SOME EXAMPLES OF DR-INDECOMPOSABLE CLOSED FIBERS OF SEMI-STABLE REDUCTIONS OVER WITT RINGS

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1. Introduction

Decomposition theorems of the de Rham complex have become a part of Hodge theory since the fundamental work of P. Deligne and L. Illusie. They proved the following fundamental decomposition theorem:

Theorem (Deligne-Illusie, [5])**.** *Let S be a scheme of characteristic p. Assume given a (flat) lifting T of S over $\mathbb{Z}/p^2\mathbb{Z}$. Let X be a smooth S-scheme and let us denote $F : X \to X'$ the relative Frobenius of X/S. Then if X' admits a (smooth) lifting over T, the complex of $\mathscr{O}_{X'}$-modules $\tau_{<p}F_*\Omega^\bullet_{X/S}$ is decomposable in the derived category $D(X')$ of $\mathscr{O}_{X'}$-modules.*

Let X be a smooth variety over a perfect field k of characteristic $p > 0$. If X is liftable to $W_2(k)$, then $\tau_{<p}F_*\Omega^\bullet_{X/k}$ is decomposable in $D(X')$. The most important applications of this result include the degeneration of the Hodge-de Rham spectral sequence and the Kodaira-Akizuki-Nakano vanishing theorem in characteristic zero ([5] Corollary 2.7, 2.11). Note that counterexamples exist otherwise ([14],[11]).

As an application of the notion of the log structure in the sense of Fontaine-Illusie [3], K. Kato has obtained the following generalization:

Theorem (Kato [3])**.** *Let $f : X \to Y$ be a smooth morphism of Cartier type between fine log schemes over $\mathbb{F}_p$. Let X' be the Frobenius base change of X over Y and $F : X \to X'$ the relative Frobenius morphism of log schemes. Let $\tilde{Y}$ be a flat lifting of Y over $\mathbb{Z}/p^2\mathbb{Z}$. Then, there exists a canonical bijection between the set of isomorphism classes of smooth liftings of X' over $\tilde{Y}$ and the set of splittings of $\tau_{\leq 1}F_*\Omega_{X/Y}$.*

The theorem of Deligne-Illusie corresponds to the case that the log structures of X and Y are trivial. The notion of *smooth morphism of Cartier type* is referred to Definition 4.8 [3]. However, one will see that some new phenomenon arises applying Kato's decomposition theorem to a semi-stable reduction over DVR with mixed characteristic. This note grows out of our study on a problem of Illusie [6] on semi-stable reductions over Witt rings: Let k be a perfect field of positive characteristic p and $W = W(k)$ be the ring of Witt vectors. Let X be a semi-stable reduction over W (Definition 2.1). Then $X_0 = X \times_{\mathrm{Spec}\ W} \mathrm{Spec}\ k$ is the closed fiber which is a reduced normal crossing divisor in X. The log de Rham complex of X_0 is defined as

$$\Omega^{\log\bullet}_{X_0} = \Omega^\bullet_X(\log X_0)|_{X_0}.$$

Problem 1.1 (Illusie, Problem 7.14 [6])**.** *Is the complex*

$$\tau_{<p}F_*\Omega^{\log\bullet}_{X_0} \tag{1}$$

decomposable in $D(X_0')$? Here $F : X_0 \to X_0'$ is the relative Frobenius morphism.

The first named author is partially supported by National Natural Science Foundation of China (Grant No. 11471298, No. 11526212). The second named author is partially supported by....

By the standard argument following [5], the decomposability of $\tau_{<p}F_*\Omega^{\log\bullet}_{X_0}$ is equivalent to the decomposability of $\tau_{\leq 1}F_*\Omega^{\log\bullet}_{X_0}$. By using the induced log structure from the natural one over X and equipping the base field k with the log structure $1\mapsto 0$ (the so-called canonical log point) , the structural morphism $X_0\to k$ extends to a smooth morphism of Cartier type (see remarks after Definition 4.8 [3]) and $\Omega^{\log\bullet}_{X_0}$ coincides with Kato's de Rham complex $\Omega^\bullet_{X_0/(k,1\mapsto 0)}$ for log schemes [3]. One might attempt to apply Kato's theorem here by taking $Y'=(W_2=W_2(k),1\mapsto p)$, but it does not work: The log absolute Frobenius morphism on k does not lift to Y'. Therefore, although X_0 admits an obvious lifting to Y' by taking the reducton of X modulo p^2, X_0' does not lift to Y' *necessarily* (note that the underlying scheme does lift however), as we shall demonstrate in below.

Indeed, we show in this short note that the answer to this problem is generally **NO**, even when the generic fiber is so nice as a projective space! The conclusion builds on the following criterion.

Theorem. *Let k be the canonical log point and $X_0\to k$ a log smooth variety of Cartier type over k. Then $\tau_{<p}F_{X_0*}\Omega^{\log\bullet}_{X_0}$ is decomposable if and only if X_0 has a log smooth lifting over $(W_2(k),1\mapsto 0)$.*

Noticing that the obstruction whether X_0 can be lifted to $(W_2,1\mapsto 0)$ lies in $H^2(X_0,T^{\log}_{X_0})$, where $T^{\log}_{X_0}$ is the log tangent sheaf of X_0 over the canonical log point $(k,1\mapsto 0)$, we have:

Corollary. *If the special fiber X_0 satisfies that $H^2(X_0,T^{\log}_{X_0})=0$, then $\tau_{<p}F_{X_0*}\Omega^{\log\bullet}_{X_0}$ is decomposable. In particular, Problem 1.1 is affirmative if*

(1) *X_0 is affine, or*
(2) *X_0 is a curve, or*
(3) *X_0 is a combinatorial K3 surface, which appears in the semi-stable the degenerations of a $K3$ surface [12].*

Let us explain some geometry related to this theorem. In the classical situation, the local (flat) deformation of a smooth point keeps smoothness. There is a new phenomenon in the deformation of log smooth singularities. For simplicity, let us consider the log smooth lifting over W_2 of a local normal crossing singularity defined by the equation $x_1\cdots x_r=0$. There exist two types of log smooth deformations of this singularity, distinguished according to which log structure we choose on the base scheme Spec W_2.

Type I: Log smooth lifting over $(W_2,1\mapsto p)$: this deformation smooths the singularity. Étale locally the log deformation looks like

$$\begin{tikzcd} \mathbb{N}^r \arrow[r, "e_i\mapsto x_i"] & W_2[x_1,\cdots,x_n]/(x_1\cdots x_r-p) \\ \mathbb{N} \arrow[u, "\Delta"] \arrow[r, "1\mapsto p"] & W_2(k). \arrow[u] \end{tikzcd}$$

Type II: Log smooth lifting over $(W_2,1\mapsto 0)$: this deformation keeps the singularity. Étale locally the log deformation looks like

$$\begin{tikzcd} \mathbb{N}^r \arrow[r, "e_i\mapsto x_i"] & W_2[x_1,\cdots,x_n]/(x_1\cdots x_r) \\ \mathbb{N} \arrow[u, "\Delta"] \arrow[r, "1\mapsto 0"] & W_2. \arrow[u] \end{tikzcd}$$

In the above two diagrams, Δ means the diagonal map. Once we view the mod p^2-reduction of a semi-stable reduction X over $W(k)$ as a log smooth deformation of the special log smooth fiber X_0, such log smooth deformation is of **Type I**. However, the above criterion shows that the truth of Illusie's problem is equivalent to the existence of a **Type II** log smooth deformation. In the next, we provide two approaches to produce semi-stable reductions over the Witt ring whose closed fibers do not admit *any* **Type II** log smooth deformation.

First Approach: Take a smooth projective variety Y_0 over k which is non W_2-liftable. Take a closed embedding $Y_0 \hookrightarrow Z_0$ over k into a smooth projective variety such that the codimension $\mathrm{Cod}_{Z_0}Y_0 \geq 2$ and Z_0 admits a smooth lifting Z over W. Set $X = \mathrm{Bl}_{Y_0}Z$, the blowup of Z along the closed subscheme Y_0. Then X is a semi-stable reduction over W whose closed fiber X_0 does not admit any **Type II** log smooth deformation.

Therefore, if we take Y_0 to be the classical counter-example of M. Raynaud [14] (see [11] for a generalization) to the Kodaira's vanishing theorem in positive characteristic and Z_0 the projective space of suitable dimension, then we get an example of DR-indecomposable closed fibers of semi-stable reductions over the Witt rings.

Second Approach: Take a closed embedding $Y_0 \hookrightarrow Z_0$ over k such that both Y_0 and Z_0 are smooth, Z_0 admits a smooth lifting Z over W, and such that the pair $(\mathrm{Bl}_{Y_0}Z_0, E)$ is non W_2-liftable, where E is the exceptional divisor. Set $X = \mathrm{Bl}_{Y_0}Z$. Then X is a semi-stable reduction over W whose closed fiber X_0 does not admit any **Type II** log smooth deformation.

Such examples exist by the recent work of Liedtke-Satriano [9]. See Theorem 1.1 (a) [9] (more specifically Theorem 2.3 (a) and Theorem 2.4 loc. cit.). This approach (with a little modification) provides examples of DR-indecomposable closed fibers of semi-stable reductions over the Witt rings $W(k)$ of relative dimension $d \geq 2$ with the algebraically closed field k arbitrary positive characteristic (see Proposition 3.5). It is desirable to find examples of minimal semi-stable reduction.

Notations: We mainly follow the notions and notations in [3] with some exceptions:

- We use the capital letters X, Y, etc. to denote log schemes. If X is a log scheme, we denote $\underline{X}$ be the underlying scheme and $\alpha_X : \mathscr{M}_X \to \mathscr{O}_X$ be the log structure. We consider a classical scheme as a log scheme with the trivial log structure.
- We denote a log cotangent sheaf by Ω instead of ω as in [3].
- Let R be a ring and $\alpha : P \to (R, \times)$ be a homomorphism of monoids. Denote (R, P_R^a) be the log scheme whose underlying scheme is $\mathrm{Spec}(R)$ and the log structure is the one associated to α (if there is no ambiguity of the homomorphism α). If α is the zero map, we use the notation $(R, P \mapsto 0)$ instead. In the case that $P = \mathbb{N}$, the monoid of nonnegative integers, we shall also use alternatively the notation $(\mathbb{N}, 1 \mapsto \alpha(1))$.

2. Semi-stable reduction, Type II deformation and DR-indecomposability

Let us recall the following

Definition 2.1. Let R be a complete discrete valuation ring (DVR) and π be a uniformizer of R. An R-scheme X is a *semi-stable reduction* over R if étale locally X is smooth over the closed subscheme of $\mathrm{Spec}(R[x_1, \cdots, x_r])$ defined by the equation $x_1 \cdots x_r = \pi$ for some $r \geq 1$.

By an R-scheme X we mean the scheme X is flat and finite type over $\mathrm{Spec}(R)$. The following characterization of semi-stable reductions over R will be used below. For a proof, see [4], 2.16.

Lemma 2.2. *Notation as above. Let K be the fractional field of R and k the residue field. Then a semi-stable reduction over R has the following two properties:*

1. *the generic fiber $X_K = X \times_R K$ is smooth over K,*
2. *the closed fiber $X_k = X \times_R k$ is a normal crossing variety over k.*

If k is perfect, then an R-scheme X is a semi-stable reduction over R if the above two properties hold.

By a normal crossing variety over k we mean a connected and geometrically reduced k-scheme which étale locally over each closed point x is isomorphic to $\mathrm{Spec}(k(x)[x_1, \cdots, x_r]/(x_1 \cdots x_r))$.

Semi-stable reductions are important examples of smooth morphisms in log geometry. Indeed, the log structure $\mathscr{M}_{X_k}$ (resp. $\mathscr{M}_{\mathrm{Spec}(k)}$) attached to the divisor X_k (resp. $\mathrm{Spec}(k)$) is fine, and the natural morphism of log schemes $f : (X, \mathscr{M}_{X_k}) \to (\mathrm{Spec}(R), \mathscr{M}_{\mathrm{Spec}(k)})$ is log smooth. If r (the number of local branches) in the definition is one everywhere, then the underlying morphism $\underline{X} \to \mathrm{Spec}(R)$ is smooth. In the special case where R is the valuation ring of a local field, they are of particular interest in p-adic Hodge theory. However, extra subtlety and difficulty arises in the case R is (very) ramified. This is related to the DR-decomposability of the closed fiber X_k. Note that there exist DR-indecomposable examples of X_k where X is smooth over R and R is ramified over $W(k)$ (see e.g. [10]). The fact that the absolute Frobenius on $\mathrm{Spec}(k)$ does not lift over $\mathrm{Spec}(R)$ contributes to the DR-indecomposability in examples. Similar phenomenon occurs in the semi-stable reduction case, due to the fact that the log absolute Frobenius on $(k, 1 \mapsto 0)$ does not lift over $(R, 1 \mapsto \pi)$, even the special case $(W(k), 1 \mapsto p)$. For this consideration, we shall restrict ourselves to the case $R = W(k)$ in the following.

In the following, we fix a finitely generated integral monoid P. For simplicity we denote $\mathbf{k} = (k, P \mapsto 0)$ and $\mathbf{W}_2 = (W_2(k), P \mapsto 0)$.

Theorem 2.3. *Let $f : X \to \mathbf{k}$ be a smooth morphism of Cartier type, then*

$$\tau_{<p} F_{X*} \Omega^*_{X/\mathbf{k}}$$

is decomposable if and only if X is liftable to $\mathbf{W}_2$, that is, X admits a Type II deformation.

Proof. Let

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ \mathbf{k} \arrow[r, "F"] & \mathbf{k} \end{tikzcd}$$

be the base change of X by the log Frobenius map of $\mathbf{k}$. After Kato's decomposition theorem, it remains to show that X' is $\mathbf{W}_2$-liftable if and only if X itself is $\mathbf{W}_2$-liftable. Notice the absolute log Frobenius morphism of $\mathbf{k}$ admits a lifting $G : \mathbf{W}_2 \to \mathbf{W}_2$ as in the diagram

$$\begin{tikzcd} W_2 \arrow[r, "F_{W_2}"] & W_2 \\ P \arrow[u, "0"] \arrow[r, "\times p"] & P. \arrow[u, "0"'] \end{tikzcd}$$

Thus, via the base change of G, one obtains a $\mathbf{W}_2$-lifting of X' from that of X. Since G is not isomorphism of log schemes, our argument is to show the converse nevertheless is still true. Let $\omega_X \in H^2(X, T_{X/\mathbf{k}})$ (resp. $\omega_{X'} \in H^2(X', T_{X'/\mathbf{k}})$) be the obstruction class of the lifting of X (resp. X') to $\mathbf{W}_2$. Recall that ω_X is constructed as follows: Let $\{U_i\}$ be an affine cover of X. Choosing for each U_i a log smooth lifting $\mathscr{U}_i$ on $\mathbf{W}_2$, then we have that on each overlap $U_{ij} = U_i \cap U_j$ there exists an isomorphism $\alpha_{ij} : \mathscr{U}_j|U_{ij} \to \mathscr{U}_i|U_{ij}$. Moreover, ω_X is represented by $\{(U_i \cap U_j \cap U_k, \alpha_{ij}\alpha_{jk}\alpha_{ki})\}$. Because of the existence of G, $\{(G^{-1}(U_i \cap U_j \cap U_k), G^*\alpha_{ij}\alpha_{jk}\alpha_{ki})\}$ represents $\omega_{X'}$. Thus we have that $\sigma^*(\omega_X) = \omega_{X'}$ through the canonical map

$$\begin{tikzcd} H^2(X', T_{X'/\mathbf{k}}) \arrow[r, phantom, "="] & H^2(X', \sigma^* T_{X/\mathbf{k}}) \, . \\ & H^2(X, T_{X/\mathbf{k}}) \arrow[u, "\sigma^*"] \end{tikzcd}$$

The above equality uses the fact that $\sigma^*\Omega_{X/\mathbf{k}} = \Omega_{X'/\mathbf{k}}$ and that both sheaves are locally free (see 1.7 and Proposition 3.10 [3]). However, since σ is an isomorphism of schemes, the map σ^* in the vertical line is bijective. It follows immediately that $\omega_X = 0$ under that assumption that X' is $\mathbf{W}_2$-liftable and hence X itself is $\mathbf{W}_2$-liftable. $\square$

As a corollary, the decomposability of $\tau_{<p}F_*\Omega^*_{X/\mathbf{k}}$ does not depend on the base field. Explicitly, we have

Corollary 2.4. *Let $f : X \to \mathbf{k}$ be a smooth morphism of Cartier type and K be a perfect field containing k. Denote by $\mathbf{K}$ the field K with the induced log structure from $\mathbf{k}$ and by $X_{\mathbf{K}}$ the log base change. Then $\tau_{<p}F_{X*}\Omega^*_{X/\mathbf{k}}$ decomposed in $D(X)$ if and only if $\tau_{<p}F_{X_{\mathbf{K}}*}\Omega^*_{X_{\mathbf{K}}/\mathbf{K}}$ decomposed in $D(X_{\mathbf{K}})$.*

Proof. By Theorem 2.3, it is enough to show that a $(W_2(K), P \mapsto 0)$-lifting of $X_{\mathbf{K}}$ induces a $(W_2(k), P \mapsto 0)$-lifting of X. By the flat base change, one has the isomorphism $H^2(X, T_{X/\mathbf{k}}) \otimes_k K = H^2(X_{\mathbf{K}}, T_{X/\mathbf{K}})$ and hence the injection $\alpha : H^2(X, T_{X/\mathbf{k}}) \to H^2(X_{\mathbf{K}}, T_{X/\mathbf{K}})$. Then, by the same arguments in Theorem 2.3, the obstruction class ob_k to lifting X to $\mathbf{W_2(k)}$ sends to the obstruction class ob_K of lifting $X_{\mathbf{K}}$ to $(W_2(K), P \mapsto 0)$ via the map α. By the condition that $\alpha(ob_k) = ob_K = 0$, it follows that $ob_k = 0$. □

As a consequence, when considering problem 1.1 one can assume in the following that k is algebraically closed.

Remark 2.5. After presenting our results, Weizhe Zheng provided us a more conceptual proof of Theorem 2.3: Denote by $\mathrm{Lift}(X)$ (resp.$\mathrm{Lift}(X')$) the groupoid of liftings of X (rsep. X') over $\mathbf{W}_2$. Denote $G : \mathbf{W}_2 \to \mathbf{W}_2$ be a lifting of the log Frobenius morphism $F : \mathbf{k} \to \mathbf{k}$. Given a lifting $X^{(1)} \in \mathrm{Lift}(X)$, the pullback of $X^{(1)}$ along G gives an object in $\mathrm{Lift}(X')$. With the obvious assignments on morphisms, one can get a functor

$$\mathrm{Lift}(X) \to \mathrm{Lift}(X'). \tag{2}$$

Conversely, let $X'^{(1)} \in \mathrm{Lift}(X')$ be a lifting of X'. Denote by $i : X' \hookrightarrow X'^{(1)}$ the canonical strict closed immersion and by $\sigma : X' \to X$ the base change of $F : \mathbf{k} \to \mathbf{k}$. Recall that $\underline{\sigma} : \underline{X'} \to \underline{X}$ is an isomorphism and $\mathscr{M}_{X'} \simeq \mathscr{M}_X \oplus_{\mathscr{K}_k} \mathscr{M}_k$. One can construct the pushout $X'^{(1)} \amalg_{X'} X$ of the diagram

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d, "i"] & X \\ X'^{(1)} & \end{tikzcd}$$

as follows:

- The underlying scheme $\underline{X'^{(1)} \amalg_{X'} X}$ is defined to be $\underline{X'^{(1)}}$,
- the log structure of $X'^{(1)} \amalg'_X X$ is defined to be $\mathscr{M}_{X'^{(1)}} \times_{\mathscr{M}_{X'}} \mathscr{M}_X$.

With the obvious assignments on morphisms, the pushout process along $\sigma : X' \to X$ gives a functor

$$\mathrm{Lift}(X') \to \mathrm{Lift}(X). \tag{3}$$

It is straightforward to check the following proposition.

Proposition 2.6. *The above two functors (2) and (3) are the equivalences of groupoids, quasi-inverse to each other.*

3. Examples

Let k be an algebraically closed field of characteristic $p > 0$. In this section, we shall use Theorem 2.3 to provide some examples of DR-indecomposable semi-stable reductions over the Witt ring W. First a simple lemma.

Lemma 3.1. *Let Z be a smooth scheme over W. Let Y_0 be a smooth closed subvariety of $Z_0 = Z \times_W k$. Set $X = Bl_{Y_0}Z$, the blowup of Z along the closed subscheme Y_0. Then X is a semi-stable reduction over W, whose closed fiber X_0 is a simple normal crossing divisor consisting of two* *smooth components $Bl_{Y_0}Z_0$ and $\mathbb{P}(N_{Y_0/Z})$ (the projective normal bundle of Y_0 in Z) which intersect transversally along $\mathbb{P}(N_{Y_0/Z_0})$ (the projective normal bundle of Y_0 in Z_0). Furthermore, if the normal crossing variety X_0 over k admits Type II deformation, then both pairs $(Bl_{Y_0}Z_0, \mathbb{P}(N_{Y_0/Z_0}))$ and $(\mathbb{P}(N_{Y_0/Z}), \mathbb{P}(N_{Y_0/Z_0}))$ are $W_2(k)$-liftable.*

Proof. The proof of the first statement is fairly standard and therefore omitted. The second statement follows from Lemma 3.4 below. □

The following proposition is due to Cynk-van Straten.

Proposition 3.2 ([1] Theorem 3.1)**.** *Let $\pi : Y \to X$ be a morphism of schemes over k and let $S = Spec(A)$, A artinian with residue field k. Assume that $\mathscr{O}_X = \pi_*\mathscr{O}_Y$ and $R^1\pi_*(\mathscr{O}_Y) = 0$. Then for every lifting $\mathscr{Y} \to S$ of Y there exists a preferred lifting $\mathscr{X} \to S$ making a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & \mathscr{Y} \arrow[d] \\ X \arrow[r, hook] & \mathscr{X} \end{tikzcd}$$

Corollary 3.3. *Notation as above. If Y_0 is non $W_2(k)$-liftable, then the closed fiber X_0 of X does not admit any Type II deformation.*

Proof. Use Lemma 3.1 and Proposition 3.2 which says that the W_2-liftability of $\mathbb{P}(N_{Y_0/Z})$ implies that of Y_0. □

Thus, one deduces from the corollary the first approach to requested examples as given in the introduction.

Lemma 3.4. *Let $X = \bigcup_{i\in I} X_i$ be a normal crossing variety over k, where $\{X_i, i \in I\}$ are irreducible components. Endow X with the canonical log structure so that it is log smooth over $(k, 1 \mapsto 0)$. Let $\mathscr{X}$ be a Type II deformation of the log scheme X. Then $\underline{\mathscr{X}} = \bigcup_{i\in I} \mathscr{X}_i$ is the schematic union of closed subschemes $\mathscr{X}_i$s such that for each nonempty $J \subseteq I$, the schematic intersection $\bigcap_{j\in J} \mathscr{X}_j$ is a lifting of $\bigcap_{j\in J} X_j$ over W_2.*

Proof. Let

$$\mathscr{I}_i = I_i + pI_i$$

where I_i denotes the ideal sheaf of X_i in X. $\mathscr{I}_i$ is an ideal sheaf of $\mathscr{O}_{X'}$. We claim that the closed subscheme $\mathscr{X}_i$ defined by $\mathscr{I}_i$ is what we want. To do this it is enough to show

(1) $\mathscr{O}_{X'}/\mathscr{I}_i$ is flat over W_2,
(2) $\bigcap \mathscr{I}_i = 0$, and
(3) for each nonempty $J \subseteq I$, $\mathscr{O}_{X'}/\cup_{j\in J} \mathscr{I}_j$ is flat over W_2.

Since $\widehat{\mathscr{O}_{X'x}}$ is faithfully flat over $\mathscr{O}_{X',x}$ for each point $x \in X'$, it is sufficient to verify the claim above after tensoring with $\widehat{\mathscr{O}_{X'x}}$ for every $x \in X'$. By ([3] Theorem 3.5, Proposition 3.14) , There is an étale morphisms $U' \to X'$ such that there is an diagram

$$\begin{tikzcd} U' \arrow[r, "f"] \arrow[dr, "\pi'|_{U'}"'] & \mathrm{Spec}(W_2(k)[x_1,\cdots,x_n]/(x_1\cdots x_r)) \arrow[d] \\ & \mathrm{Spec}(W_2(k)) \end{tikzcd},$$

where f is an étale morphism. As a consequence, there is an isomorphism

$$\alpha : \widehat{\mathscr{O}_{X'x}} \simeq W_2(k)[[x_1,\cdots,x_n]]/(x_1\cdots x_r)$$

such that each $\mathscr{I}_i\widehat{\mathscr{O}_{X'x}}$ (whenever it is nonempty) is generated by $\alpha^{-1}(\Pi_{j\in J_i}x_j)$ for some nonempty $J_i \subseteq \{1,\cdots,r\}$. Moreover, $\{1,\cdots,r\}$ is the disjoint union of J_is. Then the claim follows from direct calculations. ☐

The following proposition illustrates the second approach in the introduction.

Proposition 3.5. *For any $d \geq 2$ and any $p \geq 2$, there exists a smooth scheme Z over $W(k)$ of relative dimension d and char$(k) = p$ such that there exists a smooth closed subvariety $Y_0 \subset Z_0$ such that the pair $(Bl_{Y_0}Z_0, E)$ is non $W_2(k)$-liftable. Here E is the exceptional divisor of the blowup $Bl_{Y_0}Z_0$. As a consequence, $X = Bl_{Y_0}Z$ is a semi-stable reduction over W whose closed fiber is DR-indecomposable of dimension $d \geq 2$ over k with arbitrary positive characteristic.*

Proof. For $d \geq 3$ and p arbitrary, this is the statement of Theorem 2.4 [9]. The surface case can be also obtained by adapting its proof as follows. By [11], there is a projective smooth curve C of genus $g \geq 2$ over k, a vector bundle E on C of rank 2, and a smooth curve D in $\mathbb{P}_C(E)$ such that the composite $D \to \mathbb{P}_C(E) \to C$ is the relative Frobenius morphism $F_0 : D \to D^{(p)} = C$. Denote

$$C_0 = C, \quad Y_0 = D, \quad Z_0 = \mathbb{P}_C(E).$$

It is clear that Z_0 admits a W-lifting $Z = \mathbb{P}_{\mathscr{C}}(\mathscr{E})$, where $\mathscr{C}$ is a W-lifting of C and $\mathscr{E}$ is a W-lifting of E over $\mathscr{C}$ (for dimensional reason formal lifting exists and then apply Grothendieck's existence theorem in formal geometry to conclude the algebracity of the formal objects). Then we claim that the closed fiber X_0 of $X = Bl_{Y_0}Z$ is non W_2-liftable. If not, by Lemma 3.4, the pair (Z_0, Y_0) consisting of the component $Z_0 = Bl_{Y_0}Z_0$ of X_0 together with the divisor $Y_0 = \mathbb{P}(N_{Y_0/Z}) \cap Z_0 \subset X_0$ lift to a pair (Z_1, Y_1) over W_2 (The scheme Z_1 is not necessarily the reduction of Z over W_2). On the other hand, Proposition 3.2 implies that the projection $Z_0 \to C_0$ is the reduction of a certain W_2-morphism $Z_1 \to C_1$. Therefore, the composite $F_0 : Y_0 \hookrightarrow Z_0 \to C_0$ lifts to the composite $F_1 : Y_1 \hookrightarrow Z_1 \to C_1$ over W_2. But this leads to a contradiction: the nonzero morphism $dF_1 : F_1^*\Omega_{C_1} \to \Omega_{Y_1}$ is divisible by p and it induces a nonzero morphism over k

$$\frac{dF_1}{p} : F_0^*\Omega_{C_0} \to \Omega_{Y_0}$$

which is impossible for the degree reason. This shows the claim and hence the proposition. ☐

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