that $s_i(p_i)\neq 0$ and $s_i(p_{3-i})=0$ for $i=1,2$. For every $u\in H^0(\mathcal{E})$, by Lemma 4.9, there exist $\omega_i\in\mathcal{W}_{X,b}$ such that

$$\xi(\omega_i) = \xi(s_i u) \tag{4.126}$$

in $H^0(\Lambda, \mathcal{E}(1))$ for $i = 1, 2$. Then as (4.101), by Lemma 4.8 we have

$$\eta(\omega_i) - s_i\gamma_i \in H^0(I_Z(1)) \otimes \operatorname{Span} J_d \tag{4.127}$$

for some $\gamma_i \in \operatorname{Span} J_d$. It follows that

$$s_i\left(u - L_\lambda(\gamma_i)\right) \in W_{X,b,Z,\lambda} \tag{4.128}$$

for $i = 1, 2$, when restricted to Z. As before, we must have

$$(u - L_\lambda(\gamma_i))\Big|_{p_i} = 0 \tag{4.129}$$

for $i = 1, 2$; otherwise, (3.52) follows.

By (4.127) and $H^0(\Lambda, I_Z(1)) = 0$, $\xi\left(\eta(\omega_i) - s_i\gamma_i\right) = 0$ and hence

$$\begin{aligned} &\xi(s_i\gamma_i) = \xi\left(\eta(\omega_i)\right) = \xi\left(\eta(s_i u)\right) = \xi\left(s_i\eta(u)\right) \\ &\Rightarrow s_i(\gamma_i - \eta(u))\Big|_\Lambda = 0 \Rightarrow (\gamma_i - \eta(u))\Big|_\Lambda = 0 \Rightarrow \xi(\gamma_i) = \xi(\eta(u)) \end{aligned} \tag{4.130}$$

for $i = 1, 2$. Then $\xi(\gamma_1) = \xi(\gamma_2)$ and $\gamma_1 - \gamma_2 \in H^0(I_Z(1)) \otimes \operatorname{Span} J_{d-1}$ by Lemma 4.5. Therefore, $L_\lambda(\gamma_1) = L_\lambda(\gamma_2)$. Combining this with (4.129), we conclude that

$$u\Big|_Z = L_\lambda(\gamma_1) = L_\lambda(\gamma_2) \tag{4.131}$$

in $H^0(Z, T_P)$. This implies that the map L_λ in (4.125) is onto. Indeed, the combination of (4.130) and (4.131) tells us exactly what L_λ is:

$$\boxed{L_\lambda(\gamma) = u\Big|_Z \text{ if } \gamma\Big|_\Lambda = \eta(u)\Big|_\Lambda} \tag{4.132}$$

for $\gamma \in T_{B,b} = \operatorname{Span} J_d$ and $u \in H^0(\mathcal{E})$. Let us see the geometric implication of (4.132).

Let $\widehat{\eta}$ be the map given by the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{E}) & \xrightarrow{\eta} & H^0(\mathcal{O}_P(d)) \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\xi} \\ H^0(\Lambda, \mathcal{E}) & \xrightarrow{\widehat{\eta}} & H^0(\mathcal{O}_\Lambda(d)). \end{array} \tag{4.133}$$

Obviously, $\widehat{\eta}$ is the restriction of η to Λ and defined in the same way as η by

$$\widehat{\eta}\left(x_i \frac{\partial}{\partial x_j}\right) = x_i \frac{\partial F}{\partial x_j} \tag{4.134}$$

for all $0 \le i, j \le n+1$ with everything restricted to Λ.

Since

$$h^0(\Lambda, \mathcal{E}) - h^0(\mathcal{O}_\Lambda(d)) = 2(n+2) - (d+1) > 0 \tag{4.135}$$

for $d = 2n+2$, there exists $u \neq 0 \in h^0(\Lambda, \mathcal{E})$ such that $\widehat{\eta}(u) = 0$. By (4.132), u vanishes in $H^0(Z, T_P)$. That is, u lies in the kernel of the map

$$H^0(\Lambda, \mathcal{E}) \xrightarrow{\ \rho\ } H^0(Z, T_P). \tag{4.136}$$

Obviously, $\ker(\rho)$ is two dimensional and $\alpha \in \ker(\rho)$ for α given in (4.10).

We can make everything very explicit. If we identify Λ with $\mathbb{P}^1$ and let $p_1 = (0,1)$, $p_2 = (1,0)$ and y be the affine coordinate of $\Lambda \backslash p_2$, then

$$\widehat{\eta}\,(\ker(\rho)) = \operatorname{Span}\{f(y), yf'(y)\} \tag{4.137}$$

for $f(y) = \widehat{\eta}(\alpha) = \xi(F) \in H^0(\mathcal{O}_\Lambda(d))$. To see this, we let

$$x_i = a_i y + b_i$$

be the restriction of x_i to Λ. Then

$$\sum g_i(y) \frac{\partial}{\partial x_i} \in \ker(\rho) \Leftrightarrow \frac{g_0(y)}{a_0 y + b_0} = \frac{g_1(y)}{a_1 y + b_1} = \ldots = \frac{g_{n+1}(y)}{a_{n+1} y + b_{n+1}} \text{ at } 0, \infty$$

Therefore, $\ker(\rho)$ is spanned by

$$\alpha = \sum (a_i y + b_i) \frac{\partial}{\partial x_i} \quad \text{and} \quad \beta = \sum a_i y \frac{\partial}{\partial x_i}.$$

Then

$$\widehat{\eta}(\beta) = \sum a_i y \left.\frac{\partial F}{\partial x_i}\right|_\Lambda = y \sum \frac{d}{dy}(a_i y + b) \left.\frac{\partial F}{\partial x_i}\right|_\Lambda = y f'(y)$$

and (4.137) follows.

Since $u \neq 0 \in \ker(\rho)$ and $\widehat{\eta}(u) = 0$, we conclude that $f(y)$ and $yf'(y)$ must be two linearly dependent polynomials in y. This can only happen if $f(y) = cy^m$, i.e., $\xi(F)$ vanishes only at p_1 and p_2. Namely, X_b and Λ have no intersections other than p_1 and p_2. So we have reached our key conclusion:

Proposition 4.10. *If there are two points $p_1 \neq p_2$ on a general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $2n+2$ that are Γ-equivalent over $\mathbb{Q}$, then the line Λ joining p_1 and p_2 meets X only at p_1 and p_2.*

It remains to prove the following:

Proposition 4.11. *Let $P = \mathbb{P}^{n+1}$, $\mathbb{G}(1,P)$ be the Grassmannian of lines in P and $B = \mathbb{P}H^0(\mathcal{O}_P(d))$ be the parameter space of hypersurfaces in P of degree $d = 2n+2$. For $0 < m < d$, let W_m be the incidence correspondence*

$$\begin{aligned} W_m = \Big\{(X, \Lambda, p_1, p_2) : p_1 \neq p_2 \text{ and } X.\Lambda = mp_1 + (d-m)p_2\Big\} \\ \subset B \times \mathbb{G}(1,P) \times P \times P. \end{aligned} \tag{4.138}$$

Then

(1) *W_m is irreducible.*
(2) *W_m is generically finite over B via the projection $\pi : W_m \to B$.*
(3) *For a general $X \in B$, the fiber $\pi^{-1}([X])$ contains at least two points $(X, \Lambda_i, p_{i1}, p_{i2})$ for $i = 1, 2$ such that $p_{11} \neq p_{21}$ and the line joining p_{11} and p_{21} meet X at more than two points.*

Let us see how the above proposition implies our main theorem. We consider the incidence correspondence

$$\begin{aligned} W = \Big\{(X, \Lambda, p_1, p_2) : p_1 \neq p_2 \in X \cap \Lambda \text{ and } p_1 \sim_\Gamma p_2 \text{ over } \mathbb{Q}\Big\} \\ \subset B \times \mathbb{G}(1,P) \times P \times P \end{aligned} \tag{4.139}$$

for $B = \mathbb{P}H^0(\mathcal{O}_P(d))$. This is a locally Noetherian scheme, a priori.

If no components of W dominate B, we are done. Otherwise, by Proposition 4.10 and 4.11, W must contain some W_m as an irreducible component. Then by Proposition 4.11 again, for $X \in B$ general, there exist $(X, \Lambda_i, p_{i1}, p_{i2}) \in W_m \subset W$ for $i = 1, 2$ such that $p_{11} \neq p_{21}$ and the line joining p_{11} and p_{21} meet X at more than two points.

Since $p_{i1} \sim_\Gamma p_{i2}$ over $\mathbb{Q}$ and $X.\Lambda_i = mp_{i1} + (d-m)p_{i2}$, we have

$$X.\Lambda_1 \sim_{\mathbb{P}^1} X.\Lambda_2 \Rightarrow X.\Lambda_1 \sim_\Gamma X.\Lambda_2 \Rightarrow dp_{i1} \sim_\Gamma dp_{i2} \sim_\Gamma X.\Lambda_i \tag{4.140}$$

over $\mathbb{Q}$ on X for $i = 1, 2$. It follows that all four points p_{ij} are Γ-equivalent over $\mathbb{Q}$. Then by Proposition 4.10 again, the line joining p_{11} and p_{21} must meet X only at p_{11} and p_{21}, which is a contradiction.

It remains to prove Proposition 4.11.

Proof of Proposition 4.11. The proof of this statement is fairly standard. To see that W_m is irreducible of dimension $\dim B$, it suffices to project it to $\mathbb{G}(1,P) \times P \times P$. The fiber of W_m over (Λ, p_1, p_2) for $p_1 \neq p_2 \in \Lambda$ is a linear subspace of B of dimension $\dim B - d$. Therefore, W_m is irreducible of dimension

$$\begin{aligned} \dim W_m &= \dim\left\{(\Lambda, p_1, p_2) : p_1 \neq p_2 \in \Lambda\right\} + (\dim B - d) \\ &= \dim \mathbb{G}(1, P) + 2 - d + \dim B \\ &= \dim B + (2n + 2 - d) = \dim B \end{aligned} \tag{4.141}$$

for $d = 2n + 2$.

To show that W_m is generically finite over B, it suffices to exhibit a point $(X, \Lambda, p_1, p_2) \in W_m$ such that Λ does not deform while preserving the tangency conditions with X. By that we mean there does not exist a one-parameter family of lines Λ_t such that $\Lambda_0 = \Lambda$ and Λ_t meets X at two points with multiplicities m and $d - m$, respectively. Such deformation of Λ is governed by the standard exact sequence

$$0 \longrightarrow T_\Lambda(-p_1 - p_2) \longrightarrow T_P(-\log X)\Big|_\Lambda \longrightarrow N \longrightarrow 0. \tag{4.142}$$

It is easy to find $(X, \Lambda, p_1, p_2) \in W_m$ such that $H^0(N) = 0$. We leave the details to Appendix A.

Finally, to show (3), it again suffices to exhibit $(X, \Lambda_i, p_{i1}, p_{i2}) \in W_m$ for $i = 1, 2$ with the required properties and neither Λ_1 nor Λ_2 deforms while preserving the tangency conditions with X. Again, it is easy to find such X and Λ_i and use the exact sequence (4.142) to show that Λ_i do not deform. Again, we refer the reader to Appendix A for the details. □

This finishes the proof of our main Theorem 1.5. It remains to provide the following

Proof of Lemma 4.9. Let $\{\omega_{ijk}\}$ be the basis of $\mathcal{W}_{X,b}$ given by (4.18) with c_{ijk} given by (4.19). For $b \in B$ general, $\{c_{ijk} : 0 \le i \neq j, k \le n + 1\}$ is a general set of numbers satisfying $c_{ijk} = c_{ikj}$.

We write $u_1 \equiv u_2$ if $\xi(u_1 - u_2) \in \xi(\mathcal{W}_{X,b})$. Of course, we have $\omega_{ijk} \equiv 0$ and want to show that $u \equiv 0$ for all $u \in H^0(\mathcal{E}(1))$.

For starters, it is obvious that

$$\omega_{ijk} \equiv 0 \Rightarrow x_i x_j \frac{\partial}{\partial x_k} \equiv 0 \text{ for all } i \neq j \neq k \tag{4.143}$$

and

$$\omega_{iik} \equiv 0 \Rightarrow x_i^2 \frac{\partial}{\partial x_k} - \sum_{j \neq i} c_{ijk} x_i x_j \frac{\partial}{\partial x_i} \equiv 0 \text{ for all } i \neq k. \tag{4.144}$$

Without loss of generality, we assume (4.13). We discuss in two cases:

(1) Suppose that

$$\mathrm{Span}\{x_0, x_1\} = \mathrm{Span}\{x_1, x_i\} = \mathrm{Span}\{x_i, x_0\} = H^0(\mathcal{O}_Z(1)) \tag{4.145}$$

for some i. Without loss of generality, we may assume that $i = 2$. Namely, we have

$$\mathrm{Span}\{x_0, x_1\} = \mathrm{Span}\{x_1, x_2\} = \mathrm{Span}\{x_2, x_0\} = H^0(\mathcal{O}_Z(1)). \tag{4.146}$$

(2) Otherwise, suppose that there does not exist x_i satisfying (4.145). Namely, for each x_i, either $x_i \in \mathrm{Span}\{x_0\}$ or $x_i \in \mathrm{Span}\{x_1\}$ in $H^0(\mathcal{O}_Z(1))$. And since Z is generic, there must exist $i \neq j \neq 0, 1$ such that

$$\mathrm{Span}\{x_0, x_i\} = \mathrm{Span}\{x_1, x_j\} = H^0(\mathcal{O}_Z(1)). \tag{4.147}$$

Without loss of generality, we may assume that $i = 3$ and $j = 2$. In summary, when (4.145) fails, we may assume that

$$\begin{aligned}&\mathrm{Span}\{x_0, x_3\} = \mathrm{Span}\{x_1, x_2\} = \mathrm{Span}\{x_0, x_1\} = H^0(\mathcal{O}_Z(1)) \text{ and}\\ &\{x_2, ..., x_{n+1}\} \subset \mathrm{Span}\{x_0\} \cup \mathrm{Span}\{x_1\} \text{ in } H^0(\mathcal{O}_Z(1)).\end{aligned} \tag{4.148}$$

In the first case, we assume (4.146). Then for all $k \neq 0, 1, 2$ and all i, j,

$$x_0 x_1 \frac{\partial}{\partial x_k} \equiv x_1 x_2 \frac{\partial}{\partial x_k} \equiv x_0 x_2 \frac{\partial}{\partial x_k} \equiv 0 \tag{4.149}$$

and hence

$$x_i x_j \frac{\partial}{\partial x_k} \equiv 0 \tag{4.150}$$

since $\{x_0x_1, x_1x_2, x_0x_2\}$ spans $H^0(\mathcal{O}_\Lambda(2))$ by (4.146).

Suppose that $x_k \neq 0$ in $H^0(\mathcal{O}_Z(1))$ for some $3 \le k \le n+1$. Without loss of generality, suppose that $x_3 \neq 0$ in $H^0(\mathcal{O}_Z(1))$. Then at least two pairs among $\{x_0, x_3\}$, $\{x_1, x_3\}$ and $\{x_2, x_3\}$ are linearly independent in $H^0(\mathcal{O}_Z(1))$. Without loss of generality, let us assume that

$$\mathrm{Span}\{x_0, x_1\} = \mathrm{Span}\{x_1, x_3\} = \mathrm{Span}\{x_3, x_0\} = H^0(\mathcal{O}_Z(1)). \tag{4.151}$$

Then

$$x_0 x_1 \frac{\partial}{\partial x_2} \equiv x_1 x_3 \frac{\partial}{\partial x_2} \equiv x_0 x_3 \frac{\partial}{\partial x_2} \equiv 0 \Rightarrow x_i x_j \frac{\partial}{\partial x_2} \equiv 0 \tag{4.152}$$

for all i, j. That is, (4.150) holds for $k = 2$ as well. Thus, it holds for all $k \neq 0, 1$:

$$x_i x_j \frac{\partial}{\partial x_k} \equiv 0 \text{ if } k \neq 0, 1. \tag{4.153}$$

It remains to prove (4.150) for $k = 0, 1$.

Setting $i \neq 0, 1$ and $k = 0, 1$ in (4.144), we have

$$x_i^2 \frac{\partial}{\partial x_k} \equiv \sum_{j \neq i} c_{ijk} x_i x_j \frac{\partial}{\partial x_i} \equiv 0$$

by (4.153). Switching i and k, we can rewrite the above as

$$x_k^2 \frac{\partial}{\partial x_i} \equiv 0 \text{ for all } i = 0, 1 \text{ and } k \neq 0, 1.$$

Similarly, setting $i = 0, 1$ and $k \neq 0, 1$ in (4.144), we have

$$\sum_{j \neq i} c_{ijk} x_i x_j \frac{\partial}{\partial x_i} \equiv x_i^2 \frac{\partial}{\partial x_k} \equiv 0$$

by (4.153). In summary, by (4.144) and (4.153), we see that

$$x_k^2 \frac{\partial}{\partial x_i} \equiv x_i \sum_{j \neq i} c_{ijk} x_j \frac{\partial}{\partial x_i} \equiv 0 \text{ for all } i = 0, 1 \text{ and } k \neq 0, 1. \tag{4.154}$$

Setting $i = 0$ in (4.154) and combining it with (4.143), we have

$$x_k^2 \frac{\partial}{\partial x_0} \equiv x_0 \sum_{j \neq 0} c_{0jk} x_j \frac{\partial}{\partial x_0} \equiv x_k x_l \frac{\partial}{\partial x_0} \equiv 0 \text{ for all } k > l \geq 1. \tag{4.155}$$

If $\operatorname{Span}\{x_k, x_l\} = H^0(\mathcal{O}_Z(1))$ for some $k > l \geq 2$, then

$$x_k^2 \frac{\partial}{\partial x_0} \equiv x_l^2 \frac{\partial}{\partial x_0} \equiv x_k x_l \frac{\partial}{\partial x_0} \equiv 0 \Rightarrow x_i x_j \frac{\partial}{\partial x_0} \equiv 0 \tag{4.156}$$

for all i, j by (4.155). Otherwise, x_k and x_l are linearly dependent in $H^0(\mathcal{O}_Z(1))$ for all $k > l \geq 2$. This implies that

$$x_3, ..., x_{n+1} \in \operatorname{Span}\{x_2\} \tag{4.157}$$

in $H^0(\mathcal{O}_Z(1))$. Thus

$$x_2^2 \frac{\partial}{\partial x_0} \equiv x_1 x_2 \frac{\partial}{\partial x_0} \equiv 0 \Rightarrow x_0 x_2 \frac{\partial}{\partial x_0} \equiv 0 \Rightarrow x_0 x_j \frac{\partial}{\partial x_0} \equiv 0 \text{ for } j \geq 2 \tag{4.158}$$

since $x_0 \in \text{Span}\{x_1, x_2\}$. So we may rewrite (4.155) as

$$x_2^2 \frac{\partial}{\partial x_0} \equiv x_1 x_2 \frac{\partial}{\partial x_0} \equiv c_{01k} x_0 x_1 \frac{\partial}{\partial x_0} \equiv 0 \tag{4.159}$$

for all $k \geq 2$. Since $c_{012} \neq 0$ for general b, we have

$$x_0 x_1 \frac{\partial}{\partial x_0} \equiv 0 \Rightarrow x_1^2 \frac{\partial}{\partial x_0} \equiv 0 \tag{4.160}$$

since $x_0 = b_1 x_1 + b_2 x_2$ in $H^0(\mathcal{O}_Z(1))$ for some $b_i \neq 0$ by (4.146). Combining (4.159) and (4.160), we conclude that $x_i x_j (\partial/\partial x_0) \equiv 0$ for all i, j. This proves (4.150) for $k = 0$. The same argument works for $k = 1$. This finishes the proof of the lemma if we have (4.146) and one of $x_3, ..., x_{n+1}$ does not vanish in $H^0(\mathcal{O}_Z(1))$.

Otherwise, while we still have (4.146), $x_3 = ... = x_{n+1} = 0$ in $H^0(\mathcal{O}_Z(1))$. Then we have a system of linear equations:

$$\begin{aligned}
x_0 x_1 \frac{\partial}{\partial x_2} \equiv x_1 x_2 \frac{\partial}{\partial x_0} \equiv x_0 x_2 \frac{\partial}{\partial x_1} &\equiv 0 \\
(c_{013} x_0 x_1 + c_{023} x_0 x_2) \frac{\partial}{\partial x_0} &\equiv 0 \\
(c_{103} x_1 x_0 + c_{123} x_1 x_2) \frac{\partial}{\partial x_1} &\equiv 0 \\
(c_{203} x_2 x_0 + c_{213} x_2 x_1) \frac{\partial}{\partial x_2} &\equiv 0 \\
x_0^2 \frac{\partial}{\partial x_1} - (c_{011} x_0 x_1 + c_{021} x_0 x_2) \frac{\partial}{\partial x_0} &\equiv 0 \\
x_0^2 \frac{\partial}{\partial x_2} - (c_{012} x_0 x_1 + c_{022} x_0 x_2) \frac{\partial}{\partial x_0} &\equiv 0 \\
x_1^2 \frac{\partial}{\partial x_0} - (c_{100} x_1 x_0 + c_{120} x_1 x_2) \frac{\partial}{\partial x_1} &\equiv 0 \\
x_1^2 \frac{\partial}{\partial x_2} - (c_{102} x_1 x_0 + c_{122} x_1 x_2) \frac{\partial}{\partial x_1} &\equiv 0 \\
x_2^2 \frac{\partial}{\partial x_0} - (c_{200} x_2 x_0 + c_{210} x_2 x_1) \frac{\partial}{\partial x_2} &\equiv 0 \\
x_2^2 \frac{\partial}{\partial x_1} - (c_{201} x_2 x_0 + c_{211} x_2 x_1) \frac{\partial}{\partial x_2} &\equiv 0
\end{aligned} \tag{4.161}$$

Suppose that

$$a_0 x_0 + a_1 x_1 + a_2 x_2 = 0 \tag{4.162}$$

in $H^0(\mathcal{O}_Z(1))$ for some constants a_0, a_1, a_2, not all zero. By our hypothesis (4.146), $a_i \neq 0$ for $i = 0, 1, 2$.

Using (4.162), we can reduce (4.161) into a system of linear equations in $x_i^2(\partial/\partial x_j)$ for $0 \le i \ne j \le 2$. For example,

$$\begin{aligned} &\left.\begin{aligned} (c_{013}x_0x_1 + c_{023}x_0x_2)\frac{\partial}{\partial x_0} &\equiv 0 \\ x_1x_2\frac{\partial}{\partial x_0} &\equiv 0 \end{aligned}\right\} \Rightarrow \\ (a_1x_1 + a_2x_2)(c_{013}x_1 + c_{023}x_2)\frac{\partial}{\partial x_0} &\equiv a_1c_{013}x_1^2\frac{\partial}{\partial x_0} + a_2c_{023}x_2^2\frac{\partial}{\partial x_0} \\ &\equiv 0. \end{aligned} \tag{4.163}$$

In this way, we obtain a more manageable system of linear equations:

$$\begin{aligned} c_{013}\left(a_1x_1^2\frac{\partial}{\partial x_0}\right) + c_{023}\left(a_2x_2^2\frac{\partial}{\partial x_0}\right) &\equiv 0 \\ c_{103}\left(a_0x_0^2\frac{\partial}{\partial x_1}\right) + c_{123}\left(a_2x_2^2\frac{\partial}{\partial x_1}\right) &\equiv 0 \\ c_{203}\left(a_0x_0^2\frac{\partial}{\partial x_2}\right) + c_{213}\left(a_1x_1^2\frac{\partial}{\partial x_2}\right) &\equiv 0 \\ a_0x_0^2\frac{\partial}{\partial x_1} + c_{011}\left(a_1x_1^2\frac{\partial}{\partial x_0}\right) + c_{021}\left(a_2x_2^2\frac{\partial}{\partial x_0}\right) &\equiv 0 \\ a_0x_0^2\frac{\partial}{\partial x_2} + c_{012}\left(a_1x_1^2\frac{\partial}{\partial x_0}\right) + c_{022}\left(a_2x_2^2\frac{\partial}{\partial x_0}\right) &\equiv 0 \\ a_1x_1^2\frac{\partial}{\partial x_0} + c_{100}\left(a_0x_0^2\frac{\partial}{\partial x_1}\right) + c_{120}\left(a_2x_2^2\frac{\partial}{\partial x_1}\right) &\equiv 0 \\ a_1x_1^2\frac{\partial}{\partial x_2} + c_{102}\left(a_0x_0^2\frac{\partial}{\partial x_1}\right) + c_{122}\left(a_2x_2^2\frac{\partial}{\partial x_1}\right) &\equiv 0 \\ a_2x_2^2\frac{\partial}{\partial x_0} + c_{200}\left(a_0x_0^2\frac{\partial}{\partial x_2}\right) + c_{210}\left(a_1x_1^2\frac{\partial}{\partial x_2}\right) &\equiv 0 \\ a_2x_2^2\frac{\partial}{\partial x_1} + c_{201}\left(a_0x_0^2\frac{\partial}{\partial x_2}\right) + c_{211}\left(a_1x_1^2\frac{\partial}{\partial x_2}\right) &\equiv 0. \end{aligned} \tag{4.164}$$

We may consider (4.164) as a system of homogeneous linear equations in $a_ix_i^2(\partial/\partial x_j)$ for $0 \le i \ne j \le 2$. It is easy to show that (4.164) has only the trivial solution for c_{ijk} general. That is,

$$a_ix_i^2\frac{\partial}{\partial x_j} \equiv 0 \Rightarrow x_i^2\frac{\partial}{\partial x_j} \equiv 0 \text{ for all } i \ne j. \tag{4.165}$$

Together with (4.143), we see that (4.150) holds for all i, j, k. This finishes the proof of the lemma in the first case.

In the second case, we assume (4.148). Note that under this hypothesis, $\{x_0, x_2\}$ and $\{x_1, x_3\}$ are linearly dependent in $H^0(\mathcal{O}_Z(1))$, respectively. Then for all $k \neq 0, 1, 2, 3$,

$$\begin{aligned} x_0x_1\frac{\partial}{\partial x_k} &\equiv x_0x_2\frac{\partial}{\partial x_k} \equiv x_1x_3\frac{\partial}{\partial x_k} \equiv 0 \\ \Rightarrow x_0x_1\frac{\partial}{\partial x_k} &\equiv x_0^2\frac{\partial}{\partial x_k} \equiv x_1^2\frac{\partial}{\partial x_k} \equiv 0. \end{aligned} \tag{4.166}$$

And since $\{x_0^2, x_0x_1, x_1^2\}$ spans $H^0(\mathcal{O}_\Lambda(2))$, we see that (4.150) holds for all $k \geq 4$. It remains to prove (4.150) for $k = 0, 1, 2, 3$. We argue in a similar way to the first case.

Suppose that one of $x_4, ..., x_{n+1}$ does not vanish in $H^0(\mathcal{O}_Z(1))$. Without loss of generality, suppose that $x_4 \neq 0$ in $H^0(\mathcal{O}_Z(1))$. By (4.148), x_4 lies in either $\mathrm{Span}\{x_0\}$ or $\mathrm{Span}\{x_1\}$. Without loss of generality, we may assume that $x_4 \neq 0 \in \mathrm{Span}\{x_0\}$ in $H^0(\mathcal{O}_Z(1))$. Then

$$\begin{aligned} x_1x_4\frac{\partial}{\partial x_0} &\equiv x_2x_4\frac{\partial}{\partial x_0} \equiv x_1x_3\frac{\partial}{\partial x_0} \equiv 0 \\ \Rightarrow x_0x_1\frac{\partial}{\partial x_0} &\equiv x_0^2\frac{\partial}{\partial x_0} \equiv x_1^2\frac{\partial}{\partial x_0} \equiv 0 \text{ and} \\ x_0x_1\frac{\partial}{\partial x_2} &\equiv x_0x_4\frac{\partial}{\partial x_2} \equiv x_1x_3\frac{\partial}{\partial x_2} \equiv 0 \\ \Rightarrow x_0x_1\frac{\partial}{\partial x_2} &\equiv x_0^2\frac{\partial}{\partial x_2} \equiv x_1^2\frac{\partial}{\partial x_2} \equiv 0. \end{aligned} \tag{4.167}$$

So (4.150) holds for $k = 0, 2$ and hence for all $k \neq 1, 3$.

Let us prove (4.150) for $k = 1$. If $x_k \neq 0 \in \mathrm{Span}\{x_1\}$ in $H^0(\mathcal{O}_Z(1))$ for some $k \geq 5$, then we have (4.150) for $k = 1, 3$ by the same argument as above. Otherwise, $x_k \in \mathrm{Span}\{x_0\}$ for all $k \neq 1, 3$. Then

$$x_0x_2\frac{\partial}{\partial x_1} \equiv x_0x_3\frac{\partial}{\partial x_1} \equiv 0 \Rightarrow x_0^2\frac{\partial}{\partial x_1} \equiv x_0x_1\frac{\partial}{\partial x_1} \equiv 0 \tag{4.168}$$

and

$$x_1^2\frac{\partial}{\partial x_0} - \sum_{j\neq 1} c_{1j0}x_1x_j\frac{\partial}{\partial x_1} \equiv 0 \Rightarrow c_{130}x_1x_3\frac{\partial}{\partial x_1} \equiv 0. \tag{4.169}$$

As long as $c_{130} \neq 0$, we have

$$x_1x_3\frac{\partial}{\partial x_1} \equiv 0 \Rightarrow x_1^2\frac{\partial}{\partial x_1} \equiv 0 \tag{4.170}$$

which, together with (4.168), implies (4.150) for $k = 1$. The same argument works for $k = 3$. This proves the lemma if we have (4.148) and one of $x_4, ..., x_{n+1}$ does not vanish in $H^0(\mathcal{O}_Z(1))$.

The only remaining case is that we have (4.148) and $x_4 = \ldots = x_{n+1} = 0$ in $H^0(\mathcal{O}_Z(1))$. In this case, we have

$$\begin{aligned} x_1x_2\frac{\partial}{\partial x_0} &\equiv x_1x_3\frac{\partial}{\partial x_0} \equiv 0 \Rightarrow x_0x_1\frac{\partial}{\partial x_0} \equiv x_1^2\frac{\partial}{\partial x_0} \equiv 0 \\ x_0x_1\frac{\partial}{\partial x_2} &\equiv x_1x_3\frac{\partial}{\partial x_2} \equiv 0 \Rightarrow x_0x_1\frac{\partial}{\partial x_2} \equiv x_1^2\frac{\partial}{\partial x_2} \equiv 0 \end{aligned} \tag{4.171}$$

and

$$\begin{aligned} x_0^2\frac{\partial}{\partial x_2} - \sum_{j\neq 0} c_{0j2}x_0x_j\frac{\partial}{\partial x_0} &\equiv 0 \Rightarrow x_0^2\frac{\partial}{\partial x_2} - c_{022}x_0x_2\frac{\partial}{\partial x_0} \equiv 0 \\ x_2^2\frac{\partial}{\partial x_0} - \sum_{j\neq 2} c_{2j0}x_2x_j\frac{\partial}{\partial x_2} &\equiv 0 \Rightarrow x_2^2\frac{\partial}{\partial x_0} - c_{200}x_0x_2\frac{\partial}{\partial x_2} \equiv 0. \end{aligned} \tag{4.172}$$

Suppose that $x_2 = ax_0$ in $H^0(\mathcal{O}_Z(1))$ for some $a \neq 0$. Then (4.172) becomes

$$\begin{aligned} -ac_{022}\left(x_0^2\frac{\partial}{\partial x_0}\right) + x_0^2\frac{\partial}{\partial x_2} &\equiv 0 \\ a^2\left(x_0^2\frac{\partial}{\partial x_0}\right) - ac_{200}\left(x_0^2\frac{\partial}{\partial x_2}\right) &\equiv 0. \end{aligned} \tag{4.173}$$

For $c_{022}c_{200} \neq 1$, (4.173) has only the trivial solution as a system of homogeneous linear equations in $x_0^2(\partial/\partial x_k)$ for $k = 0, 2$. That is,

$$x_0^2\frac{\partial}{\partial x_0} \equiv x_0^2\frac{\partial}{\partial x_2} \equiv 0 \tag{4.174}$$

which, combined with (4.171), implies (4.150) for $k = 0, 2$. Similarly, we can prove (4.150) for $k = 1, 3$. This finishes the proof of the lemma. □

Appendix A. Simple facts on Γ-equivalence and multi-tangent lines to hypersurfaces

In this appendix, we prove a few simple facts. These should be more or less well known to the experts. We provide the proofs for readers' convenience.

First, we claim that Γ-equivalence is indeed an equivalence on 0-cycles. It suffices to prove that it is symmetric: Fixing a smooth projective curve Γ and two points $p \neq q$ on Γ, two 0-cycles ξ_1 and ξ_2 on a projective variety X are equivalent under (Γ, p, q) if and only if they are equivalent under (Γ, q, p).

As another interpretation of Γ-equivalence, ξ_1 and ξ_2 are equivalent under (Γ, p, q) if and only if there exists a morphism $f : \Gamma \to S^N X$ for N sufficiently large such that $f(p) = \xi_1 + \eta$ and $f(q) = \xi_2 + \eta$ for some effective zero cycle η, where $S^N X$ is the N-th symmetric product of X.

Let us first show that there exists a morphism $\phi : \Gamma \to S^d\Gamma$ for d sufficiently large such that $\phi(p) = q + D$ and $\phi(q) = p + D$ for some effective divisor D on Γ. There exists a natural map $\pi : S^d\Gamma \to \mathrm{Pic}^d(\Gamma) \cong J(\Gamma)$ sending $(p_1, p_2, ..., p_d)$ to $p_1 + p_2 + ... + p_d$. For $d \geq 2g - 1$, this is an $\mathbb{P}^{d-g}$-bundle whose fibers can be identified with the complete linear series $|L|$ for $L \in \mathrm{Pic}^d(\Gamma)$, where $g = g(\Gamma)$ is the genus of Γ.

Fixing a divisor $F \in \mathrm{Pic}^{d+1}(\Gamma)$ for $d \geq 2g$, we can embed Γ to $\mathrm{Pic}^d(\Gamma)$ by $\lambda : \Gamma \to \mathrm{Pic}^d(\Gamma)$ sending $\lambda(s) = F - s$ for all $s \in \Gamma$. Let $Z = \pi^{-1}(\lambda(\Gamma))$ be the fiber product of π and λ. Then Z is a $\mathbb{P}^{d-g}$-bundle over Γ, whose fiber over $s \in \Gamma$ is the linear series $|F - s|$. Since $\deg(F - p - q) \geq 2g - 1$, $F - p - q$ is effective and we let $D \in |F - p - q|$. Then $(p, q + D)$ and $(q, p + D)$ are two points on two fibers of Z over Γ. So there exists a section $\phi : \Gamma \to Z \subset S^d\Gamma$ passing through these two points. That is, $\phi(p) = q + D$ and $\phi(q) = p + D$.

Now we combine $f : \Gamma \to S^N X$ and $\phi : \Gamma \to S^d\Gamma$ to obtain a morphism $h : \Gamma \to S^d(S^N X) \to S^{dN} X$ sending

$$h(p) = f(q + D) = \xi_2 + f(D) \text{ and } h(q) = \xi_1 + f(D).$$

It follows that ξ_1 and ξ_2 are also equivalent under (Γ, q, p).

Next, let us explain how the sequence (4.142) governs the deformation of a line with tangency conditions with a fixed hypersurface.

More generally, let us consider the deformation of a nonconstant morphism $f : C \to P$ from a smooth projective curve C to a smooth projective variety P. Suppose that f deforms over a smooth variety B. That is, there exists a smooth family of curves W over B and a morphism $\phi : W \to P \times B$ preserving the base B such that $\phi_0 : W_0 \to P$ is exactly $f : C \to P$ when ϕ is restricted to a point $0 \in B$.

The Kodaira-Spencer map $T_{B,0} \to H^0(\mathcal{N}_f)$ associated to ϕ is given by

$$\begin{array}{ccccccccc} & & & & \phi^*\pi_B^* T_B & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & T_W & \longrightarrow & \phi^* T_{P\times B} & \longrightarrow & \mathcal{N}_\phi & \longrightarrow & 0 \end{array} \tag{A.1}$$

where $\mathcal{N}_\phi$ and $\mathcal{N}_f$ are normal bundles of the maps ϕ and f, respectively, and π_B is the projection $P \times B \to B$. Here we restrict the map $\phi^*\pi_B^* T_B \to \mathcal{N}_\phi$ to W_0 and take the global sections to obtain the map $T_{B,0} \to H^0(\mathcal{N}_f)$.

Now let us impose some tangency conditions on $f(C)$ with a fixed hypersurface $X \subset P$. For simplicity, let us assume that X is a divisor of simple normal crossings. Suppose that

$$f^* X = m_1 p_1 + m_2 p_2 + ... + m_r p_r$$

for some distinct points $p_1, p_2, ..., p_r \in C$. Let us assume that ϕ preserves the multiplicities of p_i but not necessarily the points $f(p_i)$ themselves. That is,

$$\phi^*(X \times B) = m_1 D_1 + m_2 D_2 + ... + m_r D_r$$

where D_i are disjoint sections of W/B with $D_i \cap W_0 = p_i$.

Then the map $\phi^* \pi_B^* T_B \to \mathcal{N}_\phi$ in (A.1) factors through $\mathcal{N}_{\phi,X}$ via the diagram

$$\begin{array}{ccccccccc} & & & & \phi^*\pi_B^*T_B & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & T_W(-\log D) & \longrightarrow & \phi^*T_{P\times B}(-\log(X\times B)) & \longrightarrow & \mathcal{N}_{\phi,X} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_W & \longrightarrow & \phi^*T_{P\times B} & \longrightarrow & \mathcal{N}_\phi & \longrightarrow & 0 \end{array} \tag{A.2}$$

where $D = \sum D_i$. Therefore, the Kodaira-Spencer map $T_{B,0} \to H^0(\mathcal{N}_f)$ factors through $H^0(\mathcal{N}_{f,X})$, where $\mathcal{N}_{f,X}$ is given by the exact sequence

$$0 \longrightarrow T_C(-\textstyle\sum p_i) \longrightarrow f^*T_P(-\log X) \longrightarrow \mathcal{N}_{f,X} \longrightarrow 0 \tag{A.3}$$

So the versal deformation space of $f : C \to P$ preserving the tangency conditions at $f^{-1}(X)$ with a fixed hypersurface X has dimension no more than

$$\begin{aligned} h^0(\mathcal{N}_{f,X}) &\le h^0(f^*T_P(-\log X)) - \chi(T_C(-\sum p_i)) \\ &= h^0(f^*T_P(-\log X)) + (3g(C) - 3 + r). \end{aligned} \tag{A.4}$$

Note the above inequality holds for *any* $g = g(C)$. Now let us apply (A.4) to the deformation of lines $\Lambda \hookrightarrow P$ in $P = \mathbb{P}^{n+1}$ preserving tangency conditions with a fixed hypersurface X of degree d. Our purpose is to construct pairs (X, Λ) such that the deformation Λ has the expected dimension 0 given by (A.4). That is, these multi-tangent lines to X are rigid.

Obviously, we need to compute $H^0(\Lambda, T_P(-\log X))$. This can be done via the restriction of the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_P(-\log X) & \longrightarrow & T_P & \longrightarrow & \mathcal{N}_{X/P} & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathcal{O}_X(d) & & \end{array}$$

to Λ. Since $\Lambda \not\subset X$, the above exact sequence remains exact when restricted to Λ. So $H^0(\Lambda, T_P(-\log X))$ is the kernel of the map

$$H^0(\Lambda, T_P) \xrightarrow{\eta} H^0(\Lambda, \mathcal{O}_X(d)) \tag{A.5}$$

Note the right hand side of the map is equivalent to $H^0(\Gamma, \mathcal{O}_\Gamma(d))/\langle F|_\Gamma\rangle$. This map has been used throughout the paper. Once again, it is given by

$$\eta\left(\sum L_i \frac{\partial}{\partial z_i}\right) = \sum L_i \frac{\partial F}{\partial z_i}$$

with $L_i \in H^0(\mathcal{O}_P(1))$ and $\partial F/\partial z_i$ restricted to Λ, where $(z_0, z_1, ..., z_{n+1})$ are the homogeneous coordinates of $P = \mathbb{P}^{n+1}$ and F is the defining equation of X.

In our first example, we will construct a smooth hypersurface X of degree $d = 2n + 1$ and two lines Λ_i for $i = 1, 2$ such that each Λ_i meets X at a unique point p_i with $p_1 \neq p_2$ and has rigid deformation preserving the tangency. We let

$$\begin{aligned}
&\Lambda_1 = \{z_2 = z_3 = z_4 = ... = z_{n+1} = 0\},\ p_1 = (1, -1, 0, ..., 0)\\
&\Lambda_2 = \{z_1 = z_3 = z_4 = ... = z_{n+1} = 0\},\ p_2 = (1, 0, -1, 0, ..., 0)\\
&F = (z_0 + z_1 + z_2)^d + z_1 z_2 G_0(z_0, z_1, z_2) + \sum_{j=3}^{n+1} z_j G_j(z_0, z_1, z_2)\\
&\qquad + G(z_3, z_4, ..., z_{n+1})
\end{aligned}$$

where $G_0(z_0, z_1, z_2), G_j(z_0, z_1, z_2)$ and $G(z_3, z_4, ..., z_{n+1})$ are homogeneous polynomials in (z_0, z_1, z_2) and $(z_3, z_4, ..., z_{n+1})$ of degree $d-2, d-1$ and d, respectively. For a general choice of (G_0, G_j, G), $X = \{F = 0\}$ is smooth by Bertini.

For each $\Lambda = \Lambda_i$ and a general choice of $(G_0, G_3, ..., G_{n+1})$, it is easy to check that the space

$$\begin{aligned}
&H^0(\mathcal{O}_\Lambda(1)) \otimes \operatorname{Span}\left\{\frac{\partial F}{\partial z_0}, \frac{\partial F}{\partial z_1}, ..., \frac{\partial F}{\partial z_{n+1}}\right\}\Big|_\Lambda\\
&= H^0(\mathcal{O}_\Lambda(1)) \otimes \operatorname{Span}\big\{(z_0 + z_1 + z_2)^{d-1}, z_i G_0(z_0, z_1, z_2),\\
&\qquad\qquad G_3(z_0, z_1, z_2), ..., G_{n+1}(z_0, z_1, z_2)\big\}\Big|_\Lambda
\end{aligned}$$

surjects onto $H^0(\mathcal{O}_\Lambda(d))$. Therefore, the map η in (A.5) is surjective for each $\Lambda = \Lambda_i$. Then it is easy to compute

$$h^0(f^* T_P(-\log X)) = h^0(\Lambda, T_P) - h^0(\Lambda, \mathcal{O}_X(d)) = (2n + 3) - d = 2$$

and hence each $\Lambda = \Lambda_i$ is rigid by (A.4). This construction shows that there exist a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $2n + 1$ and two lines Λ_1 and Λ_2, each tangent to X at a point p_i with multiplicity $2n + 1$ such that $p_1 \neq p_2$ and neither Λ_i can deform when preserving the tangency condition with X. Combining this with an incidence correspondence argument as in the proof of Proposition 4.11, we can prove that for a general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $2n + 1$, there are finitely many lines in $\mathbb{P}^{n+1}$ meeting X at a unique point and there exist (at least) two lines Λ_i for $i = 1, 2$, each meeting X at a unique point p_i with $p_1 \neq p_2$. This justifies our claim that the bound $2n + 2$ in Theorem 1.5 is optimal.

In our second example, we will construct $(X, \Lambda_i, p_{i1}, p_{i2})$ as claimed in the proof of Proposition 4.11, where

- X is a smooth hypersurface in $P = \mathbb{P}^{n+1}$ of degree $d = 2n + 2$,
- Λ_1 and Λ_2 are two lines, each meeting X at two points p_{i1} and p_{i2} with assigned multiplicities m and $d - m$,
- neither Λ_i deforms when preserving the tangency conditions with X,
- $p_{11} \neq p_{21}$ and the line $\overline{p_{11}p_{21}}$ meets X at more than two points.

As in the first example, we let

$$\begin{aligned}
\Lambda_1 &= \{z_2 = z_3 = z_4 = ... = z_{n+1} = 0\}, \\
\Lambda_2 &= \{z_1 = z_3 = z_4 = ... = z_{n+1} = 0\}, \\
p_{11} &= (1, -1, 0, ..., 0),\ p_{12} = (1, 1, 0, ..., 0) \\
p_{21} &= (1, 0, -1, 0, ..., 0),\ p_{22} = (1, 0, 1, 0, ..., 0) \\
F &= (z_0 + z_1 + z_2)^m (z_0 - z_1 - z_2)^{d-m} + z_1 z_2 G_0(z_0, z_1, z_2) \\
&\quad + \sum_{j=3}^{n+1} z_j G_j(z_0, z_1, z_2) + G(z_3, z_4, ..., z_{n+1})
\end{aligned}$$

where $G_0(z_0, z_1, z_2), G_j(z_0, z_1, z_2)$ and $G(z_3, z_4, ..., z_{n+1})$ are homogeneous polynomials in (z_0, z_1, z_2) and $(z_3, z_4, ..., z_{n+1})$ of degree $d-2, d-1$ and d, respectively. For a general choice of (G_0, G_j, G), $X = \{F = 0\}$ is smooth by Bertini.

For G_0 general, the line $\overline{p_{11}p_{21}}$ clearly meets X at more than two points.

As in the first example, it is easy to check that

$$H^0(\mathcal{O}_\Lambda(1)) \otimes \operatorname{Span} \left\{ \frac{\partial F}{\partial z_0}, \frac{\partial F}{\partial z_1}, ..., \frac{\partial F}{\partial z_{n+1}} \right\} \Bigg|_\Lambda$$

surjects onto $H^0(\mathcal{O}_\Lambda(d))$ for each $\Lambda = \Lambda_i$ and a general choice of (G_0, G_j). Thus,

$$h^0(f^* T_P(-\log X)) = h^0(\Lambda, T_P) - h^0(\Lambda, \mathcal{O}_X(d)) = (2n + 3) - d = 1$$

and hence each $\Lambda = \Lambda_i$ is rigid by (A.4). This proves the claim we made in the proof of Proposition 4.11.

Appendix B. Notes on algebraic invariants

Here we give another construction of the invariants defined in §2 based on Hodge theory. These algebraic invariants from Hodge theory, some of which are used in [19], are the same thing as de Rham invariants, the latter not involving Hodge theory. First let us fix some notations. For a $\mathbb{Q}$-MHS V, we put $\Gamma(V) := \hom_{\mathrm{MHS}}(\mathbb{Q}(0), V)$ and accordingly $J(V) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), V)$.

To arrive at the invariants of interest, we must introduce a natural filtration on the Chow groups of X. Let $\rho : \mathcal{X} \to S$ be a smooth and proper morphism of smooth quasi-projective varieties over a subfield k of $\mathbb{C}$ finitely generated over $\overline{\mathbb{Q}}$, and let $K = k(S)$. Fix an embedding $K \hookrightarrow \mathbb{C}$ over k, and put $X := X/\mathbb{C} = \mathcal{X}_{\eta_S} \times_K \mathbb{C}$.

Theorem B.1 *([9]). Let $X := X/\mathbb{C}$ be smooth projective of dimension d. Then for all $r \geq 0$, there is a filtration, depending on $k \subset \mathbb{C}$,*

$$\mathrm{CH}^r(X;\mathbb{Q}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^\nu \supseteq F^{\nu+1} \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = F^{r+2} = \cdots,$$

which satisfies the following

(i) $F^1 = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$.
(ii) $F^2 \subseteq \ker AJ \otimes \mathbb{Q} : \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J\big(H^{2r-1}(X(\mathbb{C}),\mathbb{Q}(r))\big)$.
(iii) $F^{\nu_1}\mathrm{CH}^{r_1}(X;\mathbb{Q}) \bullet F^{\nu_2}\mathrm{CH}^{r_2}(X;\mathbb{Q}) \subset F^{\nu_1+\nu_2}\mathrm{CH}^{r_1+r_2}(X;\mathbb{Q})$, *where $\bullet$ is the intersection product.*
(iv) F^ν *is preserved under the action of correspondences between smooth projective varieties over $\mathbb{C}$.*
(v) *Let $\mathrm{Gr}^\nu_F := F^\nu/F^{\nu+1}$ and assume that the Künneth components of the diagonal class $[\Delta_X] = \oplus_{p+q=2d}[\Delta_X(p,q)] \in H^{2d}(X \times X, \mathbb{Q}(d))$ are algebraic over $\mathbb{Q}$. Then*

$$\Delta_X(2d-2r+\ell, 2r-\ell)_*\big|_{\mathrm{Gr}^\nu_F \mathrm{CH}^r(X,m;\mathbb{Q})} = \delta_{\ell,\nu} \cdot \mathrm{Identity}.$$

[If we assume the conjecture that homological and numerical equivalence coincide, then (v) says that Gr^ν_F factors through the Grothendieck motive.]
(vi) *Let $D^r(X) := \bigcap_\nu F^\nu$, and $k = \overline{\mathbb{Q}}$. If the Bloch-Beilinson conjecture on the injectivity of the Abel-Jacobi map ($\otimes\mathbb{Q}$) holds for smooth quasi-projective varieties defined over $\overline{\mathbb{Q}}$, then $D^r(X) = 0$.*

It is instructive to briefly explain how this filtration comes about. Consider a k-spread $\rho : \mathcal{X} \to S$, where ρ is smooth and proper. Let η be the generic point of S/k, and put $K := k(\eta)$. Write $X_K := \mathcal{X}_\eta$. From [9] we introduced a decreasing filtration $\mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$, with the property that $Gr^\nu_{\mathcal{F}}\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) \hookrightarrow E^{\nu,2r-\nu}_\infty(\rho)$, where $E^{\nu,2r-\nu}_\infty(\rho)$ is the ν-th graded piece of the Leray filtration on the lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r))$ of Beilinson's absolute Hodge cohomology $H^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r))$ associated to ρ. That lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r)) \subset H^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r))$ is given by the image $H^{2r}_{\mathcal{H}}(\overline{\mathcal{X}}, \mathbb{Q}(r)) \to H^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r))$, where $\overline{\mathcal{X}}$ is a smooth compactification of $\mathcal{X}$. There is a cycle class map $\mathrm{CH}^r(\mathcal{X}; \mathbb{Q}) := \mathrm{CH}^r(\mathcal{X}/k; \mathbb{Q}) \to \underline{H}^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r))$, which is conjecturally injective if $k = \overline{\mathbb{Q}}$ under the Bloch-Beilinson conjecture assumption, using the fact that there is a short exact sequence:

$$0 \to J(H^{2r-1}(\mathcal{X}, \mathbb{Q}(r))) \to H^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r)) \to \Gamma(H^{2r}(\mathcal{X}, \mathbb{Q}(r))) \to 0.$$

(Injectivity would imply $D^r(X) = 0$.) Regardless of whether or not injectivity holds, the filtration $\mathcal{F}^\nu \mathrm{CH}^r(\mathcal{X}; \mathbb{Q})$ is given by the pullback of the Leray filtration on $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r))$ to $\mathrm{CH}^r(\mathcal{X}; \mathbb{Q})$. It is proven in [9] that the term $E^{\nu,2r-\nu}_\infty(\rho)$ fits in a short exact sequence:

$$0 \to \underline{E}^{\nu,2r-\nu}_\infty(\rho) \to E^{\nu,2r-\nu}_\infty(\rho) \to \underline{\underline{E}}^{\nu,2r-\nu}_\infty(\rho) \to 0,$$

where

$$\underline{\underline{E}}^{\nu,2r-\nu}_\infty(\rho) = \Gamma(H^\nu(S, R^{2r-\nu}\rho_*\mathbb{Q}(r))),$$

$$\underline{E}^{\nu,2r-\nu}_\infty(\rho) = \frac{J(W_{-1}H^{\nu-1}(S, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{\Gamma(Gr^0_W H^{\nu-1}(S, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}$$

$$\subset J(H^{\nu-1}(S, R^{2r-\nu}\rho_*\mathbb{Q}(r))).$$

[Here the latter inclusion is a result of the short exact sequence:

$$0 \to W_{-1}H^{\nu-1}(S, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to W_0 H^{\nu-1}(S, R^{2r-\nu}\rho_*\mathbb{Q}(r))$$
$$\to Gr^0_W H^{\nu-1}(S, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to 0.]$$

One then has (by definition)

$$F^\nu \mathrm{CH}^r(X_K; \mathbb{Q}) = \lim_{\substack{\to \\ U \subset S/k}} \mathcal{F}^\nu \mathrm{CH}^r(\mathcal{X}_U; \mathbb{Q}), \quad \mathcal{X}_U := \rho^{-1}(U)$$

$$F^\nu \mathrm{CH}^r(X_{\mathbb{C}}; \mathbb{Q}) = \lim_{\substack{\to \\ K \subset \mathbb{C}}} F^\nu \mathrm{CH}^r(X_K; \mathbb{Q})$$

Further, since direct limits preserve exactness,

$$Gr^\nu_F \mathrm{CH}^r(X_K; \mathbb{Q}) = \lim_{\substack{\to \\ U \subset S/k}} Gr^\nu_{\mathcal{F}} \mathrm{CH}^r(\mathcal{X}_U; \mathbb{Q}),$$

$$Gr^\nu_F \mathrm{CH}^r(X_{\mathbb{C}}; \mathbb{Q}) = \lim_{\substack{\to \\ K \subset \mathbb{C}}} Gr^\nu_F \mathrm{CH}^r(X_K; \mathbb{Q})$$

B.1. (Generalized) normal functions

Let us now assume that with regard to the smooth and proper map $\rho : \mathcal{X} \to S$ over a subfield $k \subset \mathbb{C}$, finitely generated over $\overline{\mathbb{Q}}$, and after possibly shrinking S, that S is affine, with $K = k(S)$. Let $V \subset S_{\mathbb{C}}$ be smooth, irreducible, closed subvariety of dimension $\nu - 1$ (note that S affine $\Rightarrow V$ affine). One has a commutative square

$$\begin{array}{ccc} \mathcal{X}_V & \hookrightarrow & \mathcal{X}_{\mathbb{C}} \\ \rho_V \downarrow & & \downarrow \rho \\ V & \hookrightarrow & S_{\mathbb{C}}, \end{array}$$

and a commutative diagram

$$\begin{array}{ccccccccc} & & & \xi \in & Gr_{\mathcal{F}}^{\nu} \operatorname{CH}^r(\mathcal{X};\mathbb{Q}) & \mapsto & Gr_F^{\nu} \operatorname{CH}^r(X_K;\mathbb{Q}) & & \\ & & & & \downarrow & & & & \\ 0 & \to & \underline{E}_{\infty}^{\nu,2r-\nu}(\rho) & \to & E_{\infty}^{\nu,2r-\nu}(\rho) & \to & \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \underline{E}_{\infty}^{\nu,2r-\nu}(\rho_V) & \to & E_{\infty}^{\nu,2r-\nu}(\rho_V) & \to & \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho_V) & \to & 0 \\ & & & & & & || & & \\ & & & & & & 0 & & \end{array}$$

where $\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho_V) = 0$ follows from the weak Lefschetz theorem for locally constant systems over affine varieties. Thus for any $\xi \in Gr_{\mathcal{F}}^{\nu} \operatorname{CH}^r(\mathcal{X};\mathbb{Q})$, we have a "normal function" η_ξ with the property that for any such smooth irreducible closed $V \subset S_{\mathbb{C}}$ of dimension $\nu - 1$, we have a value $\eta_\xi(V) \in \underline{E}_{\infty}^{\nu,2r-\nu}(\rho_V)$. Here we think of V as a point on a suitable open subset of the Chow variety of dimension $\nu - 1$ subvarieties of $S_{\mathbb{C}}$ and η_ξ defined on that subset. For example if $\nu = 1$, then we recover the classical notion of normal functions.

Definition B.2. η_ξ is called an arithmetic normal function.

Example B.3. Suppose S is affine of dimension $\nu - 1$. Then in this case $V = S$, and $\xi \in Gr_{\mathcal{F}}^{\nu} \operatorname{CH}^r(\mathcal{X};\mathbb{Q})$ induces a "single point" normal function

$$\eta_\xi(V) = \eta_\xi(S) \in J(H^{\nu-1}(S, R^{2r-\nu}\rho_*\mathbb{Q}(r))).$$

Now let $\xi \in \mathcal{F}^{\nu} \operatorname{CH}^r(\mathcal{X};\mathbb{Q})$ be given, and let $[\xi] \in \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho)$ be its image via the composite

$$\mathcal{F}^{\nu}\,\mathrm{CH}^{r}(\mathcal{X};\mathbb{Q}) \to E_{\infty}^{\nu,2r-\nu}(\rho) \to \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho).$$

B.2. The invariants

Theorem B.4 *(see [7]). The class $[\xi]$ depends only on η_ξ, and is called the topological invariant of η_ξ.*

Let us assume that S is affine. Then

$$\mathcal{O}_S \otimes_{\mathbb{C}} R^i\rho_*\mathbb{C} \xrightarrow{\nabla} \Omega_S^1 \otimes R^i\rho_*\mathbb{C} \xrightarrow{\nabla} \cdots,$$

is an acyclic resolution of $R^{2r-\nu}\rho_*\mathbb{C}$ in the analytic topology, where $\nabla := \partial \otimes Id$ is the Gauss-Manin connection. The corresponding cohomology $H^{\nu}(S, R^{2r-\nu}\rho_*\mathbb{C})$ is given by $H^0(S,-)$ of the middle cohomology in:

$$\Omega_S^{\nu-1} \otimes R^{2r-\nu}\rho_*\mathbb{C} \xrightarrow{\nabla} \Omega_S^{\nu} \otimes R^{2r-\nu}\rho_*\mathbb{C} \xrightarrow{\nabla} \Omega_S^{\nu+1} \otimes R^{2r-\nu}\rho_*\mathbb{C},$$

which is by definition the space of de Rham invariants, and is denoted by $\nabla DR^{r,\nu}(\mathcal{X}/S)$. As the map $\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho) \hookrightarrow \nabla DR^{r,\nu}(\mathcal{X}/S)$, together with the regularity of ∇, it follows that the de Rham invariant of an algebraic cycle is the same as the topological invariant. It turns out that $H^i(S, R^j\rho_*\mathbb{Q}(r))$ defines a $\mathbb{Q}$-MHS [1], hence its complexification carries a descending Hodge filtration $F^{\bullet}H^i(S, R^j\rho_*\mathbb{C})$. In particular,

$$\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho) \hookrightarrow F^r H^{\nu}(S, R^{2r-\nu}\rho_*\mathbb{C}),$$

where the latter term maps to $H^0(S,-)$ of the middle cohomology in:

$$\Omega_S^{\nu-1} \otimes F^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{C} \xrightarrow{\nabla} \Omega_S^{\nu} \otimes F^{r-\nu}R^{2r-\nu}\rho_*\mathbb{C} \xrightarrow{\nabla} \Omega_S^{\nu+1} \otimes F^{r-\nu-1}R^{2r-\nu}\rho_*\mathbb{C}, \tag{B.1}$$

which is called the space of Mumford-Griffiths invariants, and is denoted by $\nabla J^{r,\nu}(\mathcal{X}/S)$. Note that there is a natural "forgetful" map $\nabla J^{r,\nu}(\mathcal{X}/S) \to \nabla DR^{r,\nu}(\mathcal{X}/S)$, which need not be injective. Having said this, it is clear from the above discussion that

$$\mathrm{Im}\left(\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho) \to \nabla J^{r,\nu}(\mathcal{X}/S)\right) \to \mathrm{Im}\left(\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho) \to \nabla DR^{r,\nu}(\mathcal{X}/S)\right),$$

is an isomorphism. Thus when it comes to the image of algebraic cycles, the de Rham and Mumford-Griffiths invariants coincide! (All of this is based on [12] and [17].) Those cycles that have trivial Mumford-Griffiths invariant must therefore land in $\underline{E}_{\infty}^{\nu,2r-\nu}(\rho)$. In some instances, this can be an uncountable space. Note that

$$\Omega_S^{\nu-1} \otimes F^{r-\nu+2}R^{2r-\nu}\rho_*\mathbb{C} \xrightarrow{\nabla} \Omega_S^{\nu} \otimes F^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{C}$$

$$\xrightarrow{\nabla} \Omega_S^{\nu+1} \otimes F^{r-\nu} R^{2r-\nu} \rho_* \mathbb{C},$$

is a subcomplex of (B.1). The Mumford invariants are $H^0(S, -)$ of the middle cohomology of the cokernel complex:

$$\Omega_S^{\nu-1} \otimes \mathcal{H}^{r-\nu+1,r-1}(\mathcal{X}/S) \xrightarrow{\tilde{\nabla}} \Omega_S^{\nu} \otimes \mathcal{H}^{r-\nu,r}(\mathcal{X}/S)$$
$$\xrightarrow{\tilde{\nabla}} \Omega_S^{\nu+1} \otimes \mathcal{H}^{r-\nu-1,r+1}(\mathcal{X}/S),$$

and where $\tilde{\nabla}$ is induced from ∇.

Example B.5. Let us put $N := \dim S$ and n the relative dimension of ρ, with $r = n$. In this case we are studying the relative 0-cycles on each fiber of ρ. This involves $\mathcal{F}^n \operatorname{CH}^n(\mathcal{X}; \mathbb{Q})$, where we set $\nu = n$. Then

$$H^0\left(S, \frac{\Omega_S^n \otimes_{\mathcal{O}_S} \mathcal{H}^{0,n}(\mathcal{X}/S)}{\tilde{\nabla}\big(\Omega_S^{n-1} \otimes_{\mathcal{O}_S} \mathcal{H}^{1,n-1}(\mathcal{X}/S)\big)}\right)$$

is the associated space of Mumford invariants. If $n = 2$, it also appears in [19]. Note that in this case, we need $\xi \in \operatorname{CH}^2(\mathcal{X}; \mathbb{Q})$ to be Abel-Jacobi equivalent to zero fiberwise, in order that $\xi \in \mathcal{F}^2 \operatorname{CH}^2(\mathcal{X}; \mathbb{Q})$.

Question B.6.

(i) Can one characterize this filtration in terms of arithmetic normal functions?

(ii) By choosing V sufficiently general, can one characterize this filtration in terms of the corresponding Abel-Jacobi map for a fixed general variety? E.g. we know that $F^1 \operatorname{CH}^r(X; \mathbb{Q}) = \operatorname{CH}^r_{\text{hom}}(X; \mathbb{Q})$ and

$$F^2 \operatorname{CH}^r(X; \mathbb{Q}) \subseteq \operatorname{CH}^r_{AJ}(X; \mathbb{Q}) := \ker AJ_X : \operatorname{CH}^r_{\text{hom}}(X; \mathbb{Q}) \to J^r(X)_{\mathbb{Q}}.$$

Is it the case that $F^2 \operatorname{CH}^r(X; \mathbb{Q}) = \operatorname{CH}^r_{AJ}(X; \mathbb{Q})$?

(ii)′ What about the zero (or torsion) locus of such normal functions. I.e., are they sensitive to the field of definition of algebraic cycles?

Remark B.7.

- $\bullet_1$ Special cases of Question B.6(i) are worked out in [7]. Further, if both X and S are defined over k, with $\mathcal{X} = S \times_k X$, with $\rho = \Pr_1$, then the answer is yes, as shown in [11].
- $\bullet_2$ In the case where $\nu = 1$, (ii) and (ii)′ can be shown to be equivalent. (See for example [10].)

B.3. Example B.5 revisited

Let us put $N := \dim S$ and n the relative dimension of ρ.

Question B.8. Does there exist a morphism of sheaves

$$\frac{\Omega_S^n \otimes_{\mathcal{O}_S} \mathcal{H}^{0,n}}{\tilde{\nabla}\big(\Omega_S^{n-1} \otimes_{\mathcal{O}_S} \mathcal{H}^{1,n-1}\big)} \to \mathcal{H}om_{\mathcal{O}_S}\big(\rho_*(\wedge^N \Omega_{\mathcal{X}}), \omega_S\big),$$

induced by

$$(a \otimes b, \rho_*(c)) \in \big(\Omega_S^n \otimes \mathcal{H}^{0,n}, \rho_*(\wedge^N \Omega_{\mathcal{X}})\big) \mapsto a \wedge \rho_*(\rho^*(b), c) = a \wedge \int_{\mathcal{X}_t} \rho^*(b) \wedge c \in \omega_{S,t},$$

where ω_S is the canonical sheaf on S?

Remark B.9. The answer is yes if $\mathcal{X} = S \times_k X$, for in this case

$$\tilde{\nabla}\big(\Omega_S^{n-1} \otimes_{\mathcal{O}_S} \mathcal{H}^{1,n-1}\big) = 0.$$

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