RINGEL-HALL ALGEBRAS BEYOND THEIR QUANTUM GROUPS I: RESTRICTION FUNCTOR AND GREEN'S FORMULA

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ABSTRACT. In this paper, we generalize the categorifical construction of a quantum group and its canonical basis by Lusztig ([16, 17]) to the generic form of whole Ringel-Hall algebra. We clarify the explicit relation between the Green formula in [7] and the restriction functor in [17]. By a geometric way to prove Green formula, we show that the Hopf structure of a Ringel-Hall algebra can be categorified under Lusztig's framework.

1. INTRODUCTION

Based on the classical works of Hall ([8]) and Steineiz ([31]), the Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of a (small) abelian category $\mathcal{A}$ was introduced by Ringel in [24], as a model to realize the quantum group. When $\mathcal{A}$ is the category $\mathrm{Rep}_{\mathbb{F}_q}Q$ of finite dimensional representations for a simply-laced Dynkin quiver Q over a finite field $\mathbb{F}_q$, the Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is isomorphic to the positive part of the corresponding quantum group ([24]). For any acyclic quiver Q and $\mathcal{A} = \mathrm{Rep}_{\mathbb{F}_q}Q$, the composition subalgebra of $\mathcal{H}(\mathcal{A})$ generated by the elements corresponding to simple representations is isomorphic to the positive part of the quantum group of type Q. This gives the algebraic realization of the positive/negative part of a (Kac-Moody type) quantum group. This realization was achieved by Green ([7]), through solving a natural question whether there is a comultiplication on $\mathcal{H}(\mathcal{A})$ compatible with the corresponding multiplication so that the above isomorphism is an isomorphism between bialgebras. Now it is well-known that Green's comultiplication depends on a remarkable homological formula in [7] (called the Green formula in the following).

In the seminal papers [16] and [17], Lusztig gave the geometric realization of the positive/negative part of a quantum group and then constructed the canonical basis for it. Let $Q = (Q_0, Q_1)$ be a quiver and

$$\mathbb{E}_{\underline{d}} := \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{d_{s(\alpha)}}, \mathbb{K}^{d_{t(\alpha)}})$$

be the variety with the natural action of the algebraic group

$$G_{\underline{d}} := \prod_{i \in Q_0} GL(d_i, \mathbb{K})$$

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for a given dimension vector $\underline{d} \in \mathbb{N}Q_0$. Lusztig ([17]) defined the flag $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ and the subvariety

$$\widetilde{\mathbb{E}}_{\mathbf{i},\mathbf{a}} \subseteq \mathbb{E}_{\underline{d}} \times \mathcal{F}_{\mathbf{i},\mathbf{a}}.$$

Fix any type $(\mathbf{i},\mathbf{a})$, consider the canonical proper morphism $\pi_{\mathbf{i},\mathbf{a}} : \widetilde{\mathbb{E}}_{\mathbf{i},\mathbf{a}} \to \mathbb{E}_{\underline{d}}$. By the decomposition theorem of Beilinson, Bernstein and Deligne ([1]), the complex $\pi_{\mathbf{i},\mathbf{a}!}\mathbf{1}$ is semisimple, where $\mathbf{1}$ is the constant perverse sheaf on $\widetilde{\mathbb{E}}_{\mathbf{i},\mathbf{a}}$. Let $\mathcal{Q}_{\underline{d}}$ be the category of complexes isomorphic to sums of shifts of simple perverse sheaves appearing in $\pi_{\mathbf{i},\mathbf{a}!}\mathbf{1}$, $\mathcal{K}_{\underline{d}}$ the Grothendieck group of $\mathcal{Q}_{\underline{d}}$ and $\mathcal{K} = \bigoplus_{\underline{d}} \mathcal{K}_{\underline{d}}$. Lusztig ([17]) already endowed $\mathcal{K}$ with the multiplication and comultiplication structures by introducing his induction and restriction functors. He proved that the comultiplication is compatible with the multiplication in $\mathcal{K}$ and $\mathcal{K}$ is isomorphic to the positive part of the corresponding quantum group as bialgebras up to a twist.

By this isomorphism, the isomorphism classes of simple perverse sheaves in $\mathcal{Q}_{\underline{d}}$ provide a basis of the corresponding quantum group, which is called the canonical basis. The canonical basis theory of quantum algebras is crucially important in Lie theory. This basis has many remarkable properties such as such as integrality and positivity of structure constants, compatibility with all highest weight integrable representations, etc. Lusztigs approach essentially motivates the categorification of quantum groups (for example, see [12],[28] and [32]) or quantum cluster algebras (see [9],[21],[13], etc.), i.e., a quantum group/quantum cluster algebra can be viewed as the Grothendieck ring of a monoidal category and some simple objects provides a basis (see also [33]).

For a long time we have been asked what the explicit relation exists between the Green's comultiplication and Lusztig's restriction functor. As one of the main result in the present paper, the following Theorem 4.8 and the definition of the comultiplication operator Δ provide us this strong and clear link. Thanks to an embedding property as in [14] and [27], we can lift the Green formula from finite fields to the level of sheaves. This is finally suitable to apply the Lusztig's restriction functor to the larger categories of the so-called Weil complexes, whose Grothendieck groups realize the weight spaces of a generic Ringel-Hall algebra.

The second aim of this paper is to give the categorification of Ringel-Hall algebras via Lusztig's geometric method. In Section 2, we recall the theory of Ringel-Hall algebras, focusing on the Hopf structure of Ringel-Hall algebras. In Section 3, we recall Lusztig's construction of Hall algebras via functions invariant under the Frobenius map. Then we obtain an algebra $\mathcal{CF}^F(Q)$. The comultiplication over $\mathcal{CF}^F(Q)$ is just a twist of Green's comultiplication. However, the proof of the compatibility of Lusztig's comultiplication and multiplication for the whole Ringel-Hall algebra essentially depends on the proof of the Green formula. In the end of this section, we show that the twist of $\mathcal{CF}^F(Q)$ is isomorphic to $\mathcal{H}^{tw}(\mathcal{A})$ as a Hopf algebra. Hence, Lusztig's comultiplication can be applied to Ringel-Hall algebras. In Section 4, we extend the geometric realization of a quantum group to the whole Ringel-Hall algebra under Lusztig's framework. We obtain the generic Ringel-Hall algebra as the direct sum of Grothendieck groups of the derived categories of a class of Weil complexes. The simple perverse sheaves provide the caonical basis. We show that the compatibility of the induction and restriction functor holds for these perverse sheaves. Therefore the generic Ringel-Hall algebra has a structure of Hopf algebra. In Section 5, we construct the Drinfeld double of the generic Ringel-Hall algebra.

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2. A revisit of Ringel-Hall algebras as Hopf algebras

We recall the definition of the Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ for the hereditary abelian category $\mathcal{A} = \mathrm{mod}kQ = \mathrm{Rep}_k Q$, where $k = \mathbb{F}_q$ is a finite field with $q = p^e$ elements for some prime number p and Q is a finite quiver without oriented cycles.

For $M \in \mathcal{A}$, we denote by $\underline{\mathrm{dim}}M$ the dimension vector in $\mathbb{N}Q_0$ and define the Euler-Ringel form on $\mathbb{N}Q_0$ as follows:

$$\langle M, N\rangle = \mathrm{dim}_k \mathrm{Hom}_{\mathcal{A}}(M, N) - \mathrm{dim}_k \mathrm{Ext}^1_{\mathcal{A}}(M, N).$$

For M, N and $L \in \mathrm{mod}kQ$, we denote by $\mathcal{F}^L_{MN}$ the set $\{X \subset L \mid X \in \mathrm{mod}kQ, X \cong N, L/X \cong M\}$ and $\mathrm{Ext}^1_{\mathcal{A}}(M, N)_L$ the subset of $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ with the middle term isomorphic to L. Write $F^L_{MN} = |\mathcal{F}^L_{MN}|$ and $h^{MN}_L = \frac{|\mathrm{Ext}^1_{\mathcal{A}}(M,N)_L|}{|\mathrm{Hom}_{\mathcal{A}}(M,N)|}$. For $X \in \mathcal{A}$, set $a_X = |\mathrm{Aut}_{\mathcal{A}} X|$.

The ordinary Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is a $\mathbb{C}$-space with isomorphism classes $[X]$ of all kQ-modules X as a basis and the multiplication is defined by

$$[M] * [N] = \sum_{[L]} F^L_{MN}[L]$$

for M, N and $L \in \mathrm{mod}kQ$. We can endow $\mathcal{H} = \mathcal{H}(\mathcal{A})$ a comultiplication $\delta : \mathcal{H} \to \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}$ by setting

$$\delta([L]) = \sum_{[M],[N]} h^{MN}_L [M] \otimes [N].$$

The comultiplication is compatible with the multiplication via Green's theorem.

Theorem 2.1. [7] *The map δ is an algebra homomorphism with respect to the twisted multiplication on $\mathcal{H} \otimes \mathcal{H}$ as follows:*

$$([M_1] \otimes [N_1]) \circ ([M_2] \otimes [N_2]) = q^{\langle M_1, N_2\rangle}([M_1] * [M_2]) \otimes ([N_1] * [N_2]).$$

The theorem is equivalent to the following Green formula holds:

$$a_{M_1} a_{M_2} a_{N_1} a_{N_2} \sum_{[L]} F^L_{M_1 N_1} F^L_{M_2 N_2} a_L^{-1}$$

$$= \sum_{[X],[Y_1],[Y_2],[Z]} \frac{|\mathrm{Ext}^1_{\mathcal{A}}(X, Z)|}{|\mathrm{Hom}_{\mathcal{A}}(X, Z)|} F^{M_1}_{XY_1} F^{M_2}_{XY_2} F^{N_1}_{Y_2 Z} F^{N_2}_{Y_1 Z} a_X a_{Y_1} a_{Y_2} a_Z.$$

Define a symmetric bilinear form on $\mathcal{H}$ by setting $([M], [N]) = \frac{\delta_{M,N}}{a_M}$. We call the form Green's Hopf pairing. It is clear that the Green Hopf pairing is a non degenerated bilinear form over $\mathcal{H}$. The following proposition shows that the comultiplication is dual to the multiplication, i.e., the comultiplication can be viewed as the multiplication over $\mathcal{H}^* = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}, \mathbb{C})$.

Proposition 2.2. The comultiplication is left adjoint to the multiplication with respect to Green's Hopf pairing, i.e., for $a, b, c \in \mathcal{H}$, $(a, bc) = (\delta(a), b \otimes c)$.

The proposition is equivalent to the Riedtmann-Peng formula

$$F_{MN}^{L}a_Ma_N = h_L^{MN}a_L$$

holds.

It is easy to generalize the multiplication and comultiplication to the r-fold versions for $r \geq 2$. For $M_1, \cdots, M_r, M \in \mathcal{A}$, set $\mathcal{F}_{M_1,\cdots,M_r}^{M}$ to be the set

$$\{0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = M \mid X_i \in \mathcal{A}, X_{i+1}/X_i \cong M_{r-i}, i = 0, 1, \cdots, r-1\}$$

and $F_{M_1\cdots M_r}^{M} = |\mathcal{F}_{M_1,\cdots,M_r}^{M}|$. Then

$$[M_1] * [M_2] * \cdots * [M_r] = \sum_{[M]} F_{M_1\cdots M_r}^{M}[M].$$

The r-fold comultiplication δ^r can be defined inductively. For $r = 1$, $\delta^1 = \delta$ and $\delta^{r+1} = (1 \otimes \cdots \otimes 1 \otimes \delta) \circ \delta^r$ for $r \geq 1$. Set

$$\delta^{r-1}([M]) = \sum_{[M_1],\cdots,[M_r]} h_M^{M_1\cdots M_r}[M_1] \otimes \cdots \otimes [M_r]$$

for $r \geq 2$. It is clear that the Riedtmann-Peng formula can be reformulated as

$$h_M^{M_1M_2\cdots M_r} = F_{M_1\cdots M_r}^{M}a_{M_1}\cdots a_{M_r}a_M^{-1}$$

for $r \geq 2$.

Let $\sigma : \mathcal{H} \to \mathcal{H}$ be a map such that

$$\sigma([M]) = \delta_{M,0} + \sum_{r\geq 1}(-1)^r \cdot \sum_{[N],[M_1],\cdots,[M_r]\neq 0} h_M^{M_1\cdots M_r}F_{M_1\cdots M_r}^{N}[N].$$

for $M \in \mathcal{A}$. We call σ the antipode of $\mathcal{H}$.

One can also define the twisted version of the multiplication over $\mathcal{H}(\mathcal{A})$ by setting

$$[M] \cdot [N] = v^{\langle M,N\rangle}[M] * [N],$$

where $v = \sqrt{q}$. Similarly, the comultiplication, the antipode and Green's Hopf pairing can be twisted (see [34] for more details). We denote by $\mathcal{H}^{tw}(\mathcal{A})$ the twisted version of $\mathcal{H}(\mathcal{A})$.

Theorem 2.3. [24, 7, 34] *The algebra $\mathcal{H}^{tw}(\mathcal{A})$ is a Hopf algebra with the above twisted multiplication, comultiplication and antipode. Let $\mathfrak{g}_Q$ be the Kac-Moody algebra associated to the quiver Q and $U_v^+(\mathfrak{g}_Q)$ be the positive part of the quantum group $U_\nu(\mathfrak{g}_Q)$ specialized at $\nu = v$ with $v = q^{\frac{1}{2}}$. Let $\mathcal{C}^{tw}(\mathcal{A})$ be the subalgebra of the twisted Ringel-Hall algebra $\mathcal{H}^{tw}(\mathcal{A})$ generated by isomorphism classes of simple kQ-modules. Denoted by $\mathcal{C}_{\mathbb{Z}[v,v^{-1}]}^{tw}(\mathcal{A})$ the integral form of $\mathcal{C}^{tw}(\mathcal{A})$. Then there is an isomorphism of Hopf algebras*

$$\Psi : U_v^+(\mathfrak{g}_Q) \to \mathcal{C}_{\mathbb{Z}[v,v^{-1}]}^{tw}(\mathcal{A}) \bigotimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v)$$

sending E_i to $[S_i]$ for $i \in Q_0$.

3. Lusztig's construction of Hall algebras via functions

In this section, we recall Lusztig's construction of Hall algebras via functions in [19] and compare it with Ringel-Hall algebras. Let $k = \mathbb{F}_q$ as above and $\mathbb{K} = \overline{\mathbb{F}}_q$. Given a dimension vector $\underline{d} = (d_i)_{i\in Q_0}$, define the variety

$$\mathbb{E}_{\underline{d}} := \mathbb{E}_{\underline{d}}(Q) = \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{d_{s(\alpha)}}, \mathbb{K}^{d_{t(\alpha)}}).$$

Any element $x = (x_\alpha)_{\alpha\in Q_1}$ in $\mathbb{E}_{\underline{d}}(Q)$ defines a representation $M(x) = (\mathbb{K}^{\underline{d}}, x)$ of Q with $\mathbb{K}^{\underline{d}} = \bigoplus_{i\in Q_0} \mathbb{K}^{d_i}$. The algebraic group

$$G_{\underline{d}} := G_{\underline{d}}(Q) = \prod_{i\in Q_0} GL(d_i, \mathbb{K})$$

acts on $\mathbb{E}_{\underline{d}}$ by $(x_\alpha)^g_{\alpha\in Q_1} = (g_{t(\alpha)} x_\alpha g^{-1}_{s(\alpha)})_{\alpha\in Q_1}$ for $g = (g_i)_{i\in Q_0} \in G_{\underline{d}}$ and $(x_\alpha)_{\alpha\in Q_1} \in \mathbb{E}_{\underline{d}}$. The isomorphism class of a $\mathbb{K}Q$-module X is just the orbit of X. The quotient stack $[\mathbb{E}_{\underline{d}}/G_{\underline{d}}]$ parametrizes the isomorphism classes of $\mathbb{K}Q$-modules of dimension vector $\underline{d}$.

Let F be the Frobenius automorphism of $\mathbb{K}$, i.e., $F(x) = x^q$. The F-fixed subfield is just $\mathbb{F}_q$. This induces an isomorphism $\mathbb{E}_{\underline{d}} \to \mathbb{E}_{\underline{d}}$ sending $(((x_\alpha)_{ij})_{d_{s(\alpha)}\times d_{t(\alpha)}})_{\alpha\in Q_1}$ to $(((x_\alpha)^q_{ij})_{d_{s(\alpha)}\times d_{t(\alpha)}})_{\alpha\in Q_1}$. We will denote all induced map by F if it does not cause any confusion.

For any $\mathbb{K}Q$-module $M(x) = (\mathbb{K}^{\underline{d}}, x)$, set $M(x)^{[q]} = F(M(x))$. The representation $M(x) \in \mathbb{E}_{\underline{d}}$ is F-fixed if $M(x) \cong M(x)^{[q]}$. The last condition is equivalent to say that $M(x)$ is defined over $\mathbb{F}_q$, i.e., there exists a kQ-module $M_0(x)$ such that $M(x) \cong M_0(x) \otimes_{\mathbb{F}_q} \mathbb{K}$ ([10]). We denoted by $\mathbb{E}^F_{\underline{d}}$ and $G^F_{\underline{d}}$ the F-fixed subset of $\mathbb{E}_{\underline{d}}$ and $G_{\underline{d}}$ respectively. For a kQ-module $M \in \mathbb{E}^F_{\underline{d}}$, let $\mathcal{O}_M$ denote the orbit of M in $\mathbb{E}_{\underline{d}}$ and $\mathcal{O}^F_M$ the F-fixed subset of $\mathcal{O}_M$.

Let $l \neq p$ be a prime number and $\overline{\mathbb{Q}}_l$ be the algebraic closure of the field of l-adic numbers. Fix a square root $v = q^{\frac{1}{2}} \in \overline{\mathbb{Q}}_l$. Define $\mathcal{CF}^F_{\underline{d}}$ to be the $\overline{\mathbb{Q}}_l$-space generated by $G^F_{\underline{d}}$-invariant functions: $\mathbb{E}^F_{\underline{d}} \to \overline{\mathbb{Q}}_l$. We will endow the vector space $\mathcal{CF}^F(Q) = \bigoplus_{\underline{d}} \mathcal{CF}^F_{\underline{d}}$ with a multiplication and comultiplication.

First we recall two functors: the pushforward functors and the inverse image functors in [19]. Given two finite sets X, Y and a map $\phi : X \to Y$. Let $\mathcal{CF}(X)$ be the vector space of all functions $X \to \overline{\mathbb{Q}}_l$ over X. Define the pushforward of ϕ to be

$$\phi_! : \mathcal{CF}(X) \to \mathcal{CF}(Y),\ \phi_!(f)(y) = \sum_{x\in\phi^{-1}(y)} f(x)$$

and the inverse image of ϕ

$$\phi^* : \mathcal{CF}(Y) \to \mathcal{CF}(X),\ \phi^*(g)(x) = g(\phi(x)).$$

Let $\mathbb{E}^{''}$ be the variety of all pairs (x, W) where $x \in \mathbb{E}_{\alpha+\beta}$ and $(W, x|_W)$ is a $\mathbb{K}Q$-submodule of $(\mathbb{K}^{\alpha+\beta}, x)$ with dimension vector β. Let $\mathbb{E}'$ be the variety of all quadruples (x, W, ρ_1, ρ_2) where $(x, W) \in \mathbb{E}^{''}$ and $\rho_1 : \mathbb{K}^{\alpha+\beta}/W \cong \mathbb{K}^\alpha$, $\rho_2 : W \cong \mathbb{K}^\beta$ are linear isomorphisms. Consider the following diagram

$$\mathbb{E}_\alpha \times \mathbb{E}_\beta \xleftarrow{p_1} \mathbb{E}' \xrightarrow{p_2} \mathbb{E}'' \xrightarrow{p_3} \mathbb{E}_{\alpha+\beta}$$

where p_2, p_3 are natural projections and $p_1(x, W, \rho_1, \rho_2) = (x', x'')$ such that

$$x'_h(\rho_1)_{s(h)} = (\rho_1)_{t(h)} x_h \text{ and } x''_h(\rho_2)_{s(h)} = (\rho_2)_{t(h)} x_h$$

for any $h \in Q_1$.

The groups $G_\alpha \times G_\beta$ and $G_{\alpha+\beta}$ naturally act on $\mathbb{E}'$. The map p_1 is $G_{\alpha+\beta} \times G_\alpha \times G_\beta$-equivariant under the trivial action of $G_{\alpha+\beta}$ on $\mathbb{E}_\alpha \times \mathbb{E}_\beta$. The map p_2 is a principal $G_\alpha \times G_\beta$-bundle.

Applying the Frobenius map F, we can define the above diagram over $\mathbb{F}_q$ as follows:

$$\mathbb{E}_\alpha^F \times \mathbb{E}_\beta^F \xleftarrow{p_1} \mathbb{E}'^F \xrightarrow{p_2} \mathbb{E}''^F \xrightarrow{p_3} \mathbb{E}_{\alpha+\beta}^F .$$

There is a linear map (called the induction)

$$\underline{m} : \mathcal{CF}_{G_\alpha \times G_\beta}(\mathbb{E}_\alpha^F \times \mathbb{E}_\beta^F) \to \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}_{\alpha+\beta}^F) = \mathcal{CF}_{\alpha+\beta}^F$$

sending g to $|G_\alpha^F \times G_\beta^F|^{-1}(p_3)_!(p_2)_! p_1^*(g)$. Iteratively, one can define the r-fold version $\underline{m}^r$ of $\underline{m}$ for $r \geq 1$ by setting $\underline{m}^1 = id$, $\underline{m}^2 = \underline{m}$ and $\underline{m}^{r+1} = \underline{m} \circ (1 \otimes \underline{m}^r)$ for $r \geq 2$.

Now we can define the multiplication over $\mathcal{CF}^F(Q)$. For $f_\alpha \in \mathcal{CF}_\alpha^F, f_\beta \in \mathcal{CF}_\beta^F$ and $(x_1, x_2) \in \mathbb{E}_\alpha \times \mathbb{E}_\beta$, set $g(x_1, x_2) = f_\alpha(x_1) f_\beta(x_2)$. Then $g \in \mathcal{CF}_{\alpha+\beta}^F$ and define the multiplcation

$$f_\alpha * f_\beta = \underline{m}(g).$$

Lemma 3.1. [15] *Given three kQ-modules M, N and L, let $1_{\mathcal{O}_M}, 1_{\mathcal{O}_N}$ and $1_{\mathcal{O}_L}$ be the characteristic functions over orbits, respectively. Then $1_{\mathcal{O}_M^F} * 1_{\mathcal{O}_N^F}(L) = F_{MN}^L$.*

We now turn to define the comultiplication over $\mathcal{CF}^F(Q)$. Fix a subspace W of $\mathbb{K}^{\alpha+\beta}$ with $\underline{\dim} W = \beta$ and linear isomorphisms $\rho_1 : \mathbb{K}^{\alpha+\beta}/W \cong \mathbb{K}^\alpha$, $\rho_2 : W \cong \mathbb{K}^\beta$. Let $F_{\alpha,\beta}$ be the closed subset of $\mathbb{E}_{\alpha+\beta}$ consisting of all $x \in \mathbb{E}_{\alpha+\beta}$ such that $(W, x|_W)$ is a $\mathbb{K}Q$-submodule of $(\mathbb{K}^{\alpha+\beta}, x)$ with dimension vector β. Consider the diagram

$$\mathbb{E}_\alpha \times \mathbb{E}_\beta \xleftarrow{\kappa} F_{\alpha,\beta} \xrightarrow{i} \mathbb{E}_{\alpha+\beta} ,$$

where the map i is the inclusion and $\kappa(x) = p_1(x, W, \rho_1, \rho_2)$. For $(x_1, x_2) \in \mathbb{E}_\alpha \times \mathbb{E}_\beta$, the fibre $\kappa^{-1}(x_1, x_2) \cong \bigoplus_{h \in Q_1} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{\alpha_{s(h)}}, \mathbb{K}^{\beta_{t(h)}})$ and then κ is a vector bundle of dimension $\sum_{h \in Q_1} \alpha_{s(h)} \beta_{t(h)}$.

Applying the Frobenius map F, we can define the above diagram over $\mathbb{F}_q$ as follows:

$$\mathbb{E}_\alpha^F \times \mathbb{E}_\beta^F \xleftarrow{\kappa} F_{\alpha,\beta}^F \xrightarrow{i} \mathbb{E}_{\alpha+\beta}^F .$$

There is also the restriction map

$$\delta_{\alpha,\beta} : \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}_{\alpha+\beta}^F) \to \mathcal{CF}_{G_\alpha \times G_\beta}(\mathbb{E}_\alpha^F \times \mathbb{E}_\beta^F)$$

sending $f \in \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}_{\alpha+\beta}^F)$ to $\kappa_! i^*(f)$. It defines the comultiplication δ over $\mathcal{CF}^F(Q)$, i.e., for $f \in \mathcal{CF}_\gamma^F$ and $\alpha + \beta = \gamma$, $\delta(f) = \sum_{\alpha,\beta; \alpha+\beta=\gamma} \delta_{\alpha,\beta}(f)$. Iteratively, we can define δ^r for $r \geq 1$ by setting $\delta^1 = \delta$ and $\delta^{r+1} = (1 \otimes \cdots \otimes \delta) \circ \delta^r$ for $r \geq 1$.

For M, N and L in $\mathcal{A} = \mathrm{Rep}_k Q$, we set

$$D_L^{MN} = \delta(1_{\mathcal{O}_L^F})(M, N) = \kappa_! i^*(1_{\mathcal{O}_L^F})(M, N).$$

In order to compare this with the comultiplication of Ringel-Hall algebras, we define the twist $\delta_{\alpha,\beta}^{tw} = q^{-\sum_{i \in Q_0} \alpha_i \beta_i} \delta_{\alpha,\beta}$ and δ^{tw} in the same way.

Lemma 3.2. *With the notations in Lemma 3.1 and $\underline{\dim} M = \alpha, \underline{\dim} N = \beta$, we have $\delta_{\alpha,\beta}^{tw}(1_{\mathcal{O}_L^F})(M, N) = h_L^{MN}$.*

Proof. Suppose $M = (\mathbb{K}^\alpha, x_1)$ and $N = (\mathbb{K}^\beta, x_2)$. The linear isomorphisms ρ_1, ρ_2 induce the module structures of $\mathbb{K}^{\alpha+\beta}/W$ and W, denoted by $(\mathbb{K}^{\alpha+\beta}/W, y_{\mathbb{K}^{\alpha+\beta}/W})$ and (W, y_W) respectively. Consider the set

$$S = \{x \in \mathbb{E}^{\alpha+\beta} \mid (W, x|_W) = (W, y_W),$$
$$(\mathbb{K}^{\alpha+\beta}/W, x|_{\mathbb{K}^{\alpha+\beta}/W}) = (\mathbb{K}^{\alpha+\beta}/W, y_{\mathbb{K}^{\alpha+\beta}/W}), (\mathbb{K}^{\alpha+\beta}, x) \cong L\}.$$

Fix a decomposition of the vector space $\mathbb{K}^{\alpha+\beta} = W \oplus \mathbb{K}^{\alpha+\beta}/W$. Then

$$S = \{x = \begin{pmatrix} (y_W)_h & d(h) \\ 0 & (y_{\mathbb{K}^{\alpha+\beta}/W})_h \end{pmatrix}_{h\in Q_1} \mid d(h) \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{\alpha_{s(h)}}, \mathbb{K}^{\beta_{t(h)}}), (\mathbb{K}^{\alpha+\beta}, x) \cong L\}.$$

Set $D(\alpha,\beta) = \bigoplus_{h\in Q_1} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{\alpha_{s(h)}}, \mathbb{K}^{\beta_{t(h)}})$. Apply the Frobenius map F, we have the following long exact sequence (see [4])

$$0 \longrightarrow \mathrm{Hom}_{kQ}(M,N) \longrightarrow \oplus_{i\in Q_0} \mathrm{Hom}_k(k^{\alpha_i}, k^{\beta_i}) \longrightarrow$$
$$\longrightarrow D^F(\alpha,\beta) \xrightarrow{\pi} \mathrm{Ext}^1_{kQ}(M,N) \longrightarrow 0.$$

We denote by $D^F(\alpha,\beta)_L$ the inverse image of $\mathrm{Ext}^1_{kQ}(M,N)_L$ under the map π. Then $D^{MN}_L = |D^F(\alpha,\beta)_L| = q^{\sum_{i\in Q_0}\alpha_i\beta_i} h^{MN}_L$. By definition, $\delta_{\alpha,\beta}(1_{\mathcal{O}^F_L})(M,N) = |D^F(\alpha,\beta)_L| = D^{MN}_L$. This completes the proof. $\square$

In order to compare these with Lusztig's construction, we consider the subalgebra of $\mathcal{CF}^F(Q)$ generated by $1_{S_i} = 1_{\mathcal{O}^F_{S_i}}$ for all $i \in Q_0$, denoted by $\mathcal{F}^F(Q)$. Then $\mathcal{F}^F(Q) = \bigoplus_{\underline{d}} \mathcal{F}^F_{\underline{d}}$.

Lemma 3.3. *Given a sequence $\underline{i} = (i_1, i_2, \cdots, i_m)$ in Q_0 such that $i_j \neq i_k$ for $j \neq k \in \{1, 2, \cdots, m\}$ and let $f = 1_{S_{i_1}} * 1_{S_{i_2}} * \cdots * 1_{S_{i_m}} \in \mathcal{CF}^F(Q)$. Then $\delta(f) = \delta^{tw}(f)$.*

The Riedtmann-Peng formula can be reformulated to the following form, which generalizes [19, Lemma 1.13] from $\mathcal{F}^F(Q)$ to $\mathcal{CF}^F(Q)$.

Proposition 3.4. Let $f_i \in \mathcal{CF}^F_{\underline{d}_i}$ for $i = 1, 2$ and $g \in \mathcal{CF}^F_{\underline{d}}$ for $\underline{d} = \underline{d}_1 + \underline{d}_2$. Then

$$|G^F_{\underline{d}}| \sum_{x,y} f_1(x) f_2(y) \delta^{tw}_{\underline{d}_1,\underline{d}_2}(g)(x,y) = |G^F_{\underline{d}_1} \times G^F_{\underline{d}_2}| \sum_z f_1 * f_2(z) g(z)$$

where $x \in \mathbb{E}^F_{\underline{d}_1}, y \in \mathbb{E}^F_{\underline{d}_2}$ and $z \in \mathbb{E}^F_{\underline{d}_1}$.

Proof. Given a dimension vector $\underline{d}$, take $f \in \mathbb{E}^F_{\underline{d}}$, then $f = \sum_{i=1}^l a_i 1_{\mathcal{O}^F_{M_i}}$ for some $a_i \in \overline{\mathbb{Q}}_l$, $l \in \mathbb{Z}$ and kQ-modules $M_1, \cdots, M_l$. With loss of generality, we may assume that $f_1 = 1_{\mathcal{O}^F_M}$, $f_2 = 1_{\mathcal{O}^F_N}$ and $g = 1_{\mathcal{O}^F_L}$ for some kQ-modules M, N and L. Following Lemma 3.1 and 3.2, the left side of the equation is equal to

$$|G^F_{\underline{d}}| \cdot |\mathcal{O}^F_M| \cdot |\mathcal{O}^F_N| \cdot h^L_{MN}$$

and the right side of the equation is equal to

$$|G^F_{\underline{d}_1}| \cdot |G^F_{\underline{d}_2}| \cdot |\mathcal{O}^F_L| \cdot F^L_{MN}$$

Using $a_L = |G^F_{\underline{d}}|/|\mathcal{O}^F_L|$ and the Riedtmann-Peng formula, we prove the proposition. $\square$

By definition, $\dim_k G^F_{\underline{d}} = \sum_{i\in Q_0} d_i^2$ and then we obtain the following lemma ([30, Section 1.2]).

Lemma 3.5. *With the above notation, we have*

$$\dim_k G^F_{\underline{d}} - \dim_k G^F_{\underline{d}_1} - \dim_k G^F_{\underline{d}_2} = \sum_{i\in Q_0} (\underline{d}_1)_i (\underline{d}_2)_i.$$

Hence, the equation in the above proposition can also be written as

$$\frac{|G^F_{\underline{d}}|}{|\mathfrak{g}^F_{\underline{d}}|}\cdot \sum_{x,y} f_1(x) f_2(y) \delta_{\underline{d}_1,\underline{d}_2}(g)(x,y) = \frac{|G^F_{\underline{d}_1}|}{|\mathfrak{g}^F_{\underline{d}_1}|}\frac{|G^F_{\underline{d}_2}|}{|\mathfrak{g}^F_{\underline{d}_2}|}\sum_z f_1 * f_2(z) g(z)$$

by substituting δ for δ^{tw}, where $\mathfrak{g}^F_{\underline{d}}$, $\mathfrak{g}^F_{\underline{d}_1}$ and $\mathfrak{g}^F_{\underline{d}_2}$ are the Lie algebras of $G^F_{\underline{d}}$, $G^F_{\underline{d}_1}$ and $G^F_{\underline{d}_2}$, respectively.

Fix dimension vectors $\alpha, \beta, \alpha', \beta'$ with $\alpha+\beta = \alpha'+\beta' = \underline{d}$. Let $\mathcal{N}$ be the set of quadruples $\lambda = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ of dimension vectors such that $\alpha = \alpha_1+\alpha_2, \beta = \beta_1+\beta_2, \alpha' = \alpha_1+\beta_1$ and $\beta' = \alpha_2+\beta_2$. Consider the following diagram

$$\begin{tikzcd}
\mathbb{E}^F_\alpha \times \mathbb{E}^F_\beta & \mathbb{E}'^F_{\alpha,\beta} \arrow[l, "p_1"'] \arrow[r, "p_2"] & \mathbb{E}''^F_{\alpha,\beta} \arrow[r, "p_3"] & \mathbb{E}^F_{\underline{d}} \\
\coprod_{\lambda\in\mathcal{N}} F^F_\lambda \arrow[u, "i'"] \arrow[d, "\kappa'"'] & & & F^F_{\alpha',\beta'} \arrow[u, "i"'] \arrow[d, "\kappa"] \\
\coprod_{\lambda\in\mathcal{N}} E^F(\lambda) & \coprod_{\lambda\in\mathcal{N}} \mathbb{E}'^F_\lambda \arrow[l, "p'_1"'] \arrow[r, "p'_2"] & \coprod_{\lambda\in\mathcal{N}} \mathbb{E}''^F(\lambda) \arrow[r, "p'_3"] & \mathbb{E}^F_{\alpha'} \times \mathbb{E}^F_{\beta'}
\end{tikzcd}$$

where $\mathbb{E}^F(\lambda) = \mathbb{E}^F_{\alpha_1} \times \mathbb{E}^F_{\alpha_2} \times \mathbb{E}^F_{\beta_1} \times \mathbb{E}^F_{\beta_2}$, $\mathbb{E}'^F(\lambda) = \mathbb{E}'^F_{\alpha_1,\beta_1} \times \mathbb{E}'^F_{\alpha_2,\beta_2}$ and $\mathbb{E}''^F(\lambda) = \mathbb{E}''^F_{\alpha_1,\beta_1} \times \mathbb{E}''^F_{\alpha_2,\beta_2}$ for $\lambda = (\alpha_1,\alpha_2,\beta_1,\beta_2)$. This induces the maps between F-fixed subsets and then the maps between vector spaces of functions as follows:

$$\begin{tikzcd}
\mathcal{CF}^F_\alpha \times \mathcal{CF}^F_\beta \arrow[r, "\underline{m}_{\alpha,\beta}"] \arrow[d, "\delta^{tw}"] & \mathcal{CF}^F_{\underline{d}} \arrow[d, "\delta^{tw}_{\alpha',\beta'}"] \\
\mathcal{CF}^F(\prod_{\lambda\in\mathcal{N}} \mathbb{E}_\lambda) \arrow[r, "\underline{m}"] & \mathcal{CF}^F_{\alpha'} \times \mathcal{CF}^F_{\beta'}
\end{tikzcd}$$

The following theorem can be viewed as the geometric analog of Green's theorem.

Theorem 3.6. *With the above notation, the above diagram is commutative, i.e.,* $\delta^{tw}_{\alpha',\beta'}\underline{m}_{\alpha,\beta} = \underline{m}\delta^{tw}$.

In order to prove the theorem, we introduce some notations. Set

$$C'^F_{\alpha,\beta,\alpha',\beta'} = \{(x, W, \rho_1, \rho_2) \in \mathbb{E}'^F_{\alpha,\beta} \mid x \in F^F_{\alpha',\beta'}\}$$

and

$$C''^F_{\alpha,\beta,\alpha',\beta'} = \{(x, W) \in \mathbb{E}''^F_{\alpha,\beta} \mid x \in F^F_{\alpha',\beta'}\}.$$

The sets can be illustrated by the following diagram

$$\begin{tikzcd} & W' \arrow[d] & \\ W \arrow[r] & (\mathbb{K}^{\underline{d}}, x) \arrow[r] \arrow[d] & (\mathbb{K}^{\underline{d}}, x)/W \\ & (\mathbb{K}^{\underline{d}}, x)/W' & \end{tikzcd}$$

where $(x, W') \in \mathbb{E}''^F_{\alpha',\beta'}$. Consider the following diagram

$$\mathbb{E}^F_\alpha \times \mathbb{E}^F_\beta \xleftarrow{p} C'^F_{\alpha,\beta,\alpha',\beta'} \xrightarrow{q} C''^F_{\alpha,\beta,\alpha',\beta'} \xrightarrow{r} F^F_{\alpha',\beta'} \xrightarrow{\kappa} \mathbb{E}^F_{\alpha'} \times \mathbb{E}^F_{\beta'} .$$

Then by definition, we have

$$\delta^{tw}_{\alpha',\beta'} \underline{m}_{\alpha,\beta} = |G^F_\alpha \times G^F_\beta|^{-1} q^{-\sum_{i\in Q_0} \alpha'_i \beta'_i} (\kappa)_!(r)_!(q)_! p^*.$$

For $M \in \mathbb{E}^F_\alpha, N \in \mathbb{E}^F_\beta, M' \in \mathbb{E}^F_{\alpha'}, N' \in \mathbb{E}^F_{\beta'}$, one can check

$$\delta^{tw}_{\alpha',\beta'} \underline{m}_{\alpha,\beta}(1_{\mathcal{O}^F_M}, 1_{\mathcal{O}^F_N})(M', N') = \sum_{[L]\in \mathbb{E}^F_{\underline{d}}/G^F_{\underline{d}}} F^L_{MN} h^{M'N'}_L.$$

Similarly, we can define the set

$$\begin{aligned} S'^F_\lambda = \{&(x_\alpha, x_\beta, x_{\alpha'}, x_{\beta'}, W_1, W_2, \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) \mid (x_{\alpha'}, W_1, \rho_{11}, \rho_{12}) \in \mathbb{E}'^F_{\alpha_2,\beta_2}, \\ &(x_{\beta'}, W_2, \rho_{21}, \rho_{22}) \in \mathbb{E}'^F_{\alpha_1,\beta_1}, (x_\beta, W_2) \in \mathbb{E}''^F_{\beta_1,\beta_2}, (\mathbb{K}^\beta, x_\beta)/W_2 \cong (W_1, x_{\alpha'|_{W_1}}), \\ &\exists W_3, (x_\alpha, W_3) \cong (\mathbb{K}^{\beta'}, x_{\beta'})/W_2, (\mathbb{K}^\alpha, x_\alpha)/W_3 \cong (\mathbb{K}^{\alpha'}, x_{\alpha'})/W_1\} \end{aligned}$$

where $\lambda = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ and W_1, W_2 and W_3 are graded vector spaces of dimensions β_2, β_1 and α_1, respectively. The set can be illustrated by the following diagram:

$$\begin{tikzcd} W_2 \arrow[r] \arrow[d] & (\mathbb{K}^{\beta'}, x_{\beta'}) \arrow[r] & W_3 \arrow[d] \\ (\mathbb{K}^\beta, x_\beta) \arrow[d] & & (\mathbb{K}^\alpha, x_\alpha) \arrow[d] \\ W_1 \arrow[r] & (\mathbb{K}^{\alpha'}, x_{\alpha'}) \arrow[r] & (\mathbb{K}^\alpha, x_\alpha)/W_3 \cong (\mathbb{K}^{\alpha'}, x_{\alpha'})/W_1. \end{tikzcd}$$

Set

$$\begin{aligned} S''^F_\lambda = \{&(x_\alpha, x_\beta, x_{\alpha'}, x_{\beta'}, W_1, W_2) \mid \exists \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}, \\ &(x_\alpha, x_\beta, x_{\alpha'}, x_{\beta'}, W_1, W_2, \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) \in S'^F_\lambda\}. \end{aligned}$$

Then there is a projection $S'^F_\lambda \to S''^F_\lambda$ which is a principal $G_{\alpha_1} \times G_{\alpha_2} \times G_{\beta_1} \times G_{\beta_2}$-bundle. We also have the following diagram

$$\mathbb{E}^F_\alpha \times \mathbb{E}^F_\beta \xleftarrow{i'} \coprod_{\lambda\in\mathcal{N}} F^F_\lambda \xleftarrow{p'} \coprod_{\lambda\in\mathcal{N}} S'^F_\lambda \xrightarrow{q'} \coprod_{\lambda\in\mathcal{N}} S''^F_\lambda \xrightarrow{r'} \mathbb{E}^F_{\alpha'} \times \mathbb{E}^F_{\beta'} .$$

Then we have

$$\underline{m}\delta^{tw} = |G^F_{\alpha_1} \times G^F_{\alpha_2} \times G^F_{\beta_1} \times G^F_{\beta 2}|^{-1} q^{-\langle\alpha_2,\beta_1\rangle - \sum_{i\in Q_0}[(\beta_1)_i(\beta_2)_i + (\alpha_1)_i(\alpha_2)_i]} (r')_!(q')_!(p')^*(i')^*.$$

For $M\in\mathbb{E}^F_\alpha, N\in\mathbb{E}^F_\beta, M'\in\mathbb{E}^F_{\alpha'}, N'\in\mathbb{E}^F_{\beta'}$, one can check

$$\underline{m}\delta^{tw}(1_{\mathcal{O}^F_M},1_{\mathcal{O}^F_N})(M',N')=\sum_{[X],[Y_1],[Y_2],[Z]}F^{M'}_{XY_2}F^{N'}_{Y_1Z}h^{XY_1}_M h^{Y_2Z}_N.$$

where $[X]\in\mathbb{E}^F_{\alpha_2}/G^F_{\alpha_2},[Y_1]\in\mathbb{E}^F_{\alpha_1}/G^F_{\alpha_1},[Y_2]\in\mathbb{E}^F_{\beta_2}/G^F_{\beta_2},[Z]\in\mathbb{E}^F_{\beta_1}/G^F_{\beta_1}$.

We obtain the following diagram

$$\begin{tikzcd}
\mathbb{E}^F_\alpha\times\mathbb{E}^F_\beta & C'^F \arrow[l, "p"'] \arrow[d, "\kappa rq"] \\
\coprod_{\lambda\in\mathcal{N}}S'^F_\lambda \arrow[u, "p'i'"] \arrow[r, "r'q'"] & \mathbb{E}^F_{\alpha'}\times\mathbb{E}^F_{\beta'}
\end{tikzcd}$$

and the proof of Theorem 3.6 is deduced to prove the identity

$$\sum_{[L]\in\mathbb{E}^F_{\underline{d}}/G^F_{\underline{d}}}F^L_{MN}h^{M'N'}_L=\sum_{[X],[Y_1],[Y_2],[Z]}q^{-\langle X,Z\rangle}F^{M'}_{XY_2}F^{N'}_{Y_1Z}h^{XY_1}_M h^{Y_2Z}_N.$$

Applying the Riedtmann-Peng formula, it is equivalent to the Green formula in Section 2. The following we refer the reformulation of the proof of Green's theorem in [29].

The left side of the identity is the number counting the set of crossings with the group action. More precisely, fix M,N,M',N' and consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & N' \arrow[d, "a'"] & & \\
0 \arrow[r] & N \arrow[r, "a"] & L \arrow[r, "b"] \arrow[d, "b'"] & M \arrow[r] & 0 \\
 & & M' \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Set

$$Q=\{(a,b,a',b')\mid a,b,a',b' \text{ as in the above crossing}\}.$$

By calculation, $|Q|=\sum_{[L]\in\mathbb{E}^F_{\underline{d}}/G^F_{\underline{d}}}F^L_{MN}h^{M'N'}_L|G^F_{\underline{d}}||G^F_\alpha||G^F_\beta|$. Consider the natural action of $G^F_{\underline{d}}$ on Q, and the orbit space is denoted by $\widetilde{Q}$. The fibre of the map $Q\to\widetilde{Q}$ has cardinality $\frac{|G^F_{\underline{d}}|}{|\mathrm{Hom}(\mathrm{Coker}b'a,\mathrm{Ker}b'a)|}$.

The right side of the identity is the number counting the squares with the group action. Consider the diagram

$$\begin{tikzcd} & 0 \arrow[d] & & 0 \arrow[d] & \\ 0 \arrow[r] & Z \arrow[r, "e_1"] \arrow[d, "u'"] & N' \arrow[r, "e_2"] & Y_1 \arrow[r] \arrow[d, "x"] & 0 \\ & N \arrow[d, "v'"] & & M \arrow[d, "y"] & \\ 0 \arrow[r] & Y_2 \arrow[r, "e_3"] \arrow[d] & M' \arrow[r, "e_4"] & X \arrow[r] \arrow[d] & 0 \\ & 0 & & 0 & \end{tikzcd}$$

Set

$$\mathcal{O} = \{(e_1, e_2, e_3, e_4, u', v', x, y) \mid \text{ all morphisms occur in the above diagram}\}.$$

The group $G^F_{\alpha_1} \times G^F_{\alpha_2} \times G^F_{\beta_1} \times G^F_{\beta 2}$ freely acts on $\mathcal{O}$ with orbit space $\widetilde{\mathcal{O}}$. Note that

$$|\widetilde{\mathcal{O}}| = \sum_{[X],[Y_1],[Y_2],[Z]} F^{M'}_{XY_2} F^{N'}_{Y_1 Z} h^{XY_1}_M h^{Y_2 Z}_N |G^F_\alpha||G^F_\beta|.$$

There is a canonical map $\widetilde{f} : \widetilde{Q} \to \widetilde{\mathcal{O}}$, which the cardinality of fibre is $|\mathrm{Ext}^1(X, Z)|$.

As Ringel-Hall algebras, we can define the analogue $\sigma : \mathcal{CF}^F(Q) \to \mathcal{CF}^F(Q)$ of the antipode by setting

$$\sigma(f) = \sum_{r \geq 1 \in \mathbb{Z}} (-1)^r \sum_{\alpha_1, \cdots, \alpha_r \neq 0} \underline{m}^r_{\alpha_1, \cdots, \alpha_r} \circ \delta^{tw,r}_{\alpha_1, \cdots, \alpha_r}(f)$$

for $f \neq 0 \in \mathcal{CF}^F(Q)$.

In order to compare Lusztig's Hall algebras with twisted Ringel-Hall algebras. We twist $\mathcal{CF}^F(Q)$ by setting $\underline{m}^t_{\alpha,\beta} = v^{\langle \alpha, \beta \rangle} \underline{m}_{\alpha,\beta}$, $\delta^t_{\alpha,\beta} = v^{\langle \alpha, \beta \rangle} \delta^{tw}_{\alpha,\beta}$ and

$$\sigma^t(f) = \sum_{r \geq 1 \in \mathbb{Z}} (-1)^r \sum_{\alpha_1, \cdots, \alpha_r \neq 0} \underline{m}^{t,r}_{\alpha_1, \cdots, \alpha_r} \circ \delta^{t,r}_{\alpha_1, \cdots, \alpha_r}(f).$$

We denote the twist version by $\mathcal{CF}^{F,tw}(Q)$. By applying Lemma 3.1, 3.2 and Theorem 3.6, we obtain the following result.

Theorem 3.7. *The algebra $\mathcal{CF}^{F,tw}(Q)$ is a Hopf algebra with the multiplication $\underline{m}^t$, comultiplication δ^t and antipode σ^t. Fix an isomorphism $\tau : \overline{\mathbb{Q}_l} \to \mathbb{C}$, there is an isomorphism of Hopf algebra $\Phi : \mathcal{CF}^{F,tw}(Q) \to \mathcal{H}^{tw}(\mathcal{A})$ where $\mathcal{A} = \mathrm{Rep}_k Q$ by setting $\Phi(1_{\mathcal{O}^F_M}) = [M]$ for $M \in \mathcal{A}$.*

4. The categorification of Ringel-Hall algebras

Let $\mathbb{F}_q$ be a finite field with q elements. In the following, $\mathbb{K}$ is an algebraic closure of $\mathbb{F}_q$.

Let X be a scheme of finite type over $\mathbb{K}$. We say that X has an $\mathbb{F}_q$-structure if there exists a variety X_0 over $\mathbb{F}_q$ such that $X = X_0 \times_{spec(\mathbb{F}_q)} Spec(k)$. Let $F_{X_0} : X_0 \to X_0$ be the Frobenius morphism. It can be extended to the morphism $F_X : X \to X$. Let X^F be the set of closed points of X fixed by F, i.e., the set of $\mathbb{F}_q$-rational points. For any $n \in \mathbb{N}$, let X^{F^n} be the set of closed points of X fixed by F^n. Note that $X^{F^1} = X^F$.

Denote by $\mathcal{D}^b(X) = \mathcal{D}^b(X, \overline{\mathbb{Q}_l})$ the bounded derived category of $\overline{\mathbb{Q}_l}$-constructible complexes on X.

The morphism $F_X : X \to X$ naturally induces a functor $F_X^* : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$. A Weil complex is a pair $(\mathcal{F}, j)$ such that $\mathcal{F} \in \mathcal{D}^b(X)$ and $j : F_X^*(\mathcal{F}) \to \mathcal{F}$ is an isomorphism. Let $\mathcal{D}^b_w(X)$ be the triangulated subcategory of $\mathcal{D}^b(X)$ of Weil complexes and $K_w(X)$ be the Grothendieck group of $\mathcal{D}^b_w(X)$.

Given a Weil complex $\mathcal{F} = (\mathcal{F}, j)$[1] in $\mathcal{D}^b_w(X)$, let $x \in X^F$ be a closed point. We get automorphisms

$$F_{i,x} : \mathcal{H}^i(\mathcal{F})_{|x} \to \mathcal{H}^i(\mathcal{F})_{|x}.$$

One can define a F-invariant function $\chi^F_{\mathcal{F}} : X^F \to \overline{\mathbb{Q}_l}$ via defining

$$\chi^F_{\mathcal{F}}(x) = \sum_i (-1)^i tr(F_{i,x}, \mathcal{H}^i(\mathcal{F})_{|x}) = \sum_i (-1)^i tr(F_{i,x}).$$

Similarly, one can define $\chi^{F^n}_{\mathcal{F}} : X^{F^n} \to \overline{\mathbb{Q}_l}$ via defining

$$\chi^{F^n}_{\mathcal{F}}(x) = \sum_i (-1)^i tr(F^n_{i,x}, \mathcal{H}^i(\mathcal{F})_{|x}) = \sum_i (-1)^i tr(F^n_{i,x}).$$

In particular, $\chi^{F^1} = \chi^F$.

Theorem 4.1. [14, Theorem 12.1] *Let X be as above. Then we have*

1. *Let* $\mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}[1]$ *be a distinguished triangle in $\mathcal{D}^b_w(X)$. Then $\chi^F_{\mathcal{K}} + \chi^F_{\mathcal{M}} = \chi^F_{\mathcal{L}}$.*
2. *Let $g : X \to Y$ be a morphism. Then for $\mathcal{K} \in \mathcal{D}^b_w(X)$ and $\mathcal{L} \in \mathcal{D}^b_w(Y)$, we have $\chi^F_{Rg_!\mathcal{K}} = g_!(\chi^F_{\mathcal{K}})$ and $\chi^F_{g^*\mathcal{L}} = g^*(\chi^F_{\mathcal{L}})$.*
3. *For $\mathcal{K} \in \mathcal{D}^b_w(X)$, we have $\chi^F_{\mathcal{K}[d]} = (-1)^d \chi^F_{\mathcal{K}}$ and $\chi^F_{\mathcal{K}(n)} = q^{-n}\chi^F_{\mathcal{K}}$.*

By Theorem 4.1(1), the function $\chi^{F^n}_{\mathcal{F}}$ only depends on the isomorphism class of $\mathcal{F}$ in $\mathcal{D}^b_w(X)$. Let $\mathcal{CF}(X^{F^n})$ be the vector space of all functions $X^{F^n} \to \overline{\mathbb{Q}_l}$. Hence, we obtain a map $\chi^{F^n} : K_w(X) \to \mathcal{CF}(X^{F^n})$.

Let G be an algebraic group over $\mathbb{K}$ and X be a scheme of finite type over $\mathbb{K}$ together with an G-action. Assume that X and G have $\mathbb{F}_q$-structures and $X = X_0 \times_{spec(\mathbb{F}_q)} Spec(k)$, $G = G_0 \times_{spec(\mathbb{F}_q)} Spec(k)$. Let $F_{G_0} : G_0 \to G_0$ be the Frobenius morphism. It can be extended to the morphism $F_G : G \to G$. Denote by $\mathcal{D}^b_G(X) = \mathcal{D}^b_G(X, \overline{\mathbb{Q}_l})$ the G-equivariant bounded derived category of $\overline{\mathbb{Q}_l}$-constructible complexes on X and $\mathcal{D}^b_{G,w}(X)$ the subcategory of $\mathcal{D}^b_G(X)$ consisting of Weil complexes. Let $K_{G,w}(X)$ be the Grothendieck group of $\mathcal{D}^b_{G,w}(X)$.

[1] All Weil complexes considered here are induced by the complexes in $\mathcal{D}^b(X_0, \overline{\mathbb{Q}_l})$.

Assume that we have the following commutative diagram

$$\begin{tikzcd} G\times X \arrow[r] \arrow[d, "F_G\times F_X"'] & X \arrow[d, "F_X"] \\ G\times X \arrow[r] & X \end{tikzcd}$$

Then, the morphism $F_X : X \to X$ naturally induces a functor

$$F_X^* : \mathcal{D}^b_{G,w}(X) \to \mathcal{D}^b_{G,w}(X).$$

Let $x \in X^F$ be a closed point. For any $(\mathcal{F}, j)$ in $\mathcal{D}^b_{G,w}(X)$, we get automorphisms

$$F_{i,x} : \mathcal{H}^i_G(\mathcal{F})_{|x} \to \mathcal{H}^i_G(\mathcal{F})_{|x}.$$

In the same way, one can define the G-equivariant version of χ^{F^n} for $n \in \mathbb{N}$ as follows:

$$\chi^F_{\mathcal{F}}(x) = \sum_i (-1)^i tr(F_{i,x}, \mathcal{H}^i_G(\mathcal{F})_{|x})$$

and

$$\chi^{F^n}_{\mathcal{F}}(x) = \sum_i (-1)^i tr(F^n_{i,x}, \mathcal{H}^i_G(\mathcal{F})_{|x}).$$

In particular, $\chi^{F^1} = \chi^F$.

Lemma 4.2. *For any $(\mathcal{F}, j) \in \mathcal{D}^b_{G,w}(X)$, $\chi^{F^n}_{\mathcal{F}}$ is a G-equivariant function.*

Proof. For any x and y in the same G-orbit of X, there exists an element $g \in G$ such that $g.x = y$. Consider the following commutative diagram

$$\begin{tikzcd} \mathcal{H}^i_G(\mathcal{F})_{|x} \arrow[r, "F_x"] \arrow[d, "g^*"] & \mathcal{H}^i_G(\mathcal{F})_{|x} \arrow[d, "g^*"] \\ \mathcal{H}^i_G(g^*\mathcal{F})_{|y} \arrow[r, "F_y"] & \mathcal{H}^i_G(g^*\mathcal{F})_{|y} \end{tikzcd}$$

Since $g^*\mathcal{F} \simeq \mathcal{F}$, we have

$$\begin{tikzcd} \mathcal{H}^i_G(\mathcal{F})_{|x} \arrow[r, "F_x"] \arrow[d, "g^*"] & \mathcal{H}^i_G(\mathcal{F})_{|x} \arrow[d, "g^*"] \\ \mathcal{H}^i_G(\mathcal{F})_{|y} \arrow[r, "F_y"] & \mathcal{H}^i_G(\mathcal{F})_{|y} \end{tikzcd}$$

By the definition of χ^{F^n}, $\chi^{F^n}_{\mathcal{F}}(x) = \chi^{F^n}_{\mathcal{F}}(y)$. That is $\chi^{F^n}_{\mathcal{F}}$ is a G-equivariant function. $\square$

In the G-equivariant case, we also have Theorem 4.1. Hence the function $\chi^{F^n}_{\mathcal{F}}$ also only depends on the isomorphism class of $\mathcal{F}$ in $\mathcal{D}^b_{G,w}(X)$. Hence, we obtain a map $\chi^{F^n} : K_{G,w}(X) \to \mathcal{CF}_G(X^{F^n})$.

Let Q be a finite quiver without loops. Given a dimension vector $\underline{d} = (d_i)_{i\in Q_0}$, the variety $\mathbb{E}_{\underline{d}}$ and the algebraic group $G_{\underline{d}}$ are defined in Section 3. Both of them have natural $\mathbb{F}_q$-structures. Consider the following diagram

$$\mathbb{E}_\alpha \times \mathbb{E}_\beta \xleftarrow{p_1} \mathbb{E}' \xrightarrow{p_2} \mathbb{E}'' \xrightarrow{p_3} \mathbb{E}_{\alpha+\beta}\ .$$

This induces a functor $\mathfrak{m}:\mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta)\to\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})$ described as the composition of the following functors:

$$\mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta)\xrightarrow{\mathfrak{p}_1^*}\mathcal{D}^b_{G_\alpha\times G_\beta\times G_{\alpha+\beta},w}(\mathbb{E}')\xrightarrow{(\mathfrak{p}_2)_b}\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}'')\xrightarrow{(\mathfrak{p}_3)_!}\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta}),$$

where $(\mathfrak{p}_2)_b$ is the inverse of the pull-back functor

$$\mathfrak{p}_2^*:\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}'')\to\mathcal{D}^b_{G_\alpha\times G_\beta\times G_{\alpha+\beta},w}(\mathbb{E}'),$$

which is an equivalence of derived categories. By definition,

$$\chi^F_{(\mathfrak{p}_2)_b(\mathcal{K})}=\frac{1}{|G_\alpha\times G_\beta|}\chi^F_{(\mathfrak{p}_2)_!(\mathcal{K})}$$

for $\mathcal{K}\in\mathcal{D}^b_{G_{\alpha+\beta},m}(\mathbb{E}'')$ since p_2 is a principal $G_\alpha\times G_\beta$-bundle.

Applying Theorem 4.1, we obtain the following commutative diagrams

$$\begin{array}{ccccccc}
\mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta) & \xrightarrow{\mathfrak{p}_1^*} & \mathcal{D}^b_{G_\alpha\times G_\beta\times G_{\alpha+\beta},w}(\mathbb{E}') & \xrightarrow{(\mathfrak{p}_2)_b} & \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}'') & \xrightarrow{(\mathfrak{p}_3)_!} & \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathcal{CF}_{G_\alpha\times G_\beta}(\mathbb{E}^F_\alpha\times\mathbb{E}^F_\beta) & \xrightarrow{p_1^*} & \mathcal{CF}_{G_\alpha\times G_\beta\times G_{\alpha+\beta}}(\mathbb{E}'^F) & \xrightarrow{\iota} & \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}''^F) & \xrightarrow{(p_3)_!} & \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}^F_{\alpha+\beta})
\end{array}$$

where $\iota=\frac{1}{|G_\alpha\times G_\beta|}(p_2)_!$. Hence, the linear functor $\mathfrak{m}:\mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta)\to\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta) & \xrightarrow{\mathfrak{m}} & \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})\\
\downarrow{\scriptstyle\chi^F} & & \downarrow{\scriptstyle\chi^F}\\
\mathcal{CF}_{G_\alpha\times G_\beta}(\mathbb{E}^F_\alpha\times\mathbb{E}^F_\beta) & \xrightarrow{\underline{m}} & \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}^F_{\alpha+\beta}).
\end{array}\tag{4.1}$$

Lemma 4.3. *For simple perverse sheaves $\mathbb{P}\in\mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta)$, $\mathfrak{m}(\mathbb{P})$ is still semisimple in $\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})$.*

Proof. Since $\mathfrak{p}_1$ is smooth with connect fibres, $\mathfrak{p}_1^*(\mathbb{P})$ is still semisimple by Section 4.2.4 and 4.2.5 in [1]. Since $(\mathfrak{p}_2)_b$ is a equivalence of categories, $(\mathfrak{p}_2)_b\mathfrak{p}_1^*(\mathbb{P})$ is still semisimple. At last, $\mathfrak{p}_3$ is proper implies that $\mathfrak{m}(\mathbb{P})=(\mathfrak{p}_3)_!(\mathfrak{p}_2)_b\mathfrak{p}_1^*(\mathbb{P})$ is also semisimple. $\square$

By Lemma 4.3, the linear functor $\mathfrak{m}:\mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta)\to\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})$ induces a linear map

$$\mathfrak{m}:K_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta)\to K_{G_{\alpha+\beta}}(\mathbb{E}_{\alpha+\beta})$$

such that the following diagram is commutative

$$\begin{array}{ccc}
K_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha\times\mathbb{E}_\beta) & \xrightarrow{\mathfrak{m}} & K_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})\\
\downarrow{\scriptstyle\chi^F} & & \downarrow{\scriptstyle\chi^F}\\
\mathcal{CF}_{G_\alpha\times G_\beta}(\mathbb{E}^F_\alpha\times\mathbb{E}^F_\beta) & \xrightarrow{\underline{m}} & \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}^F_{\alpha+\beta}).
\end{array}\tag{4.2}$$

Set $K_{G,w}=\bigoplus_{\underline{d}}K_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})$ and $\mathcal{CF}^F(Q)=\bigoplus_{\underline{d}}\mathcal{CF}_{G_{\underline{d}}}(\mathbb{E}^F_{\underline{d}})$. There is a linear map from $K_{G,w}$ to $\mathcal{CF}^F(Q)$ induced by χ^F. For simplicity, we also denote it by χ^F. For

$\mathcal{M} \in \mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$ and $\mathcal{N} \in \mathcal{D}^b_{G_\beta,w}(\mathbb{E}_\beta)$, define $[\mathcal{M}] * [\mathcal{N}] := \mathfrak{m}(\mathcal{M} \boxtimes \mathcal{N})$. Then the linear maps $\mathfrak{m}$ and $\underline{m}$ endow $K_{G,w}$ and $\mathcal{CF}^F(Q)$ with multiplication structures, respectively. Using Diagram (4.2), we obtain the following result.

Lemma 4.4. *The $\mathbb{Z}$-linear map $\chi^F : K_{G,w} \to \mathcal{CF}^F(Q)$ is an algebra homomorphism.*

Consider the diagram

$$\mathbb{E}_\alpha \times \mathbb{E}_\beta \xleftarrow{\kappa} F_{\alpha,\beta} \xrightarrow{i} \mathbb{E}_{\alpha+\beta} .$$

This induces a functor $\Delta_{\alpha,\beta} : \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta}) \to \mathcal{D}^b_{G_\alpha \times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta)$ as the composition of functors:

$$\mathcal{D}^b_{G_\alpha \times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta) \xleftarrow{\kappa_!} \mathcal{D}^b_{G_{\alpha+\beta},w}(F_{\alpha,\beta}) \xleftarrow{i^*} \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta}) .$$

Applying Theorem 4.1, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta}) \arrow[r, "\Delta_{\alpha,\beta}"] \arrow[d, "\chi^F"] & \mathcal{D}^b_{G_\alpha \times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta) \arrow[d, "\chi^F"] \\
\mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}^F_{\alpha+\beta}) \arrow[r, "\delta_{\alpha,\beta}"] & \mathcal{CF}_{G_\alpha \times G_\beta}(\mathbb{E}^F_\alpha \times \mathbb{E}^F_\beta).
\end{tikzcd}$$

Lemma 4.5. [3][2] *For simple perverse sheaves $\mathbb{P} \in \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})$, $\Delta_{\alpha,\beta}(\mathbb{P})$ is still semisimple in $\mathcal{D}^b_{G_\alpha \times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta)$.*

Proof. Since $\Delta_{\alpha,\beta}$ is a hyperbolic localization ([3]). $\square$

By Lemma 4.5, the linear functor $\Delta_{\alpha,\beta} : \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta}) \to \mathcal{D}^b_{G_\alpha \times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta)$ induces a linear map

$$\Delta_{\alpha,\beta} : K_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta}) \to K_{G_\alpha \times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta)$$

such that the following diagram is commutative

$$\begin{tikzcd}
K_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta}) \arrow[r, "\Delta^{tw}_{\alpha,\beta}"] \arrow[d, "\chi^F"] & K_{G_\alpha \times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta) \arrow[d, "\chi^F"] \\
\mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}^F_{\alpha+\beta}) \arrow[r, "\delta^{tw}_{\alpha,\beta}"] & \mathcal{CF}_{G_\alpha \times G_\beta}(\mathbb{E}^F_\alpha \times \mathbb{E}^F_\beta).
\end{tikzcd}$$

where $\Delta^{tw}_{\alpha,\beta} = \kappa_! i^* [\sum_{i \in Q_0} \alpha_i \beta_i](\frac{\sum_{i \in Q_0} \alpha_i \beta_i}{2})$.

The maps $\Delta^{tw}_{\alpha,\beta}$ and $\delta^{tw}_{\alpha,\beta}$ induce the comultiplication structures over $K_{G,w}$ and $\mathcal{CF}^F(Q)$, respectively. In the same way as Lemma 4.4, we obtain the following result.

Lemma 4.6. *The $\mathbb{Z}$-linear map $\chi^F : K_{G,w} \to \mathcal{CF}^F(Q)$ is a coalgebra homomorphism.*

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[2] The authors thank Hiraku Nakajima for pointing out Reference [3].

In Section 3, we have shown that there exists a comultiplication structure over $\mathcal{CF}^F(Q)$. By Green's theorem, the comultiplication is compatible with the multiplication structure and then $\mathcal{CF}^F(Q)$ is a bialgebra. Naturally, one would like to check whether $K_{G,w}$ is a bialgebra. Let $\mathcal{CF}^{F^n}(Q)=\bigoplus_{\underline{d}}\mathcal{CF}_{G_{\underline{d}}}(\mathbb{E}_{\underline{d}}^{F^n})$. Similarly to $\chi^F: K_{G,w}\to\mathcal{CF}^F(Q)$, we have $\chi^{F^n}: K_{G,w}\to\mathcal{CF}^{F^n}(Q)$ for any $n\in\mathbb{N}$.

Theorem 4.7. [27, Theorem 3.5] *The ring homomorphism*

$$\chi=\prod_{n\in\mathbb{N}}\chi^{F^n}: K_{G,w}\to\prod_{n\in\mathbb{N}}\mathcal{CF}^{F^n}(Q)$$

is injective.

The following theorem can be viewed as the categorification of the Green formula in Section 1.

Theorem 4.8. *Let $\mathbb{P}_1\in\mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$ and $\mathbb{P}_2\in\mathcal{D}^b_{G_\beta,w}(\mathbb{E}_\beta)$ be two simple perverse sheaves. Then we have*

$$\Delta(\mathbb{P}_1*\mathbb{P}_2)=\Delta(\mathbb{P}_1)*\Delta(\mathbb{P}_1).$$

Proof. For simple perverse sheaves $\mathbb{P}_1\in\mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$ and $\mathbb{P}_2\in\mathcal{D}^b_{G_\beta,w}(\mathbb{E}_\beta)$, $\mathbb{P}_1*\mathbb{P}_2$ is still semisimple by Lemma 4.3 and then $\Delta(\mathbb{P}_1*\mathbb{P}_2)$ is semisimple by Lemma 4.5 . The rightside $\Delta(\mathbb{P}_1)*\Delta(\mathbb{P}_1)$ of the equation is also semisimple by Lemma 4.3 and 4.5. Hence, it is enough to prove the equation holds in $K_{G,w}$. Applying Theorem 4.7, it is equivalent to show that

$$\chi^{F^n}(\mathbb{P}_1*\mathbb{P}_2)=\chi^{F^n}(\mathbb{P}_1)*\chi^{F^n}(\mathbb{P}_2)$$

for $n\in\mathbb{N}$. The last statement follows from the Green formula. □

As a corollary, we have the following theorem.

Theorem 4.9. *The algebra $K_{G,w}$ is a bialgebra and the $\mathbb{Z}$-linear map $\chi^F: K_{G,w}\to\mathcal{CF}^F(Q)$ is a bialgebra homomorphism.*

There is a natural $\mathbb{Z}[v,v^{-1}]$-module structure on $K_{G,w}$ by $v^{\pm1}[\mathcal{K}]=[\mathcal{K}[\pm1](\pm\frac{1}{2})]$ for any dimension vector α and $\mathcal{K}\in\mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$. Given any dimension vector α and a simple perverse sheaf $\mathbb{P}\in\mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$, we set $T_{\mathbb{P}}=\{\mathbb{P}[m](\frac{m}{2})\mid m\in\mathbb{Z}\}$ and

$$\mathcal{T}=\{T_{\mathbb{P}}\mid\exists\alpha,\mathbb{P}\in\mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)\text{ is a simple perverse sheaf}\}.$$

Fix an assignment $S:\mathcal{T}\to K_{G,w}$. Then $K_{G,w}$ is a free $\mathbb{Z}[v,v^{-1}]$-module with a basis $S(\mathcal{T})$.

We also consider the twisted version of $K_{G,w}$ in order to preserve the subcategories of perverse sheaves by defining

$$\mathfrak{m}^t_{\alpha,\beta}=\mathfrak{m}_{\alpha,\beta}[\{\alpha,\beta\}](\frac{\{\alpha,\beta\}}{2}),$$

where $\{\alpha,\beta\}=\sum_{i\in Q_0}\alpha_i\beta_i+\sum_{h\in Q_1}\alpha_{s(h)}\beta_{t(h)}$ and

$$\Delta^t_{\alpha,\beta}=\Delta^{tw}_{\alpha,\beta}[-\langle\alpha,\beta\rangle](-\frac{\langle\alpha,\beta\rangle}{2}).$$

We denote by $K^{tw}_{G,w}$ the twist of $K_{G,w}$ with the multiplication and comultiplication induced by $\mathfrak{m}^t_{\alpha,\beta}$ and $\Delta^t_{\alpha,\beta}$, respectively.

The following lemma is the simple generalization of Lusztig's construction over quantum groups ([30, Theorem 3.24]).

Lemma 4.10. *There is a bialgebra homomorphism $\chi^{F,tw}: K^{tw}_{G,w} \to \mathcal{CF}^{F,tw}(Q)$ by sending $[\mathbb{P}]$ for $\mathbb{P} \in \mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$ to $v^{dim G_\alpha}\chi^F(\mathbb{P})$.*

Corollary 4.11. The ring homomorphism

$$\chi^{tw} = \prod_{n\in\mathbb{N}} \chi^{F^n,tw} : K^{tw}_{G,w} \to \prod_{n\in\mathbb{N}} \mathcal{CF}^{F^n,tw}(Q)$$

is an injective homomorphism.

We will endow $K^{tw}_{G,w}$ with the structure of the antipode map as an analogue of the antipode over a Ringel-Hall algebra. Let $\mathfrak{m}^r$ and Δ^r be the r-fold multiplication and comultiplication for $r \geq 1$. More explicitly, we have the following functors:

$$\mathfrak{m}^r_{\alpha_1,\cdots,\alpha_r} : \mathcal{D}^b_{\prod_{i=1}^r G_{\alpha_i},w}(\prod_{i=1}^r \mathbb{E}_{\alpha_i}) \to \mathcal{D}^b_{G_{\alpha_1+\cdots+\alpha_r},w}(\mathbb{E}_{\alpha_1+\cdots+\alpha_r})$$

and

$$\Delta^r_{\alpha_1,\cdots,\alpha_r} : \mathcal{D}^b_{G_{\alpha_1+\cdots+\alpha_r},w}(\mathbb{E}_{\alpha_1+\cdots+\alpha_r}) \to \mathcal{D}^b_{\prod_{i=1}^r G_{\alpha_i},w}(\prod_{i=1}^r \mathbb{E}_{\alpha_i}).$$

such that $\mathfrak{m}^{r+1} = \mathfrak{m} \circ (1 \otimes \mathfrak{m}^r)$ and $\Delta^{r+1} = (1 \otimes \cdots \otimes \Delta) \circ \Delta^r$ for $r \geq 1$.

Now we define the functor $S : \mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha) \to \mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$ by setting

$$S(\mathbb{P}) = \bigoplus_{r\geq 1} \bigoplus_{\alpha_1,\cdots,\alpha_r \neq 0} \mathfrak{m}^{t,r}_{\alpha_1,\cdots,\alpha_r} \circ \Delta^{t,r}_{\alpha_1,\cdots,\alpha_r}(\mathbb{P})[r] \text{ and } S(0) = 0.$$

For simplicity, we denote still by S the induced map over $K^{tw}_{G_\alpha,w}(\mathbb{E}_\alpha)$, even $K^{tw}_{G,w}$.

Lemma 4.12. *The map $\chi^{F,tw}$ satisfies that $\chi^{F,tw}(S(\mathbb{P})) = \sigma^t(\chi^{F,tw}(\mathbb{P}))$ for $\mathbb{P} \in \mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$.*

This follows from $\chi^{F,tw}\mathfrak{m}^{t,r} = \underline{m}^{t,r}\chi^{F,tw}$ and $\chi^{F,tw}\Delta^{t,r} = \delta^{t,r}\chi^{F,tw}$.

Proposition 4.13. With the above notation, we have

$$\mathfrak{m}^t(S \otimes 1)\Delta^t(\mathbb{P}) = \mathfrak{m}^t(1 \otimes S)\Delta^t(\mathbb{P}) = 0$$

for $\mathbb{P} \neq 0$ and $[0]$ for $\mathbb{P} = 0$.

Proof. By Theorem 4.7, we only need to prove the image of this identity under $\chi^{F^k,tw}$. Since we have $\chi^{F^k,tw}\mathfrak{m}^{t,r} = \underline{m}^{t,r}\chi^{F^k,tw}$ and $\chi^{F^k,tw}\Delta^{t,r} = \delta^{t,r}\chi^{F^k,tw}$, the equation is deduced to

$$\underline{m}^t(\sigma \otimes 1)\delta^t(f) = \underline{m}^t(1 \otimes \sigma)\delta^t(f) = 0$$

for $f \neq 0$. $\square$

As a consequence of Theorem 4.9 and Proposition 4.13, we obtain the main theorem in this section.

Theorem 4.14. *The algebra $K^{tw}_{G,w}$ is a Hopf algebra.*

5. Green's Hopf pairing

Let $Q=(Q_0,Q_1)$ as in Section 3. Given $\underline{d}\in\mathbb{N}Q_0$ and $\mathbb{P},\mathbb{Q}\in\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})$, we set the following geometric pair:

$$\{\mathbb{P},\mathbb{Q}\}=\sum_i(\dim H^i_{G_{\underline{d}}}(\mathbb{P}\otimes\mathbb{Q},\mathbb{E}_{\underline{d}}))v^i.$$

One can refer to [30] for more details and the definition of $H^i_G(-,X)$.

The geometric pair has the following basic properties:

Lemma 5.1. [16] *For any $\mathbb{P}\in\mathcal{D}^b_{G_{\underline{d}_1},w}(\mathbb{E}_{\underline{d}_1})$, $\mathbb{Q}\in\mathcal{D}^b_{G_{\underline{d}_2},w}(\mathbb{E}_{\underline{d}_2})$ and $\mathbb{R}\in\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})$ such that $\underline{d}_1+\underline{d}_2=\underline{d}\in\mathbb{N}Q_0$, we have*

$$\{\mathfrak{m}(\mathbb{P}\boxtimes\mathbb{Q}),\mathbb{R}\}=\{\mathbb{P}\boxtimes\mathbb{Q},\Delta(\mathbb{R})\}.$$

Applying this lemma iteratively, we have the following corollary.

Corollary 5.2. For any $\mathbb{P}_i\in\mathcal{D}^b_{G_{\underline{d}_i},w}(\mathbb{E}_{\underline{d}_i})(i\in\{1,2,\dots,r\})$ and $\mathbb{R}\in\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})$ such that $\sum_{i=1}^r\underline{d}_i=\underline{d}\in\mathbb{N}Q_0$, we have

$$\{\mathfrak{m}^{t,r}(\mathbb{P}_1\boxtimes\cdots\boxtimes\mathbb{P}_r),\mathbb{R}\}=\{\mathbb{P}_1\boxtimes\cdots\boxtimes\mathbb{P}_r,\Delta^{t,r}(\mathbb{R})\}$$

for $r\geq 2$.

Proposition 5.3. Let $\mathcal{K}\longrightarrow\mathcal{L}\longrightarrow\mathcal{M}\longrightarrow\mathcal{K}[1]$ be a distinguished triangle in $\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})$. Then

$$\{\mathcal{L},\mathcal{R}\}_{v=-1}=\{\mathcal{K},\mathcal{R}\}_{v=-1}+\{\mathcal{M},\mathcal{R}\}_{v=-1},$$

for any $\mathcal{R}\in\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})$.

Proof. By definition,

$$\{\mathcal{L},\mathcal{R}\}=\sum_i(dimH^i_{G_{\underline{d}}}(\mathcal{L}\otimes\mathcal{R},\mathbb{E}_{\underline{d}}))v^i=\sum_i(dimExt^i_{\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})}(\mathcal{L},D\mathcal{R}))v^i,$$

where D is the Verdier duality. Applying the functor $Hom(-,D\mathcal{R})$ to the distinguished triangle, we obtain

$$\cdots\to Ext^i_{\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})}(\mathcal{K},D\mathcal{R})\to Ext^i_{\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})}(\mathcal{L},D\mathcal{R})\to Ext^i_{\mathcal{D}^b_{G_{\underline{d}},w}(\mathbb{E}_{\underline{d}})}(\mathcal{M},D\mathcal{R})\to\cdots$$

Hence, we have $\{\mathcal{L},\mathcal{R}\}_{v=-1}=\{\mathcal{K},\mathcal{R}\}_{v=-1}+\{\mathcal{M},\mathcal{R}\}_{v=-1}$. □

The proof of the proposition also implies that [16]

$$\{\mathcal{K}\oplus\mathcal{M},\mathcal{R}\}=\{\mathcal{K},\mathcal{R}\}+\{\mathcal{M},\mathcal{R}\}\quad\text{and}\quad\{\mathcal{P}[n](\frac{n}{2}),\mathcal{Q}\}=v^n\{\mathcal{P},\mathcal{Q}\}.$$

The following we consider the relation between the geometric pair and Green's pair. For $M\in modkQ$ with dimension vector $\underline{d}$, let $\mathcal{O}_M$ be the orbit of M. Consider the natural embedding $j:\mathcal{O}_M\to\mathbb{E}_{\underline{d}}$. Let $\mathcal{C}_M$ be the pushforward of the constant sheaf over $\mathcal{O}_M$.

Proposition 5.4. Let M,N be kQ-modules of dimension vector $\underline{d}$ and $M\ncong N$. Then we have

$$\{\mathcal{C}_M,\mathcal{C}_N\}=0.$$

Proof. Since $\mathcal{C}_M$ is supported over $\mathcal{O}_M$. If $M\ncong N$, then $\mathcal{O}_M\cap\mathcal{O}_N=\emptyset$. Hence, for any $x\in\mathbb{E}_{\underline{d}}$, $(\mathcal{C}_M\otimes\mathcal{C}_N)_x=(\mathcal{C}_M)_x\otimes(\mathcal{C}_N)_x=0$. □

Recall Green's pair $(1_{\mathcal{O}_M}, 1_{\mathcal{O}_N}) = 0$ for $\mathcal{O}_M \neq \mathcal{O}_N$ and $\frac{1}{|G_{\underline{d}}|}$ for $\mathcal{O}_M = \mathcal{O}_N$. The proposition says that the geometric pair has almost orthognality as Green's pair.

Proposition 5.5. Let Q be a Dynkin quiver and M, N be kQ-modules of dimension vector $\underline{d}$. Then we have

$$\{\mathcal{C}_M, \mathcal{C}_N\} = (\chi^F_{\mathcal{C}_M}, \chi^F_{\mathcal{C}_N}).$$

Proof. If M, N are two simple modules, then the identity in the proposition holds by [16]. Using Lemma 5.1, the identity in the proposition holds for the isomorphism classes of M, N belonging to the composition subalgebra of Q. Since Q is of Dynkin type, the Ringel-Hall algebra of Q and its composition subalgebra coincides and then the proposition follows. $\square$

Since $K^{tw}_{G,w}$ is the free $\mathbb{Z}[v, v^{-1}]$-module with a basis in which the base elements are simple perverse sheaves. Now we define the bilinear form $\{-,-\} : K^{tw}_{G,w} \times K^{tw}_{G,w} \to \mathbf{R} = N((v))$ by setting

$$\{\mathbb{P}, \mathbb{Q}\} = \sum_i (\mathrm{dim} H^i_{G_{\underline{d}}}(\mathbb{P} \otimes \mathbb{Q}, \mathbb{E}_{\underline{d}}))v^i. \tag{5.1}$$

for two simple perverse sheaves $\mathbb{P}, \mathbb{Q}$.

Fix a basis $\mathbf{B}$ of $K_{G,w}$ as a free $\mathbb{Z}[v, v^{-1}]$-module. We extend the algebra $K^{tw}_{G,w}$ by adding the free $\mathbb{Z}[v, v^{-1}]$-module $\mathbf{K} = \bigoplus_{\alpha \in \mathbb{Z}[Q_0]} A\mathbf{k}_\alpha$ and set $\widetilde{K}^{tw}_{G,w}$ to be the free $\mathbb{Z}[v, v^{-1}]$-module with the basis $\{k_\alpha[\mathbb{P}] \mid \alpha \in \mathbb{Z}[Q_0], [\mathbb{P}] \in \mathbf{B}\}$. The multiplication $\widetilde{\mathfrak{m}}$, comultiplication $\widetilde{\Delta}$ and antipode structures $\widetilde{S}$ over $\widetilde{K}^{tw}_{G,w}$ by adding the following relations to $K^{tw}_{G,w}$:

(1) $\widetilde{\mathfrak{m}}(\mathbb{P} \boxtimes \mathbb{Q}) = \mathfrak{m}^t(\mathbb{P} \boxtimes \mathbb{Q})$;
(2) $\mathbf{k}_\alpha[\mathbb{P}] = v^{(\alpha,\beta)}[\mathbb{P}]\mathbf{k}_\alpha \quad \text{for} \quad \mathbb{P} \in \mathcal{D}^b_{G_\beta,w}(\mathbb{E}_\beta)$;
(3) $k_\alpha k_\beta = k_{\alpha+\beta}$;
(4) $\widetilde{\Delta}([\mathbb{P}]) = \sum_{\alpha,\beta} \Delta^t_{\alpha,\beta}([\mathbb{P}]) \cdot (\mathbf{k}_\beta \otimes 1)$;
(5) $\widetilde{\Delta}(\mathbf{k}_\alpha) = \mathbf{k}_\alpha \otimes \mathbf{k}_\alpha$;
(6) $\widetilde{S}([\mathbb{P}]) = \bigoplus_{r\geq 1} \bigoplus_{\alpha_1,\cdots,\alpha_r \neq 0} \mathbf{k}_{-\alpha_1-\cdots-\alpha_r} \mathfrak{m}^{t,r}_{\alpha_1,\cdots,\alpha_r} \circ \Delta^{t,r}_{\alpha_1,\cdots,\alpha_r}(\mathbb{P})[r]$ for $\mathbb{P} \neq 0$;
(7) $\widetilde{S}(\mathbf{k}_\alpha) = \mathbf{k}_{-\alpha}$.

We can endow $\widetilde{K}^{tw}_{G,w}$ with the different multiplication $\widetilde{\mathfrak{m}}^*$, comultiplication $\widetilde{\Delta}^*$ and antipode $\widetilde{S}^*$ structures by the following relations:

(1) $\widetilde{\mathfrak{m}}^*(\mathbb{P} \boxtimes \mathbb{Q}) = \widetilde{\mathfrak{m}}(\mathbb{P} \boxtimes \mathbb{Q})$;
(2) $\mathbf{k}_\alpha[\mathbb{P}] = v^{-(\alpha,\beta)}[\mathbb{P}]\mathbf{k}_\alpha \quad \text{for} \quad \mathbb{P} \in \mathcal{D}^b_{G_\beta,w}(\mathbb{E}_\beta)$;
(3) $k_\alpha k_\beta = k_{\alpha+\beta}$;
(4) $\widetilde{\Delta}^*([\mathbb{P}]) = \sum_{\alpha,\beta} (\Delta^{op}_{\alpha,\beta})^t([\mathbb{P}]) \cdot (1 \otimes \mathbf{k}_{-\beta})$;
(5) $\widetilde{\Delta}^*(\mathbf{k}_\alpha) = \mathbf{k}_\alpha \otimes \mathbf{k}_\alpha$;
(6) $\widetilde{S}^*([\mathbb{P}]) = \bigoplus_{r\geq 1} \bigoplus_{\alpha_1,\cdots,\alpha_r \neq 0} \mathfrak{m}^r_{\alpha_1,\cdots,\alpha_r} \circ (\Delta^{op}_{\alpha_1,\cdots,\alpha_r})^{tw,r}(\mathbb{P})[r]\mathbf{k}_{\alpha_1+\cdots+\alpha_r}$ for $\mathbb{P} \neq 0$;
(7) $\widetilde{S}^*(\mathbf{k}_\alpha) = \mathbf{k}_{-\alpha}$.

We denote $\widetilde{K}^{tw}_{G,w}$ with the above new structures by $\widetilde{K}^{tw,*}_{G,w}$ and then the basis is denoted by $\{k_\alpha[\mathbb{P}]^* \mid \alpha \in \mathbb{Z}[Q_0], [\mathbb{P}] \in \mathbf{B}\}$ in case causing ambiguity. The opposite of $\widetilde{\Delta}^*$ satisfies that

$$(\widetilde{\Delta}^*)^{op}([\mathbb{P}]) = \sum_{\alpha,\beta}(\Delta^t_{\alpha,\beta})([\mathbb{P}]) \cdot (\mathbf{k}_{-\beta} \otimes 1).$$

The inverse of $\widetilde{S}^*$ satisfies that

$$(\widetilde{S}^*)^{-1}([\mathbb{P}]) = \bigoplus_{r\geq 1} \bigoplus_{\alpha_1,\cdots,\alpha_r \neq 0} \mathbf{k}_{\alpha_1+\cdots+\alpha_r} \mathfrak{m}^{t,r}_{\alpha_1,\cdots,\alpha_r} \circ \Delta^{t,r}_{\alpha_1,\cdots,\alpha_r}(\mathbb{P})[r] \text{ for } \mathbb{P} \neq 0.$$

Definition 5.6. Given two Hopf algebras A and B, a skew-Hopf pairing of A and B is a $\mathbf{R}$-bilinear function $\varphi : A \times B \to \mathbf{R}$ such that

(1) $\varphi(1, b) = \varepsilon_B(b)$ and $\varphi(a, 1) = \varepsilon_A(a)$;
(2) $\varphi(a, bb') = \varphi(\Delta_A(a), b \otimes b')$;
(3) $\varphi(aa', b) = \varphi(a \otimes a', \Delta^{op}_B(b))$;
(4) $\varphi(\sigma_A(a), b) = \varphi(a, \sigma_B^{-1}(b))$.

Proposition 5.7. [11] Let (A, B, φ) be a skew-Hopf pairing. Then $A \otimes B$ is a Hopf algebra, called the Drinfeld double of (A, B, φ).

The definition in 5.1 can be extended to define a bilinear form $\varphi : \widetilde{K}^{tw}_{G,w} \times \widetilde{K}^{tw,*}_{G,w} \to \mathbf{R}$ by setting

$$\varphi(\mathbf{k}_\alpha[\mathbb{P}], \mathbf{k}_\beta[\mathbb{Q}]^*) = v^{-(\alpha,\beta)-(\alpha',\beta)+(\alpha,\beta')}\{\mathbb{P}, \mathbb{Q}\}$$

for $\mathbb{P} \in \mathcal{D}^b_{G_{\alpha'},w}(\mathbb{E}_{\alpha'})$ and $\mathbb{Q} \in \mathcal{D}^b_{G_{\beta'},w}(\mathbb{E}_{\beta'})$.

Theorem 5.8. *The bilinear form φ is a skew-Hopf pairing.*

The proof is a direct consequence of Lemma 5.1 and Corollary 5.2 and very similar to [34, Proposition 5.3]. We omit it.

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