

Sweeping out 3-manifold of positive Ricci curvature by short 1-cycles via estimates of min-max surfaces

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Abstract: We prove that given a three manifold with an arbitrary metric (M^3, g) of positive Ricci curvature, there exists a sweepout of M by surfaces of genus ≤ 3 and areas bounded by $C \text{Vol}(M^3, g)^{2/3}$. We use this result to construct a sweepout of M by 1-cycles of length at most $C \text{Vol}(M^3, g)^{1/3}$, giving a partial answer to a question of L. Guth.

The sweepout of surfaces is generated from a min-max minimal surface. If further assuming a positive scalar curvature lower bound, we can get a diameter upper bound for the min-max surface.

1 Introduction

Let M be a 3-manifold with positive Ricci curvature. In this paper we obtain quantitative results about sweepouts of M by 1-cycles and surfaces.

Theorem 1.1. *Every closed 3-manifold M of positive Ricci curvature admits a map $f : M \rightarrow \mathbb{R}^2$ with fibers of length at most $C \text{Vol}(M)^{1/3}$ for a universal constant $C > 0$.*

Moreover, the family of 1-cycles $\{f^{-1}(x)\}_{x \in \mathbb{R}^2}$ is continuous in a strong sense. We define this precisely in Section 3.

The geometric invariant that we bound from above in Theorem 1.1 is called the k -waist of a Riemannian manifold. For an n -dimensional manifold M and $k < n$ the k -waist, $\text{waist}_k(M)$, is defined as $\inf\{\sup_{x \in \mathbb{R}^{n-k}} \{\text{Vol}_k f^{-1}(x)\}\}$, where the infimum is taken over all proper functions $f : M \rightarrow \mathbb{R}^{n-k}$. Waists have been defined by Gromov and extensively studied in [Gro83], [Gro88], [Gro03], [Gro09], [Gro10], [Gro15]. Gromov proved deep theorems about waists of manifolds (see [Gu14] and [M11] for exposition of some of his work), yet many significant questions about waists remain open. In particular, existence of upper bounds for $\text{waist}_k(M)$ when $k < n - 1$ is a widely open question. In [Gu10] Guth asked if for every Riemannian metric g on a 3-torus T the 1-waist $\text{waist}_1(T, g)$ is bounded from above by $C \text{Vol}(T, g)^{1/3}$. More generally, one may ask analogous questions for any Riemannian 3-manifold, that is, does there exist a constant C , such that for any metric g there exists a map $f : M \rightarrow \mathbb{R}^2$ with fibers of length at most $C \text{Vol}(M, g)^{1/3}$? If this is true, it would be a strong generalization of the systolic inequality. In Theorem 1.1 we affirmatively answer this question for

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3-manifolds M under an additional assumption of $Ric > 0$. It is to our knowledge the first occasion where such a generalization is proved.

Theorem 1.1 generalizes, in the setting of positive Ricci curvature, some fundamental theorems about the length of closed geodesics and stationary one-cycles in Riemannian manifolds due to Gromov and Nabutovsky-Rotman. As consequence of Theorem 1.1 we obtain

Corollary 1.2. *Let M be a closed 3-manifold of positive Ricci curvature. If M is homeomorphic to $\mathbb{R}P^3$ then it contains a non-contractible closed geodesic of length at most $CVol(M)^{\frac{1}{3}}$. If $M = (S^3, g)$ then it contains a geodesic net of length at most $CVol(M)^{\frac{1}{3}}$.*

The first statement of Corollary 1.2 is a special case of Gromov's systolic inequality [Gro83]. The systole of a Riemannian manifold is defined as the length of the shortest non-contractible geodesic loop. In the same paper Gromov conjectured that every manifold contains a non-trivial closed geodesic of length at most $C_n Vol(M)^{1/n}$. Nabutovsky and Rotman [NR04] proved that every Riemannian manifold contains a stationary 1-cycle of length at most $C_n Vol(M)^{\frac{1}{n}}$ (a stationary 1-cycle need not be a closed geodesic; it may look, for example, like a bouquet of geodesic loops all intersecting at a point with tangent vectors at that point summing up to 0, see [NR04] for more examples).

If M is topologically a sphere, then a min-max argument yields an upper bound for the length of a stationary 1-cycle, giving an alternative proof of a special case of the result of Nabutovsky and Rotman.

Here we present a short (and incomplete) overview of previously known estimates for sweepouts of manifolds. In [Gu07] Guth proved that every open subset of Euclidean space $U \subset \mathbb{R}^n$ admits a sweepout by relative k -cycles of volume at most $C_n Vol(U)^{\frac{k}{n}}$. In general such inequalities do not hold for Riemannian manifolds (see Appendix 5 in [Gu07] and [PS15]). However, we may control volumes of $(n-1)$ -cycles if we impose an additional requirement on the metric. In [GL], among other results, it was shown that if M is conformally equivalent to a manifold with non-negative Ricci curvature then it admits a sweepout by $(n-1)$ -cycles of volume at most $C_n Vol(U)^{\frac{n-1}{n}}$ (in [S15] Sabourau independently constructed a sweepout of M with $Ric(M) \geq 0$ by $(n-1)$ -cycles of controlled volume).

When $Ric > 0$ and $n = 3$ we show that we can simultaneously control the area and the genus of surfaces in the sweepout, which will be essential in the proof of Theorem 1.1.

Theorem 1.3. *Given a three manifold with an arbitrary metric (M^3, g) of positive Ricci curvature, i.e. $Ric_g > 0$, there exists a minimal surface Σ_0^2 , such that $Area(\Sigma_0) \leq C vol(M^3, g)^{2/3}$, for a universal constant $C > 0$. Also we have*

- *If Σ_0 is orientable, then the genus g_0 of Σ satisfies $g_0 \leq 3$, and there exists a smooth sweepout $\{\Sigma_t\}_{t \in [-1,1]}$ of (M^3, g) , such that*
 - *$\{\Sigma_t\}$ forms a Heegaard splitting of M^3 , i.e. Σ_t is an embedded surface of genus g_0 , for $t \in (-1, 1)$, and Σ_{-1} and Σ_1 are graphs;*
 - *$Area(\Sigma_t) < Area(\Sigma_0)$ for $t \neq 0$.*

- If Σ_0 is non-orientable, then the genus \tilde{g}_0 of its double cover $\tilde{\Sigma}_0$ satisfies $\tilde{g}_0 \leq 3$. Moreover, by removing Σ from M , we get a manifold with boundary \tilde{M} with $\partial\tilde{M} = \tilde{\Sigma}_0$, and there exists a smooth sweepout $\{\Sigma_t\}_{t \in [0,1]}$ of \tilde{M} , such that
 - $\{\Sigma_t\}$ forms a Heegaard splitting of \tilde{M} , i.e. Σ_t is an embedded surface of genus g_0 lying in the interior of \tilde{M} , for $t \in (0, 1)$, and $\Sigma_0 = \partial\tilde{M}$;
 - $Area(\Sigma_t) < 2Area(\Sigma_0)$ for $t \neq 0$.

Remark 1.4. After this paper was submitted for publication Ketover, Marques and Neves [KMN] proved that the min-max minimal surface Σ_0 in 3-manifolds with positive Ricci curvature must be orientable. This result follows from a clever application of the catenoid estimate they derive. Therefore, the second case of Theorem 1.3 does not occur. Note that this simplifies our construction of a continuous sweepout of M by 1-cycles as it rules out Case 2 in the proof of Theorem 5.1 (see Section 5).

We would like to compare our result with that of F. Marques and A. Neves [MN11]. In [MN11], assuming $Ric_g > 0$ and the scalar curvature lower bound $Scal_g \geq 6$, Marques-Neves produced a smooth sweepout $\{\Sigma_t\}_{t \in [0,1]}$, where the genus of Σ_t is the Heegaard genus¹, and $Area(\Sigma_t) \leq 4\pi$. The advantage of [MN11] is that they have better estimates for the genus. However, from the point of view of area estimates (e.g. for the application to prove Theorem 1.1), our result can be much better than that in [MN11] while we still have a relatively good genus estimate. An example illustrating this fact is a long and thin 3-dimensional ellipsoid; when we normalize the scalar curvature lower bound to be 6, the width can be very small (compared to 4π). The difference between our method with [MN11] is that we use the Almgren-Pitts min-max theory [AF62, P81] for general sweepouts constructed in [GL], while Marques-Neves used the Colding-De Lellis [CD03] (or Simon-Smith [Sm82]) min-max method for smooth sweepouts given by Heegaard splittings. We refer to §6 for more discussion.

The sweepout $\{\Sigma_t\}$ in Theorem 1.3 is used to construct a sweepout by 1-cycles of controlled length in Theorem 1.1. An important open question is whether one can construct a sweepout by *closed curves* of controlled length rather than 1-cycle (see more discussion in §6). One approach in this direction is to first construct a sweepout of M by spheres or tori of controlled area and *diameter*. For this purpose, we derive the following partial result. In particular, if we further assume a scalar curvature lower bound, we can get a uniform diameter upper bound for the min-max minimal surface.

Theorem 1.5. *Let (M^3, g) be as in Theorem 1.3; if the scalar curvature of (M^3, g) is bounded from below, i.e. $Scal_g \geq 2\Lambda$, for some $\Lambda > 0$, then the diameter of Σ_0^2 (when it is orientable) or the diameter of its double cover (when it is non-orientable) is bounded from above by $\sqrt{6} \frac{\pi}{\sqrt{\Lambda}}$.*

The main idea of proving Theorem 1.1 is a dimension reduction type argument. We first construct a nice sweepout by 2-surfaces with controlled area and genus by Theorem 1.3. Then we continuously

¹Heegaard genus is the least genus of a Heegaard surface; so Heegaard genus is the best one we can expect in Theorem 1.3.

sweep out these 2-surfaces by 1-cycles. A large portion of the argument (in particular, proof of Lemma 5.1) is devoted to making this family continuous in a strong sense (cf. Section 3). This continuity is needed to obtain a geodesic net of controlled length as a limit of a min-max sequence of 1-cycles.

Theorem 1.3 is proved by combining several ingredients. We apply the Almgren-Pitts min-max theory to the sweepout constructed in [GL] and get a min-max minimal surface of controlled area. By using one of the authors Morse index bound [Z12], we can get the desired genus bound via Schoen-Yau genus estimates [Y87]. The existence of good Heegaard splitting follows from Meeks-Simon-Yau [MSY]. The diameter estimates (Theorem 1.5) for the min-max surface comes from Schoen-Yau diameter estimates [SY83] and the Morse index estimate.

Our paper is organized as follows. In §2, we prove Theorem 1.3 and Theorem 1.5. In §3 we give a precise definition of the sweepout by 1-cycles. In §4, we show how to sweep out a family of surfaces simultaneously by continuous 1-cycles with lengths controlled by the genus and area. In §5 we prove Theorem 1.1 by combining results in §2 and §4. Finally, we summarize several interesting open questions in §6.

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2 Area and diameter estimates for the min-max minimal surface

We outline the proof of Theorem 1.3.

Proof. Since (M, g) has positive Ricci curvature, we can start with a sweepout constructed in [GL], and, independently, [S15], i.e. $\Phi : S^1 \rightarrow \mathcal{Z}_2(M^3)$, such that for a universal constant $C > 0$,

$$\sup_{t \in S^1} \mathcal{H}^2(\Phi(t)) \leq C \text{vol}(M^3, g)^{2/3}.$$

Now we can adapt such a sweepout to the Almgren-Pitts theory [AF62, AF65, P81] as in [MN12, Z12]. Application of the Almgren-Pitts theory produces a min-max minimal surface $\bar{\Sigma}_0$, such that an integer multiple $n_0 \bar{\Sigma}_0$, $n_0 \in \mathbb{N}$ achieves the min-max value W , i.e.

$$n_0 \text{Area}(\bar{\Sigma}_0) = W \leq \sup_{t \in S^1} \mathcal{H}^2(\Phi(t)) \leq C \text{vol}(M^3, g)^{2/3}.$$

Using [Z12, Theorem 1.1], there exists a minimal surface Σ_0 , such that Σ_0 has least area among all closed, embedded, minimal hypersurfaces in the following sense. Define (see [Z12, (1.1)])

$$W_M = \inf \left\{ \begin{array}{ll} \text{Area}(\Sigma), & \text{if } \Sigma \text{ is an orientable minimal surface} \\ 2\text{Area}(\Sigma), & \text{if } \Sigma \text{ is a non-orientable minimal surface} \end{array} \right\}.$$

Then $\text{Area}(\Sigma_0) = W_M$ when it is orientable, or $2\text{Area}(\Sigma_0) = W_M$ when it is non-orientable.

By comparing the area of $\bar{\Sigma}_0$ with that of Σ_0 , we have that:

- $Area(\Sigma_0) \leq 2Area(\bar{\Sigma}_0) \leq Cvol(M^3, g)^{2/3}$.

Moreover, when Σ_0 is orientable, it is proven in [Z12, Theorem 1.1] that

- The Morse index of Σ_0 is one.

When Σ_0 is non-orientable, it is shown by [MR15, Theorem A] that

- The Morse index of the double cover $\tilde{\Sigma}_0$ of Σ_0 is one.

Remark 2.1. Our definition of W_M is the same as $\mathcal{A}_1(M)$ in [MR15]; and when $Ric_M > 0$, [MR15, Theorem A] reduces to [Z12, Theorem 1.1], except that [MR15, Theorem A] showed that the double cover of a non-orientable min-max surface has Morse index 1.

By [Y87, §4][F88, Theorem 2], when $Ric_g > 0$, The Morse index equal to one implies that the genus $g_0 = g(\Sigma_0)$ of Σ_0 (when it is orientable) or the genus $\tilde{g}_0 = g(\tilde{\Sigma}_0)$ of its double cover $\tilde{\Sigma}_0$ (when it is non-orientable) is bounded by 3.²

Next we show the existence of good sweepouts generated by Σ_0 . When Σ_0 is orientable, we claim that Σ_0 must be a Heegaard splitting; or equivalently, $M^3 \setminus \Sigma_0$ is a union of two connected component M_+ and M_- , such that M_+ and M_- are both handlebodies. This is essentially due to [MSY] (see also [MN11, Lemma 3.2]). In fact, Σ_0 separates M into two connected components M_+ and M_- by [Z12, Proposition 3.5] (see also [MN11, Lemma 3.2]). By minimizing area in the isotopy class of Σ_0 inside M_+ using [MSY, Theorem 1’], we either get another minimal surface Σ' in the interior of M_+ , or we get an empty set, i.e Σ_0 can be isotopically changed to a surface of arbitrarily small area. The first case will violate the Frankel’s Theorem [F66] which says that every two closed minimal surfaces must intersect when $Ric_g > 0$; while the second case implies that M_+ is a handlebody by [MSY, Proposition 1]. Similarly M_- is also a handlebody. By [MN11, Lemma 3.5]³, we can construct a heegaard splitting $\{\Sigma_t\}_{t \in [-1,1]}$ satisfying the requirement of Theorem 1.3.

When Σ_0 is non-orientable, by removing Σ_0 from M , we get a manifold \tilde{M} with boundary ∂M . The boundary ∂M is a minimal surface, and is a double cover of Σ_0 . Similar argument as above shows that \tilde{M} is a handlebody. Again by the same method in [MN11, Lemma 3.5], we can construct a Heegaard splitting $\{\Sigma_t\}_{t \in [0,1]}$ satisfying the requirement. \square

To prove Theorem 1.5, we will adapt the Schoen-Yau [SY83] diameter estimates via scalar curvature lower bound for stable minimal surfaces as follows. Let Σ be a two-sided minimal surface possibly with boundary. Here “two-sided” means that Σ has a unit normal vector field ν . Given a function $u \in C_0^1(\Sigma)$, the second variation of area functional along normal deformation in the direction of $u(x)\nu(x)$ is given by [CM11, Chap. 1, §8]:

$$\delta^2 \Sigma(u, u) = \int_{\Sigma} |\nabla_{\Sigma} u|^2 - (Ric(\nu, \nu) + |A|^2)u^2 d\mu = - \int_{\Sigma} u L_{\Sigma} u d\mu,$$

²It is conjectured that the optimal upper bound is 2. See also [N14].

³The construction there only used the fact that there is no non-intersecting minimal surfaces, and it is true here by Frankel’s Theorem [F66] as $Ric_g > 0$.

where $L_\Sigma u = \Delta_\Sigma u + (\text{Ric}(\nu, \nu) + |A|^2)u$ is the Jacobi operator, and A is the second fundamental form of Σ . Σ is stable if $\delta^2 \Sigma(u, u) \geq 0$ for all $u \in C_0^1(\Sigma)$.

Proposition 2.2. *Given a three manifold (M^3, g) , assume that the scalar curvature is bounded from below $R^M \geq 2\Lambda$, $\Lambda > 0$. Let Σ be a two-sided stable minimal surface with boundary $\partial\Sigma$, then the inf-radius $\rho(\Sigma)$ of Σ is bounded from below by $\sqrt{\frac{3}{2} \frac{\pi}{\Lambda}}$.⁴*

Proof. The fact that Σ is stable implies that the Jacobi operator L_Σ is non-positive. Let φ be the first Dirichlet eigenfunction of L_Σ , i.e. $L_\Sigma \varphi = -\lambda \varphi$, $\lambda \geq 0$. Then $\varphi > 0$ in the interior of Σ . Using [SY79, page 193], by rewriting the Jacobi operator, we have

$$L_\Sigma \varphi = \Delta_\Sigma \varphi + \frac{1}{2}(R^M - R^\Sigma + |A|^2)\varphi = -\lambda \varphi \leq 0, \quad (2.1)$$

where R^M and R^Σ are respectively the scalar curvatures of M and Σ .

Take a point p in the interior of Σ , such that the distance of p to $\partial\Sigma$ achieves the inf-radius $\rho(\Sigma)$ of Σ . Now consider the problem of minimizing the following functional

$$\mathcal{L}(\tilde{\gamma}) = \int_{\tilde{\gamma}} \varphi ds,$$

among all curves $\tilde{\gamma}$ connecting p to $\partial\Sigma$. Assume that γ achieves a minimum, then

$$\int_\gamma ds = \text{Length}(\gamma) \geq \rho(\Sigma).$$

The first variation of \mathcal{L} at γ vanishes:

$$\delta \mathcal{L}(\gamma) = \int_\gamma \langle \nabla \varphi(\gamma(s)), V(s) \rangle ds + \int_\gamma \varphi(\gamma(s)) \langle \nabla_v V(s), v \rangle ds = 0. \quad (2.2)$$

Here ∇ is the Riemannian connection of Σ , and $V(s)$ is an arbitrary variational vector field along $\gamma(s)$ vanishing at the end points of γ , and $v(s)$ in the unit tangent vector field along $\gamma(s)$, and ds is the length parameter. Integrating by parts shows that

$$\int_\gamma \langle V(s), (\nabla \varphi)^\perp - \varphi(\gamma(s)) \nabla_v v \rangle ds = 0,$$

where $(\nabla \varphi)^\perp$ is normal component (with respect to the tangent vector of γ in Σ) of $\nabla \varphi$. Therefore the weighted geodesic equation is

$$\varphi(\gamma(s)) \nabla_v v - (\nabla \varphi)^\perp = 0. \quad (2.3)$$

The second variation of \mathcal{L} is non-negative:

$$\begin{aligned} \delta^2 \mathcal{L}(\gamma) &= \int_\gamma (Hess \varphi(V, V) + \langle \nabla \varphi, \nabla_V V \rangle) ds + 2 \int_\gamma \langle \nabla \varphi, V \rangle \langle \nabla_v V, v \rangle ds \\ &\quad + \int_\gamma \varphi (\langle \nabla_v \nabla_V V, v \rangle - K^\Sigma(V, v, V, v) + \langle \nabla_v V, \nabla_v V \rangle - \langle \nabla_v V, v \rangle^2) ds \geq 0. \end{aligned}$$

⁴When preparing the manuscript, the authors learned that A. Carlotto also did something similar [Ca15, Proposition 2.12].

Here K^Σ is the curvature tensor of Σ . Denote ν by the unit normal vector field along γ , and let $V(s) = f(s)\nu(s)$ for some function f which vanishes at the end points of γ . Using (2.2) and (2.3) we have,

$$\begin{aligned} \delta^2 \mathcal{L}(\gamma) &= \int_\gamma [Hess\varphi(\nu, \nu) - 2\varphi(\gamma(s))\langle \nabla_\nu \nu, \nu \rangle^2] f^2 ds \\ &\quad + \int_\gamma \varphi(\gamma(s))(|\nabla_\nu f|^2 - K^\Sigma(\nu, \nu, \nu, \nu)f^2) ds \geq 0. \end{aligned} \quad (2.4)$$

Using the fact that $\Delta_\Sigma \varphi = Hess\varphi(\nu, \nu) + Hess\varphi(v, v) = Hess\varphi(\nu, \nu) + \nu\nu\varphi - \nabla_{\nabla_\nu \nu}\varphi$ and (2.3), (2.1) can be re-written as

$$Hess\varphi(\nu, \nu) - \langle \nabla_\nu \nu, \nu \rangle^2 \varphi - \frac{1}{2}R^\Sigma \varphi \leq -\nu\nu\varphi - \frac{1}{2}(R^M + |A|^2)\varphi.$$

Combing this with (2.4) and using the fact that $\varphi > 0$ and that $R^M \geq 2\Lambda$, we have,

$$\int_\gamma \varphi |\nabla_\nu f|^2 - \Lambda \varphi f^2 - \nu\nu\varphi f^2 ds \geq 0.$$

Now parametrize γ by the length parameter on $[0, l]$, with $l = \text{length}(\gamma)$, and using integration by part, we get

$$\int_0^l -\varphi f f'' - f \varphi' f' - (\Lambda \varphi + \varphi'') f^2 ds \geq 0.$$

This implies that the following operator L_0 is non-negative⁵:

$$L_0 f = -\frac{d^2 f}{ds^2} - \frac{1}{\varphi} \frac{d\varphi}{ds} \frac{df}{ds} - \left(\Lambda + \frac{1}{\varphi} \frac{d^2 \varphi}{ds^2} \right) f.$$

Let $h(s)$ be the first Dirichlet eigen-function of L_0 on $[0, l]$, then $h(s) > 0$, and

$$\frac{h''}{h} + \frac{\varphi' h'}{\varphi h} + \Lambda + \frac{\varphi''}{\varphi} \leq 0.$$

Multiply the above inequality with any f^2 , $f \in C_0^1([0, l])$, and use integration by part, then

$$\int_0^l \frac{(h')^2}{h^2} f^2 - 2\frac{h'}{h} f f' + \frac{(\varphi')^2}{\varphi^2} f^2 - 2\frac{\varphi'}{\varphi} f f' + \frac{\varphi' h'}{\varphi h} f^2 + \Lambda f^2 ds \leq 0.$$

Re-arranging, we get

$$\int_0^l \frac{1}{2} \left(\frac{h'}{h} + \frac{\varphi'}{\varphi} \right)^2 f^2 + \frac{1}{2} \left(\frac{(h')^2}{h^2} + \frac{(\varphi')^2}{\varphi^2} \right) f^2 + \Lambda f^2 ds \leq 2 \int_0^l f f' \left(\frac{h'}{h} + \frac{\varphi'}{\varphi} \right) ds.$$

By the Cauchy-Schwartz inequality,

$$2f f' \left(\frac{h'}{h} + \frac{\varphi'}{\varphi} \right) \leq \frac{1}{2} \left(\frac{h'}{h} + \frac{\varphi'}{\varphi} \right)^2 f^2 + \frac{1}{2} \left(\frac{(h')^2}{h^2} + \frac{(\varphi')^2}{\varphi^2} \right) f^2 + \frac{3}{2} (f')^2.$$

⁵ L_0 is the same as that in [SY83, p577] where the "f" used in [SY83] is a constant in our setting.

So

$$\int_0^l \Lambda f^2 ds \leq \frac{3}{2} \int_0^l (f')^2 ds.$$

It implies that the operator $-\frac{d^2}{ds^2} - \frac{2}{3}\Lambda$ is non-negative on $[0, l]$, so ODE comparison implies that

$$l \leq \pi / \sqrt{\frac{2}{3}\Lambda} = \sqrt{\frac{3}{2}} \frac{\pi}{\sqrt{\Lambda}}.$$

□

Next we prove Theorem 1.5. Let us first assume that Σ_0 is orientable, and hence is two-sided (c.f. [Z12, Proposition 3.5]). Pick two points p, q on Σ such that the distance $d = d(p, q)$ achieves the diameter. Consider the geodesic balls $B(p, d/2)$ and $B(q, d/2)$ of Σ . As the Morse index of Σ_0 is one, at least one geodesic ball, say $B(p, d/2)$ is a stable minimal surface with smooth boundary. Then the proof of Proposition 2.2 implies that $d/2 \leq \sqrt{\frac{3}{2}} \frac{\pi}{\sqrt{\Lambda}}$. When Σ_0 is non-orientable, its double cover $\tilde{\Sigma}_0$ is then a two-side minimal surface of Morse index 1 by Theorem 1.3, so we finish the proof.

3 Families of 1-cycles

In this section we define what we mean by a family of 1-cycles and a sweepout of a manifold by 1-cycles.

Following [CC92], [NR04] for $k \in \mathbb{N}$ let $\Gamma_k(M)$ denote the space of all k -tuples $(\gamma^1, \dots, \gamma^k)$ of Lipschitz maps of $[0, 1]$ to M such that $\sum_i \gamma^i(0) = \sum_i \gamma^i(1)$ with the following topology. Using Nash embedding theorem we embed M isometrically into a Euclidian space and define the distance by the formula $d_\Gamma((\gamma^1, \dots, \gamma^k), (\bar{\gamma}^1, \dots, \bar{\gamma}^k)) = \max_{i,t} d_M(\gamma^i(t), \bar{\gamma}^i(t)) + \sum \sqrt{\int_0^1 |\dot{\gamma}^i(t) - \dot{\bar{\gamma}}^i(t)|^2 dt}$. We let $\Gamma = \bigcup \Gamma_k$. Observe that the induced topology on Γ_k is finer than the flat topology on the space of integer 1-cycles [Si83, §31] and that the length functional is continuous on Γ_k .

Let $\Gamma^0(M) \subset \Gamma(M)$ denote the space of all constant curves (points). Let K be an $(n-1)$ -polyhedral complex and K_0 be a subcomplex of K . We say that a family of 1-cycles $\{z_t\}_{t \in K} \subset \Gamma_k(M)$ is a sweepout of M if

- For each $t \in K_0$ the cycle z_t has zero length
- $(\{z_t\}_{t \in K}, \{z_t\}_{t \in K_0})$ is not contractible in (Γ, Γ^0)

As noted above, constructing a family of cycles that is continuous in Γ_k is a stronger result than constructing a continuous family of flat cycles. Alternatively, we could start with a family that is only continuous in the flat norm and then use Almgren-Pitts theory of almost minimizing 1-dimensional varifolds as it was done in [P74] (see [P81] for a much more general and detailed exposition). The first step in this construction is to modify this family so that there is no cancellation of mass, that is, so that the family is continuous both in the mass and the flat topology, while controlling the mass of the maximal slice. This is analogous to the procedure that we perform in the proof of Lemma 5.1.

Both approaches lead to a min-max sequence that converges in the appropriate sense (in $\Gamma_k(M)$ topology in the first case or as varifolds for Almgren-Pitts theory) to a stationary geodesic net.

The min-max argument for families in $\Gamma_k(M)$ has the advantage of being simpler (see [NR04] and Appendix of [CC92]) and this is why we follow this approach.

It is often of interest to consider families of cycles that arise as fibers of a certain well-behaved mapping from M to a space of lower dimension. For example, if Σ is a 2-dimensional closed surface and $f : M \rightarrow [0, 1]$ is an onto Morse function then we consider the family $\{f^{-1}(t)\}_{t \in [0,1]}$. For this family we have the following result.

Lemma 3.1. *There exists a sweepout of Σ by 1-cycles $\{z_t\}_{t \in [0,1]} \subset \Gamma$, such that for each t the image of z_t coincides with $f^{-1}(t)$ except possibly for a finite collection of points.*

Proof. Let k be the maximum number of connected components of $f^{-1}(t)$, $t \in [0, 1]$. We define $\{z_t\} \subset \Gamma_k$ by induction on the number of singular points of f .

Since f is an onto Morse function we have that $f^{-1}(0)$ consists of $k_0 \leq k$ points p_1, \dots, p_{k_0} . Define the first k_0 components of $z_0 = (\gamma_0^1, \dots, \gamma_0^{k_0})$ to be $\gamma^i([0, 1]) = p_i$ and for $i > k_0$ set $\gamma_0^i([0, 1]) = p_{k_0}$. Let $t_1 > 0$ be the smallest critical value of f . For each $i \leq k_0$ and $t < t_1$ we can define a homotopy $\{\gamma_t^i\}_{0 \leq t \leq 1}$, so that $\gamma_t^i([0, 1])$ is a connected component of $f^{-1}(t)$ (we have $\gamma_t^i(0) = \gamma_t^i(1)$).

Let t' be a critical point of f and assume that z_t is defined for all $t < t'$. The singularity that occurs at t' may be a destruction/creation of a connected component of $f^{-1}(t)$ or a splitting/merging of two connected components. In the first case we proceed in the obvious way. Consider the case of a splitting. Choose a small $\epsilon > 0$ so that f has no critical values in $[t' - \epsilon, t')$. Since the number of connected components of $f^{-1}(t)$ is less than k for $t \in [t' - \epsilon, t')$ there exists a constant component $\gamma_t^{k'} = p$ of z_t . Let γ^m denote the component that splits into two at time t' . For $t \in [t' - \epsilon, t' - \epsilon/2)$ we deform homotopically $\gamma^{k'}$ to the point $\gamma_t^m(0) = \gamma_t^m(1)$. For $t \in [t' - \epsilon/2, t)$ we homotop γ^m and $\gamma^{k'}$ so that they form two arcs of the same connected component of $f^{-1}(t)$ and their endpoints approach the singular point of f at t' . For $t \geq t'$ we can split the two arcs into two distinct connected components. This ensure continuity of the family of cycles in $\Gamma(M)$. We deal with a merging of two components in a similar way.

This finishes the construction of a family of 1-cycles $\{z_t\}_{t \in [0,1]} \subset \Gamma(\Sigma)$ corresponding to f . To see that this family is a sweepout recall a result of Almgren [AF62] about homotopy groups of the space of 1-cycles.

Let $Z_1(M, \mathbb{Z})$ denote the space of integer flat cycles in M . Almgren constructed an isomorphism between homotopy groups of the space of cycles $\pi_k(Z_1(M, \mathbb{Z}), \{0\})$ and homology groups $H_{k+1}(M, \mathbb{Z})$ of the space M . Let $\Phi : \Gamma \rightarrow Z_1(M, \mathbb{Z})$ be the map that sends each cycle in Γ to the corresponding cycle in $Z_1(M, \mathbb{Z})$. We will show that $\Phi(\{z_t\}, \{z_0, z_1\})$ is not contractible in the space of flat cycles and hence, it is not be contractible in $\Gamma(M)$.

Recall the definition of Almgren's map. We pick a fine subdivision t_1, \dots, t_n of $[0, 1]$ and for each i consider a Lipschitz chain c_i filling $z_{t_i} - z_{t_{i-1}}$. The Almgren's map then sends the family $\{z_t\}$ to the homology class of $\sum_i c_i$. As long as the area of each filling c_i is sufficiently close to the minimal area, the exact choice of c_i does not matter. Hence, by our construction of $\{z_t\}$ it is immediately clear that $\sum_i c_i$ represents the generator of $H_2(\Sigma, \mathbb{Z})$. \square

In this paper we construct two families of cycles in a Riemannian 3-manifold of positive Ricci curvature: the family of fibers of a mapping $f : M \rightarrow \mathbb{R}^2$ and the corresponding family $\{z_t\}$ of cycles in $\Gamma(M)$, where z_t and $f^{-1}(t)$ coincide except possibly for a finite number of constant curves. Because of this correspondence we will often talk about them as if they are the same family.

4 Parametric sweepouts of surfaces

In this section we will prove a parametric version of the following theorem of Balacheff and Sabourau [BS10].

Theorem 4.1. *Let Σ be a Riemannian surface of genus γ , area A and with a (possibly empty) piecewise smooth boundary of length L . There exists a Morse function $f : \Sigma \rightarrow [0, 1]$, such that $f^{-1}(0) = \partial\Sigma$, and the length of $f^{-1}(x)$ is at most $C\sqrt{\gamma+1}\sqrt{A} + L$ for all $x \in [0, 1]$ and some universal $C < 1000$.*

This version of the theorem is slightly more general and the upper bound for constant C is better than in [BS10]. These improvements follow from the methods of [L13], [GL] and [L14].

Proof. To each boundary component of Σ we glue a very small disc to obtain a closed surface Σ' of area $A + \epsilon'$.

Let Σ_0 denote the unique surface of constant Gaussian curvature -1 , 0 or 1 , which lies in the conformal class of Σ and let $\phi : \Sigma_0 \rightarrow \Sigma'$ be a conformal diffeomorphism.

For each $U \subset \Sigma'$ we will construct a Morse function $f_U : U \rightarrow \mathbb{R}$, such that $f^{-1}(0) = \partial U$, and the length of $f^{-1}(x)$ is at most $616\sqrt{\gamma+1}\sqrt{Area(U)} + length(\partial U)$ for all $x \in [0, 1]$.

Step 1. Choose $\epsilon > 0$ much smaller than the injectivity radius of Σ' and suppose $Area(U) < \epsilon^2$. It follows from the systolic inequality on surfaces that the genus of U is 0. Let $\delta > 0$ be a small number. By Lemma 19 from [L13] if ϵ is sufficiently small then there exists a Morse function $f : U \rightarrow [0, 1]$ with $f^{-1}(0) = \partial U$ and the length of fibers at most $length(\partial U) + \delta$.

Step 2. Now we prove that for every open subset $U \subset \Sigma$ there exists a relative cycle c (with $\partial c \subset \partial U$), subdividing U into two subsets of area at least $\frac{1}{24}Area(U)$ and such that $length(c) < 6.48 \max\{1, \sqrt{\gamma}\}\sqrt{Area(U)}$. Let r be the smallest radius such that there exists a ball $B_r(x) \subset \Sigma_0$ (on the constant curvature conformal representative), such that $Area_{\Sigma'}(\phi(B_r(x)) \cap U) = \frac{Area_{\Sigma'}(U)}{12}$.

We consider two cases. Suppose first that $r \leq 1$ then it follows by comparison with a constant curvature space that the annulus $B_{3r/2}(x) \setminus B_r(x)$ can be covered by 10 discs of radius r in Σ_0 . Let x be such that $Vol_{\Sigma'}(\phi(B_r(x)) \cap U)$ is maximized. It follows from the choice of x and r that $Vol_{\Sigma'}(\phi(B_{3r/2}(x) \setminus B_r(x)) \cap U) \leq \frac{10}{12}Area_{\Sigma'}(U)$. Using the length-area method (cf. [L14]) we find that there exists a cycle in the image of the annulus of length

$$\leq \frac{1}{0.5} \sqrt{Area_{\Sigma_0}(B_{3r/2}(x) \setminus B_r(x))} \sqrt{\frac{10}{12}Area_{\Sigma'}(U)} \leq 4.12 \sqrt{Area_{\Sigma'}(U)}.$$

Now suppose $r \geq 1$. In this case we use an idea of [CM08] (cf. proof of Lemma 3.3 in [GL]) of considering systems of balls of radius 1. Let k be the smallest integer, such that there exist $x_1, \dots, x_k \in \Sigma_0$ with $Area_{\Sigma'}(\phi(\bigcup B_1(x_i)) \cap U) \geq \frac{Area_{\Sigma'}(U)}{24}$. Assume that x_1, \dots, x_k are

chosen in such a way that this quantity is maximized. Observe that by the choice of k we have $Area_{\Sigma'}(\phi(\bigcup_{i=1}^k B_1(x_i)) \cap U) < \frac{Area_{\Sigma'}(U)}{12}$. For each $i = 1, \dots, k$, let $B_1(y_i^1), \dots, B_1(y_i^{10})$ be a collection of 10 balls of radius 1 covering the annulus $B_{3/2}(x) \setminus B_1(x) \subset \Sigma_0$. By our choice of x_i for each $j = 1, \dots, 10$ we have $Area_{\Sigma'}(\phi(\bigcup_{i=1}^k B_1(y_i^j)) \cap U) \leq Area_{\Sigma'}(\phi(\bigcup_{i=1}^k B_1(x_i)) \cap U) < \frac{Area_{\Sigma'}(U)}{12}$. As in the case $r \leq 1$ we use coarea formula and the length-area method to find a relative cycle c in the image of $1/2$ -neighbourhood $\phi(\{x : 0 < dist_{\Sigma_0}(x, \bigcup_{i=1}^k B_1(x_i)) < 1/2\}) \cap U$ of length at most $\leq \frac{1}{0.5} \sqrt{Area_{\Sigma_0}(\Sigma_0)} \sqrt{\frac{10}{12} Area_{\Sigma'}(U)}$. Cycle c subdivides U into two parts each of area at least $\frac{1}{24} Area_{\Sigma'}(U)$. For a surface of genus γ and constant curvature 1, 0 or -1 we have $Area_{\Sigma_0}(\Sigma_0) \leq 4\pi \max\{1, \gamma - 1\}$. We conclude that the length of c is bounded above by $6.48 \max\{1, \sqrt{\gamma}\} \sqrt{Area_{\Sigma'}(U)}$.

Step 3. Let U_1 and U_2 be two open subsets of Σ with disjoint interiors and let $f_i : U_i \rightarrow [0, 1]$, $i = 1, 2$, be Morse functions, such that $f_i^{-1}(0) = \partial U_i$ and $length(f_i^{-1}(t)) \leq length(\partial U_i) + C$. By Lemma 18 in [L13] there exists a Morse function $f : U_1 \cup U_2 \rightarrow [0, 1]$ with $length(f_i^{-1}(t)) \leq length(\partial(U_1 \cup U_2)) + 2length(\partial U_1 \cap \partial U_2) + C$ and $f^{-1}(0) = \partial(U_1 \cup U_2)$.

Step 4. To finish the proof we combine three steps above in the following inductive argument. We claim that for each integer $0 \leq n \leq \log_{\frac{24}{23}} \frac{Area(\Sigma')}{\epsilon}$ and for every U with $(\frac{24}{23})^{n-1} \epsilon < Area(U) \leq (\frac{24}{23})^n \epsilon$ there exists a Morse function $f_U : U \rightarrow \mathbb{R}$, such that $f^{-1}(0) = \partial U$, and the length of $f^{-1}(x)$ is at most $616\sqrt{\gamma + 1} \sqrt{Area(U)} + length(\partial U) + \delta$ for all $x \in [0, 1]$.

By Step 1 the claim is true for $n = 0$. Suppose that every subset of Σ' of area at most $(\frac{24}{23})^{n-1} \epsilon$ satisfies the inductive hypothesis. By Step 2 we can subdivide U into two subsets of area $\leq (\frac{24}{23})^{n-1} \epsilon$ by a cycle of length $6.48 \max\{1, \sqrt{\gamma}\} \sqrt{Area(U)}$. By Step 3 there exists a desired Morse function with length of fibers $\leq 616\sqrt{\gamma + 1} \sqrt{\frac{23}{24} Area(U)} + 2 * 6.48\sqrt{\gamma + 1} \sqrt{Area(U)} + length(\partial U) + \delta$. δ can be chosen much smaller than $Area(U)$. This finishes the inductive argument. \square

We state a parametric version of this result.

Theorem 4.2. *Let Σ be a surface of genus γ and let $\{g_t\}_{t \in [0,1]}$ be a smooth family of Riemannian metrics on Σ , such that the area $A(\Sigma, g_t) \leq A$ for some constant A . There exists a continuous family of functions $f_t : (\Sigma, g_t) \rightarrow \mathbb{R}$, $t \in [0, 1]$, such that for each $x \in \mathbb{R}$ we have that $f_t^{-1}(x)$ is a 1-cycle in (Σ, g_t) of length at most $2000\sqrt{(\gamma + 1)A}$. All functions f_t are Morse except possibly for finitely many with a finite number of ordinary cusp singularities. There exists a corresponding continuous family of sweepouts in $\Gamma(\Sigma)$ (see Section 3).*

Theorem 4.2 easily follows from the following proposition conjectured by A. Nabutovsky in a conversation with one of the authors.

Proposition 4.3. *Let Σ be a closed Riemannian surface and let $f_i : \Sigma \rightarrow \mathbb{R}$, $i = 0, 1$, be two Morse functions, such that the length of $f_i^{-1}(x)$ is bounded above by L for all x . Then f_0 and f_1 are homotopic through functions f_t , $0 \leq t \leq 1$, with the length of $f_t^{-1}(x)$ bounded above by $2L + \epsilon$ for all x and t and arbitrarily small $\epsilon > 0$. Functions f_t are all Morse, except for finitely many, which may have a finite number of ordinary cusp singularities. There exists a corresponding continuous family of sweepouts in $\Gamma(M)$.*

Note that cusp singularities occur in a generic 1-parameter family of functions, so we can not expect them all to be Morse.

Proof. Without any loss of generality we may assume that $f_i(\Sigma) = [0, 1]$ for $i = 0, 1$. Let $z_s^i = f_i^{-1}(s)$, $s \in [0, 1]$, be the 1-parametric family of 1-cycles given by the level sets of f_i . Since f_i is a Morse function we have that the family $\{z_s^i\}$ is a foliation with finitely many singular leaves. The singularities are either constant curves or curves with transverse self-intersections.

The set of singular points of f_1 and f_2 is finite. We can make finitely many local perturbations so that the lengths of preimages changes by less than ϵ and the singular points of f_1 and f_2 are disjoint. Moreover, by Thom's Multijet Transversality we can make a perturbation so that each pre-image of f_1 has only finitely many tangency points with preimages of f_2 .

To summarize, without any loss of generality we may assume that families z_s^i satisfy the following:

1. If $z_{s'}^1$ is a singular leaf and x is a singular point of $z_{s'}^1$, then it is disjoint from singular points of z_s^0 for all s .
2. For each s' all but finitely many z_s^0 intersect $z_{s'}^1$ transversely; z_s^0 and $z_{s'}^1$ have at most one non-transverse touching away from the singular points of z_s^0 and $z_{s'}^1$.

Let $t \in [0, 1]$. We define a 1-parameter family z_s^t , $s \in [0, 1]$, as follows.

For $s \leq t$ we set $z_s^t = z_s^1$. For $s > t$ we set $z_s^t = \partial(f_1^{-1}((-\infty, t]) \cup f_0^{-1}((-\infty, \frac{s-t}{1-t})))$. The family of cycles $\{z_s^t\}$ is illustrated on Figure 1. Note that for each s and t the cycle z_s^t consists of arcs of cycles z_s^1 and $z_{\frac{s-t}{1-t}}^0$. Moreover, by properties (1) and (2) each z_s^t is a collection of finitely many piecewise smooth curves with a finite number of corners. We can smooth out the corners by a small perturbation. Using properties (1) and (2) we can perturb family z_s^t so that it contains only finitely many singular leaves. Hence, we have constructed a continuous family of sweepouts by 1-cycles in $\Gamma(\Sigma)$.

Now we would like to construct a continuous family of Morse functions with short fibers. Note that the above family $\{z_s^t\}$ (for t fixed) is not a foliation because cycles $z_{s_1}^t$ and $z_{s_2}^t$ may overlap. Also, we can not just glue together different fibers of f_0 and f_1 because they may correspond to different values.

We can fix this as follows. Choose $\epsilon_0 > 0$ sufficiently small, so that every ball of radius ϵ_0 is $(1 + \epsilon)$ -bilipschitz diffeomorphic to a disc on the plane of the same radius. Let $\delta > 0$ be such that cycles $f_0^{-1}(t)$ and $f_1^{-1}(t)$ have length less than ϵ_0 for all $t \in [0, 2\delta] \cup [1 - \delta, 1]$.

Let $t \in [\delta, 1 - \delta]$. For $x \in f_1^{-1}([0, t])$ we set $f_t(x) = f_1(x)$. Consider a foliation of $f_1^{-1}([t, 1])$ by cycles $z_{t+\frac{s}{\delta}(1-t)}^{t+s}$ for $s \in [0, \delta]$. This foliation will be the level sets of f_t on $f_1^{-1}([t, 1])$. We set $f_t(x) = t + s$ for $x \in z_{t+\frac{s}{\delta}(1-t)}^{t+s}$. In particular, for $f_1^{-1}([t + \delta, 1])$ we have $f_t(x) = t + \delta f_0(x)$.

For $t = \delta$ we have that $f_t(x) = \delta + \delta f_0(x)$ everywhere except for a small set $U = f_1^{-1}([0, 2\delta])$. We linearly homotop $f_\delta(x)$ to $\frac{f_\delta(x) - \delta}{\delta}$. On $\Sigma \setminus U$ the function now coincides with f_0 . Since U is contained in a collection of small discs $(1 + \epsilon)$ -bilipschitz to discs in the plane, we can homotop the function to f_0 in U while controlling the lengths of the fibers.

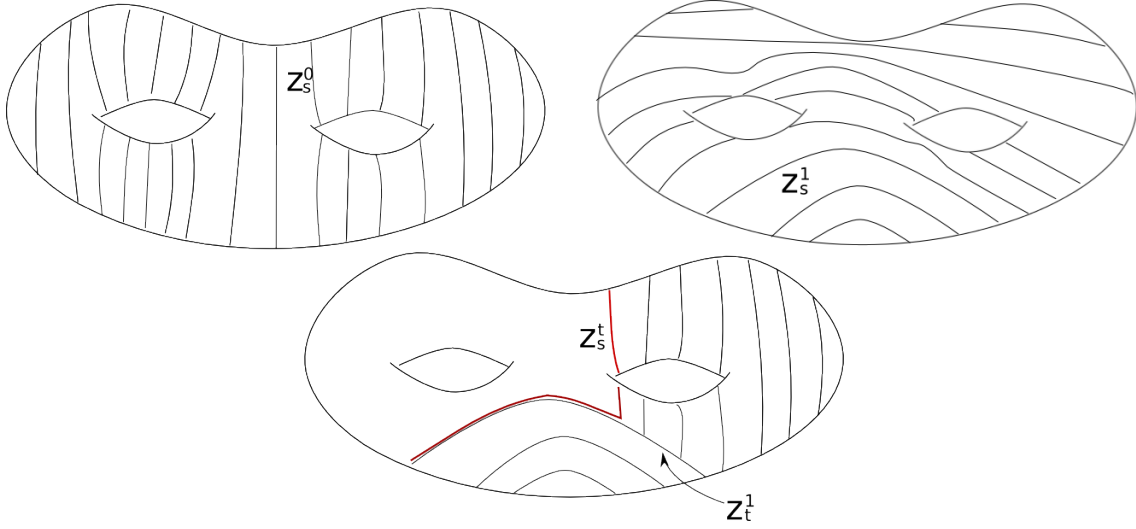


Figure 1: Homotopy through short sweepouts

When $t = 1 - \delta$ function f_t coincides with f_1 everywhere except for a small set $f_1^{-1}([1 - \delta, 1])$. Similarly, we homotop the function to f_1 while controlling the lengths of the fibers. \square

We now prove Theorem 4.2. Let C be the constant from Theorem 4.1. Fix a small $\epsilon > 0$. Subdivide $[0, 1]$ into n sufficiently small intervals $[t_i, t_{i+1}]$, so that for any $t_a, t_b \in [t_i, t_{i+1}]$ we have $(1 - \epsilon)^2 g_{t_a} \leq g_{t_b} \leq (1 + \epsilon)^2 g_{t_a}$.

For each i let $f_{t_i} : (\Sigma, g_{t_i}) \rightarrow \mathbb{R}$ be a Morse function from Theorem 4.1 with fibers of length at most $C\sqrt{(\gamma + 1)A}$. If $t \in [t_i, t_{i+1}]$ then we have that the functions $f_t : (\Sigma, g_t) \rightarrow \mathbb{R}$ and $f_{t_{i-1}} : (\Sigma, g_t) \rightarrow \mathbb{R}$ have fibers of length at most $(1 + \epsilon)C\sqrt{(\gamma + 1)A}$. By Proposition 4.3 there exists a family of Morse functions $\{h_r : \Sigma \rightarrow \mathbb{R} : r \in [0, 1]\}$, such that $h_0 = f_{t_{i-1}}$ and $h_1 = f_t$. Moreover, for any $r \in [0, 1]$ the fibers of h_r have length at most $2(1 + \epsilon)C\sqrt{(\gamma + 1)A} + \epsilon < 2000\sqrt{(\gamma + 1)A}$, when measured with respect to g_t . We then set $f_t = h_{\frac{t-t_{i-1}}{t_i-t_{i-1}}}$ for $t \in [t_i, t_{i+1}]$. This finishes the proof of Theorem 4.2.

5 Proof of Theorems 1.1 and Corollary 1.2

In this section we give a proof of Theorem 1.1.

Theorem 5.1. *Given a 3-dimensional Riemannian manifold M with an arbitrary metric of positive Ricci curvature, there exists a smooth map $f : M \rightarrow \mathbb{R}^2$ with fibers of length at most $C \text{Vol}(M)^{\frac{1}{3}}$. The family of pre-images $\{f^{-1}(x)\}_{x \in \mathbb{R}^2}$ corresponds to a continuous sweepout in $\Gamma(M)$.*

Proof. Let M be a 3-manifold of positive Ricci curvature. By Theorem 1.3 we have the following two possibilities:

Case 1. There exist a smooth function $f_0 : M \rightarrow [-1, 1]$, such that the fibers of f for $t \in (-1, 1)$ form a family of smooth diffeomorphic surfaces of genus $\gamma \leq 3$ and area $\leq C \text{vol}(M)^{2/3}$ and $f_0^{-1}(-1)$ and $f_0^{-1}(1)$ are graphs. Decomposition into cycles of controlled length then immediately follows by Theorem 4.2. The fact that these 1-cycle are continuous in $\Gamma(M)$ follows from the fact that $\{\Sigma_t\}$ are continuous in the smooth topology.

Now we consider Case 2 of Theorem 1.3. Let $\Sigma_0 \subset M$ be a non-orientable min-max minimal surface as in the theorem. Let $\gamma \leq 3$ be the genus of the double cover $\tilde{\Sigma}_0$ of Σ_0 . Let $S_t = \{x \in M : \text{dist}(x, \Sigma_0) = t\}$ be the set of all points at a distance t from Σ_0 . We have that for a sufficiently small $\delta > 0$ and all $0 < t \leq \delta$, the surface S_t is bi-Lipschitz diffeomorphic to the double cover $\tilde{\Sigma}_0$ of Σ_0 with bi-Lipschitz constant $1 + \epsilon$. Let $U = \Sigma_0 \cup \{S_t\}_{t \in (0, \delta]}$ denote the tubular neighborhood of Σ_0 .

Remark 5.2. Suppose we are interested in constructing a function $f : M \rightarrow \mathbb{R}^2$ with fibers forming a family of 1-cycles of controlled length and continuous in the *flat norm*, but not necessarily continuous in $\Gamma(M)$. Then we can argue as follows.

Let $f_0 : \Sigma_0 \rightarrow [0, 1] \times \{0\} \subset \mathbb{R}^2$ be the Morse function from Theorem 4.1. Composing with the covering map we obtain a map from the double cover $\tilde{f}_0 : \tilde{\Sigma}_0 \rightarrow [0, 1] \times \{0\}$. Let \tilde{M} denote the manifold with boundary from Theorem 1.3, such that interior of \tilde{M} is isometric to $M \setminus \Sigma_0$ and $\partial \tilde{M} = \tilde{\Sigma}_0$. By Theorem 4.2 we construct a function $\tilde{h} : \tilde{M} \rightarrow [0, 1]^2$, such that the restriction of \tilde{h} to $\partial \tilde{M}$ is \tilde{f}_0 . We then define $h(x) = \tilde{h}(x)$ for all $x \in M \setminus \Sigma_0$ and $h(x) = f_0(x)$ for $x \in \Sigma_0$.

In the remainder of the proof we will modify this construction in order to produce a family, which is continuous in $\Gamma(M)$. Our argument is similar to constructions of families of cycles with no cancellation of mass in the works of Almgren and Pitts (see [P81]).

By Theorem 1.3 there exists a sweepout of $M \setminus U$ by surfaces of controlled area. As in the first case we construct a map $f : M \setminus U \rightarrow [0, 1] \times [0, 1]$ with preimages of controlled length, and such that the preimages form a continuous family of 1-cycles. Moreover, we can do it in such a way so that f restricted to ∂U is a Morse function and $\{f^{-1}(t, 0)\}_{t \in [0, 1]}$ is a family of 1-cycles sweeping out ∂U .

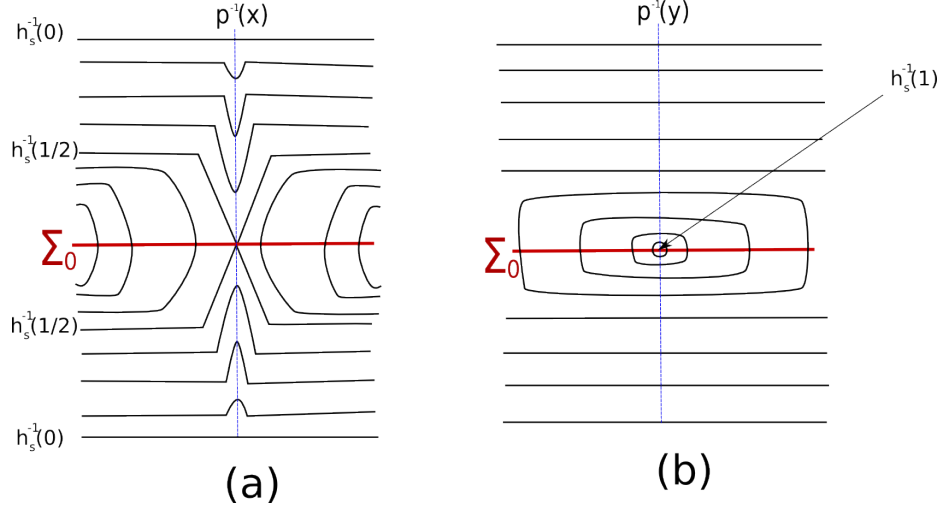
Next we construct an extension of this map to U .

Lemma 5.3. *There exists a map $h : U \rightarrow [0, 1] \times [-1, 0]$, such that the length of $h^{-1}(t, s)$, $(t, s) \in [0, 1] \times [-1, 0]$ is at most $10^4 \sqrt{\text{Area}(\Sigma_0)}$; $h(\cdot, 0) : \partial U \rightarrow [0, 1] \times 0$ is a Morse function; and the family of cycles $\{h^{-1}(t, s) : (t, s) \in [0, 1] \times [-1, 0]\}$ is continuous in Γ .*

Proof. Let $p : U \rightarrow \Sigma_0$ be the projection map, i.e. p is the identity map on Σ_0 and it sends $x \in S_t \subset U$ to the unique point $y \in \Sigma_0$ with $\text{dist}(x, y) = t$ for $t \in (0, \delta]$. Let p_t denote the restriction of p to S_t . Observe that p_t is locally $(1 + \epsilon)$ -bi-Lipschitz.

By Theorem 4.1 there exists a Morse function $g : \Sigma_0 \rightarrow \mathbb{R}$, such that all preimages of g are bounded in length by $1600\sqrt{\gamma + 1}\sqrt{A}$. We may assume that $g(\Sigma_0) = [0, 1]$. Let $c_s \subset \Sigma_0$ denote the 1-cycle $g^{-1}(s)$. For each $s \in [0, 1]$ we will define a function $h_s : p^{-1}(c_s) \rightarrow [0, 1]$ with fibers of controlled length.

Suppose first that c_s is the pre-image of a regular value of g . We can write c_s as a union of finitely many disjoint embedded circles $c_s = \bigsqcup c_s^i$ in Σ_0 .

Figure 2: Foliation of V_x and V_y by fibers of function h_s .

The set $p^{-1}(c_s^i)$ is diffeomorphic to a cylinder or a Mobius band. We will construct a foliation of $p^{-1}(c_s^i)$ by 1-cycles and use it to define function h . Choose two distinct points x and y on c_s^i and consider a small tubular neighborhood V_x (resp. V_y) of $p^{-1}(x)$ (resp. $p^{-1}(y)$) in $p^{-1}(c_s^i)$. We foliate V_x by 1-cycles as depicted on Figure 2(a). Clearly we can define a smooth function from V_x to $[0, 1]$, whose fibers are 1-cycles in the foliation and which is non-degenerate everywhere except for a saddle point at x . Call this type of foliation of V_x a *saddle foliation*. Similarly, Figure 2(b) depicts a foliation of V_y and we define the corresponding function from V_y to $[0, 1]$ with a singularity of index 2 at y (a maximum point for h_s). Call this type of foliation of V_y as *node foliation*. We can extend V_x and V_y so that they cover all of $p^{-1}(c_s^i)$ and extend the corresponding foliations and functions in the obvious way. Observe that the lengths of the preimages are bounded above by $2\text{length}(c_s^i) + O(\epsilon)$. For $u \in p^{-1}(c_s)$ we define the function $h(u) = (s, -h_s(u)) \in [0, 1] \times [-1, 0]$. Hence, we described a construction of h on each connected component of $p^{-1}(c_s)$ in the case when c_s is non-singular. As s varies we can vary x and y and the corresponding foliations continuously and extend the map h to $p^{-1}(g^{-1}([s, s+a]))$, where $s+a$ is the first singular value of g after s .

Let c_s^i be a singular connected component of c_s . Consider the case when c_s^i is a point. This occurs when we have a creation or a destruction of a connected component. Suppose first that $c_{s+\epsilon}$ has one less connected components than $c_{s-\epsilon}$. The case of creation of a connected component is treated similarly. We modify the function h on a neighborhood $V = p^{-1}(\{c_{s'}^i\}_{s' \in [s-\epsilon, s]}) \subset U$ as follows. Let $D_\epsilon \subset \mathbb{R}^2$ denote a disc of radius ϵ . Observe that there exists a diffeomorphism $\phi : D_\epsilon \times [-\delta, \delta] \rightarrow V$, such that for a concentric circle $S_r \subset D_\epsilon$ of radius r and $t \in (0, \delta]$ we have $\phi(S_r \times \{t, -t\}) = \Sigma_t \cap p^{-1}(c_{s-r}^i)$. Let us consider the set $\phi(S_r \times \{t\})$. For $|t| \geq \frac{\delta}{\epsilon}r$ we define h on $\phi(S_r \times \{t\})$ to be equal to $(s-r, \frac{|t|}{\delta}-1)$. In particular, if $x \in p^{-1}(c_s^i)$ and $x \neq c_s^i$ then $h^{-1}(h(x))$ consists of two points. For $|t| < \frac{\delta}{\epsilon}r$ we construct $h(x)$ exactly as how we constructed h_s on the pre-image under p of a non-singular curve, but we scale the image so that h is continuous along the diagonal $|t| = \frac{\delta}{\epsilon}r$. We do it as follows.

Fix $r \in [0, \epsilon)$ and let $W = \phi(S_r \times (-\frac{r}{\epsilon}\delta, \frac{r}{\epsilon}\delta))$. Construct a foliation of W by 1-cycles with two singularities as on Figure 2. We then define h on W so that $h(W) = \{s - r\} \times [-1, -1 + \frac{r}{\epsilon}]$ and its preimages are given by 1-cycles in the foliation.

Suppose now that c_{s+a}^i is a figure-8 curve with a self-intersection at a point $z \in c_{s+a}^i$. This means that we either have a splitting of one component into two or a merging of two components into one (for otherwise we could perturb this family of cycles so that no singularity would occur). We consider the case of two components merging and the argument for the other case is analogous. For $s' < s + a$ let $c_{s'}^i$ and $c_{s'}^{i+1}$ be two components that merge into c_{s+a}^i at time $s + a$. We arrange $h_{s'}$ on $p^{-1}(c_{s'}^j)$ ($j = i, i + 1$) so that $h_{s'}$ takes on its maximum at a point $y_j(s')$ on $p^{-1}(c_{s'}^j)$ with $y_j(s')$ converging to the self-intersection point of the figure-8: $y_j(s') \rightarrow z$ as $s' \rightarrow s + a$. For $p^{-1}(c_{s+a}^i)$ we obtain two node foliations (Figure 2(b)) glued along $p^{-1}(z)$. Observe that we can arrange the foliations to match properly so that they correspond to preimages of a smooth function on $p^{-1}(c_{s+a}^i)$ and, moreover, we can extend it to a foliation of $p^{-1}(c_{s+a-\epsilon}^i)$ by cycles satisfying $2\text{length}(c_{s+a-\epsilon}^i) + O(\epsilon)$ upper bound on their lengths. This foliation has two node foliations and two saddle foliations. We can arrange for one node foliation and one saddle foliation to collide and annihilate in a saddle-node bifurcation. This can be done without increasing the length bounds by more than $O(\epsilon)$. (Thinking of h_s as a height function one can picture saddle-node bifurcation as smoothing out a hill). This completes the construction of h . It is clear from the construction that the corresponding family of cycles is continuous in $\Gamma(M)$. \square

From the construction in the proof it follows that $\{h^{-1}(s, 0)\}$, $s \in [0, 1]$, is a family of 1-cycles sweeping out ∂U . Moreover h has only finitely many singularities on ∂U , all of them non-degenerate. By Proposition 4.3 $\{h^{-1}(s, 0)\}$ and $\{f^{-1}(s, 0)\}$ can be connected by a family of sweepouts of controlled length. After a small perturbation this produces the desired map from M to \mathbb{R}^2 with fibers of controlled length. \square

We now use Theorem 1.1 to give alternative proofs of results of Gromov and Nabutovsky-Rotman stated in Corollary 1.2. First, let $M \cong \mathbb{R}P^3$. By Theorem 1.1 there exists a function $f : M \rightarrow [0, 1]^2$ and a continuous sweepout $\{z_t = f^{-1}(t)\}_{t \in [0, 1]^2} \subset \Gamma$ of M by 1-cycles of length at most $CVol(M)^{\frac{1}{3}}$. Corollary 1.2 follows from the lemma below.

Lemma 5.4. *A connected component of some cycle z_t in the sweepout that we constructed is a non-contractible loop in M .*

Proof. The idea of the proof of this lemma was suggested to us by Nabutovsky and Rotman (see also [GZ], where a version of this lemma is proved for sweepouts of 2-dimensional tori). For contradiction we assume that every connected component of z_t is contractible for all t .

Let $f : M \rightarrow [0, 1]^2$ be the map constructed in Theorem 1.1. From [Gro83, p.128] recall the definition of a connected map $\bar{f} : M \rightarrow X$ associated to f . The set X is defined as the quotient of M by the equivalence relation $x \sim y$ if x and y are in the same connected component of $f^{-1}(t)$ for some $t \in [0, 1]^2$ and $\bar{f} : M \rightarrow X$ is the quotient map. There is a unique map $\tilde{f} : X \rightarrow [0, 1]^2$, such that $\tilde{f} \circ \bar{f} = f$. Observe that X must be connected since M is. Also, by our construction of f we can endow X with the structure of a polyhedral complex, such that for an interior point x of

every face we have that every pre-image $\bar{f}^{-1}(x)$ is a simple closed curve in M and for each point x contained in the 1-skeleton of X the pre-image $\bar{f}^{-1}(x)$ is a point or a closed curve with a finite number of self-intersections.

We claim that X is simply-connected. Indeed, since \bar{f} is a connected map every loop in X lifts to a loop in M , so the induced map on the fundamental groups is surjective. On the other hand, the cohomology ring structure of $\mathbb{R}P^3$ forces the induced map on homology to be trivial.

Let \bar{M} denote the universal cover of M and $p : \bar{M} \rightarrow M$ be the covering map. Consider the composition $F = \bar{f} \circ p$. By our assumption for each x we have that $\bar{f}^{-1}(x)$ is a contractible closed curve in M , so $F^{-1}(x)$ consists of k disjoint closed curves in \bar{M} . Let $\bar{F} : \bar{M} \rightarrow \bar{X}$ be the connected map associated to F . We obtain that \bar{X} is a k -fold covering space for X . This contradicts the fact that X is simply-connected. \square

Now by applying Brikhoff curve-shortening process we obtain a closed geodesic in M of length at most $CVol(M)^{\frac{1}{3}}$ proving Corollary 1.2 for the case when $M \cong \mathbb{R}P^3$.

If M is homeomorphic to a 3-sphere then by the min-max argument described in [NR04] existence of a sweepout, which is continuous in $\Gamma(M)$ by short 1-cycles implies existence of a geodesic net with the desired length bound.

6 Further discussion

The first open question is the relation between the Almgren-Pitts min-max minimal surface and the Simon-Smith (also Colding-De Lellis) min-max minimal surface in (S^3, g) with positive Ricci curvature (or even in any 3-manifold with positive Ricci curvature). The Almgren-Pitts minimal surface, which we use in this paper, has area bounded by the $\frac{2}{3}$ -power of the volume up to a universal constant, and genus ≤ 3 ; while the Simon-Smith min-max minimal surface has genus 0 (Heegaard genus for S^3) but no a priori area bound in terms of the volume of the ambient manifold. It is then a natural question to compare them.

The second open question is whether we could have diameter bound for the whole min-max family constructed in Theorem 1.3 when we assume the scalar curvature lower bound instead of just getting a diameter bound for the min-max surface as in Theorem 1.5.

These two questions are related to the problem of finding an upper bound for the length of the shortest non-trivial closed geodesic in manifold (S^3, g) . The methods of this paper produce a sweepout of (S^3, g) by short 1-cycles which yields a stationary geodesic net in (S^3, g) of controlled length, but they do not give any bound for the length of the shortest closed geodesic. For this purpose one would need to consider sweepouts by loops instead of 1-cycles. If a manifold (S^3, g) admits a sweepout by 2-spheres or 2-tori of controlled area A and diameter d then it seems plausible that using methods of [LNR] one could bound the length of the shortest closed geodesic in (S^3, g) in terms of A and d .

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