

MIN-MAX THEORY FOR CONSTANT MEAN CURVATURE HYPERSURFACES

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ABSTRACT. In this paper, we develop a min-max theory for the construction of constant mean curvature (CMC) hypersurfaces of prescribed mean curvature in an arbitrary closed manifold. As a corollary, we prove the existence of a nontrivial, smooth, closed, almost embedded, CMC hypersurface of any given mean curvature c . Moreover, if c is nonzero then our min-max solution always has multiplicity one.

0. INTRODUCTION

A hypersurface Σ^n in a Riemannian manifold M^{n+1} has constant mean curvature if it is a critical point of the area functional amongst variations preserving the enclosed volume. Equivalently, hypersurfaces of constant mean curvature c are critical points of the Lagrange-multiplier functional

$$(0.1) \quad \mathcal{A}^c = \text{Area} - c \text{Vol}.$$

Constant mean curvature (CMC) hypersurfaces constitute a classical and extensively-studied topic in differential geometry, and play an essential role in many areas, from isoperimetric problems [45] to the modeling of interface phenomena [30, 31] and to general relativity [23, 44, 10]. Despite the classical nature of the problem, relatively few examples of closed CMC hypersurfaces were known, even for $n = 2$, until the breakthrough work of Wente [56]. Many attempts have been made to construct more CMC hypersurfaces, especially with *prescribed* constant mean curvature, c.f. [20, 21, 54, 24, 59, 15, 42, 32, 47]. However, these works left wide open the question of which values may be prescribed - that is, for which constants c does there exist a closed hypersurface of constant mean curvature c ?

In this article we construct, via a min-max approach, nontrivial closed CMC hypersurfaces of *any* prescribed mean curvature, in any smooth closed Riemannian manifold M^{n+1} of dimension at most seven. (The dimensional restriction arises from, and matches with, the well-known regularity theory for minimal hypersurfaces [50, 49]; see also [58].)

Theorem 0.1. *Let M^{n+1} be a smooth, closed Riemannian manifold of dimension $3 \leq n + 1 \leq 7$. Given any $c \in \mathbb{R}$, there exists a nontrivial, smooth, closed, almost embedded hypersurface Σ^n of constant mean curvature c .*

Here we say that an immersed hypersurface Σ is almost embedded if Σ locally decomposes into smoothly embedded components that (pairwise) lie to one side of each other. That is, the sheets may touch but not cross; the example of touching spheres shows that this regularity is optimal. In particular, almost embedded hypersurfaces are automatically Alexandrov embedded.

We want to compare our result with a classical problem by Arnold [5, page 395] and Novikov [41, Section 5] on the periodic orbits of a charged particle in a magnetic field on a topological two sphere. It is conjectured that there exist closed embedded curves of any prescribed constant geodesic curvature. This conjecture remains open, and we refer to [16, 46, 48] for more backgrounds and some partial results of this conjecture. Our result can be viewed as a complete resolution of the higher dimensional analog of Arnold-Novikov conjecture.

Remark 0.2. For $c \neq 0$, we can prove that our min-max procedure converges to the constructed hypersurface Σ with multiplicity 1. This is a stark and surprising contrast to the minimal ($c = 0$) case, for which the min-max multiplicity 1 conjecture is a fundamental open problem [34].

The existence problem for CMC hypersurfaces has been studied from a number of perspectives. The boundary value problems were substantially developed by Heinz [20], Hildebrandt [21], Struwe [54, 55], etc. using the mapping method, and by Duzaar-Steffen [15] using geometric measure theory, while both methods can only produce CMC hypersurfaces whose mean curvatures satisfy certain upper bound. For the case of closed CMC hypersurfaces, the more classical approach is to minimize the area functional amongst volume-preserving variations, that is, to solve the isoperimetric problem for a given volume. Indeed, for each fixed volume there exists a smooth minimizer (up to a singular set of codimension 7; see for instance [2, 39, 45]). However, this approach does not yield any control on the value of the mean curvature.

Another class of approaches relies on perturbative methods. Given a closed minimal hypersurface, one may deform it to a CMC hypersurface, but only for very small values of the mean curvature. On the other end, one attempts to construct foliations by closed CMC hypersurfaces near minimal submanifolds of strictly lower dimension. This program was carefully implemented in various cases by Ye [59], Mahmoudi-Mazzeo-Pacard [32], and others (see the survey article [42]). The hypersurfaces produced by this approach necessarily have large mean curvatures, which in fact diverge as the hypersurfaces condense onto the minimal submanifold.

We also mention the delicate gluing procedures pioneered by Kapouleas [24] as well as the degree theory developed by Rosenberg-Smith [47]. These provide important examples of CMC hypersurfaces, but the former is typically restricted by the availability of known solutions, whilst the latter can only produce CMC hypersurfaces of fairly large mean curvature greater than some threshold depending on ambient manifolds. Finally, we remark that Meeks-Mira-Perez-Ros [36, 37] were able to determine, in the special case of homogeneous ambient 3-manifolds, precisely the values for which there exists a CMC 2-sphere with the specified mean curvature.

In order to prove Theorem 0.1, we instead study CMC hypersurfaces from the perspective of the \mathcal{A}^c -functional. It is easy to see that the simplest method, minimization, does not succeed in detecting a nontrivial critical point for the \mathcal{A}^c -functional. In fact, the minimizer of \mathcal{A}^c among domains Ω in M with smooth boundary is always the total manifold M , as $\mathcal{A}^c(M) = -c \text{Vol}(M) \leq \mathcal{A}^c(\Omega)$. Therefore, the min-max method becomes the natural way to find nontrivial critical points of \mathcal{A}^c . (Note that minimization method does succeed for the Plateau problem with fixed boundary [15].)

For finding critical points of the area functional - that is, minimal hypersurfaces - the min-max method has been greatly successful. In [3], Almgren initiated a celebrated program to develop a variational theory for minimal submanifolds in Riemannian manifolds of any dimension and co-dimension using geometric measure theory, namely the min-max theory for minimal submanifolds. He was able to prove the existence of a nontrivial weak solution as stationary integral varifolds [4]. Higher regularity was established in the co-dimension-one case by the seminal work of Pitts [43] (for $2 \leq n \leq 5$) and later extended by Schoen-Simon [37] (for $n \geq 6$). Colding-De Lellis [11] established the corresponding theory using smooth sweepouts based on ideas of Simon-Smith [52]. Indeed, the preceding body of work completely resolved the $c = 0$ case of Theorem 0.1.

Very recently, in a series of works, Marques-Neves [33, 1, 35] found surprising applications of the Almgren-Pitts min-max theory to solve a number of longstanding open problems in geometry, including their celebrated proof of the Willmore conjecture. Due to these tremendous successes, there have been a vast number of developments of this program in various contexts, including [14, 38, 18,

26, 34, 29, 9, 13, 28, 25, 53]. In this regard, our work represents a natural extension of the min-max method to the CMC setting.

0.1. Min-max procedure. We now give a heuristic overview of our min-max method. In the main proofs, for technical reasons we will work with discrete families as in Almgren-Pitts, but here we will describe the key ideas using continuous families to elucidate those ideas.

Let M, c be as in Theorem 0.1. We only need to consider $c > 0$. The \mathcal{A}^c functional (0.1) is defined on open sets Ω with smooth boundary by $\mathcal{A}^c(\Omega) = \text{Area}(\partial\Omega) - c \text{Vol}(\Omega)$. Denote $I = [0, 1]$. Consider a continuous 1-parameter family of sets with smooth boundary

$$\{\Omega_x : x \in I\}, \text{ with } \Omega_0 = \emptyset \text{ and } \Omega_1 = M.$$

Fix such a family $\{\Omega_x^0\}$, and consider its homotopy class $[\{\Omega_x^0\}] = \{\{\Omega_x\} \sim \{\Omega_x^0\}\}$. The *c-min-max value* is defined as

$$\mathbf{L}^c = \inf_{\{\Omega_x\} \sim \{\Omega_x^0\}} \max\{\mathcal{A}^c(\Omega_x) : x \in I\}.$$

A sequence $\{\{\Omega_x^i\} : i \in \mathbb{N}\}$ with $\max_{x \in I} \mathcal{A}^c(\Omega_x^i) \rightarrow \mathbf{L}^c$ is typically called a *minimizing sequence*, and any sequence $\{\Omega_{x_i}^i : x_i \in (0, 1), i \in \mathbb{N}\}$ with $\mathcal{A}^c(\Omega_{x_i}^i) \rightarrow \mathbf{L}^c$ is called a *min-max sequence*.

Our main result (for the precise statement see Theorem 3.8) then says that there is a nice minimizing sequence $\{\{\Omega_x^i\} : i \in \mathbb{N}\}$, and some min-max sequence $\{\Omega_{x_i}^i : x_i \in (0, 1), i \in \mathbb{N}\}$, such that:

Theorem 0.3. *The sequence $\partial\Omega_{x_i}^i$ converges as varifolds with multiplicity one to a nontrivial, smooth, closed, almost embedded hypersurface Σ of constant mean curvature c .*

Our proof broadly follows the Almgren-Pitts scheme, but with several important difficulties. This scheme proceeds generally as follows:

- Construct a sweepout with positive width, and extract a minimizing sequence;
- Apply a ‘tightening’ map to construct a new sequence whose varifold limit satisfies a variational property and an ‘almost-minimizing’ property;
- Use these properties to construct ‘replacements’ on annuli which must be regular;
- Apply successive concentric annular replacements to the min-max limit and show that they coincide with each other, and hence extend to the center;
- Show that the min-max limit coincides with the replacement near the center.

We in fact show that \mathbf{L}^c is positive on any sweepout, as a consequence of the isoperimetric inequality for small volumes (see Theorems 2.15 and 3.9).

For the tightening, in the minimal ($c = 0$) case one shows that for the tightened sequence, any min-max (varifold) limit must be stationary, that is, a weak solution in the sense of first variations. In the CMC setting it is an important, yet subtle, issue to determine the correct variational property to replace stationarity. For instance, the \mathcal{A}^c functional is not well-defined on varifolds, so it is difficult to formulate a notion of weak solution for its critical points. To overcome this issue, we utilize the property of *c-bounded first variation*. This notion is a generalisation of bounded mean curvature, and is loose enough to be satisfied by our min-max limit V (after tightening) whilst providing enough control to develop the regularity theory. In particular, varifolds with *c-bounded first variation* satisfy a uniform monotonicity formula, and any blowups are stationary. Furthermore, we formulate the property of being *c-almost minimizing*, which is inspired by the almost-minimizing property of Almgren-Pitts, with the area functional replaced by the \mathcal{A}^c functional (see Definition 5.1).

To construct our so-called *c-replacements*, we solve a series of constrained minimization problems for the \mathcal{A}^c -functional in a subset $U \subset M$. Each \mathcal{A}^c -minimizer is an open set Ω_i^* with stable, regular CMC boundary in U , and the *c-replacement* V^* is obtained as the varifold limit $\lim |\partial\Omega_i^*|$. At this

point, V^* gains regularity by classical curvature estimates, but in contrast to the minimal case, V^* is merely almost embedded since we only have a *one-sided* maximum principle for CMC hypersurfaces (see Lemma 2.7). Nevertheless, the one-sided maximum principle implies that V^* has multiplicity one in U , and this is the key ingredient to obtain the multiplicity-one-property for V .

Another new difficulty in the CMC setting is that the total mass of the replacement V^* may differ from the total mass of the original varifold V . A key observation is that the mass defect is controlled by $c \text{Vol}(U)$, which is of higher order than the mass and hence converges to zero under any blowup process. Using this insight, we are able to prove that any blowup of the min-max limit V has the good replacement property of Colding-De Lellis, and is therefore regular (see Lemma 5.10). The same observation allows us to show that the tangent cones of V are always planes.

For the regularity of V , the proof structure is inspired by Pitts. Namely, we first apply successive replacements V^* and V^{**} on two overlapping concentric annuli A_1 and A_2 , with the goal of showing that they match smoothly on the overlapping region and may thus be extended all the way to the center by taking further replacements. However, in the CMC case two main issues arise: the presence of a nontrivial touching set, and the need to show that the orientations match to give the same sign for the mean curvatures. These difficulties are overcome by keeping track of the approximate replacements $\{\Omega_i^*\}$ and careful analysis of their convergence behaviors (see Section 2.1). In particular, a key step is to show that the second replacement V^{**} may be represented by a boundary in $A_1 \cup A_2$, so that the orientations of V^* , V^{**} match as desired. Near the touching set, we use the graphical decomposition into embedded sheets together with the gluing along regular part to properly match the sheets together.

Finally, to prove that the min-max limit V coincides with the extended replacement V^* near the center, we face one more obstacle since in the minimal case one typically appeals to the Constancy Theorem, which does not have an analog in the CMC setting. Instead, we first directly prove the removability of the center singularity for V^* , and then use a moving sphere argument to show that the densities of V and V^* are the same in the annular region.

0.2. Outline of the paper. We describe our notation and basic background material in Section 1. Then in Section 2 we gather some requisite results including compactness theorems and maximum principles, in particular for almost embedded CMC hypersurfaces.

In Section 3 we formulate the technical setup for our min-max procedure, and prove the existence of nontrivial (positive width) sweepouts (Theorem 3.9). Then for any minimizing sequence we can extract a min-max sequence $\{\partial\Omega_{x_i}^i\}$ that converges in the measure-theoretic sense to a nontrivial varifold V .

We then present the tightening process in Section 4. In particular, we show that there exists a nice minimizing sequence $\{\Omega_x^i\}$, such that every min-max sequence converges to a varifold with c -bounded first variation (Proposition 4.4). To do this, using the set of varifolds with c -bounded first variation as the central set, we construct a discrete gradient flow for \mathcal{A}^c , namely the *tightening map* (see section 4.3). Applying this map to any given minimizing sequence will result in a nice minimizing sequence by a standard contradiction argument. To deal with the issue that \mathcal{A}^c is not defined for varifolds, we derive a quantitative tightening inequality (4.5) inspired by Colding-De Lellis.

In Section 5, we first show that V may further be taken to be c -almost minimizing in the sense of Definition 5.1 (see Theorem 5.6); that is, V is the limit of boundaries of so-called (ϵ, δ) - c -almost minimizing sets Ω . The remainder of Section 5 records important properties of c -almost minimizing varifolds, including the existence and regularity of c -replacements (see Proposition 5.8 and Lemma 5.9). Namely, for each (ϵ, δ) - c -a.m. Ω , one can construct a \mathcal{A}^c -minimizer Ω^* among certain admissible deformations, whose boundary is a stable embedded CMC hypersurface in U by classical regularity theory. We then construct the c -replacement V^* of V as the varifold limit $\lim |\partial\Omega^*|$.

Finally, in Section 6 we prove the regularity of the min-max varifold V (Theorem 6.1). Namely, given two c -replacements V^* and V^{**} of V in two overlapping concentric annuli, we prove that they match in C^1 along the boundary sphere of the smaller annulus, in order to apply the unique continuation. Using the key observation that blowups are regular and the classical maximum principle, we first show that the blowups of V^* , V^{**} are identical along this sphere. The C^1 gluing follows readily on the regular part, after which we require a much more careful analysis to extend the gluing across the touching set. We then continue the c -replacement V^* smoothly as an almost embedded CMC hypersurface all the way to the center of the annuli and complete the proof after removing the center singularity.

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1. NOTATION

In this section, we collect some notions. We refer to [51] and [43, §2.1] for further materials in geometric measure theory.

Let (M^{n+1}, g) denote a closed, oriented, smooth Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Assume that (M, g) is embedded in some \mathbb{R}^L , $L \in \mathbb{N}$. $B_r(p), \tilde{B}_r(p)$ denote respectively the Euclidean ball of \mathbb{R}^L or the geodesic ball of (M, g) . We denote by \mathcal{H}^k the k -dimensional Hausdorff measure; $\mathbf{I}_k(M)$ the space of k -dimensional integral currents in \mathbb{R}^L with support in M ; $\mathcal{Z}_k(M)$ the space of integral currents $T \in \mathbf{I}_k(M)$ with $\partial T = 0$; $\mathcal{V}_k(M)$ the closure, in the weak topology, of the space of k -dimensional rectifiable varifolds in \mathbb{R}^L with support in M ; $G_k(M)$ the Grassmannian bundle of un-oriented k -planes over M ; \mathcal{F} and \mathbf{M} respectively the flat norm [51, §31] and mass norm [51, 26.4] on $\mathbf{I}_k(M)$; \mathbf{F} the varifold \mathbf{F} -metric on $\mathcal{V}_k(M)$ and currents \mathbf{F} -metric on $\mathbf{I}_k(M)$, [43, 2.1(19)(20)]; $\mathcal{C}(M)$ or $\mathcal{C}(U)$ the space of sets $\Omega \subset M$ or $\Omega \subset U \subset M$ with finite perimeter (Caccioppoli sets), [51, §14][17, §1.6]; and $\mathfrak{X}(M)$ or $\mathfrak{X}(U)$ the space of smooth vector fields in M or supported in U .

We also utilize the following definitions:

- a) Given $T \in \mathbf{I}_k(M)$, $|T|$ and $\|T\|$ denote respectively the integral varifold and Radon measure in M associated with T ;
- b) Given $c > 0$, a varifold $V \in \mathcal{V}_k(M)$ is said to have *c -bounded first variation in an open subset $U \subset M$* , if

$$|\delta V(X)| \leq c \int_M |X| d\mu_V, \quad \text{for any } X \in \mathfrak{X}(U);$$

here the first variation of V along X is $\delta V(X) = \int_{G_k(M)} \text{div}_S X(x) dV(x, S)$, [51, §39];

- c) $U_r(V)$ denotes the ball in $\mathcal{V}_k(M)$ under \mathbf{F} -metric with center $V \in \mathcal{V}_k(M)$ and radius $r > 0$;
- d) Given $p \in \text{spt } \|V\|$, $\text{VarTan}(V, p)$ denotes the space of tangent varifolds of V at p , [51, 42.3];
- e) Given a smooth, immersed, closed, orientable hypersurface Σ in M , or a set $\Omega \in \mathcal{C}(M)$ with finite perimeter, $[[\Sigma]]$, $[[\Omega]]$ denote the corresponding integral currents with the natural orientation, and $[\Sigma]$, $[\Omega]$ denote the corresponding integer-multiplicity varifolds;
- f) $\partial\Omega$ denotes the (reduced)-boundary of $[[\Omega]]$ as an integral current, and $\nu_{\partial\Omega}$ denotes the outward pointing unit normal of $\partial\Omega$, [51, 14.2].

In this paper, we are interested in the following weighted area functional defined on $\mathcal{C}(M)$. Given $c > 0$, define the \mathcal{A}^c -functional on $\mathcal{C}(M)$ as

$$(1.1) \quad \mathcal{A}^c(\Omega) = \mathcal{H}^n(\partial\Omega) - c\mathcal{H}^{n+1}(\Omega).$$

The *first variation formula* for \mathcal{A}^c along $X \in \mathfrak{X}(M)$ is (see [51, 16.2])

$$(1.2) \quad \delta\mathcal{A}^c|_{\Omega}(X) = \int_{\partial\Omega} \operatorname{div}_{\partial\Omega} X d\mu_{\partial\Omega} - c \int_{\partial\Omega} X \cdot \nu d\mu_{\partial\Omega},$$

where $\nu = \nu_{\partial\Omega}$ is the outward unit normal on $\partial\Omega$.

When the boundary $\partial\Omega = \Sigma$ is a smooth immersed hypersurface, we have

$$\operatorname{div}_{\Sigma} X = H X \cdot \nu,$$

where H is the mean curvature of Σ with respect to ν ; if Ω is a critical point of \mathcal{A}^c , then (1.2) directly implies that $\Sigma = \partial\Omega$ has constant mean curvature c with respect to the outward unit normal ν . In this case, we can calculate the *second variation formula* for \mathcal{A}^c along normal vector fields $X \in \mathfrak{X}(M)$ such that $X = \varphi\nu$ along $\partial\Omega = \Sigma$ where $\varphi \in C^\infty(\Sigma)$, [6, Proposition 2.5],

$$(1.3) \quad \delta^2\mathcal{A}^c|_{\Omega}(X, X) = II_{\Sigma}(\varphi, \varphi) = \int_{\Sigma} (|\nabla\varphi|^2 - (\operatorname{Ric}^M(\nu, \nu) + |A^{\Sigma}|^2) \varphi^2) d\mu_{\Sigma}.$$

In the above formula, $\nabla\varphi$ is the gradient of φ on Σ ; Ric^M is the Ricci curvature of M ; A^{Σ} is the second fundamental form of Σ .

2. PRELIMINARIES

In this section, we collect some preliminary results. First, we study the compactness properties of stable CMC hypersurfaces. In particular, we describe the structure of the touching sets which appear naturally when one takes the limit of embedded stable CMC hypersurfaces. We also present a maximum principle for varifolds with bounded first variation, a regularity result for boundaries that minimize the \mathcal{A}^c -functional, and a result on isoperimetric profile for small volumes.

2.1. Compactness of stable CMC hypersurfaces.

Definition 2.1. Let Σ be a smooth, immersed, two-sided hypersurface with unit normal vector ν , and $U \subset M$ an open subset. We say that Σ is a *stable c -hypersurface* in U if

- the mean curvature H of $\Sigma \cap U$ with respect to ν equals to c ; and
- $II_{\Sigma}(\varphi, \varphi) \geq 0$ for all $\varphi \in C^\infty(\Sigma)$ with $\operatorname{spt} \varphi \subset \Sigma \cap U$, where II_{Σ} is as in (1.3).

Definition 2.2. Let Σ_i , $i = 1, 2$, be connected embedded hypersurfaces in a connected open subset $U \subset M$, with $\partial\Sigma_i \cap U = \emptyset$ and unit normals ν_i . We say that Σ_2 *lies on one side of Σ_1* if Σ_1 divides U into two connected components $U_1 \cup U_2 = U \setminus \Sigma_1$, where ν_1 points into U_1 , and either:

- $\Sigma_2 \subset \operatorname{Clos}(U_1)$, which we write as $\Sigma_1 \leq \Sigma_2$ or that Σ_2 lies on the positive side of Σ_1 ; or
- $\Sigma_2 \subset \operatorname{Clos}(U_2)$, which we write as $\Sigma_1 \geq \Sigma_2$ or that Σ_2 lies on the negative side of Σ_1 .

Definition 2.3 (Almost embedding). Let $U \subset M^{n+1}$ be an open subset, and Σ^n be a smooth n -dimensional manifold. A smooth immersion $\phi : \Sigma \rightarrow U$ is said to be an *almost embedding* if at any point $p \in \phi(\Sigma)$ where Σ fails to be embedded, there is a small neighborhood $W \subset U$ of p , such that

- $\Sigma \cap \phi^{-1}(W)$ is a disjoint union of connected components $\cup_{i=1}^l \Sigma_i$;
- $\phi(\Sigma_i)$ is an embedding for each $i = 1, \dots, l$;
- for each i , any other component $\phi(\Sigma_j)$, $j \neq i$, lies on one side of $\phi(\Sigma_i)$ in W .

We will simply denote $\phi(\Sigma)$ by Σ and denote $\phi(\Sigma_i)$ by Σ_i . The subset of points in Σ where Σ fails to be embedded will be called the *touching set*, and denoted by $\mathcal{S}(\Sigma)$. We will call $\Sigma \setminus \mathcal{S}(\Sigma)$ the regular set, and denote it by $\mathcal{R}(\Sigma)$.

Remark 2.4. From the definition, the collection of components $\{\Sigma_i\}$ meet tangentially along $\mathcal{S}(\Sigma)$.

Definition 2.5 (Almost embedded c -boundary). (1) An almost embedded hypersurface $\Sigma \subset U$ is said to be a *boundary* if there is an open subset $\Omega \in \mathcal{C}(U)$, such that Σ is equal to the boundary $\partial\Omega$ (in U) in the sense of currents;
 (2) The *outer unit normal* ν_Σ of Σ is the choice of the unit normal of Σ which points outside of Ω along the regular part $\mathcal{R}(\Sigma)$;
 (3) Σ is called a *stable c -boundary* if Σ is a boundary as well as a stable immersed c -hypersurface.

We have the following variant of the famous Schoen-Simon-Yau (for $2 \leq n \leq 5$) [50] and Schoen-Simon ($n = 6$) [49] curvature estimates.

Theorem 2.6 (Curvature estimates for stable c -hypersurfaces). *Let $2 \leq n \leq 6$, and $U \subset M$ be an open subset. If $\Sigma \subset U$ is a smooth, immersed (almost embedded when $n = 6$), two-sided, stable c -hypersurface in U with $\partial\Sigma \cap U = \emptyset$, and $\text{Area}(\Sigma) \leq C$, then there exists C_1 depending only on n, M, c, C , such that*

$$|A^\Sigma|^2(x) \leq \frac{C_1}{\text{dist}_M^2(x, \partial U)} \quad \text{for all } x \in \Sigma.$$

Moreover if $\Sigma_k \subset U$ is a sequence of smooth, immersed (almost embedded when $n = 6$), two-sided, stable c -hypersurfaces in U with $\partial\Sigma_k \cap U = \emptyset$ and $\sup_k \text{Area}(\Sigma_k) < \infty$, then up to a subsequence, Σ_k converges locally smoothly (possibly with multiplicity) to some stable c -hypersurface Σ_∞ in U .

Proof. The compactness statement follows in the standard way from the curvature estimates. The curvature estimates follow from standard blowup arguments together with the Bernstein Theorem [50, Theorem 2] and [49, Theorem 3], the key being that the blowup will be a stable minimal hypersurface, and when $n = 6$, the blowup of a sequence of almost embedded c -hypersurfaces will be embedded by the classical maximum principle for embedded minimal hypersurfaces (c.f. [12]). \square

We need the following maximum principle.

Lemma 2.7 (Maximum principle for embedded c -hypersurfaces). *Given a connected open subset $U \subset M$, let $\Sigma_i \subset U$ be two connected embedded hypersurfaces with $\partial\Sigma_i \cap U = \emptyset$ for $i = 1, 2$. Suppose that the mean curvature of each Σ_i is a given constant $c > 0$ with respect to the respective unit normal ν_i . Assume that Σ_2 lies on one side of Σ_1 . Then we have the following:*

- (i) *If there exists $p \in \Sigma_1 \cap \Sigma_2$ such that $\nu_1(p) = \nu_2(p)$, then $\Sigma_1 = \Sigma_2$;*
- (ii) *Suppose Σ_2 lies on the negative side of Σ_1 . Then if $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, we must have $\nu_1(p) = \nu_2(p)$ for any $p \in \Sigma_1 \cap \Sigma_2$, and hence $\Sigma_1 = \Sigma_2$. In particular, either $\Sigma_1 \cap \Sigma_2 = \emptyset$ or $\Sigma_1 = \Sigma_2$.*

Proof. This follows directly from the classical maximum principle as follows.

Consider $p \in \Sigma_1 \cap \Sigma_2$. Since Σ_2 lies on one side of Σ_1 , the tangent planes must coincide at any point of their intersection. So without loss of generality we may assume that U is a small ball around p for which Σ_1, Σ_2 may be written as graphs u_1, u_2 in the ν_1 -direction over the tangent plane $T_p\Sigma_1 = T_p\Sigma_2$.

Let $u = u_1 - u_2$, then a standard computation shows that u satisfies a linear elliptic equation of the form: $Lu = 0$, if $\nu_2(p) = \nu_1(p)$; or $Lu = 2c$ if $\nu_2(p) = -\nu_1(p)$. Here L is a positive elliptic operator with smooth coefficients. Moreover, if $\Sigma_2 \subset \text{Clos}(U_1)$ then $u \leq 0$; if $\Sigma_2 \subset \text{Clos}(U_2)$ then $u \geq 0$. Both items then follow from the maximum principle for nonpositive (or nonnegative) functions. \square

Lemma 2.8. *Let Ω be a domain in \mathbb{R}^m and suppose that u is a classical solution on Ω of a linear inhomogenous elliptic PDE with smooth coefficients:*

$$(2.1) \quad Lu = a^{ij}D_{ij}u + b^jD_ju + qu = f,$$

where f has no zeroes on Ω . Then the zero set $\{u = 0\}$ is contained in a countable union of connected, embedded $(m - 1)$ -dimensional submanifolds.

Proof. Let K be a compact subset of Ω .

First, the implicit function theorem implies that the zero set is smooth away from the critical set. In particular, for any $\epsilon > 0$ the compact set $\{u = 0, |Du| \geq \epsilon\} \cap K$ is contained in the union of finitely many connected, smoothly embedded $(m - 1)$ -dimensional submanifolds.

Now consider $x \in \{u = 0, Du = 0\}$. Then we have $a^{ij}(x)D_{ij}u(x) = f(x) \neq 0$, so by ellipticity, the Hessian D^2u must have rank at least 1. Thus for some j , the gradient $D(D_ju) \neq 0$, so again by the implicit function theorem there is an $r > 0$ such that $B_r(x) \cap \{D_ju = 0\}$ is an embedded $(m - 1)$ -dimensional submanifold, which clearly contains $B_r(x) \cap \{u = 0\} \cap \{Du = 0\}$. It follows that the compact set $\{u = 0, Du = 0\} \cap K$ is also contained in a finite union of connected, embedded $(m - 1)$ -dimensional submanifolds.

Taking $\epsilon = 1/j \rightarrow 0$ and $K = K_j$, where K_j is an exhaustion of Ω , then completes the proof. \square

Proposition 2.9 (Touching sets for almost embedded c -hypersurface). *If the metric on U^{n+1} is smooth, then for any almost embedded hypersurface $\Sigma^n \subset U$ of constant mean curvature c , the touching set $\mathcal{S}(\Sigma)$ is contained in a countable union of connected, embedded $(n - 1)$ -dimensional submanifolds.*

In particular, the regular set $\mathcal{R}(\Sigma)$ is open and dense in Σ .

Proof. Let $p \in \mathcal{S}(\Sigma)$. As in the proof of Lemma 2.7, there is a small neighborhood W of p so that the image $\Sigma \cap W$ decomposes as graphs $\{u_i\}_{i=1}^k$, ordered by height, over the common tangent plane $T_p\Sigma$. In fact by the conclusions of that lemma, after possibly shrinking W there will be exactly two distinct graphs $u_1 \leq u_2$, for which the difference $u = u_1 - u_2$ satisfies $Lu = 2c$. The zero set of u corresponds to the touching set $\mathcal{S}(\Sigma) \cap W$. The proposition then follows from the previous lemma. \square

Remark 2.10. In the case that the metric on M is real analytic, we have the stronger statement that the touching set is a finite union of real analytic subvarieties $\bigcup_{k=0}^{n-1} S^k$ of respective dimension k . This follows from [27, Theorem 5.2.3], since in this setting the operator L will have analytic coefficients, and hence the solution u is also real analytic.

Theorem 2.11 (Compactness theorem for almost embedded stable c -hypersurfaces). *Let $2 \leq n \leq 6$. Suppose $\Sigma_k \subset U$ is a sequence of smooth, almost embedded, two-sided, stable c_k -hypersurfaces in U , with $\sup_k \text{Area}(\Sigma_k) < \infty$ and $\sup_k c_k < \infty$. Then the following hold:*

- (i) *if $\inf c_k > 0$, then up to a subsequence, $\{\Sigma_k\}$ converges locally smoothly (with multiplicity) to some almost embedded stable c -hypersurface Σ_∞ in U ;*
- (ii) *if additionally $\{\Sigma_k\}$ are all boundaries, then Σ_∞ is also a boundary, and the density of Σ_∞ is 1 along $\mathcal{R}(\Sigma_\infty)$ and 2 along $\mathcal{S}(\Sigma_\infty)$;*
- (iii) *if $c_k \rightarrow 0$, then up to a subsequence, $\{\Sigma_k\}$ converges locally smoothly (with multiplicity) to some smooth embedded stable minimal hypersurface Σ_∞ in U .*

Remark 2.12. We learned that Bellettini-Wickramasekera [7] also have similar compactness results for stable CMC varifolds.

Proof of Theorem 2.11. Case (i) follows straightforwardly from Theorem 2.6, the almost embedded assumption, together with the maximum principle Lemma 2.7.

Now we prove Case (ii). Denote $\Sigma_k = \partial\Omega_k$ for some $\Omega_k \in \mathcal{C}(U)$. By standard compactness [51, Theorem 6.3], a subsequence of $\partial\Omega_k$ converges weakly as currents to some $\partial\Omega_\infty$ with $\Omega_\infty \in \mathcal{C}(U)$. We claim that $\partial\Omega_\infty = \Sigma_\infty$ as varifold. To show this, we only need to check that the density of Σ_∞ along $\mathcal{R}(\Sigma_\infty)$ is one, and then by Lemma 2.7 and Proposition 2.9, the density of Σ_∞ along the touching set $\mathcal{S}(\Sigma_\infty)$ is automatically two.

To show that the density along $\mathcal{R}(\Sigma_\infty)$ is 1, take an arbitrary point $p \in \mathcal{R}(\Sigma_\infty)$. If the density at p is larger than 1, then by the locally smooth convergence of Σ_k to Σ_∞ , there is a neighborhood $\tilde{B}_p \subset U$ of p , such that for k large enough $\Sigma_k \cap \tilde{B}_p$ has a graphical decomposition as $\cup_{i=1}^{l_k} \Sigma_k^i$ with $l_k \geq 2$. Moreover, by Lemma 2.7 we have $\Sigma_k^1 < \Sigma_k^2 < \dots < \Sigma_k^{l_k}$, and the outward unit normals ν_k^i of Σ_k^i all point to the same direction. With out loss of generality, we may assume $l_k = 2$ and omit the sub-index k . Then $\tilde{B}_p \setminus (\Sigma^1 \cup \Sigma^2)$ has three connected components U_0, U_1, U_2 with, counting orientation, $(\partial U_0) \llcorner \tilde{B}_p = \Sigma^1$, $(\partial U_1) \llcorner \tilde{B}_p = \Sigma^2 - \Sigma^1$, and $(\partial U_2) \llcorner \tilde{B}_p = -\Sigma^2$.

On the other hand, for each i the Constancy Theorem [51, Theorem 26.27] applied to $\Omega_k \llcorner U_i$ implies that $\Omega_k \llcorner U_i$ is identical to either \emptyset or U_i . That is, $\Omega_k \llcorner \tilde{B}_p = \sum_{i=0}^2 a_i U_i$, where each $a_i = 0, 1$. It is then easy to see that any choice of the a_i will contradict the fact that, counting orientation, $\partial(\Omega_k \llcorner \tilde{B}_p) \llcorner \tilde{B}_p = \Sigma_k \cap \tilde{B}_p = \Sigma^1 + \Sigma^2$.

Case (iii) follows directly from Theorem 2.6, the almost embedded assumption, and the classical maximum principle for embedded minimal hypersurfaces (c.f. [12]). \square

2.2. Maximum principle for varifolds with c -bounded first variation. We will need the following maximum principle which is essentially due to White [57, Theorem 5].

Proposition 2.13 (Maximum principle for varifolds with c -bounded first variation). *Suppose $V \in \mathcal{V}_n(M)$ has c -bounded first variation in a open subset $U \subset M$. Let $K \subset U$ be an open subset with compact closure in U , such that $\text{spt}(\|V\|) \subset K$, and*

- (i) ∂K is smoothly embedded in M ,
- (ii) the mean curvature of ∂K with respect to the outward pointing normal is greater than c .

Then $\text{spt}(\|V\|) \cap \partial K = \emptyset$.

2.3. Regularity for boundaries which minimize the \mathcal{A}^c functional. The following result about regularity of boundaries which minimize the \mathcal{A}^c functional can be found in [39].

Theorem 2.14. *Given $\Omega \in \mathcal{C}(M)$, $p \in \text{spt} \|\partial\Omega\|$, and some small $r > 0$, suppose that $\Omega \llcorner \tilde{B}_r(p)$ minimizes the \mathcal{A}^c -functional: that is, for any other $\Lambda \in \mathcal{C}(M)$ with $\text{spt} \|\Lambda - \Omega\| \subset \tilde{B}_r(p)$, we have $\mathcal{A}^c(\Lambda) \geq \mathcal{A}^c(\Omega)$. Then except for a set of Hausdorff dimension at most $n - 7$, $\partial\Omega \llcorner \tilde{B}_r(p)$ is a smooth and embedded hypersurface, and is real analytic if the ambient metric on M is real analytic.*

Proof. Since $\Omega \llcorner \tilde{B}_r(p)$ minimizes the \mathcal{A}^c -functional, for all $\Lambda \in \mathcal{C}(M)$ as in the supposition, we have

$$\mathcal{H}^n(\partial\Lambda) - \mathcal{H}^n(\partial\Omega) \geq -c|\mathcal{H}^{n+1}(\Lambda) - \mathcal{H}^{n+1}(\Omega)|.$$

This is precisely condition [39, 3.1(1)]. The regularity then follows from [39, Corollary 3.7, 3.8]. \square

2.4. Isoperimetric profiles for small volume. We have the following lower bound for the isoperimetric profiles for small volumes, which is a consequence of the fact that the isoperimetric profile is asymptotically Euclidean for small volumes [8] (see also [40, Theorem 3]). Note that although it was only stated for domains with smooth boundary, the result indeed holds for any $\Omega \in \mathcal{C}(M)$ by using the regularity theory for isoperimetric domains (c.f. Theorem 2.14).

Theorem 2.15. *There exists constants $C_0 > 0$ and $V_0 > 0$ depending only on M such that*

$$\text{Area}(\partial\Omega) \geq C_0 \text{Vol}(\Omega)^{\frac{n}{n+1}}, \text{ whenever } \Omega \in \mathcal{C}(M) \text{ and } \text{Vol}(\Omega) \leq V_0.$$

3. THE c -MIN-MAX CONSTRUCTION

In this section, we present the setups of the min-max construction mainly followed Pitts [43]. We also prove the existence of a non-trivial sweepout with positive \mathcal{A}^c -min-max value.

3.1. Homotopy sequences. We will introduce the min-max construction using the scheme developed by Almgren and Pitts [3, 4, 43].

Definition 3.1. (cell complex.)

- (1) Denote $I = [0, 1]$, $I_0 = \partial I = I \setminus (0, 1)$;
- (2) For $j \in \mathbb{N}$, $I(1, j)$ is the cell complex of I , whose 1-cells are all intervals of form $[\frac{i}{3^j}, \frac{i+1}{3^j}]$, and 0-cells are all points $[\frac{i}{3^j}]$;
- (3) For $p = 0, 1$, $\alpha \in I(1, j)$ is a p -cell if $\dim(\alpha) = p$. 0-cell is also called a vertex;
- (4) $I(1, j)_p$ denotes the set of all p -cells in $I(1, j)$, and $I_0(1, j)_0$ denotes the set $\{[0], [1]\}$;
- (5) Given a 1-cell $\alpha \in I(1, j)_1$, and $k \in \mathbb{N}$, $\alpha(k)$ denotes the 1-dimensional sub-complex of $I(1, j+k)$ formed by all cells contained in α . For $q = 0, 1$, $\alpha(k)_q$ and $\alpha_0(k)_q$ denote respectively the set of all q -cells of $I(1, j+k)$ contained in α , or in the boundary of α ;
- (6) The boundary homeomorphism $\partial : I(1, j) \rightarrow I(1, j)$ is given by $\partial[a, b] = [b] - [a]$ if $[a, b] \in I(1, j)_1$, and $\partial[a] = 0$ if $[a] \in I(1, j)_0$;
- (7) The distance function $\mathbf{d} : I(1, j)_0 \times I(1, j)_0 \rightarrow \mathbb{N}$ is defined as $\mathbf{d}(x, y) = 3^j |x - y|$;
- (8) The map $\mathbf{n}(i, j) : I(1, i)_0 \rightarrow I(1, j)_0$ is defined as: $\mathbf{n}(i, j)(x) \in I(1, j)_0$ is the unique element of $I(1, j)_0$, such that $\mathbf{d}(x, \mathbf{n}(i, j)(x)) = \inf \{\mathbf{d}(x, y) : y \in I(1, j)_0\}$.

Consider a map to the space of Caccioppoli sets: $\phi : I(1, j)_0 \rightarrow \mathcal{C}(M)$. The *fineness* of ϕ is defined as:

$$(3.1) \quad \mathbf{f}(\phi) = \sup \left\{ \frac{\mathbf{M}(\partial\phi(x) - \partial\phi(y))}{\mathbf{d}(x, y)} : x, y \in I(1, j)_0, x \neq y \right\}.$$

Similarly we can define the fineness of ϕ with respect to the \mathcal{F} -norm and \mathbf{F} -metric. We use $\phi : I(1, j)_0 \rightarrow (\mathcal{C}(M), \{0\})$ to denote a map such that $\phi(I(1, j)_0) \subset \mathcal{C}(M)$ and $\partial\phi|_{I_0(1, j)_0} = 0$, i.e. $\phi([0]), \phi([1]) = \emptyset$ or M .

Definition 3.2. Given $\delta > 0$ and $\phi_i : I(1, k_i)_0 \rightarrow (\mathcal{C}(M), \{0\})$, $i = 0, 1$, we say ϕ_1 is *1-homotopic to ϕ_2 in $(\mathcal{C}(M), \{0\})$ with fineness δ* , if $\exists k_3 \in \mathbb{N}$, $k_3 \geq \max\{k_1, k_2\}$, and

$$\psi : I(1, k_3)_0 \times I(1, k_3)_0 \rightarrow \mathcal{C}(M),$$

such that

- $\mathbf{f}(\psi) \leq \delta$;
- $\psi([i], x) = \phi_i(\mathbf{n}(k_3, k_i)(x))$, $i = 0, 1$;
- $\partial\psi(I(1, k_3)_0 \times I_0(1, k_3)_0) = 0$.

Definition 3.3. A $(1, \mathbf{M})$ -homotopy sequence of mappings into $(\mathcal{C}(M), \{0\})$ is a sequence of mappings $\{\phi_i\}_{i \in \mathbb{N}}$,

$$\phi_i : I(1, k_i)_0 \rightarrow (\mathcal{C}(M), \{0\}),$$

such that ϕ_i is 1-homotopic to ϕ_{i+1} in $(\mathcal{C}(M), \{0\})$ with fineness δ_i , and

- $\lim_{i \rightarrow \infty} \delta_i = 0$;

- $\sup_i \{\mathbf{M}(\partial\phi_i(x)) : x \in I(1, k_i)_0\} < +\infty$.

Remark 3.4. Note that the second condition implies that $\sup_i \{\mathcal{A}^c(\phi_i(x)) : x \in I(1, k_i)_0\} < +\infty$.

Definition 3.5. Given two $(1, \mathbf{M})$ -homotopy sequences of mappings $S_1 = \{\phi_i^1\}_{i \in \mathbb{N}}$ and $S_2 = \{\phi_i^2\}_{i \in \mathbb{N}}$ into $(\mathcal{C}(M), \{0\})$, S_1 is *homotopic to* S_2 if $\exists \{\delta_i\}_{i \in \mathbb{N}}$, such that

- ϕ_i^1 is 1-homotopic to ϕ_i^2 in $(\mathcal{C}(M), \{0\})$ with fineness δ_i ;
- $\lim_{i \rightarrow \infty} \delta_i = 0$.

It is easy to see that the relation “is homotopic to” is an equivalence relation on the space of $(1, \mathbf{M})$ -homotopy sequences of mappings into $(\mathcal{C}(M), \{0\})$. An equivalence class is a $(1, \mathbf{M})$ -homotopy class of mappings into $(\mathcal{C}(M), \{0\})$. Denote the set of all equivalence classes by $\pi_1^\#(\mathcal{C}(M, \mathbf{M}), \{0\})$.

3.2. Min-max construction.

Definition 3.6. (Min-max definition) Given $\Pi \in \pi_1^\#(\mathcal{C}(M, \mathbf{M}), \{0\})$, define: $\mathbf{L}^c : \Pi \rightarrow \mathbb{R}^+$ as a function given by:

$$\mathbf{L}^c(S) = \mathbf{L}^c(\{\phi_i\}_{i \in \mathbb{N}}) = \limsup_{i \rightarrow \infty} \max \{\mathcal{A}^c(\phi_i(x)) : x \text{ lies in the domain of } \phi_i\}.$$

The \mathcal{A}^c -min-max value of Π is defined as

$$(3.2) \quad \mathbf{L}^c(\Pi) = \inf \{\mathbf{L}^c(S) : S \in \Pi\}.$$

A sequence $S = \{\phi_i\} \in \Pi$ is called a *critical sequence* if $\mathbf{L}^c(S) = \mathbf{L}^c(\Pi)$.

Given a critical sequence S , then $K(S) = \{V = \lim_{j \rightarrow \infty} |\partial\phi_{i_j}(x_j)| : x_j \text{ lies in the domain of } \phi_{i_j}\}$ is a compact subsets of $\mathcal{V}_n(M^{n+1})$. The *critical set* of S is the subset $C(S) \subset K(S)$ defined by

$$C(S) = \{V = \lim_{j \rightarrow \infty} |\partial\phi_{i_j}(x_j)| : \text{with } \lim_{j \rightarrow \infty} \mathcal{A}^c(\phi_{i_j}(x_j)) = \mathbf{L}^c(S)\}.$$

Note that by [43, 4.1(4)], we immediately have:

Lemma 3.7. *Given any $\Pi \in \pi_1^\#(\mathcal{C}(M, \mathbf{M}), \{0\})$, there exists a critical sequence $S \in \Pi$.*

The main theorem of this paper is as follows:

Theorem 3.8. *Let $2 \leq n \leq 6$. Given a smooth closed Riemannian manifold M^{n+1} and $c > 0$, there exists $\Pi \in \pi_1^\#(\mathcal{C}(M, \mathbf{M}), \{0\})$ and a critical sequence $S \in \Pi$ such that:*

- $\mathbf{L}^c(\Pi) = \mathbf{L}^c(S) > 0$;
- *There exists an element of $C(S)$ induced by a nontrivial, smooth, almost embedded, closed hypersurface $\Sigma^n \subset M$ of constant mean curvature c with multiplicity one.*

Proof of Theorem 3.8. This follows from combining Theorem 3.9, Theorem 5.6 and Theorem 6.1. \square

3.3. Existence of nontrivial sweepouts.

Theorem 3.9. *There exists $\Pi \in \pi_1^\#(\mathcal{C}(M, \mathbf{M}), \{0\})$, such that for any $c > 0$, we have $\mathbf{L}^c(\Pi) > 0$.*

Remark 3.10. Let us first describe a heuristic argument using smooth sweepouts which will help to reveal the key idea. Let $C_0 > 0$ and $V_0 > 0$ to be the constants in Theorem 2.15, and fix $0 < V \leq V_0$ such that $V^{\frac{-1}{n+1}} > 2c/C_0$. Note that V only depends on c, C_0, V_0 .

Then for any Ω with $\text{Vol}(\Omega) = V$, we have

$$(3.3) \quad \mathcal{A}^c(\Omega) \geq C_0 V^{\frac{n}{n+1}} - cV > cV > 0.$$

Now consider any smooth 1-parameter family $\{\Omega_x : x \in [0, 1]\}$ satisfying $\Omega_0 = \emptyset$ and $\Omega_1 = M$. Since $\{\Omega_x\}$ sweeps out M , there must exist some $x_0 \in (0, 1)$ such that $\text{Vol}(\Omega_{x_0}) = V$, whence $\max_{x \in [0, 1]} \mathcal{A}^c(\Omega_x) \geq cV > 0$. Since this holds for any sweepout we then have $\mathbf{L}^c(\Pi) \geq cV > 0$.

Proof of Theorem 3.9. Take a Morse function $\phi : M \rightarrow [0, 1]$, and consider the sub-level sets $\Phi : [0, 1] \rightarrow \mathcal{C}(M)$, given by $\Phi(t) = \{x \in M : \phi(x) < t\}$. By the interpolation theorem of the first author [60, Theorem 5.1], Φ can be discretized to a $(1, \mathbf{M})$ -homotopy sequence $\overline{S} = \{\overline{\phi}_i\}$ where $\overline{\phi}_i : I(1, k_i)_0 \rightarrow (\mathcal{C}(M), \{0\})$. Moreover, under Almgren's isomorphism F_A [3, §3.2] (see also [60, §4.2]), $\overline{\phi}_i$ is mapped to the fundamental class in $H_{n+1}(M)$ for i large, i.e. $F_A(\overline{\phi}_i) = [[M]]$. Consider $\Pi = [\overline{S}]$, then by [3, Theorem 7.1] for any $S = \{\phi_i\} \in \Pi$, we have $F_A(\phi_i) = [[M]]$ for i large. In particular, this means that $\sum_{j \in I(1, k_i)_0} Q_j = M$ as currents; here for given j and $\alpha_j = [\frac{j-1}{3^{k_i}}, \frac{j}{3^{k_i}}] = [x_{j-1}, x_j]$, $Q_j \in \mathbf{I}_{n+1}(M)$ is the isoperimetric choice (c.f. [3, 1.14]) of $\partial\phi_i(x_j) - \partial\phi_i(x_{j-1})$, i.e.

$$\mathbf{M}(Q_j) = \mathcal{F}(\partial\phi_i(x_j) - \partial\phi_i(x_{j-1})), \text{ and } \partial Q_\alpha = \partial\phi_i(x_j) - \partial\phi_i(x_{j-1}).$$

Denote $\Omega_l = \sum_{j=1}^l Q_j$. Then we have

$$\partial\Omega_l = \partial\phi_i(x_l) - \partial\phi_i(0) = \partial\phi_i(x_l),$$

and this implies, by the Constancy Theorem [51, 26.27], that Ω_l is a Caccioppoli set (possibly with a negative orientation) when $\mathbf{M}(\Omega_l) < \text{Vol}(M)$. Note that $\mathbf{M}(Q_j) < \mathbf{f}(\phi_i) = \delta_i$, hence by continuity there exists some $l_c \in \mathbb{N}$, $l_c < 3^{k_i}$, such that $\mathbf{M}(\Omega_{l_c}) \in [V - \delta_i, V + \delta_i]$, where V is as in Remark 3.10, and $\phi_i(x_{l_c}) = \Omega_{l_c}$. Then the same argument as in the remark above gives a uniform positive lower bound for $\mathcal{A}^c(\phi_i(x_{l_c}))$, and this finishes the proof. \square

4. TIGHTENING

In this section, we construct the tightening map adapted to the \mathcal{A}^c functional and prove that after applying the tightening map to a critical sequence, every element in the critical set has uniformly bounded first variation. Our approach is adapted from those in [11, §4] and [43, §4.3].

4.1. Annular decomposition. Given $L > 0$, consider the set of varifolds in $\mathcal{V}_n(M)$ with $2L$ -bounded mass: $A^L = \{V \in \mathcal{V}_n(M) : \|V\|(M) \leq 2L\}$. Denote

$$A_\infty^c = \{V \in A^L : |\delta V(X)| \leq c \int |X| d\mu_V, \text{ for any } X \in \mathfrak{X}(M)\}.$$

Consider the concentric annuli around A_∞^c under the \mathbf{F} -metric, i.e.

$$A_j = \{V \in A^L : \frac{1}{2^j} \leq \mathbf{F}(V, A_\infty^c) \leq \frac{1}{2^{j-1}}\}, \quad j \in \mathbb{N}.$$

Since c -bounded first variation is a closed condition, we have

Lemma 4.1. A_∞^c is a compact subset of A^L under the \mathbf{F} -metric.

It is easy to show (by contradiction, for instance) that for any varifold in A_j , we can find a vector field satisfying the following condition.

Lemma 4.2. For any $V \in A_j$, there exists $X_V \in \mathfrak{X}(M)$, such that

$$(4.1) \quad \|X_V\|_{C^1(M)} \leq 1, \quad \delta V(X_V) - c \int_M |X_V| d\mu_V \leq -c_j < 0,$$

where c_j depends only on j .

4.2. A map from A^L to the space of vector fields. In this part, we will construct a map $X : A^L \rightarrow \mathfrak{X}(M)$, which is continuous with respect to the C^1 topology on $\mathfrak{X}(M)$.

Given $V \in A_j$, let X_V be given in Lemma 4.2. Since $\text{div}_S X_V$ is Lipschitz on $G_n(M)$ for fixed X_V , the map

$$W \rightarrow \delta W(X_V) - c \int_M |X_V| d\mu_W = \int_{G_n(M)} \text{div}_S(X_V) dW(x, S) - c \int_M |X_V| d\mu_W$$

is continuous with respect to the **F**-metric. Therefore for any $V \in A_j$, there exists $0 < r_V < \frac{1}{2^{j+1}}$, such that for any $W \in U_{r_V}(V)$, i.e. $\mathbf{F}(W, V) < r_V$,

$$(4.2) \quad \delta W(X_V) - c \int |X_V| d\mu_W \leq \frac{1}{2} \left(\delta V(X_V) - c \int |X_V| d\mu_V \right) \leq -\frac{1}{2} c_j < 0.$$

Now $\{U_{r_V/2}(V) : V \in A_j\}$ is an open covering of A_j . By the compactness of A_j , we can find finitely many balls $\{U_{r_{j,i}}(V_{j,i}) : V_{j,i} \in A_j, 1 \leq i \leq q_j\}$, where $r_{j,i} = r_{V_{j,i}}$, such that

- (i) The balls $U_{r_{j,i}/2}(V_{j,i})$ with half radii cover A_j ;
- (ii) The balls $U_{r_{j,i}}(V_{j,i})$ are disjoint from A_k if $|j - k| \geq 2$.

In the following, we denote $U_{r_{j,i}}(V_{j,i})$, $U_{r_{j,i}/2}(V_{j,i})$ and $X_{V_{j,i}}$ by $U_{j,i}$, $\tilde{U}_{j,i}$ and $X_{j,i}$ respectively.

Now we can construct a partition of unity $\{\varphi_{j,i} : j \in \mathbb{N}, 1 \leq i \leq q_j\}$ sub-coordinate to the covering $\{\tilde{U}_{j,i}, 1 \leq i \leq q_j, j \in \mathbb{N}\}$ by

$$\varphi_{j,i}(V) = \frac{\psi_{j,i}(V)}{\sum \{\psi_{p,q}(V), p \in \mathbb{N}, 1 \leq q \leq q_p\}},$$

where $\psi_{j,i}(V) = \mathbf{F}(V, A^L \setminus \tilde{U}_{j,i})$.

The map $X : A^L \rightarrow \mathfrak{X}(M)$ is defined by

$$(4.3) \quad X(V) = \mathbf{F}(V, A_\infty^c) \sum_{j \in \mathbb{N}, 1 \leq i \leq q_j} \varphi_{j,i}(V) X_{j,i}.$$

The following lemma is a straightforward consequence of the construction.

Lemma 4.3. *The map $X : V \rightarrow X(V)$ is continuous with respect to the C^1 topology on $\mathfrak{X}(M)$.*

4.3. A map from A^L to the space of isotopies. In this part, we will associate each $V \in A$ with an isotopy of M in a continuous manner. The isotopy will be generated by the vector field $X(V)$. In particular, given $V \in A^L$, we use $\Phi_V : \mathbb{R}^+ \times M \rightarrow M$ to denote the one parameter group of diffeomorphisms generated by $X(V)$.

Given $\Omega \in \mathcal{C}(M)$ with $\partial\Omega \in A^L$, we will deform Ω by $\Phi_{|\partial\Omega|}(t)$ to get a 1-parameter family of sets of finite perimeter $\Omega_t = \Phi_{|\partial\Omega|}(t)(\Omega)$, and we will show that the \mathcal{A}^c functional of Ω_t for some $t > 0$ can be deformed down by a fixed amount depending only on $\mathbf{F}(|\partial\Omega|, A_\infty^c)$.

In fact, given $V \in A_j$, let $\rho(V)$ be the smallest radii of the balls $\tilde{U}_{k,i}$ which contain V . As there are only finitely many balls $\tilde{U}_{k,i}$ which intersect A_j nontrivially, we know that $\rho(V) \geq r_j > 0$, where r_j depends only on j ; moreover, by construction the sub-index k of these $\tilde{U}_{k,i}$ can only be $j-1, j$, or $j+1$. Then by (4.2) and (4.3), we have for any $W \in U_{\rho(V)}(V)$ that

$$\delta W(X(V)) - c \int |X(V)| d\mu_W \leq -\frac{1}{2^{j+1}} \min\{c_{j-1}, c_j, c_{j+1}\}.$$

Therefore we can find two continuous functions $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that $\rho(0) = 0$ and

$$\delta W(X(V)) - c \int |X(V)| d\mu_W \leq -g(\mathbf{F}(V, A_\infty^c)), \quad \text{if } \mathbf{F}(W, V) \leq \rho(\mathbf{F}(V, A_\infty^c)).$$

In particular, by (1.2),

$$(4.4) \quad \delta \mathcal{A}^c|_\Omega(X(V)) \leq -g(\mathbf{F}(V, A_\infty^c)), \quad \text{if } \Omega \in \mathcal{C}(M), \mathbf{F}(|\partial\Omega|, V) \leq \rho(\mathbf{F}(V, A_\infty^c)).$$

Next, we will construct a continuous time function $T : [0, \infty) \rightarrow [0, \infty)$, such that

- (i) $\lim_{t \rightarrow 0} T(t) = 0$, and $T(t) > 0$ if $t \neq 0$;
- (ii) For any $V \in A^L$, denote $\gamma = \mathbf{F}(V, A_\infty^c)$; then $V_t = (\Phi_V(t))_\# V \in U_{\rho(\gamma)}(V)$ for all $0 \leq t \leq T(\gamma)$.

In fact, given $V \in A_j$, and $\rho = \rho(\mathbf{F}(V, A_\infty^c)) > 0$, there exists $T_V > 0$, such that $V_t \in U_\rho(V)$ for all $0 \leq t \leq T_V$. Moreover, by the compactness of A_j and the continuity of $(\Phi_V(t))_\# V$ in V and t , we may choose T_V such that $T_V \geq T_j > 0$ for all $V \in A_j$, where T_j depends only on j . Interpolating between the T_j yields the desired continuous function T depending only on $\mathbf{F}(V, A_\infty^c)$.

In summary, given $V \in A^L \setminus A_\infty^c$, denote $\gamma = \mathbf{F}(V, A_\infty^c) > 0$,

$$\Psi_V(t, \cdot) = \Phi_V(T(\gamma)t, \cdot), \quad \text{for } t \in [0, 1],$$

and $L : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, with $L(\gamma) = T(\gamma)g(\gamma)$; then $L(0) = 0$ and $L(\gamma) > 0$ if $\gamma > 0$. We can deform V through a continuous family $\{V_t = (\Psi_V(t))_\# V : t \in [0, 1]\} \subset U_{\rho(\gamma)}(V)$, such that

- (i) The map $(t, V) \rightarrow V_t$ is continuous under the \mathbf{F} -metric;
- (ii) Using (4.4), when $V = \partial\Omega$, $\Omega \in \mathcal{C}(M)$, $\gamma = \mathbf{F}(|\partial\Omega|, A_\infty^c) > 0$, we have

$$(4.5) \quad \begin{aligned} \mathcal{A}^c(\Omega_1) - \mathcal{A}^c(\Omega) &\leq \int_0^{T(\gamma)} [\delta \mathcal{A}^c|_{\Omega_t}](X(|\partial\Omega|)) dt \leq -T(\gamma)g(\gamma) \\ &= -L(\gamma) < 0. \end{aligned}$$

Finally note that the flow $\Psi_V(t, \cdot)$ is generated by the vector field

$$(4.6) \quad \tilde{X}(V) = T(\gamma)X(V).$$

4.4. Deforming sweepouts by the tightening map. Applying our tightening map constructed above in place of [43, §4.3] to a critical sequence provided by Lemma 3.7, we can deduce the following result.

Proposition 4.4 (Tightening). *Let $\Pi \in \pi_1^\sharp(\mathcal{C}(M, \mathbf{M}), \{0\})$, and assume $\mathbf{L}^c(\Pi) > 0$. For any critical sequence S^* for Π , there exists another critical sequence S for Π such that $C(S) \subset C(S^*)$ and each $V \in C(S)$ has c -bounded first variation.*

Proof. Take $S^* = \{\phi_i^*\}$, where $\phi_i^* : I(1, k_i)_0 \rightarrow (\mathcal{C}(M), \{0\})$, and ϕ_i^* is 1-homotopic to ϕ_{i+1}^* in $(\mathcal{C}(M), \{0\})$ with fineness $\delta_i \searrow 0$. Let $\Xi_i : I(1, k_i)_0 \times [0, 1] \rightarrow \mathcal{C}(M)$ be defined as

$$\Xi_i(x, t) = \Psi_{|\partial\phi_i^*(x)|}(t)(\phi_i^*(x)).$$

Denote $\phi_i^t(\cdot) = \Xi_i(\cdot, t)$. Heuristically, one would like to set $\phi_i = \phi_i^1$ as the desired sequence, but since the isotopies $\Psi_{|\partial\phi_i^*(x)|}$ depend on x , the fineness of $\{\phi_i^1\}$ could be large even if $\mathbf{f}(\phi_i^*)$ is small. Thus we need to interpolate ϕ_i^1 to get the desired ϕ_i , but we need to make sure the values of ϕ_i after interpolation are \mathbf{F} -close to those of ϕ_i^1 . Similar difficulties appeared in the same way in [33, §15]. The

authors in [33] used a discrete-to-continuous interpolation argument. Unfortunately we cannot adapt their argument, since their constructions involve currents which may not be boundaries of Caccioppoli sets. Instead, we develop another interpolation method in Claim 2. Before that, we pause to prove:

Claim 1: if $\lim_{i \rightarrow \infty} \mathcal{A}^c(\phi_i^1(x_i)) = \mathbf{L}^c(\Pi)$, then (up to relabeling) there is a subsequence $\{\phi_i^1(x_i)\}$ converging (as varifolds) to a varifold in $C(S^)$ of c -bounded first variation.*

Proof of Claim 1: By (4.5),

$$(4.7) \quad \mathcal{A}^c(\phi_i^1(x_i)) - \mathcal{A}^c(\phi_i^*(x_i)) = -L(\gamma_i),$$

where $\gamma_i = \mathbf{F}(|\partial\phi_i^*(x_i)|, A_\infty^c)$. Therefore,

$$\mathbf{L}^c(\Pi) = \lim \mathcal{A}^c(\phi_i^1(x_i)) = \lim \mathcal{A}^c(\phi_i^*(x_i)) - L(\lim \gamma_i) \leq \mathbf{L}^c(\Pi) - L(\lim \gamma_i),$$

so actually we must have $\lim \gamma_i = 0$ and this implies that $\lim |\partial\phi_i^*(x_i)| \in A_\infty^c$. Moreover, by our construction of the tightening map, each $|\partial\phi_i^1(x_i)|$ had to be $\rho(\gamma_i)$ -close to $|\partial\phi_i^*(x_i)|$ under the \mathbf{F} -metric, therefore

$$\lim |\partial\phi_i^1(x_i)| = \lim |\partial\phi_i^*(x_i)| \in A_\infty^c \cap C(S^*),$$

and this finishes the proof of the claim.

Claim 2: there exist integers $l_i > k_i$ and maps $\phi_i : I(1, l_i)_0 \rightarrow (\mathcal{C}(M), \{0\})$ for each i , such that $S = \{\phi_i\}$ is homotopic to S^ , and*

- (a) $\phi_i^1 = \phi_i \circ \mathbf{n}(l_i, k_i)$ on $I(1, k_i)_0$;
- (b) $\mathbf{f}(\phi_i) \rightarrow 0$, as $i \rightarrow \infty$;
- (c) $\mathcal{A}^c(\phi_i(x)) - \max\{\mathcal{A}^c(\phi_i^1(y)) : \alpha \in I(1, k_i)_1, x, y \in \alpha\} \rightarrow 0$, uniformly in $x \in I(1, l_i)_0$ as $i \rightarrow \infty$.
- (d) $\max\{\mathbf{F}(\partial\phi_i(x), \partial\phi_i^1(y)) : \alpha \in I(1, k_i)_1, x, y \in \alpha\} \rightarrow 0$, as $i \rightarrow \infty$.

Proof of Claim 2: The idea is to extend ϕ_i^1 to a piecewise continuous (with respect to the \mathbf{F} -metric) map on I and then apply the discretization result in [60, Theorem 5.1]. However, since this procedure is somewhat technical, the proof is deferred to Appendix B. \square

In particular, S is a valid sequence in Π , and we now check that it satisfies the requirements of the proposition. First, property (c) and the fact that S^* is a critical sequence directly imply that S is also a critical sequence. It remains to show that every element in $C(S)$ must lie in $C(S^*)$ and have c -bounded first variation. Given $V \in C(S)$, one can find a subsequence (without relabeling) $\{\phi_i(\bar{x}_i) : \bar{x}_i \in I(1, l_i)_0\} \subset \mathcal{C}(M)$, such that $V = \lim |\partial\phi_i(\bar{x}_i)|$ as varifolds, and

$$\lim \mathcal{A}^c(\phi_i(\bar{x}_i)) = \mathbf{L}^c(\Pi).$$

We will need to first consider $\phi_i(x_i) = \phi_i^1(x_i)$, where x_i is the nearest point to \bar{x}_i in $I(1, k_i)_0$. By (c) and (d), we have $\lim \mathcal{A}^c(\phi_i^1(x_i)) = \mathbf{L}^c(\Pi)$ and also $\lim |\partial\phi_i(\bar{x}_i)| = \lim |\partial\phi_i^1(x_i)|$ as varifolds. Then by Claim 1, we conclude that $V \in A_\infty^c \cap C(S^*)$. This completes the proof. \square

5. c -ALMOST MINIMIZING

In this section, we introduce the notion of c -almost minimizing varifolds, and prove the existence of such a varifold from min-max construction. We prove the existence of a c -replacement for any c -almost minimizing varifold. Using this property, we show that every blowup of such varifold is regular. As an easy consequence, the tangent cones of such varifolds are always integer multiples of planes.

Definition 5.1 (*c*-almost minimizing varifolds). Let ν be the \mathcal{F} , \mathbf{M} -norms or the \mathbf{F} -metric. For any given $\epsilon, \delta > 0$ and an open subset $U \subset M$, we define $\mathcal{A}_n^c(U; \epsilon, \delta; \nu)$ to be the set of all $\Omega \in \mathcal{C}(M)$ such that if $\Omega = \Omega_0, \Omega_1, \Omega_2, \dots, \Omega_m \in \mathcal{C}(M)$ is a sequence with:

- (i) $\text{spt}(\Omega_i - \Omega) \subset U$;
- (ii) $\nu(\partial\Omega_{i+1}, \partial\Omega_i) \leq \delta$;
- (iii) $\mathcal{A}^c(\Omega_i) \leq \mathcal{A}^c(\Omega) + \delta$, for $i = 1, \dots, m$,

then $\mathcal{A}^c(\Omega_m) \geq \mathcal{A}^c(\Omega) - \epsilon$.

We say that a varifold $V \in \mathcal{V}_n(M)$ is *c*-almost minimizing in U if there exist sequences $\epsilon_i \rightarrow 0$, $\delta_i \rightarrow 0$, and $\Omega_i \in \mathcal{A}_n^c(U; \epsilon_i, \delta_i; \mathcal{F})$, such that $\mathbf{F}(|\partial\Omega_i|, V) \leq \epsilon_i$.

The following simple fact says that *c*-almost minimizing implies *c*-bounded first variation.

Lemma 5.2. *Let $V \in \mathcal{V}_n(M)$ be c-almost minimizing in U , then V has c-bounded first variation in U .*

Proof. Suppose by contradiction that V does not have *c*-bounded first variation, then there exist $\epsilon_0 > 0$ and a smooth vector field $X \in \mathfrak{X}(U)$ compactly supported in U , such that

$$\left| \int_{G_n(M)} \text{div}_S X(x) dV(x, S) \right| \geq (c + \epsilon_0) \int_M |X| d\mu_V > 0.$$

By changing the sign of X if necessary, we have

$$\int_{G_n(M)} \text{div}_S X(x) dV(x, S) \leq -(c + \epsilon_0) \int_M |X| d\mu_V.$$

By continuity, we can find $\epsilon_1 > 0$ small enough depending only on ϵ_0, V, X , such that if $\Omega \in \mathcal{C}(M)$ with $\mathbf{F}(|\partial\Omega|, V) < 2\epsilon_1$, then

$$\delta \mathcal{A}^c|_{\Omega}(X) \leq \int_{\partial\Omega} \text{div}_{\partial\Omega} X d\mu_{\partial\Omega} + c \int_{\partial\Omega} |X| d\mu_{\partial\Omega} \leq -\frac{\epsilon_0}{2} \int_M |X| d\mu_V < 0.$$

If $\mathbf{F}(|\partial\Omega|, V) < \epsilon_1$, then by deforming Ω along the 1-parameter flow $\{\Phi^X(t) : t \in [0, \tau]\}$ of X for a uniform short time $\tau > 0$, we can obtain a 1-parameter family $\{\Omega_t \in \mathcal{C}(M) : t \in [0, \tau]\}$, such that $t \rightarrow \partial\Omega_t$ is continuous under the \mathcal{F} -topology, with $\text{spt}(\Omega_t - \Omega) \subset U$, $\mathbf{F}(|\partial\Omega_t|, V) < 2\epsilon_1$ and $\mathcal{A}^c(\Omega_t) \leq \mathcal{A}^c(\Omega_0) = \mathcal{A}^c(\Omega)$ for all $t \in [0, \tau)$, but with $\mathcal{A}^c(\Omega_\tau) \leq \mathcal{A}^c(\Omega) - \epsilon_2$ for some $\epsilon_2 > 0$ depending only on $\epsilon_0, \epsilon_1, V, X$.

Summarizing the above, given any $\epsilon < \min\{\epsilon_1, \epsilon_2\}$ and $\delta > 0$, if $\Omega \in \mathcal{C}(M)$ and $\mathbf{F}(|\partial\Omega|, V) < \epsilon$, then $\Omega \notin \mathcal{A}_n^c(U; \epsilon, \delta; \mathcal{F})$; this contradicts the *c*-almost minimizing property of V . \square

We will need the following equivalence result among several almost minimizing concepts using the three different topology. In particular, we can actually use the \mathbf{M} -norm instead at the expense of shrinking the open subset $U \subset M$.

Proposition 5.3. *Given $V \in \mathcal{V}_n(M)$, then the following statements satisfy (a) \implies (b) \implies (c) \implies (d):*

- (a) V is *c*-almost minimizing in U ;
- (b) For any $\epsilon > 0$, there exists $\delta > 0$ and $\Omega \in \mathcal{A}_n^c(U; \epsilon, \delta; \mathbf{F})$ such that $\mathbf{F}(V, |\partial\Omega|) < \epsilon$;
- (c) For any $\epsilon > 0$, there exists $\delta > 0$ and $\Omega \in \mathcal{A}_n^c(U; \epsilon, \delta; \mathbf{M})$ such that $\mathbf{F}(V, |\partial\Omega|) < \epsilon$;
- (d) V is *c*-almost minimizing in W for any relatively open subset $W \subset\subset U$.

Remark 5.4. The proof was originally due to Pitts [43, Theorem 3.9]. In our context, we work with boundaries instead of general integral currents. Furthermore, in Definition 5.1(iii), we use the \mathcal{A}^c functional instead of the mass \mathbf{M} .

Proof. It is easy to see $(a) \implies (b) \implies (c)$. The last implication $(c) \implies (d)$ is an interpolation process which was originally established in Pitts [43, Proposition 3.8] using integral cycles. The corresponding interpolation process using boundaries of Caccioppoli sets was obtained by the first author in [60, Proposition 5.3]. The detailed description of this process is given in Lemma A.1 in Appendix A. \square

Definition 5.5. A varifold $V \in \mathcal{V}_n(M)$ is said to be *c-almost minimizing in small annuli* if for each $p \in M$, there exists $r_{am}(p) > 0$ such that V is *c-almost minimizing* in $A_{s,r}(p) \cap M$ for all $0 < s < r \leq r_{am}(p)$, where $A_{s,r}(p) = B_r(p) \setminus B_s(p)$.

Theorem 5.6 (Existence of *c-almost minimizing varifold*). *Let $\Pi \in \pi_1^\sharp(\mathcal{C}(M, \mathbf{M}), \{0\})$, and assume that $\mathbf{L}^c(\Pi) > 0$. There exists a nontrivial $V \in \mathcal{V}_n(M)$, such that*

- (i) $V \in C(S)$ for some critical sequence S of Π ;
- (ii) V has *c-bounded first variation*;
- (iii) V is *c-almost minimizing in small annuli*.

Proof. First we can pick a critical sequence S of Π which has been pulled-tight by Proposition 4.4, so that every $V \in C(S)$ has *c-bounded first variation*. Suppose for the sake of contradiction that for each $V \in C(S)$, there exists a $p \in M$, such that there are arbitrarily small annuli centered at p on which V is not *c-almost minimizing*. Then by Proposition 5.3, V is also not *c-almost minimizing* with respect to the mass norm on these annuli (i.e. $\nu = \mathbf{M}$).

Specifically, for any $\tilde{r} > 0$, there exists $r, s > 0$ with $\tilde{r} > r + 2s > r - 2s > 0$, and $\epsilon > 0$, such that for any $\delta > 0$, and $\Omega \in \mathcal{C}(M)$ with $\mathbf{F}(|\partial\Omega|, V) < \epsilon$, then $\Omega \notin \mathcal{A}_n^c(A_{r-2s, r+2s}(p) \cap M; \epsilon, \delta; \mathbf{M})$. Now using the same argument as in [43, 4.10] by changing the mass functional \mathbf{M} to the \mathcal{A}^c -functional, one can construct a new 1-homotopic sequence \tilde{S} which is homotopic to S , and $\mathbf{L}^c(\tilde{S}) < \mathbf{L}^c(S)$; but this contradicts the criticality of S . \square

Now we formulate and solve a natural constrained minimization problem which will be used in the construction of *c-replacements*.

Lemma 5.7 (A constrained minimization problem). *Given $\epsilon, \delta > 0$, $U \subset M$ and any $\Omega \in \mathcal{A}_n^c(U; \epsilon, \delta; \mathcal{F})$, fix a compact subset $K \subset U$. Let \mathcal{C}_Ω be the set of all $\Lambda \in \mathcal{C}(M)$ such that there exists a sequence $\Omega = \Omega_0, \Omega_1, \dots, \Omega_m = \Lambda$ in $\mathcal{C}(M)$ satisfying:*

- (a) $\text{spt}(\Omega_i - \Omega) \subset K$;
- (b) $\mathcal{F}(\partial\Omega_i - \partial\Omega_{i+1}) \leq \delta$;
- (c) $\mathcal{A}^c(\Omega_i) \leq \mathcal{A}^c(\Omega) + \delta$, for $i = 1, \dots, m$.

Then there exists $\Omega^ \in \mathcal{C}_\Omega$ such that:*

- (i) $\Omega^* \in \mathcal{C}_\Omega$, and

$$\mathcal{A}^c(\Omega^*) = \inf\{\mathcal{A}^c(\Lambda) : \Lambda \in \mathcal{C}_\Omega\},$$
- (ii) Ω^* is locally \mathcal{A}^c -minimizing in $\text{int}(K)$,
- (iii) $\Omega^* \in \mathcal{A}_n^c(U; \epsilon, \delta; \mathcal{F})$.

Proof. Proof of (i): Take any minimizing sequence $\{\Lambda_j\} \subset \mathcal{C}_\Omega$, i.e.

$$\lim_{j \rightarrow \infty} \mathcal{A}^c(\Lambda_j) = \inf\{\mathcal{A}^c(\Lambda) : \Lambda \in \mathcal{C}_\Omega\}.$$

Notice that $\text{spt}(\Lambda_j - \Omega) \subset K$ and $\mathcal{A}^c(\Lambda_j) \leq \mathcal{A}^c(\Omega) + \delta$ for all j . By standard compactness [51, Theorem 6.3], after passing to a subsequence, $\partial\Lambda_j$ converges weakly to some $\partial\Omega^*$ with $\Omega^* \in \mathcal{C}(M)$

and $\text{spt}(\Omega^* - \Omega) \subset K$. We will show that Ω^* is our desired minimizer. Since $\partial\Lambda_j$ converges weakly to $\partial\Omega^*$, we have that $\mathcal{H}^n(\partial\Omega^*) \leq \lim_{j \rightarrow \infty} \mathcal{H}^n(\partial\Lambda_j)$ and $\mathcal{H}^{n+1}(\Omega^*) = \lim_{j \rightarrow \infty} \mathcal{H}^{n+1}(\Lambda_j)$. Therefore,

$$(5.1) \quad \mathcal{A}^c(\Omega^*) \leq \inf\{\mathcal{A}^c(\Lambda) : \Lambda \in \mathcal{C}_\Omega\}.$$

It remains to show that $\Omega^* \in \mathcal{C}_\Omega$. For j sufficiently large, we have $\mathcal{F}(\partial\Lambda_j - \partial\Omega^*) < \delta$. Since $\Lambda_j \in \mathcal{C}_\Omega$, there exists a sequence $\Omega = \Omega_0, \Omega_1, \dots, \Omega_m = \Lambda_j$ in $\mathcal{C}(M)$ satisfying conditions (a-c) above. Consider now the sequence $\Omega = \Omega_0, \Omega_1, \dots, \Omega_m = \Lambda_j, \Omega_{m+1} = \Omega^*$ in $\mathcal{C}(M)$; it trivially satisfies conditions (a) and (b). Moreover, using (5.1), we also have

$$\mathcal{A}^c(\Omega^*) \leq \mathcal{A}^c(\Lambda_j) \leq \mathcal{A}^c(\Omega) + \delta.$$

Therefore, $\Omega^* \in \mathcal{C}_\Omega$ and hence (i) has been proved.

Proof of (ii): For $p \in \text{int}(K)$, we claim that there exists a small $\tilde{B}_r(p) \subset \text{int}(K)$ such that

$$(5.2) \quad \mathcal{A}^c(\Omega^*) \leq \mathcal{A}^c(\Lambda),$$

for any $\Lambda \in \mathcal{C}(M)$ with $\text{spt}(\Lambda - \Omega^*) \subset \tilde{B}_r(p)$. To establish (5.2), first choose $r > 0$ small so that $c \cdot \text{Vol}(\tilde{B}_r(p)) < \delta/4$ and $\mathbf{M}(\partial\Omega^* \llcorner \tilde{B}_r(p)) < \delta/4$ (this is possible since $\partial\Omega^*$ is rectifiable). Suppose (5.2) were false, then there exists $\Omega' \in \mathcal{C}(M)$ with $\text{spt}(\Omega' - \Omega^*) \subset \tilde{B}_r(p)$ such that $\mathcal{A}^c(\Omega') < \mathcal{A}^c(\Omega^*)$. We will show that $\Omega' \in \mathcal{C}_\Omega$, which contradicts that Ω^* is a minimizer from part (i).

To see that $\Omega' \in \mathcal{C}_\Omega$, take a sequence $\Omega = \Omega_0, \Omega_1, \dots, \Omega_m = \Omega^*$ in $\mathcal{C}(M)$ satisfying (a-c) above, and append $\Omega_{m+1} = \Omega'$ to the sequence. Since $\text{spt}(\Omega^* - \Omega) \subset K$ and $\text{spt}(\Omega' - \Omega^*) \subset K$, we have $\text{spt}(\Omega' - \Omega) \subset K$. By the facts that $\text{spt}(\Omega' - \Omega^*) \subset \tilde{B}_r(p)$ and $\mathcal{A}^c(\Omega') < \mathcal{A}^c(\Omega^*)$, we have

$$\begin{aligned} \mathbf{M}(\partial\Omega' \llcorner \tilde{B}_r(p)) &\leq \mathbf{M}(\partial\Omega^* \llcorner \tilde{B}_r(p)) + c[\mathcal{H}^{n+1}(\Omega^* \cap \tilde{B}_r(p)) + \mathcal{H}^{n+1}(\Omega' \cap \tilde{B}_r(p))] \\ &\leq \mathbf{M}(\partial\Omega^* \llcorner \tilde{B}_r(p)) + 2c \text{Vol}(\tilde{B}_r(p)); \end{aligned}$$

hence

$$\mathbf{M}(\partial\Omega' - \partial\Omega^*) \leq \mathbf{M}(\partial\Omega' \llcorner \tilde{B}_r(p)) + \mathbf{M}(\partial\Omega^* \llcorner \tilde{B}_r(p)) < \delta.$$

So $\mathcal{F}(\partial\Omega' - \partial\Omega^*) \leq \mathbf{M}(\partial\Omega' - \partial\Omega^*) < \delta$. Finally note $\mathcal{A}^c(\Omega') < \mathcal{A}^c(\Omega^*) \leq \mathcal{A}^c(\Omega) + \delta$. Therefore $\Omega' \in \mathcal{C}_\Omega$, and this proves part (ii).

Proof of (iii): Suppose that the claim is false. Then by Definition 5.1 there exists a sequence $\Omega^* = \Omega_0^*, \Omega_1^*, \dots, \Omega_\ell^*$ in $\mathcal{C}(M)$ satisfying

- $\text{spt}(\Omega_i^* - \Omega^*) \subset U$;
- $\mathcal{F}(\partial\Omega_i^* - \partial\Omega_{i+1}^*) \leq \delta$;
- $\mathcal{A}^c(\Omega_i^*) \leq \mathcal{A}^c(\Omega^*) + \delta$, for $i = 1, \dots, \ell$,

but $\mathcal{A}^c(\Omega_\ell^*) < \mathcal{A}^c(\Omega^*) - \epsilon$. Since $\Omega^* \in \mathcal{C}_\Omega$ by part (i), there exists a sequence $\Omega = \Omega_0, \Omega_1, \dots, \Omega_m = \Omega^*$ satisfying conditions (a-c) above. Then the sequence $\Omega = \Omega_0, \Omega_1, \dots, \Omega_m, \Omega_1^*, \dots, \Omega_\ell^*$ in $\mathcal{C}(M)$ still satisfies those conditions (a-c), since $\mathcal{A}^c(\Omega^*) \leq \mathcal{A}^c(\Omega)$ implies that $\mathcal{A}^c(\Omega_i^*) \leq \mathcal{A}^c(\Omega) + \delta$. Therefore $\Omega \in \mathcal{A}_n^c(U; \epsilon, \delta; \mathcal{F})$ implies that $\mathcal{A}^c(\Omega_\ell^*) \geq \mathcal{A}^c(\Omega) - \epsilon \geq \mathcal{A}^c(\Omega^*) - \epsilon$, which is a contradiction. This proves part (iii). \square

Proposition 5.8 (Existence and properties of replacements). *Let $V \in \mathcal{V}_n(M)$ be c -almost minimizing in an open set $U \subset M$ and $K \subset U$ be a compact subset, then there exists $V^* \in \mathcal{V}_n(M)$, called a c -replacement of V in K such that*

- (i) $V \llcorner (M \setminus K) = V^* \llcorner (M \setminus K)$;
- (ii) $-c \text{Vol}(K) \leq \|V\|(M) - \|V^*\|(M) \leq c \text{Vol}(K)$;
- (iii) V^* is c -almost minimizing in U ;

- (iv) moreover, $V^* = \lim_{i \rightarrow \infty} |\partial\Omega_i^*|$ as varifolds for some $\Omega_i^* \in \mathcal{C}(M)$ such that $\Omega_i^* \in \mathcal{A}_n^c(U; \epsilon_i, \delta_i; \mathcal{F})$ with $\epsilon_i, \delta_i \rightarrow 0$; furthermore Ω_i^* locally minimizes \mathcal{A}^c in $\text{int}(K)$;
- (v) if V has c -bounded first variation in M , then so does V^* .

Proof. Let $V \in \mathcal{V}_n(M)$ be c -almost minimizing in U . By definition there exists a sequence $\Omega_i \in \mathcal{A}_n^c(U; \epsilon_i, \delta_i; \mathcal{F})$ with $\epsilon_i, \delta_i \rightarrow 0$ such that V is the varifold limit of $|\partial\Omega_i|$. By Lemma 5.7 we can construct a c -minimizer $\Omega_i^* \in \mathcal{C}_{\Omega_i}$ for each i . Since $\mathbf{M}(\partial\Omega_i^*)$ is uniformly bounded, by compactness there exists a subsequence $|\partial\Omega_i^*|$ converging as varifolds to some $V^* \in \mathcal{V}_n(M)$. We claim that V^* satisfies items (i)-(v) in Proposition 5.8 and thus is our desired c -replacement.

- First, by part (i) of Lemma 5.7, we have $\Omega_i^* \in \mathcal{C}_{\Omega_i}$ and thus $\text{spt}(\Omega_i^* - \Omega_i) \subset K$. Hence the varifold limits satisfy $V^* \llcorner (M \setminus K) = V \llcorner (M \setminus K)$.
- Second, as $\Omega_i \in \mathcal{A}_n^c(U, \epsilon_i, \delta_i; \mathcal{F})$ and $\Omega_i^* \in \mathcal{C}_{\Omega_i}$, we have

$$\mathcal{A}^c(\Omega_i) - \epsilon_i \leq \mathcal{A}^c(\Omega_i^*) \leq \mathcal{A}^c(\Omega_i);$$

thus by (1.1),

$$\mathbf{M}(\partial\Omega_i) - c\mathcal{H}^{n+1}(\Omega_i) - \epsilon_i \leq \mathbf{M}(\partial\Omega_i^*) - c\mathcal{H}^{n+1}(\Omega_i^*) \leq \mathbf{M}(\partial\Omega_i) - c\mathcal{H}^{n+1}(\Omega_i).$$

Note that $|\mathcal{H}^{n+1}(\Omega_i) - \mathcal{H}^{n+1}(\Omega_i^*)| \leq \text{Vol}(K)$; taking $i \rightarrow \infty$, we have $-c\text{Vol}(K) \leq \|V\|(M) - \|V^*\|(M) \leq c\text{Vol}(K)$.

- Since each $\Omega_i^* \in \mathcal{A}_n^c(U; \epsilon_i, \delta_i; \mathcal{F})$ by Lemma 5.7(iii), by definition V^* is c -almost minimizing in U .
- (iv) follows from Lemma 5.7(ii).
- Finally by (iii) and Lemma 5.2, V^* has c -bounded first variation in U . By (i) and a standard cutoff trick it is easy to show that V^* has c -bounded first variation in M whenever V does.

□

Lemma 5.9 (Regularity of c -replacement). *Let $2 \leq n \leq 6$. Under the same hypotheses as Proposition 5.8, if $\Sigma = \text{spt} \|V^*\| \cap \text{int}(K)$, then*

- (1) Σ is a smooth, almost embedded, stable c -boundary;
- (2) the density of V^* is 1 along $\mathcal{R}(\Sigma)$ and 2 along $\mathcal{S}(\Sigma)$;
- (3) the restriction of the c -replacement $V^* \llcorner \text{int}(K) = \Sigma$.

Proof. By the regularity for local minimizers of the \mathcal{A}^c functional (Theorem 2.14), we know that each $\partial\Omega_i^*$ is a smooth, embedded, stable c -boundary in $\text{int}(K)$ by Proposition 5.8(iv). The lemma then follows from the compactness Theorem 2.11. □

Using Proposition 5.8, we can obtain the following preliminary lemma. It will be essential for proving that various blowups of the c -min-max varifold are planar, particularly in the proposition to follow, and in Section 6. Given $p \in M, r > 0$, let $\eta_{p,r} : \mathbb{R}^L \rightarrow \mathbb{R}^L$ be the dilation defined by $\eta_{p,r}(x) = \frac{x-p}{r}$.

Lemma 5.10. *Let $2 \leq n \leq 6$, and $V \in \mathcal{V}_n(M)$ be a c -almost minimizing varifold in U . Given a sequence $p_i \in U$ with $p_i \rightarrow p \in U$ and, a sequence $r_i > 0$ with $r_i \rightarrow 0$, let $\bar{V} = \lim(\eta_{p_i, r_i})_{\#} V$ be the varifold limit. Then \bar{V} is an integer multiple of some complete embedded minimal hypersurface Σ in $T_p M$, and moreover, Σ is proper.*

Proof. By Lemma 5.2, V has c -bounded first variation in U , so the blowup \bar{V} is stationary in $T_p M$. We will show that \bar{V} satisfies the good replacement property (Definition C.2) in any open subset

$W \subset T_p M$, which will imply that \bar{V} is regular by Proposition C.3. The properness follows from the Euclidean volume growth of \bar{V} , which is a direct corollary of the monotonicity formula.

Now fix a bounded open subset $W \subset T_p M$ and an arbitrary $x \in W$. Consider an arbitrary annulus $An = A_{s,t}(x) \subset W$ of outer radius $t \leq 1$. We will show that \bar{V} has a replacement in An in the sense of Definition C.1. Since $\eta_{p_i, r_i}(M) \rightarrow T_p M$ locally uniformly, we can identify $\eta_{p_i, r_i}(M)$ with $T_p M$ on compact subsets for i large. Denote $An_i = \eta_{p_i, r_i}^{-1}(An)$, then $\bar{An}_i \subset U$ for i large. For each i large, we may apply Proposition 5.8 to obtain a c -replacement V_i^* of V in \bar{An}_i . Denote $\bar{V}_i^* = (\eta_{p_i, r_i})_{\#} V_i^*$, $\bar{V}_i = (\eta_{p_i, r_i})_{\#} V$, then up to taking subsequences we have

$$\bar{V}' = \lim_{i \rightarrow \infty} \bar{V}_i^*, \text{ for some varifold } \bar{V}' \text{ in } T_p M.$$

Moreover, we can deduce the following for \bar{V}' :

- Proposition 5.8(i) $\implies \bar{V}_i^* \llcorner (W \setminus \bar{An}) = \bar{V}_i \llcorner (W \setminus \bar{An})$, hence

$$\bar{V}' \llcorner (W \setminus \bar{An}) = \bar{V} \llcorner (W \setminus \bar{An});$$

- Proposition 5.8(ii) $\implies -c \text{Vol}(\bar{An}_i) \cdot r_i^{-n} \leq \|\bar{V}_i^*\|(W) - \|\bar{V}_i\|(W) \leq c \text{Vol}(\bar{An}_i) \cdot r_i^{-n}$.

Since the outer radius of \bar{An}_i is at most r_i , for large i there exists some $C_0 > 0$ depending only on M such that $\text{Vol}(\bar{An}_i) \leq C_0 r_i^{n+1}$. This implies that

$$\|\bar{V}'\|(W) = \|\bar{V}\|(W);$$

- Proposition 5.8(iii) $\implies V_i^*$ has c -bounded first variation in U , hence \bar{V}' is stationary in W ;
- By Lemma 5.9, the restriction $V_i^* \llcorner An_i$ is a smooth, almost embedded, stable c -boundary Σ_i^* . Consider the rescalings: $\bar{\Sigma}_i = \eta_{p_i, r_i}(\Sigma_i^*) \subset An$. By Proposition 5.8(ii) and the monotonicity formula [51, 40.2], $\bar{\Sigma}_i$ have uniformly bounded mass. This together with the compactness Theorem 2.11(iii) implies that $\bar{V}' \llcorner An$ is an embedded stable minimal hypersurface.

Therefore, \bar{V}' is a good replacement of \bar{V} in An . By Proposition 5.8(iii), each V_i^* is still c -almost minimizing in U . Hence for any other annulus $An' \subset W$ of outer radius ≤ 1 , we can repeat the above process and produce a good replacement \bar{V}'' of \bar{V}' in An' . In fact, we may repeat this process any finite number of times. In particular, \bar{V} satisfies the good replacement property (Definition C.2) in W , and this completes the proof. \square

Proposition 5.11 (Tangent cones are planes). *Let $2 \leq n \leq 6$. Suppose $V \in \mathcal{V}_n(M)$ has c -bounded first variation in M and is c -almost minimizing in small annuli. Then V is integer rectifiable. Moreover, for any $C \in \text{VarTan}(V, p)$ with $p \in \text{spt} \|V\|$,*

$$(5.3) \quad C = \Theta^n(\|V\|, p) |S| \text{ for some } n\text{-plane } S \subset T_p M \text{ where } \Theta^n(\|V\|, p) \in \mathbb{N}.$$

Proof. Let $r_i \rightarrow 0$ be a sequence such that C is the varifold limit:

$$C = \lim_{i \rightarrow \infty} (\eta_{p, r_i})_{\#} V.$$

First we know C is stationary in $T_p M$. Since V is c -almost minimizing in small annuli centered at p , by the same argument as in Lemma 5.10, we can show that C satisfies the good replacement property (Definition C.2) in $T_p M$. (Note that Definition C.2 only requires the existence of good replacements in small annuli.) Therefore, by Proposition C.3, C is an integer multiple of some embedded minimal hypersurface of $T_p M$, and moreover, it is a cone by [51, 19.3]. In particular C is smooth and hence $\text{spt} \|C\|$ must be a plane. This finishes the proof. \square

6. REGULARITY FOR c -MIN-MAX VARIFOLD

In this section, we prove the regularity of our min-max varifolds. In particular we prove that every varifold which has c -bounded variation and is c -almost minimizing in small annuli is a smooth, closed, almost embedded, CMC hypersurface with multiplicity one.

Theorem 6.1 (Main regularity). *Let $2 \leq n \leq 6$, and (M^{n+1}, g) be an $(n + 1)$ -dimensional smooth, closed Riemannian manifold. Suppose $V \in \mathcal{V}_n(M)$ is a varifold which*

- (1) *has c -bounded first variation in M and*
- (2) *is c -almost minimizing in small annuli,*

then V is induced by Σ , where

- (i) *Σ is a closed, almost embedded c -hypersurface (possibly disconnected);*
- (ii) *the density of V is exactly 1 at the regular set $\mathcal{R}(\Sigma)$ and 2 at the touching set $\mathcal{S}(\Sigma)$.*

Proof. The conclusion is purely local, so we only need to prove the regularity of V near an arbitrary point $p \in \text{spt } \|V\|$. Fix a $p \in \text{spt } \|V\|$, then there exists $0 < r_0 < r_{am}(p)$ such that for any $r < r_0$, the mean curvature H of $\partial B_r(p) \cap M$ in M is greater than c . Here $r_{am}(p)$ is as in Definition 5.5.

In particular, if $r < r_0$ and $W \in \mathcal{V}_n(M)$ has c -bounded first variation in $B_r(p) \cap M$ and $W \neq 0$ in $B_r(p)$, then by the maximum principle (Proposition 2.13)

$$(6.1) \quad \emptyset \neq \text{spt } \|W\| \cap \partial B_r(p) = \text{Clos}(\text{spt } \|W\| \setminus \text{Clos}(B_r(p))) \cap \partial B_r(p).$$

Note that in the second equality we need a localized version of Proposition 2.13 which holds true by the remark after [57, Theorem 2].

We will show that $V \llcorner B_{r_0}(p)$ is an almost embedded hypersurface of constant mean curvature c with density equal to 2 along its touching set. The argument consists of five steps:

Step 1: *Constructing successive c -replacements V^* and V^{**} on two overlapping concentric annuli.*

Step 2: *Gluing the c -replacements smoothly (as immersed hypersurfaces) on the overlap.*

Step 3: *Extending the c -replacements down to the point p to get a c -‘replacement’ \tilde{V} on the punctured ball.*

Step 4: *Showing that the singularity of \tilde{V} at p is removable, so that \tilde{V} is regular.*

Step 5: *V coincides with the almost embedded hypersurface \tilde{V} on a small neighborhood of p .*

We now proceed to the proof.

Step 1. We first describe the construction of c -replacements on overlapping annuli; a key property will be that the replacements are also boundaries in the chosen annulus (see Claim 1).

Fix $0 < s < t < r_0$. By the choice of r_0 , we can apply Proposition 5.8 to V to obtain a c -replacement V^* in $K = \text{Clos}(A_{s,t}(p) \cap M)$. By (6.1) and Lemma 5.9, the restriction

$$\Sigma_1 = V^* \llcorner (A_{s,t}(p) \cap M)$$

is a nontrivial, smooth, almost embedded, stable c -boundary with outer unit normal ν_1 .

By Proposition 2.9, the touching set $\mathcal{S}(\Sigma_1)$ is contained in a countable union of $(n - 1)$ -dimensional connected submanifolds $\bigcup S_1^{(k)}$. Since a countable union of sets of measure zero still has measure zero, it follows from Sard’s theorem that we may choose $s_2 \in (s, t)$ such that $\partial(B_{s_2}(p) \cap M)$ intersects Σ_1 and all the $S_1^{(k)}$ transversally.

Given any $s_1 \in (0, s)$, by Proposition 5.8(iii), we can apply Proposition 5.8 again to get a c -replacement V^{**} of V^* in $K = \text{Clos}(A_{s_1, s_2}(p) \cap M)$. By (6.1) and Lemma 5.9 again, the restriction

$$\Sigma_2 = V^{**} \llcorner (A_{s_1, s_2}(p) \cap M)$$

is also a nontrivial, smooth, almost embedded, stable c -boundary with outer unit normal ν_2 . Note that by Proposition 5.8(v), both V^* and V^{**} have c -bounded first variation.

We can choose the second c -replacement V^{**} so that it satisfies:

*Claim 1: there exists a set $\Omega^{**} \in \mathcal{C}(M)$, such that*

- a) $\Sigma_1 \cap A_{s_2,t}(p)$ and Σ_2 are the boundaries of Ω^{**} in $A_{s_2,t}(p)$ and $A_{s_1,s_2}(p)$ respectively;
- b) ν_1, ν_2 coincide with the outer unit normal of Ω^{**} in $A_{s_2,t}(p)$ and $A_{s_1,s_2}(p)$ respectively;
- c) if $\|V^{**}\|(\partial B_{s_2}(p)) = 0$, then V^{**} is identical to $|\partial\Omega^{**}|$ in $A_{s_1,t}(p) \cap M$.

Proof of Claim 1: Fix $0 < \tau < s_1$, then by Proposition 5.8(iv), $V^* = \lim_{i \rightarrow \infty} |\partial\Omega_i^*|$ as varifolds for some $\Omega_i^* \in \mathcal{C}(M)$, where $\Omega_i^* \in \mathcal{A}_n^c(A_{\tau,r_0}(p) \cap M; \epsilon_i, \delta_i; \mathcal{F})$ with $\epsilon_i, \delta_i \rightarrow 0$. The regularity of the $\partial\Omega_i^*$ as in the proof of Lemma 5.9, together with the compactness Theorem 2.11 imply that $\partial\Omega_i^* \llcorner A_{s,t}(p)$ converges locally smoothly to Σ_1 in $A_{s,t}(p) \cap M$.

Applying Lemma 5.7 for each Ω_i^* in $K = \text{Clos}(A_{s_1,s_2}(p) \cap M)$, we can construct new c -minimizers Ω_i^{**} , such that

- $\text{spt}(\Omega_i^* - \Omega_i^{**}) \subset K = \text{Clos}(A_{s_1,s_2}(p) \cap M)$;
- Ω_i^{**} is locally \mathcal{A}^c -minimizing in $A_{s_1,s_2}(p) \cap M$;
- $V^{**} = \lim |\partial\Omega_i^{**}|$ as varifolds;
- $\partial\Omega_i^{**} \llcorner A_{s_1,s_2}(p)$ converges locally smoothly to Σ_2 (again as in the proof of Lemma 5.9).

By the weak compactness [51, Theorem 6.3], up to a subsequence, $\partial\Omega_i^{**}$ converges weakly as currents to some $\partial\Omega^{**}$ with $\Omega^{**} \in \mathcal{C}(M)$. Claims (a) and (b) follow from the locally smooth convergence. The weak convergence implies that $\|\partial\Omega^{**}\|(A_{s_1,t}(p)) \leq \|V^{**}\|(A_{s_1,t}(p))$. If $\|V^{**}\|(\partial B_{s_2}(p)) = 0$, then together with the locally smooth convergence, we have $\|\partial\Omega^{**}\|(A_{s_1,t}(p)) = \|V^{**}\|(A_{s_1,t}(p))$; moreover, $V^{**} \llcorner (A_{s_1,t}(p) \cap M) = |\partial\Omega^{**}| \llcorner (A_{s_1,t}(p) \cap M)$ by [43, 2.1(18)(f)]. This confirms (c). \square

Step 2. We now show that Σ_1 and Σ_2 glue smoothly (as immersed hypersurfaces) across $\partial(B_{s_2}(p) \cap M)$. Indeed, define the intersection set

$$(6.2) \quad \Gamma = \Sigma_1 \cap \partial(B_{s_2}(p) \cap M), \quad \mathcal{S}(\Gamma) = \Gamma \cap \mathcal{S}(\Sigma_1).$$

Then by transversality, Γ is an almost embedded hypersurface in $\partial(B_{s_2}(p) \cap M)$, and $\mathcal{S}(\Gamma)$ is its touching set. Notice that

$$(6.3) \quad \mathcal{S}(\Gamma) \text{ is closed, and } \mathcal{R}(\Gamma) = \Gamma \setminus \mathcal{S}(\Gamma) \text{ is open in } \Gamma.$$

It follows from the maximum principle that

$$\text{Clos}(\Sigma_2) \cap \partial(B_{s_2}(p) \cap M) \subset \Gamma.$$

Indeed, (6.1) implies that any $y \in \text{Clos}(\Sigma_2) \cap \partial(B_{s_2}(p) \cap M)$ is also a limit point of $\text{spt} \|V^{**}\|$ from the outer side of $\partial B_{s_2}(p)$, on which V^{**} coincides with Σ_1 . In fact, with a little more work we have

Claim 2: $\text{Clos}(\Sigma_2) \cap \partial(B_{s_2}(p) \cap M) = \Gamma$, and then Σ_1 glues together continuously with Σ_2 .

Proof of Claim 2: By Proposition 5.8(i), we have

$$(6.4) \quad V^* = V^{**} = \Sigma_1, \quad \text{in } A_{s_2,t}(p) \cap M.$$

Given any $x \in \Gamma$, using (6.4), Proposition 5.11 and the fact that Σ_1 meets $\partial(B_{s_2}(p) \cap M)$ transversally, we have

$$(6.5) \quad \text{VarTan}(V^{**}, x) = \{\Theta^n(\|V^*\|, x) | T_x \Sigma_1\}.$$

This implies that x is a limit point of $\text{spt} \|V^{**}\|$ from inside of $\partial B_{s_2}(p)$, and thus completes the proof of the claim. \square

As a direct corollary of (6.5), [51, Theorem 3.2(2)] and Claim 1(c), we have

$$(6.6) \quad \|V^{**}\|(\partial B_{s_2}(p)) = 0, \text{ and hence } V^{**} = |\partial\Omega^{**}| \text{ in } A_{s_1,t}(p) \cap M.$$

Furthermore, we will show that Σ_1 glues with Σ_2 in C^1 , i.e. the tangent spaces of Σ_1 and Σ_2 agree along Γ , with matching normals. Take an arbitrary $q \in \Gamma$. We will need to divide to two sub-cases:

Sub-case (A): q is a regular point of Σ_1 , i.e. $q \in \mathcal{R}(\Gamma)$.

First we have the following.

Claim 3(A): Fix $x \in \mathcal{R}(\Gamma)$, for any sequence of $x_i \rightarrow x$ with $x_i \in \mathcal{R}(\Gamma)$ and $r_i \rightarrow 0$, we have

$$\lim_{i \rightarrow \infty} (\boldsymbol{\eta}_{x_i, r_i})_{\#} V^{**} = T_x \Sigma_1 \quad \text{as varifolds.}$$

Proof of Claim 3(A): By the weak compactness of Radon measures, after passing to a subsequence,

$$(6.7) \quad \lim_{i \rightarrow \infty} (\boldsymbol{\eta}_{x_i, r_i})_{\#} V^{**} = C \in \mathcal{V}_n(T_x M).$$

By Lemma 5.10, C is a regular, proper, complete minimal hypersurface in $T_x M$. By (6.4), C coincides with $T_x \Sigma_1$ on a half space of $T_x M$. The classical maximum principle implies that $C \supset T_x \Sigma_1$. It then follows from the half space theorem for minimal hypersurfaces [22, Theorem 3] that there are no other connected components of C . Thus $C = T_x \Sigma_1$ and the proof is complete. \square

Since $\{(\boldsymbol{\eta}_{x_i, r_i})_{\#} V^{**} : i \in \mathbb{N}\}$ have uniformly bounded first variation, a standard argument using the monotonicity formula implies that

$$(6.8) \quad \text{spt} \|(\boldsymbol{\eta}_{x_i, r_i})_{\#} V^{**}\| \rightarrow T_x \Sigma_1 \text{ in the Hausdorff topology.}$$

To show that Σ_1 and Σ_2 glue together along Γ in C^1 near q . We need to show that:

Claim 4(A): For each $x \in \mathcal{R}(\Gamma)$, we have

$$\lim_{z \rightarrow x, z \in \Sigma_2} \nu_2(z) = \nu_1(x).$$

Moreover, the convergence is uniform in x on compact subsets of Γ near q .

Proof of Claim 4(A): The uniformity follows from the fact that ν_1 is continuous on Γ , so we only need to establish the convergence to ν_1 .

So consider a sequence $z_i \in \Sigma_2$ converging to some $x \in \mathcal{R}(\Gamma)$. Since x is a regular point of Σ_1 , by Claim 3(A) and the upper semi-continuity of density function for varifolds with bounded first variation [51, 17.8], we know that z_i is also a regular point of Σ_2 for i large enough.

Take $x_i \in \Gamma$ to be the nearest point projection (in \mathbb{R}^L) of z_i to Γ and $r_i = |z_i - x_i|$. Note that $x_i \rightarrow x \in \mathcal{R}(\Gamma)$ and $r_i \rightarrow 0$, so we are in the situation of Claim 3(A). Note that $\Sigma_2 \cap B_{r_i/2}(z_i)$ is an embedded, stable c -hypersurface in M , so by Theorem 2.11 a subsequence of the blow-ups $\boldsymbol{\eta}_{x_i, r_i}(\Sigma_2 \cap B_{r_i/2}(z_i))$ converges smoothly to a smooth, embedded, stable, minimal hypersurface Σ_{∞} contained in a half-space of $T_x M$.

On the other hand, Claim 3(A) and (6.8) imply that $\boldsymbol{\eta}_{x_i, r_i}(\Sigma_2 \cap B_{r_i/2}(z_i))$ converges in the Hausdorff topology to a domain in $T_x \Sigma_1$. Therefore, we have $\Sigma_{\infty} \subset T_x \Sigma_1$. The smooth convergence then implies that $\nu_2(z_i)$ converges to one of the unit normals $\pm \nu_1(x)$ of $T_x \Sigma_1$. By Claim 1 and (6.6), we know that $V^{**} = |\partial\Omega^{**}|$ in $A_{s_1,t}(p) \cap M$ and $\|\partial\Omega^{**}\|(\partial B_{s_2}(p)) = 0$, therefore the limit of the $\nu_2(z_i)$ must be $\nu_1(x)$, so *Claim 4(A)* is proved. \square

Thus we have proven that near any regular point $q \in \mathcal{R}(\Gamma)$, Σ_1 and Σ_2 glue together along Γ as a C^1 hypersurface with matching outer unit normals ν_1, ν_2 . The higher regularity follows from a standard elliptic PDE argument. More precisely, Σ_1 and Σ_2 can be written as graphs of some functions u_1, u_2 over $T_q \Sigma_1$ respectively. Since the mean curvatures of both Σ_1 and Σ_2 are identical to $c > 0$ with respect to some unit normals pointing to the same side of $T_q \Sigma_1$, they satisfy the same mean curvature type elliptic PDE with in-homogenous term equal to c . The higher regularity follows from the elliptic regularity of this PDE. This finishes **Sub-case (A)**.

At this point we have proven that Σ_2 glues smoothly with $\Sigma_1 \cap A_{s_2, t}(p)$ along $\mathcal{R}(\Gamma)$. Moreover, by the unique continuation for elliptic PDE, we know that Σ_2 is identical to Σ_1 in a neighborhood of $\mathcal{R}(\Gamma)$ in $A_{s_2, t}(p) \cap M$. We will need to show that the smooth gluing extends to the touching set $\mathcal{S}(\Gamma)$.

Sub-case (B): q is a touching point of Σ_1 , i.e. $q \in \mathcal{S}(\Gamma) \subset \mathcal{S}(\Sigma_1)$.

By Lemma 5.9, in some small neighborhood $U \subset M$ of q , $\Sigma_1 \cap U$ is the union of two connected, embedded c -hypersurfaces $\Sigma_{1,1} \cup \Sigma_{1,2}$ with unit normals $\nu_{1,1}$ and $\nu_{1,2}$, such that $\Sigma_{1,2}$ lies on one side of $\Sigma_{1,1}$ and they touch tangentially at $\mathcal{S}(\Sigma_1) \cap U = \Sigma_{1,1} \cap \Sigma_{1,2}$. By Lemma 2.7, $\nu_{1,1} = -\nu_{1,2}$ along the touching set $\mathcal{S}(\Sigma_1) \cap U$. Denote $\Gamma \cap \Sigma_{1,1} = \Gamma_1$ and $\Gamma \cap \Sigma_{1,2} = \Gamma_2$, then as embedded submanifolds of $\partial(B_{s_2}(p) \cap U)$, Γ_2 lies on one-side of Γ_1 and they touch tangentially along $\mathcal{S}(\Gamma) \cap U$.

Claim 3(B): Fix $x \in \mathcal{S}(\Gamma)$ and denote $P_x = T_x \Sigma_{1,1} = T_x \Sigma_{1,2}$. For any sequence of $x_i \rightarrow x$ with $x_i \in \Gamma$ and $r_i \rightarrow 0$, up to a subsequence we have

$$\lim_{i \rightarrow \infty} (\boldsymbol{\eta}_{x_i, r_i})_{\#} V^{**} = \begin{cases} P_x + \tau_v P_x & \text{for Type I convergence} \\ 2P_x & \text{for Type II convergence} \end{cases},$$

where τ_w denotes translation by a vector w , and $v \in (P_x)^\perp$ is a vector in $T_x M$ orthogonal to P_x (v may be ∞ , in which case $\tau_v P$ is understood to be the empty set). The two convergence scenarios are:

- Type I: $\liminf_{i \rightarrow \infty} \text{dist}_{\mathbb{R}^L}(x_i, \mathcal{S}(\Gamma))/r_i = \infty$,
- Type II: $\liminf_{i \rightarrow \infty} \text{dist}_{\mathbb{R}^L}(x_i, \mathcal{S}(\Gamma))/r_i < \infty$.

Proof of Claim 3(B): First we determine the blowup limit $C' = \lim_{i \rightarrow \infty} (\boldsymbol{\eta}_{x_i, r_i})(\Sigma_1)$.

In Type I convergence, for any $R > 0$, $\Gamma \cap B_{r_i R}(x_i) \subset \mathcal{R}(\Gamma)$ for i large enough. Up to a subsequence, we can assume that all x_i belong to Γ_1 , then $(\boldsymbol{\eta}_{x_i, r_i})(\Sigma_{1,1})$ converges locally smoothly to P_x . Let x'_i be the nearest point projection of x_i to $\Sigma_{1,2}$, and let $v = \lim_{i \rightarrow \infty} \frac{x'_i - x_i}{r_i}$ (up to taking a subsequence), which maybe ∞ . If v is finite, then $(\boldsymbol{\eta}_{x_i, r_i})(\Sigma_{1,2})$ converges locally smoothly to $P_x + v$; if v is infinite, then $(\boldsymbol{\eta}_{x_i, r_i})(\Sigma_{1,2})$ disappears in the limit. So in this case $C' = P_x + \tau_v P_x$. In the Type II scenario, the touching set does not disappear in the limit and we have $C' = \lim (\boldsymbol{\eta}_{x_i, r_i})(\Sigma_1) = 2P_x$.

Now let $C = \lim_{i \rightarrow \infty} (\boldsymbol{\eta}_{x_i, r_i})_{\#} V^{**}$ be the varifold limit as in (6.7). By Lemma 5.10, C is a regular, proper, complete minimal hypersurface in $T_x M$. Again C must coincide with $C' = \lim_{i \rightarrow \infty} (\boldsymbol{\eta}_{x_i, r_i})(\Sigma_1)$ on a halfspace of $T_x M$. Since C' consists of one or two parallel planes, the classical maximum principle implies that C contains these planes, and again the halfspace theorem [22, Theorem 3] rules out any other components of C . Thus C is identical to C' , which completes the proof. \square

By Claim 3(B) and the same argument in Claim 4(A), we know that

$$(6.9) \quad \lim_{z \rightarrow x, z \in \Sigma_2} [T_z \Sigma_2] = [T_x \Sigma_1],$$

where $[T_z \Sigma_2]$ and $[T_x \Sigma_1]$ denote the un-oriented tangent planes of Σ_2 and Σ_1 (without counting multiplicity) respectively. Moreover, the convergence is uniform in x on compact subsets of $\mathcal{S}(\Gamma)$ near q .

Therefore using the regularity of Σ_2 in Lemma 5.9, near q the hypersurface Σ_2 can be written as a set of graphs $\{\Sigma_{2,i} : i = 1 \cdots l\}$ over the half space $[T_q(\Sigma_1 \cap B_{s_2}(p))]$.

Indeed, take ρ small so that $B_\rho(q) \cap M$ may be identified with the tangent space $T_q M$, and for $z \in \Sigma_2 \cap B_\rho(q)$ let $C_\epsilon(z)$ denote (the image of) the cylinder in $T_q M$ with axis perpendicular to $T_q \Sigma_1$ and radius ϵ . For small enough ρ, ϵ , by almost-embeddedness and the uniform convergence of tangent planes, $\Sigma_2 \cap C_\epsilon(z) \cap B_\rho(q)$ decomposes as a finite number of ordered graphs over $T_q \Sigma_1 \cap C_\epsilon(z)$, which have uniformly bounded gradient $\delta \ll 1$. The uniform gradient bound, together with unique continuation for CMC hypersurfaces imply that this graphical decomposition may be extended all the way to the boundary Γ , preserving the ordering and the gradient bound.

Now since Σ_2 glues smoothly with Σ_1 along $\mathcal{R}(\Gamma)$, and since $\mathcal{R}(\Gamma)$ is an open and dense subset of Γ , we know that the set $\{\Sigma_{2,i} : i = 1 \cdots l\}$ consists of exactly two elements: one of them, denoted by $\Sigma_{2,1}$, glues smoothly with $\Sigma_{1,1}$ along $\Gamma_1 \setminus \mathcal{S}(\Gamma)$; the other one, denoted by $\Sigma_{2,2}$, glues smoothly with $\Sigma_{1,2}$ along $\Gamma_2 \setminus \mathcal{S}(\Gamma)$. Now (6.9) implies that the pairs $(\Sigma_{1,1}, \Sigma_{2,1})$ and $(\Sigma_{1,2}, \Sigma_{2,2})$ glue together in C^1 near q respectively. Again higher regularity follows from the elliptic PDE argument as in Sub-case (A). This finishes **Sub-case (B)**.

Step 3. We now wish to extend the replacements, via unique continuation, all the way to the center p .

Henceforth we denote V^{**} by $V_{s_1}^{**}$ and Σ_2 by Σ_{s_1} to indicate the dependence on s_1 . By varying $s_1 \in (0, s)$, we obtain a family of nontrivial, smooth, almost embedded, stable c -boundaries $\{\Sigma_{s_1} \subset A_{s_1, s_2}(p) \cap M\}$. Since unique continuation holds for immersed CMC hypersurfaces, by Step 2 we have $\Sigma_{s_1} = \Sigma_1$ in $A_{s, s_2}(p)$, and moreover, for any $s'_1 < s_1 < s$, we have $\Sigma_{s'_1} = \Sigma_{s_1}$ in $A_{s_1, s_2}(p)$. Hence

$$\Sigma := \bigcup_{0 < s_1 < s} \Sigma_{s_1}$$

is a nontrivial, smooth, almost embedded, stable c -hypersurface in $(B_{s_2}(p) \setminus \{p\}) \cap M$. Also

$$V_{s_1}^{**} = \Sigma, \text{ in } A_{s_1, s_2}(p), V_{s_1}^{**} = V^* \text{ in } A_{s, s_2}(p),$$

$$\text{and for any } s'_1 < s_1 < s, V_{s'_1}^{**} = V_{s_1}^{**} \text{ in } A_{s_1, s_2}(p).$$

By Proposition 5.8, $V_{s_1}^{**}$ has c -bounded first variation and uniformly bounded mass for all $0 < s_1 < s$, so the monotonicity formula [51, 40.2] implies that $\|V_{s_1}^{**}\|(B_r(p)) \leq Cr^n$ for some uniform $C > 0$. Therefore as $s_1 \rightarrow 0$, the family $V_{s_1}^{**}$ will converge to a varifold $\tilde{V} \in \mathcal{V}_n(M)$, i.e.

$$\tilde{V} = \lim_{s_1 \rightarrow 0} V_{s_1}^{**}, \quad \text{such that}$$

$$(6.10) \quad \tilde{V} = \begin{cases} \Sigma & \text{in } (B_{s_2}(p) \setminus \{p\}) \cap M \\ V^* & \text{in } M \setminus B_s(p) \end{cases}, \quad \text{and } \|\tilde{V}\|(\{p\}) = 0.$$

Since $p \in \text{spt } \|V_{s_1}^{**}\|$, by the upper semi-continuity of density function for varifolds with bounded first variation [51, 17.8], we know that $p \in \text{spt } \|\tilde{V}\|$.

Step 4. We now determine the regularity of \tilde{V} at p .

First, since \tilde{V} is the varifold limit of a sequence $V_{s_1}^{**}$ which all have c -bounded first variation, we know that \tilde{V} also has c -bounded first variation. Second, \tilde{V} , when restricted to any small annulus $A_{r, 2r}(p) \cap M$, already coincides with a smooth, almost embedded, stable c -boundary Σ . Using these two ingredients, we can use a blowup argument as in the proofs of Lemma 5.10 and Proposition 5.11

(without the need for replacements) to show that every tangent varifold of \tilde{V} at p is an integer multiple of some n -plane, i.e. for any $C \in \text{VarTan}(V, p)$,

$$C = \Theta^n(\|\tilde{V}\|, p)|S| \text{ for some } n\text{-plane } S \subset T_p M \text{ where } \Theta^n(\|\tilde{V}\|, p) \in \mathbb{N}.$$

Now the removability of the singularity of \tilde{V} at p (as an almost embedded hypersurface) follows similarly to [43, Theorem 7.12]. We include the details for completeness. We can assume that

$$\Theta^n(\|\tilde{V}\|, p) = m$$

for some $m \in \mathbb{N}$. Since Σ is stable in a punctured ball of p , by Theorem 2.11, for any sequence $r_i \rightarrow 0$,

$$\eta_{p, r_i}(\Sigma) \rightarrow m \cdot S$$

locally smoothly in $\mathbb{R}^L \setminus \{0\}$ for some n -plane $S \subset T_p M$. However, S may depend on the sequence r_i . By the convergence and the regularity of Σ , there exists $\sigma_0 > 0$ small enough, such that for any $0 < \sigma \leq \sigma_0$, Σ has an m -sheeted, ordered (in the sense of Definition 2.2), graphical decomposition in $A_{\sigma/2, \sigma}(p)$:

$$(6.11) \quad \tilde{V} \llcorner A_{\sigma/2, \sigma}(p) = \sum_{i=1}^m |\Sigma_i(\sigma)|.$$

Here $\Sigma_i(\sigma)$ is a graph over $A_{\sigma/2, \sigma}(p) \cap S$ for some n -plane $S \subset T_p M$.

Since (6.11) holds for all σ , by continuity of Σ we can continue each $\Sigma_i(\sigma_0)$ to $(B_{\sigma_0}(p) \setminus \{p\}) \cap M$, and we denote the continuation by Σ_i . Since each piece Σ_i has constant mean curvature c , by a standard extension argument (c.f. the proof in [19, Theorem 4.1]), each Σ_i can be extended as a varifold with c -bounded first variation in $B_{\sigma_0}(p) \cap M$. Given $C_i \in \text{VarTan}(\Sigma_i, p)$, to see that C_i has multiplicity one, first notice that

$$(6.12) \quad \Theta^n(\|C_i\|, p) \geq 1,$$

since each Σ_i is c -stable and thus its re-scalings converge with multiplicity to a smooth, embedded, stable, minimal hypersurface by Theorem 2.11. If equality does not hold for some i in (6.12), this will derive a contradiction since

$$\tilde{V} \llcorner B_{\sigma_0}(p) = \sum_{i=1}^m |\Sigma_i|.$$

Therefore, each Σ_i has c -bounded first variation in $B_{\sigma_0}(p) \cap M$ and $\Theta^n(\|\Sigma_i\|, p) = 1$; by the Allard regularity theorem [51, Theorem 24.2] and elliptic regularity, Σ_i extends as a smooth, embedded c -hypersurface across p . Finally, by the maximum principle (Lemma 2.7), we have $m = 1$ or $m = 2$ and this shows that \tilde{V} extends as an almost embedded c -hypersurface across p .

Step 5. Finally, to complete the proof we show that V coincides with \tilde{V} on a small ball about p .

We will need the following simple corollary of the first variation formula.

Lemma 6.2. *For small enough r the set*

$$Tr_p^V = \left\{ y \in \text{spt } \|V\| \cap (B_r(p) \setminus \{p\}) : \begin{array}{l} \text{VarTan}(V, y) \text{ consists of an integer multiple of an} \\ n\text{-plane transverse to } \partial(B_{\text{dist}(y, p)}(p) \cap M) \end{array} \right\}$$

is a dense subset of $\text{spt } \|V\| \cap B_r(p)$.

Proof. Assume to the contrary that there exists $y \in \text{spt} \|V\| \cap (B_r(p) \setminus \{p\})$, and some neighborhood U of p in $B_r(p) \cap M$ such that the tangent plane S_z of V at each point $z \in U \cap \text{spt} \|V\|$ is tangential to $\partial(B_{\text{dist}(z,p)}(p) \cap M)$. Since M is embedded in \mathbb{R}^L , the family $\{\partial(B_\rho(p) \cap M) : 0 < \rho < r\}$ forms a smooth foliation of $(B_r(p) \cap M) \setminus \{p\}$ for small enough r . Let ν be the outer unit normal of this smooth foliation. By our choice of r_0 , the mean curvature on the foliation is

$$\text{div}_{T_y(\partial(B_{\text{dist}(y,p)}(p) \cap M))} \nu > c.$$

Consider the vector field $X = \varphi \cdot \nu$, where $\varphi \geq 0$ is a smooth cutoff function supported in U . Plugging X into the first variation formula, we get

$$\begin{aligned} \delta V(X) &= \int \text{div}_{S_z}(\varphi \nu) d\|V\|(z) = \int \varphi \cdot \text{div}_{S_z}(\nu) d\|V\|(z) \\ &> c \cdot \int \varphi d\|V\|(z) = c \int |X| d\|V\|(z). \end{aligned}$$

In the second equality we have used the fact that S_z is tangential to the foliation and hence perpendicular to ν . This contradicts that V has c -bounded first variation and thus finishes the proof. \square

Claim 5: For small enough r , $\text{spt} \|V\| = \Sigma$ in the punctured ball $(B_r(p) \setminus \{p\}) \cap M$.

Proof of Claim 5: We first prove that $\text{Tr}_p^V \subset \Sigma$, which combined with Lemma 6.2 will imply that $\text{spt} \|V\| \cap (B_r(p) \setminus \{p\}) \subset \Sigma$. Fix $y \in \text{Tr}_p^V \cap (B_r(p) \setminus \{p\})$, and let $\rho = \text{dist}(y, p)$. Consider V_ρ^{**} . By transversality we have $y \in \text{Clos}(\text{spt} \|V\| \cap B_\rho(p))$. On the other hand, $V_\rho^{**} = V^* = V$ inside $B_\rho(p)$, so by (6.1) we have

$$\begin{aligned} \text{Clos}(\text{spt} \|V\| \cap B_\rho(p)) \cap \partial B_\rho(p) &= \text{Clos}(\text{spt} \|V_\rho^{**}\| \cap B_\rho(p)) \cap \partial B_\rho(p) \\ &\subset \text{Clos}(\text{spt} \|V_\rho^{**}\| \setminus \text{Clos}(B_\rho(p))) \cap \partial B_\rho(p). \end{aligned}$$

Since $\text{spt} \|V_\rho^{**}\| = \Sigma$ on $A_{\rho, s_2}(p)$, we therefore have $y \in \Sigma$.

Next we show the reverse inclusion $\Sigma \subset \text{spt} \|V\|$. Since Σ extends across p as an almost embedded hypersurface, we know that $T_y \Sigma$ is transverse to $\partial(B_{\text{dist}(y,p)}(p) \cap M)$ for all $y \in \Sigma \cap B_r(p)$ for small enough r . Let ρ and V_ρ^{**} be as above, then $y \in \text{spt} \|V_\rho^{**}\|$. By Proposition 5.11, $\text{VarTan}(V_\rho^{**}, y) = \{\Theta^n(\|V_\rho^{**}\|, y) | T_y \Sigma |\}$. By the transversality, we then have $y \in \text{Clos}(\text{spt} \|V_\rho^{**}\| \cap B_\rho(p))$, so since $V_\rho^{**} = V$ inside $B_\rho(p)$ we conclude that $y \in \text{Clos}(\text{spt} \|V\| \cap B_\rho(p)) \subset \text{spt} \|V\|$ as desired. \square

Note that we do not have the Constancy Theorem (c.f. [51, 41.1]) for varifolds with bounded first variation. In order to show that V coincides with Σ near p , our strategy is to show that $V = \tilde{V}$ as varifolds in a neighborhood of p . By the transversality argument as above, we can first show that the densities of V and \tilde{V} are identical along $\Sigma \cap (B_r(p) \setminus \{p\})$, where r is chosen as in Claim 5.

Claim 6: $\Theta^n(\|V\|, \cdot) = \Theta^n(\|\tilde{V}\|, \cdot)$ on $\Sigma \cap B_r(p) \setminus \{p\}$.

Proof of Claim 6: Let $y \in \Sigma$ and $\rho = \text{dist}(y, p) < r$ be as above. Then since $V_\rho^{**} = V$ inside $B_\rho(p)$, by transversality and Proposition 5.11 we have $\text{VarTan}(V, y) = \text{VarTan}(V_\rho^{**}, y)$. But $V_\rho^{**} = \tilde{V}$ on $A_{\rho, s_2}(p)$, so we must have $\text{VarTan}(V_\rho^{**}, y) = \{\Theta^n(\|\tilde{V}\|, y) | T_y \Sigma |\}$. Thus $\Theta^n(\|V\|, y) = \Theta^n(\|\tilde{V}\|, y)$. \square

Combining Claims 5 and 6 yields that $V = \tilde{V}$ on $B_r(p) \cap M$. This finishes the proof of Step 5, and hence also completes the proof of the main Theorem 6.1. \square

APPENDIX A. AN INTERPOLATION LEMMA

The first lemma below was essentially due to Pitts [43, Lemma 3.8], but the modification to find the interpolation sequence using boundaries of Caccioppoli sets was completed by the first author [60, Proposition 5.3].

Lemma A.1. *Suppose $L > 0$, $\eta > 0$, W is a compact subset of U , and $\Omega \in \mathcal{C}(M)$. Then there exists $\delta = \delta(L, \eta, U, W, \Omega) > 0$, such that for any $\Omega_1, \Omega_2 \in \mathcal{C}(M)$ satisfying*

- (a) $\text{spt}(\Omega_i - \Omega) \subset W$, $i = 1, 2$,
- (b) $\mathbf{M}(\partial\Omega_i) \leq L$, $i = 1, 2$,
- (c) $\mathcal{F}(\partial\Omega_1 - \partial\Omega_2) \leq \delta$,

there exist a sequence $\Omega_1 = \Lambda_0, \Lambda_1, \dots, \Lambda_m = \Omega_2 \in \mathcal{C}(M)$ such that for each $j = 0, \dots, m-1$,

- (i) $\text{spt}(\Lambda_j - \Omega) \subset U$,
- (ii) $\mathcal{A}^c(\Lambda_j) \leq \max\{\mathcal{A}^c(\Omega_1), \mathcal{A}^c(\Omega_2)\} + \eta$,
- (iii) $\mathbf{M}(\partial\Lambda_j - \partial\Lambda_{j+1}) \leq \eta$.
- (iv)

$$\mathbf{M}(\partial\Lambda_j) \leq \max\{\mathbf{M}(\partial\Omega_1), \mathbf{M}(\partial\Omega_2)\} + \frac{\eta}{2}.$$

(v)

$$\mathbf{M}(\Lambda_j - \Omega_i) \leq \frac{\eta}{2c}, \text{ for } i = 1, 2.$$

Proof. Note that by a covering argument, one only needs to prove the case when $\partial\Omega_2$ is fixed, and in this case $\delta = \delta(L, \eta, U, W, \Omega, \Omega_2)$.

Under our assumptions on Ω_1, Ω_2 , there are two issues for us that differ from the conclusions of Pitts [43, Lemma 3.8]:

- (1) we require the interpolating sequence $\{\partial\Lambda_j\}$ to consist of boundaries of Caccioppoli sets, while in [43] the interpolating sequence consists of integral currents $\{T_j \in \mathcal{Z}_n(M)\}$ (using notations therein);
- (2) for point (ii), we require that $\mathcal{A}^c(\Lambda_j)$ does not increase much from $\mathcal{A}^c(\Omega_1), \mathcal{A}^c(\Omega_2)$, while in [43] it was only proven that $\mathbf{M}(T_j) \leq \max\{\mathbf{M}(\partial\Omega_1), \mathbf{M}(\partial\Omega_2)\} + \eta$.

These two minor points can be easily deduced from the mentioned work of Zhou [60], and now we point out the necessary details. In fact, with regards to point (1), the proof of [60, Proposition 5.3] already proceeds using boundaries, and we will first justify properties (i, iii, iv).

Specifically, given $L, \eta, \Omega, \Omega_2$ satisfying the assumptions in the Lemma, [60, Proposition 5.3] (applied to the case when $m = 1, l = 0$ therein) gives the desired $\delta > 0$, such that if Ω_1 satisfies the assumptions (a-c), then there exists a sequence $\Omega_1 = \Lambda_0, \Lambda_1, \dots, \Lambda_m = \Omega_2 \in \mathcal{C}(M)$ such that for each $j = 0, \dots, m-1$, properties (iii) and (iv) are satisfied.

Although in [60, Proposition 5.3] Ω_i were not assumed to satisfy the additional condition $\text{spt}(\Omega_i - \Omega) \subset W$. It can be seen that the construction in [60, Proposition 5.3] indeed satisfies the required property (i) under this additional condition.

Note that properties (iv, v) immediately imply (ii), so it remains only to show (v). Indeed, from the construction in [60, Proposition 5.3], one knows that the symmetric difference $\Lambda_j \Delta \Omega_i$ is a subset of the union of the symmetric difference $\Omega_1 \Delta \Omega_2$ together with finitely many (depending only on L and η) balls of arbitrarily small radii in U . Since $\text{Vol}(\Omega_1 \Delta \Omega_2) = \mathbf{M}(\Omega_1 - \Omega_2) = \mathcal{F}(\partial\Omega_1, \partial\Omega_2)$ by the isoperimetric lemma [60, Lemma 7.3], one thus obtains (v) by taking δ small enough. \square

APPENDIX B. INTERPOLATION PROCESS

Proof of Claim 2 in Proposition 4.4. Here we describe the construction of $\{\phi_i\}$ by interpolating $\{\phi_i^1\}$.

Fix $i \in \mathbb{N}$ and consider a 1-cell $\alpha \in I(1, k_i)$. We only need to show how to interpolate ϕ_i^1 when restricted to α_0 . For notational simplicity we write $\alpha = [0, 1]$. For $x \in \alpha$ let $\tilde{X}(x)$ be the linear interpolation between $\tilde{X}_i(0) = \tilde{X}(|\partial\phi_i^*(0)|)$ and $\tilde{X}_i(1) = \tilde{X}(|\partial\phi_i^*(1)|)$. The continuity of the map $V \rightarrow \tilde{X}(V)$ implies that $\|\tilde{X}_i(x) - \tilde{X}_i(0)\|_{C^1(M)} \rightarrow 0$ uniformly as $i \rightarrow \infty$. Define $\bar{Q}_i(x)$ to be the push-forward of $\phi_i^*(0)$ by the flow of $\tilde{X}_i(x)$ up to time 1; this gives a map $\bar{Q}_i : \alpha \rightarrow \mathcal{C}(M)$. Note that $\partial\bar{Q}_i : \alpha \rightarrow \mathcal{Z}_n(M)$ is continuous under the \mathbf{F} -metric.

Since $\bar{Q}_i(x)$ and $\phi_i^1(0)$ are the push-forwards of the same initial set $\phi_i^*(0)$ under the flows of $\tilde{X}_i(x)$ and $\tilde{X}_i(0)$ respectively, we have

$$(B.1) \quad \begin{aligned} \mathbf{F}(\partial\bar{Q}_i(x), \partial\phi_i^1(0)) &\rightarrow 0 \\ \mathbf{M}(\bar{Q}_i(x) - \phi_i^1(0)) &\rightarrow 0 \end{aligned} \quad , \text{ uniformly in } x, \alpha \text{ as } i \rightarrow \infty.$$

As $\bar{Q}_i(1)$ and $\phi_i^1(1)$ are the respective push-forwards of $\phi_i^*(0)$ and $\phi_i^*(1)$ under the same flow of $\tilde{X}_i(1)$, we have

$$(B.2) \quad \mathbf{M}(\partial\bar{Q}_i(1) - \partial\phi_i^1(1)) \rightarrow 0, \text{ uniformly in } \alpha \text{ as } i \rightarrow \infty.$$

Now we can apply the interpolation result [60, Theorem 5.1] (see also [33, Theorem 13.1]) to \bar{Q}_i , which gives that for any $\eta > 0$, there exist $l_\eta > 0$ and $Q_i : \alpha(l_\eta)_0 \rightarrow \mathcal{C}(M)$, such that

(i) given $x \in \alpha(l_\eta)_0$,

$$\mathbf{M}(\partial Q_i(x)) \leq \mathbf{M}(\partial\bar{Q}_i(x)) + \eta/2,$$

so by the same argument as in the proof of point (v) of Lemma A.1,

$$\mathbf{M}(Q_i(x) - \bar{Q}_i(x)) \leq \eta/(2c),$$

and hence

$$\mathcal{A}^c(Q_i(x)) \leq \mathcal{A}^c(\bar{Q}_i(x)) + \eta;$$

(ii) $\mathbf{f}(Q_i) \leq \eta$;

(iii) $\sup\{\mathcal{F}(\partial Q_i(x) - \partial\bar{Q}_i(x)) : x \in \alpha(l_\eta)_0\} < \eta$.

When $\eta \rightarrow 0$, by (i, iii) and [43, 2.1(20)] (see also [33, Lemma 4.1]), we have

$$\lim_{\eta \rightarrow 0} \sup\{\mathbf{F}(\partial Q_i(x), \partial\bar{Q}_i(x)) : x \in \alpha(l_\eta)_0\} = 0.$$

Take a sequence $\eta_i \rightarrow 0$, and denote $l_i = k_i + l_{\eta_i} + 1$, then we construct $\phi_i : I(1, k_i + l_{\eta_i} + 1) \rightarrow \mathcal{C}(M)$ by defining ϕ_i on each $\alpha(l_{\eta_i} + 1)_0$ by

$$\phi_i(x) = \begin{cases} Q_i(3x) & \text{for } x \in [0, 1/3] \cap \alpha(l_{\eta_i} + 1)_0 \\ \phi_i^1(1) & \text{otherwise.} \end{cases}$$

The desired properties (a, b, c, d) of ϕ_i follow straightforwardly from (B.1)(B.2) and the properties of Q_j . Since \bar{Q}_i is obtained from a continuous deformation from ϕ_i^* , a further interpolation argument shows that S is homotopic to S^* , and hence we finish the proof of Claim 2. \square

APPENDIX C. GOOD REPLACEMENT PROPERTY AND REGULARITY

Here we record the notions of good replacements and the good replacement property. Recall the following definitions by Colding-De Lellis [11]. Consider two open subsets $W \subset\subset U \subset M^{n+1}$.

Definition C.1. [11, Definition 6.1]. Let $V \in V_n(U)$ be stationary in U . A stationary varifold $V' \in V_n(U)$ is said to be a *replacement for V in W* if

$$V' \llcorner (U \setminus \overline{W}) = V \llcorner (U \setminus \overline{W}), \quad \|V'\|(U) = \|V\|(U), \quad \text{and} \\ V' \llcorner W \text{ is an embedded stable minimal hypersurface } \Sigma \text{ with } \partial \Sigma \subset \partial W.$$

Definition C.2. [11, Definition 6.2]. Let $V \in V_n(U)$ be stationary in U . V is said to have the *good replacement property in W* if

- (a) there is a positive function $r : W \rightarrow \mathbb{R}$ such that for every annulus $A_{s,t}(x) \cap M \subset W$ with $0 < s < t < r(x)$, there is a replacement V' for V in $A_{s,t}(x) \cap M$;
- (b) the replacement V' has a replacement V'' in every annulus $A_{s,t}(y) \cap M \subset W$ with $0 < s < t < r(y)$;
- (c) V'' has a replacement V''' in every annulus $A_{s,t}(z) \cap M \subset W$ with $0 < s < t < r(z)$.

Note that our formulations are local compared to those in [11]. Indeed, the proofs of [11, Proposition 6.3] and [14, Theorem 2.8] are purely local, so the following proposition still holds:

Proposition C.3. [11, Proposition 6.3], [14, Proposition 2.8]. *When $2 \leq n \leq 6$, if $V \in V_n(U)$ has the good replacement property in W , then $V \llcorner W$ is an integer multiple of some smooth embedded minimal hypersurface Σ .*

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