# ON THE DENSITY OF ABELIAN $\ell$-EXTENSIONS

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Abstract. We derive an asymptotic formula which counts the number of abelian extensions of prime degrees over rational function fields. Specifically, let $\ell$ be a rational prime and $K$ a rational function field $\mathbb{F}_q(t)$ with $\ell \nmid q$. Let $\mathrm{Disc}_f\,(F/K)$ denote the finite discriminant of $F$ over $K$. Denote the number of abelian $\ell$-extensions $F/K$ with $\deg\,(\mathrm{Disc}_f(F/K)) = (\ell-1)\alpha n$ by $a_\ell(n)$, where $\alpha = \alpha(q,\ell)$ is the order of $q$ in the multiplicative group $(\mathbb{Z}/\ell\mathbb{Z})^\times$. We give a explicit asymptotic formula for $a_\ell(n)$. In the case of cubic extensions with $q \equiv 2 \pmod 3$, our formula gives an exact analogue of Cohn's classical formula.

## 1. Introduction

In arithmetic statistics, the problem of counting number fields is of particularly interest. Let $N_n(X)$ denote the number of isomorphism classes of number fields of degree $n$ over $\mathbb{Q}$ having absolute discriminant at most $X$. It is conjectured that $N_n(X)/X$ tends to a finite limit as $X$ tends to infinity and it is positive for $n > 1$. This conjecture is trivial for $n = 1$ and it is well-known for $n = 2$. For the degree $n = 3$ it is a theorem of Davenport and Heilbronn [5]. In the past decade, Bhargava [1, 2] proved this conjecture for $n = 4$ and 5.

Prompted by the work of Davenport and Heilbronn [5] on the density of cubic fields. Cohn showed that the cyclic cubic fields are rare compared to all cubic fields over $\mathbb{Q}$. More precisely, let $G$ be a fixed finite abelian group of order $m$ and let $F$ range over all abelian number fields with Galois group $\mathrm{Gal}\,(F/\mathbb{Q}) = G$. Denote by $\mathrm{Disc}\,(F/\mathbb{Q})$ the absolute value of the discriminant of $F$ over $\mathbb{Q}$ and define $N_G(X)$ to be the number of abelian number fields $F$ with $\mathrm{Disc}\,(F/\mathbb{Q}) \le X$. When $G = \mathbb{Z}/3\mathbb{Z}$, Cohn [3] proved that

$$N_G(X) \sim \frac{11\pi}{72\sqrt{3}\zeta(2)} \prod_{p\equiv 1(6)} \frac{(p+2)\,(p-1)}{p\,(p+1)}\sqrt{X}, \quad \text{as } X \to \infty, \tag{1}$$

where the product is taking over all primes $p \equiv 1 \pmod 6$. For arbitrary given finite abelian groups $G$, an asymptotic formula of $N_G(X)$ has been worked out by Mäki [7]. In

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particular, fix $\ell$ a prime and $G = \mathbb{Z}/\ell\mathbb{Z}$, there is a constant $c > 0$ such that (cf. Wright [9, Theorem I.2])

$$N_G(X) \sim c \cdot X^{\frac{1}{\ell-1}}, \quad \text{as } X \to \infty. \tag{2}$$

In this paper, we prove very precise versions of the asymptotic formulas (1), (2) for function fields. Let $K$ be a rational function field $\mathbb{F}_q(t)$ over the finite field $\mathbb{F}_q$. Fix a rational prime $\ell \nmid q$. Let $F$ be an abelian $\ell$-extension of $K$ and denote by $\mathcal{O}_F$ the integral closure of $A = \mathbb{F}_q[t]$ in $F$. Let $\mathrm{Disc}_f\,(F/K)$ denote the finite discriminant of $F$ over $K$ which means the discriminant of $\mathcal{O}_F$ over $A$. Note that $\mathrm{Disc}_f\,(F/K)$ is a $(\ell-1)$-th power for some square-free polynomial in $A$ since all ramified primes are totally tamely ramified in $F/K$. Denote $\alpha = \alpha(q,\ell)$ to be the order of $q$ in the multiplicative group $(\mathbb{Z}/\ell\mathbb{Z})^\times$. Then the degree of ramified primes in $F$ must be divided by $\alpha$. Adapting an idea of Cohn, we shall study the partial Euler product

$$f(s) := \prod_{\alpha|\deg(P)} \left(1 + (\ell-1)q^{-\deg(P)s}\right),$$

where the product is taking over all finite primes $P$ of $K$ such that $\alpha \mid \deg(P)$.

We will prove that $f(s)$ converges absolutely for $\mathrm{Re}\, s > 1$ and is analytic in the region $\{s \in B : \mathrm{Re}\, s = 1\}$ except for a pole of order $w := (\ell-1)/\alpha$ at $s = 1$ where $B := \{s \in \mathbb{C} : -\pi/\log q^\alpha \le \mathrm{Im}\, s < \pi/\log q^\alpha\}$. The last statement implies that

$$r(K,\ell) := \lim_{s\to 1}(s-1)^w f(s) > 0.$$

Then we have the following main result

**Theorem 1.** *Let $\ell$ be a rational prime and $K = \mathbb{F}_q(t)$ with $\ell \nmid q$. Write $\alpha = \alpha(q,\ell)$ for the order of $q$ in the multiplicative group $(\mathbb{Z}/\ell\mathbb{Z})^\times$ and set $w = (\ell-1)/\alpha$. Denote the number of abelian $\ell$-extensions $F/K$ with $\deg(\mathrm{Disc}_f(F/K)) = (\ell-1)\alpha n$ by $a_\ell(n)$. Then, for any $\epsilon > 0$, we have*

$$a_\ell(n) = \begin{cases} 2r(K,\ell)\log q \cdot q^{\alpha n} + O(q^{(\frac{\alpha}{2}+\epsilon)n}) & \text{if } w = 1, \\ \frac{2r(K,\ell)\log^w q^\alpha}{(\ell-1)(w-1)!} q^{\alpha n} P(n) + O(q^{(\frac{\alpha}{2}+\epsilon)n}) & \text{if } w > 1, \end{cases}$$

*where $r(K,\ell)$ is defined as above and $P(X) \in \mathbb{R}[X]$ is a monic polynomial of degree $w-1$.*

**Remark 1.** Note that, as in formula (2), the growth order of the number $a_\ell(n)$ is a $(\ell-1)$-root of the number of polynomials of degree $(\ell-1)\alpha n$ when $w = 1$.

**Remark 2.** By our much sharper Wiener-Ikehara Tauberian theorem (see Appendix for a function field version), the constants $r(K,\ell)$ and $P(X)$ can be explicitly determined (see Section 2 for details).

In the case of cyclic cubic fields, we derive explicit formulas for $r(K,\ell)$ and $P(X)$ as the following

**Theorem 2.** *Let $K=\mathbb{F}_q(t)$ with $3\nmid q$ and $a_3(n)$ the number of cyclic cubic extensions $F/K$ with $\deg(\mathrm{Disc}_f(F/K))=2\alpha n$ where $\alpha=1$ if $q\equiv 1 \pmod 3$ or $2$ otherwise. Let $P$ denote the finite primes of $K$ and $\zeta_A(s)$ is the zeta function of $A$. Then, for any $\epsilon>0$, we have*

(i) *If $q\equiv 1 \pmod 3$, let*

$$g(s)=\prod_P\left(1-3q^{-2\deg Ps}+2q^{-3\deg Ps}\right)\quad \textit{for } \mathrm{Re}\, s>1/2,$$

*where the product is taking over all finite primes $P$ of $K$. Then one has*

$$a_3(n)=g(1)q^n\left(n+1+\frac{g'(1)}{g(1)\log q}\right)+O(q^{(\frac{1}{2}+\epsilon)n})\quad \textit{as } n\to\infty.$$

(ii) *If $q\equiv 2 \pmod 3$, one has*

$$a_3(n)=\frac{1}{\zeta_A(2)}\left[\prod_{\deg(P):even}\frac{\left(q^{\deg(P)}+2\right)\left(q^{\deg(P)}-1\right)}{q^{\deg(P)}\left(q^{\deg(P)}+1\right)}\right]q^{2n}+O(q^{(1+\epsilon)n})\quad \textit{as } n\to\infty.$$

Note that the asymptotic formula (ii) of Theorem 2 is a function filed analogue of Cohn's result. This occurs because, when $q\equiv 2 \pmod 3$, the infinite prime $\infty=1/t$ is unramified in cyclic cubic fields which happen to be the same situation as for the number field $\mathbb{Q}$. We are also able to treat the case $q\equiv 1 \pmod 3$ in (i) of Theorem 2 with a bigger main term and exact second order term.

We now briefly describe the contents of this paper. In the next section, we use a version of Tauberian theorem (Theorem 4) to prove Theorem 1. In Section 3, we adapt an idea of Cohn to prove Theorem 2. In the appendix, we derive the needed function field version of Wiener-Ikehara Tauberian theorem.

## 2. Counting Abelian $\ell$-extensions

We fix the following notations in this paper:

$$
\begin{aligned}
\ell &: \text{a fixed rational prime.}\\
K &: \text{the rational function field } \mathbb{F}_q(t) \text{ of the characteristic } p \neq \ell.\\
\alpha = \alpha(q,\ell) &: \text{the order of } q \text{ in the multiplicative group } (\mathbb{Z}/\ell\mathbb{Z})^\times.\\
A &: \text{the polynomial ring } \mathbb{F}_q[t].\\
\infty &: \text{the infinite place } 1/t.\\
\mathcal{O}_\infty &: \text{the valuation ring of } \infty.\\
P &: \text{a finie prime (place) of } K.\\
K_P &: \text{the completion of } K \text{ at } P.\\
\mathcal{O}_P^\times &: \text{the group of } P\text{-units in } K_P.\\
\mathbb{A}_K^\times &: \text{the idèle group of } K.\\
F &: \text{an abelian } \ell\text{-extension over } K.\\
\mathrm{Disc}_f\,(F/K) &: \text{the finite discriminant of } F \text{ over } K \text{ identifies as a polynomial in } A.
\end{aligned}
$$

Note that we always have $\mathrm{Disc}_f\,(F/K) = D^{\ell-1}$ for some square-free polynomial $D = D(F) \in A$ since all ramified primes are totally tamely ramified in $F/K$. From class field theory, we know that each abelian $\ell$-extension $F$ over $K$ corresponds exactly to $\ell - 1$ surjective homomorphisms

$$\phi : \mathbb{A}_K^\times/K^\times \to \mathrm{Gal}\,(F/K) \cong \mathbb{Z}/\ell\mathbb{Z}.$$

Recall that the image of $\mathcal{O}_P^\times$ is equal to the inertia group of $P$ in $\mathrm{Gal}\,(F/K)$ by local class field theory. Let $K_\infty^+ = \{1/t\} \times (1 + 1/t\mathcal{O}_\infty)$ be the direct product of the cyclic group $\{1/t\}$ generated by $1/t$ and the pro-$p$ group $(1 + 1/t\mathcal{O}_\infty)$. Since $\mathbb{A}_K^\times$ can be written as a direct product of $K^\times$ and $\left(\prod_P \mathcal{O}_P^\times \times K_\infty^+\right)$, the number of abelian $\ell$-extensions $F$ over $K$ with $\mathrm{Disc}_f\,(F/K) = D^{\ell-1}$ is equal to

$$\frac{2}{\ell-1} \cdot \#\{\phi : \prod_{P|D} \mathcal{O}_P^\times \to \mathbb{Z}/\ell\mathbb{Z} : \text{the restriction of } \phi \text{ to } \mathcal{O}_P^\times \text{ is surjective for all } P \mid D\}.$$

Since the kernel of the canonical map $\mathcal{O}_P^\times \to (\mathcal{O}_P/P)^\times$ is a pro-$p$ group and $p \neq \ell$, a map $\mathcal{O}_P^\times \to \mathbb{Z}/\ell\mathbb{Z}$ must factor through $(\mathcal{O}_P/P)^\times \to \mathbb{Z}/\ell\mathbb{Z}$. As the restriction of $\phi$ from $\prod_{P|D} \mathcal{O}_P^\times$ to $\mathcal{O}_P^\times$ is surjective, we note that $\alpha(q,\ell)$ must divide $\deg(P)$. Combining these discussions, we conclude that the number of abelian $\ell$-extensions $F$ over $K$ with $D(F) = P_1 \cdots P_m$ and $\alpha(q,\ell) \mid \deg(P_i)$ for all $i$ is equal to $(\ell-1)^{m-1}$.

We are interested in the partial Euler product

$$f(s) := \prod_{\alpha \mid \deg(P)} \left(1 + (\ell-1) q^{-\deg(P)s}\right) \quad \text{for } \operatorname{Re} s > 1.$$

Write $f(s) = \sum b_{\alpha n} q^{-\alpha n s}$, then $a_\ell(n) = 2b_{\alpha n}/(\ell-1)$ for all $n > 0$, where $a_\ell(n)$ is the number of abelian $\ell$-extensions $F$ over $K$ with $\deg(\mathrm{Disc}_f(F/K)) = (\ell-1)\alpha n$. Let $\zeta_A(s)$ be the zeta function of $A$ which is defined by

$$\begin{aligned} \zeta_A(s) : &= \prod_P \left(1 - q^{-\deg(P)s}\right)^{-1} \quad \text{for } \operatorname{Re} s > 1 \\ &= \frac{1}{1-q^{1-s}}. \end{aligned}$$

We have

**Lemma 3.** *Let $\zeta_{A_\alpha}(s)$ be the zeta function of $A_\alpha = \mathbb{F}_{q^\alpha}[t]$. Then*

$$f(s) = \zeta_{A_\alpha}^{\frac{\ell-1}{\alpha}}(s) \cdot g(s) \quad \textit{for } \operatorname{Re} s > 1,$$

*where $g(s)$ is an analytic function for $\operatorname{Re} s \geq 1$ and non-vanishing at $s = 1$.*

*Proof:* For each $d \mid \alpha$, we consider the infinite product

$$\mathcal{L}_d(s) := \prod_{P:(\alpha,\deg(P))=d} \left(1 - q^{-\frac{\alpha \deg(P)}{d} s}\right)^{-d} \quad \text{for } \operatorname{Re} s > \tfrac{d}{\alpha},$$

where the product is taking over the finite primes $P$ in $K$ such that $\gcd(\alpha, \deg(P)) = d$. Then we know that

$$\zeta_{A_\alpha}(s) = \prod_{d \mid \alpha} \mathcal{L}_d(s) = \prod_{\alpha \mid \deg(P)} \left(1 - q^{-\deg(P)s}\right)^{-\alpha} \prod_{\substack{d \mid \alpha \\ d \neq \alpha}} \mathcal{L}_d(s) \quad \text{for } \operatorname{Re} s > 1.$$

Let $w = (\ell-1)/\alpha$, then

$$f(s) = \zeta_{A_\alpha}^{w}(s) \prod_{\alpha \mid \deg(P)} \left[\left(1 + (\ell-1) q^{-\deg(P)s}\right)\left(1 - q^{-\deg(P)s}\right)^{\ell-1}\right] \prod_{\substack{d \mid \alpha \\ d \neq \alpha}} \mathcal{L}_d^{-w}(s) \quad \text{for } \operatorname{Re} s > 1.$$

We know that $\mathcal{L}_d^{-1}(s)$ is analytic and non-vanishing at $s = 1$ since $\mathcal{L}_d^{-1}(1) \geq \zeta_{A_\alpha}^{-1}(2) > 0$ for all $d \neq \alpha$ (cf. [4, Section 2]). On the other hand, one computes

$$\begin{aligned} h(s) &:= \prod_{\alpha \mid \deg(P)} \left[ \left(1 + (\ell-1) q^{-\deg(P)s}\right)\left(1 - q^{-\deg(P)s}\right)^{\ell-1} \right] \\ &= \prod_{\alpha \mid \deg(P)} \left(1 + c_2 q^{-2\deg(P)s} + \cdots + c_\ell q^{-(\ell)\deg(P)s}\right) \quad \text{for } \operatorname{Re} s > 1, \end{aligned}$$

where

$$c_i = \begin{cases} (-1)^i \binom{\ell-1}{i} + (-1)^{i-1}(\ell-1)\binom{\ell-i}{i-1} & \text{if } i < \ell, \\ (-1)^{\ell-1}(\ell-1) & \text{if } i = \ell. \end{cases}$$

It is easy to check that $h(s)$ is analytic for $\operatorname{Re} s > 1/2$ and non-vanishing at $s = 1$. This completes the proof of the lemma. $\square$

Before we prove the main theorem, we recall a function field version of Wiener-Ikehara Tauberian theorem which we will prove in the appendix.

**Theorem 4.** *Let $f(u) = \sum_{n \geq 0} b_n u^n$ with $b_n \in \mathbb{C}$ be convergent in the region $\{u \in \mathbb{C} : |u| < q^{-a}\}$. Assume that in the domain of convergence $f(u) = g(u)(u - q^{-a})^{-w} + h(u)$ holds, where $w \in \mathbb{N}$ and $h(u)$, $g(u)$ are analytic functions in $\{u \in \mathbb{C} : |u| < q^{-a+\delta}\}$ for some $\delta > 0$, $g(q^{-a}) \neq 0$. Then, for any $\epsilon > 0$, we have*

$$b_n = q^{an} Q(n) + O(q^{(a-\delta+\epsilon)n}),$$

*where $Q(X) \in \mathbb{C}[X]$ is the polynomial of degree $w-1$ given by*

$$Q(X) := \sum_{j=0}^{w-2} \frac{g^{(j)}(q^{-a})(-1)^{w-j-1}}{j!(w-j-1)!} \left[ \prod_{\ell=1}^{w-j-1} (X + \ell) \right] \cdot q^{a(w-j)} + \frac{g^{(w-1)}(q^{-a})}{(w-1)!} q^a.$$

*In particular, if we write $Q(X) = \sum_{i=1}^{w-1} c_i X^{w-i}$, then we have*

$$c_1 = (-1)^{-w} \frac{g(q^{-a}) q^{aw}}{\Gamma(w)} \quad \text{and} \quad c_2 = \frac{(-1)^{-w}}{\Gamma(w-1)} \cdot q^{aw} \cdot \left[ g(q^{-a})(\frac{w}{2}) - \frac{g'(q^{-a})}{q^a} \right].$$

We are now ready to prove Theorem 1. Let $w = (\ell - 1)/\alpha$ and define the region $B := \{s \in \mathbb{C} : -\pi/\log q^\alpha \leq \operatorname{Im} s < \pi/\log q^\alpha\}$. From Lemma 3, we have proved that $f(s) = \zeta_{A_\alpha}^{\frac{\ell-1}{\alpha}}(s) \cdot g(s)$ for $\operatorname{Re} s > 1$, where

$$g(s) = \prod_{\alpha \mid \deg(P)} \left[ \left(1 + (\ell-1) q^{-\deg(P)s}\right)\left(1 - q^{-\deg(P)s}\right)^{\ell-1} \right] \prod_{\substack{d \mid \alpha \\ d \neq \alpha}} \mathcal{L}_d^{-w}(s)$$

is an analytic function for $\operatorname{Re} s \geq 1/2$ and non-vanishing at $s = 1$. Note that $f(s)$ converges absolutely for $\operatorname{Re} s > 1$ and is analytic in the region $\{s \in B : \operatorname{Re} s = 1\}$ except for a pole of order $w$ at $s = 1$. The last statement implies that

$$r(K, \ell) := \lim_{s \to 1} (s-1)^w f(s) = \frac{g(1)}{\alpha \log q} > 0.$$

Replacing the variable $s$ by $-\frac{\log u}{\log q^\alpha}$, define the function $Z_f(u) := f\left(-\frac{\log u}{\log q^\alpha}\right) = \sum b_{\alpha n} u^n$. Then we have $Z_f(u) = Z_g(u)\left(u - q^{-\alpha}\right)^{-w}$ where $Z_g(u)$ is an analytic function in the region $\{u \in \mathbb{C} : |u| < q^{-\alpha/2}\}$, and

$$\begin{aligned} Z_g\left(q^{-\alpha}\right) &= \lim_{u \to q^{-\alpha}} \left(u - q^{-\alpha}\right)^w Z_f(u) \\ &= \lim_{s \to 1} \frac{\left(q^{-\alpha s} - q^{-\alpha}\right)^w}{(s-1)^w} (s-1)^w f(s) \\ &= \left(-\frac{\log q^\alpha}{q^\alpha}\right)^w r(K, \ell). \end{aligned}$$

Applying Theorem 4, for any $\epsilon > 0$, as $n \to \infty$,

$$b_{\alpha n} = q^{\alpha n} Q(n) + O(q^{(\frac{\alpha}{2}+\epsilon)n}), \tag{3}$$

where $Q(X) \in \mathbb{R}[X]$ is a polynomial of degree $w - 1$ given by

$$Q(X) = \sum_{j=0}^{w-2} \frac{Z_g^{(j)}(q^{-a})(-1)^{w-j-1}}{j!(w-j-1)!} \left[\prod_{\ell=1}^{w-j-1} (X+\ell)\right] \cdot q^{a(w-j)} + \frac{Z_g^{(w-1)}(q^{-a})}{(w-1)!} q^a.$$

Note that the leading coefficient $c_1$ of $Q(X)$ is given by

$$c_1 = \frac{r(K, \ell)(\log q^\alpha)^w}{(w-1)!}. \tag{4}$$

Combining (3), (4) and $a_\ell(n) = 2b_{\alpha n}/(\ell - 1)$, we complete the proof of Theorem 1.

## 3. Counting cyclic cubic extensions

In this section, we derive an explicit formula of $r(K, \ell)$ for the case of cyclic cubic extensions. Moreover, we give the second order term for the number $a_3(n)$ when $q \equiv 1 \pmod 3$. In Lemma 3, we have proved that $f(s)$ can be written as a product of the zeta function of $A_\alpha$ and an analytic function $g(s)$. We will write down $g(s)$ explicitly and compute its derivative to obtain Theorem 2. In the general case, it is also possible to compute $r(K, \ell)$ and small order terms for $a_\ell(n)$ using the same method.

In the case of $q \equiv 1 \pmod 3$, we consider the partial Euler product

$$f(s) := \prod_P \left(1 + 2q^{-\deg(P)s}\right) \quad \text{for } \operatorname{Re} s > 1.$$

Write $f(s) = \sum b_n q^{-ns}$, then $a_3(n) = b_n$ for all $n > 0$. Recall that $\zeta_A(s)$ is the zeta function of $A$ which is defined by

$$\zeta_A(s) := \prod_P \left(1 - q^{-\deg(P)s}\right)^{-1} \quad \text{for } \operatorname{Re} s > 1.$$

Then one computes that $f(s) = \zeta_A^2(s) g(s)$ where

$$g(s) = \prod_P \left(1 - 3q^{-2\deg(P)s} + 2q^{-3\deg(P)s}\right) \quad \text{for } \operatorname{Re} s > 1/2$$

is analytic for $\operatorname{Re} s > 1/2$ and non-vanishing at $s = 1$.

Replacing the variable $s$ by $-\frac{\log u}{\log q}$, define the function $Z_f(u) := f\left(-\frac{\log u}{\log q}\right) = \sum b_n u^n$. Then $Z_f(u) = Z_g(u)\left(u - q^{-1}\right)^{-2}$ where

$$Z_g(u) = q^{-2} \prod_P \left(1 - 3u^{2\deg(P)} + 2u^{3\deg(P)}\right)$$

is an analytic function in the region $\{u \in \mathbb{C} : |u| \le q^{-1}\}$. Applying Theorem 4 to $Z_f(u)$ and $a_3(n) = b_n$, there is a constant $\delta < 1$ such that

$$a_3(n) = q^{2n}\left(q^2 Z_g(q^-1)n + q^2 Z_g(q^{-1}) - qZ_g'(q^{-1})\right) + O\left(q^{2n\delta}\right) \quad \text{as } n \to \infty.$$

In the case of $q \equiv 2 \pmod 3$, we consider the partial Euler product

$$f(s) := \prod_{2|\deg(P)} \left(1 + 2q^{-\deg(P)s}\right) \quad \text{for } \operatorname{Re} s > 1.$$

We now write $f(s) = \sum b_{2n} q^{-2ns}$, then $a_3(n) = b_{2n}$ for all $n > 0$. Let $\zeta_{A_2}(s)$ be the zeta function of $A_2 = \mathbb{F}_{q^2}[t]$. One has

$$\zeta_{A_2}(s) = \prod_{\deg(P):even} \left(1 - q^{-\deg(P)s}\right)^{-2} \prod_{\deg(Q):odd} \left(1 - q^{-2\deg(Q)s}\right)^{-1}$$

where $P$ are the primes in $A$ of odd degree and $Q$ are the primes in $A$ of odd degree. Then we have the identity

$$f(s) = \zeta_{A_2}(s)\zeta_A^{-1}(2s) \prod_{\deg(P):even} \frac{\left(q^{\deg(P)s} + 2\right)\left(q^{\deg(P)s} - 1\right)}{q^{\deg(P)s}(q^{\deg(P)s} + 1)} \quad \text{for } \operatorname{Re} s > 1.$$

We now replace the variable $s$ by $-\frac{\log u}{\log q^2}$ and define the function $Z_f(u) := f\left(-\frac{\log u}{\log q^2}\right) = \sum b_{2n} u^n$. Then $Z_f(u) = Z_g(u)\left(u - q^{-2}\right)^{-1}$ where

$$Z_g(u) = -q^{-2}(1-qu) \prod_{\deg(P):even} \frac{\left(u^{\frac{-\deg(P)s}{2}}+2\right)\left(u^{\frac{-\deg(P)s}{2}}-1\right)}{u^{\frac{-\deg(P)s}{2}}\left(u^{\frac{-\deg(P)s}{2}}-1\right)}$$

is an analytic function in the region $\{u \in \mathbb{C} : |u| \leq q^{-2}\}$. From Theorem 4 and $a_3(n) = b_{2n}$, there is a $\delta < 1$ such that

$$a_3(n) = \frac{1}{\zeta_A(2)}\left[\prod_{\deg(P):even} \frac{\left(q^{\deg(P)}+2\right)\left(q^{\deg(P)}-1\right)}{q^{\deg(P)}\left(q^{\deg(P)}+1\right)}\right] q^{2n} + O\left(q^{2n\delta}\right) \quad \text{as } n \to \infty.$$

This completes the proof of Theorem 2.

## Appendix

In the appendix, we derive a function field version of Wiener-Ikehara Tauberian Theorem with a main term much sharper then the standard one [8, Corollary of Theorem 17.4]. Before we prove the main theorem, we first record some lemmas.

**Lemma 5.** *Let $\Gamma(t)$ be Gamma function. Then we have*

$$\frac{n!n^t}{\prod_{i=0}^{n}(t+i)} = \Gamma(t)\left[1 - \left(\frac{t^2+t}{2}\right)\frac{1}{n} + O(\frac{1}{n^2})\right] \quad \text{as } n \to \infty$$

*for all complex numbers $t$, except the non-positive integers.*

*Proof.* Just use Stirling's formula. ∎

**Lemma 6.** *Let $\delta, a > 0$ be fixed real numbers and $q \geq 2$. Assume $w < 1$ and $w \in \mathbb{R} - \mathbb{Z}$. Then we have*

$$\int_{q^{-a}}^{q^{-a(1-\delta)}} \frac{du}{u^{n+1}\cdot(u-q^{-a})^w} = \frac{2\pi i \cdot e^{-w\pi i}}{1-e^{-2\pi i w}} \cdot \frac{1}{\Gamma(w)} \cdot \left[1 + \left(\frac{w^2-w}{2}\right)\frac{1}{n} + O(\frac{1}{n^2})\right] q^{a(n+w)} n^{w-1} \quad \text{as } n \to \infty.$$

*Here $(u-q^{-a})^w := e^{w\log(u-q^{-a})}$, analytic branch of $\log u$ is fixed with $0 < \operatorname{Arg} u < 2\pi$. Thus $\log(u-q^{-a})$ is an analytic function defined on $\mathbb{C} - [q^{-a}, \infty)$.*

*Proof.* Substituting $q^{-a}u$ for $u$, the integral changes to

$$q^{a(n+w)} \cdot \int_1^{q^{a\delta}} \frac{du}{u^{n+1} \cdot (u-1)^w} = q^{a(n+w)} \cdot \left[ \int_0^{\infty} \frac{du}{(u+1)^{n+1} \cdot u^w} - \int_{q^{a\delta}-1}^{\infty} \frac{du}{(u+1)^{n+1} \cdot u^w} \right]$$
$$:= q^{a(n+w)} (I_1 + I_2).$$

The integral $I_2$ simply equals to $O\left(\frac{q^{-a\delta n}}{n}\right)$. The integral $I_1$ can be checked to be

$$\frac{2\pi i}{1 - e^{-2\pi i w}} \cdot \mathrm{Res}\left( \frac{u^{-w}}{(u+1)^{n+1}}, -1 \right).$$

Furthermore, we have

$$\mathrm{Res}\left( \frac{u^{-w}}{(u+1)^{n+1}}, -1 \right) = \frac{1}{n! \cdot n^w} \left( \prod_{i=0}^{n-1} (-w-i)(w+n) \cdot (-1)^{-w-n} \right) \cdot \frac{n^w}{n+w}$$
$$= \frac{e^{-\pi i w}}{\Gamma(w)} \left[ 1 + \left( \frac{w^2 - w}{2} \right) \frac{1}{n} + O(\frac{1}{n^2}) \right] n^{w-1}.$$

This gives the desired identity.

$\square$

Let $q \geq 2$, our Tauberian Theorem is

**Theorem 7.** *Let* $f(u) := \sum_{n \geq 0} a_n u^n$ *with the numbers* $a_n \in \mathbb{C}$ *for all* $n$*, be convergent in*

$$\{u \in \mathbb{C} : |u| < q^{-a}\}$$

*for a fixed real number* $a > 0$*. Assume that in the above domain*

$$f(u) = g(u)(u - q^{-a})^{-w} + h(u)$$

*holds, where* $w > 0$ *and* $h(u)$*,* $g(u)$ *are analytic functions in* $\{u \in \mathbb{C} : |u| < q^{-a+\delta}\}$ *for some* $\delta > 0$*. Then*

$$a_n \sim (-1)^{-w} \frac{g(q^{-a}) q^{aw}}{\Gamma(w)} \cdot q^{an} n^{w-1} \quad \text{as } n \to \infty.$$

*Moreover, if* $w$ *is a positive integer, then, for any* $\epsilon > 0$*, one has*

$$a_n = Q(n) \cdot q^{an} + O(q^{(a-\delta+\epsilon)n}), \text{ as } n \to \infty, \tag{5}$$

*where* $Q(X) \in \mathbb{C}[X]$ *is the polynomial of degree* $w-1$ *given by*

$$Q(X) := \sum_{j=0}^{w-2} \frac{g^{(j)}(q^{-a})(-1)^{w-j-1}}{j!(w-j-1)!} \left[ \prod_{\ell=1}^{w-j-1} (X + \ell) \right] \cdot q^{a(w-j)} + \frac{g^{(w-1)}(q^{-a})}{(w-1)!} q^a.$$

*In particular, if we write* $Q(X) = \sum_{i=1}^{w-1} c_i X^{w-i}$*, then we have*

$$c_1 = (-1)^{-w} \frac{g(q^{-a})q^{aw}}{\Gamma(w)} \quad \textit{and} \quad c_2 = \frac{(-1)^{-w}}{\Gamma(w-1)} \cdot q^{aw} \cdot \left[ g(q^{-a})(\frac{w}{2}) - \frac{g'(q^{-a})}{q^a} \right].$$

*When* $0 < w < 1$*, we also have*

$$a_n = c_1 q^{an} n^{w-1} + c_2 q^{an} n^{w-2} + O\left(q^{an} n^{w-3}\right) \quad \textit{as } n \to \infty.$$

The proof will be divided into three steps.
1. If $w$ is a positive integer, then we adapt an idea of Rosen [8, Theorem 17.4]. But we remove the condition that the pole lie at $u = q^{-1}$. In this way, we are able to obtain the polynomial of $Q(n)$ precisely for $w \geq 1$.
2. If $0 < w < 1$, then we use a keyhole shape contour to carry out needed estimations.
3. If $w > 0$ is not a positive integer, then we proceed by induction on $\lfloor w \rfloor$, since we have proved the case $\lfloor w \rfloor = 0$ in step 2.

**Step** 1:

*Proof.* The case $w = 1$ and the equation (5) is covered in [8, Theorem 17.1] and [8, Theorem 17.4]; we give here exact formula for $c_i$ for $1 \leq i \leq w$ when $w > 1$. We take a $0 < \delta < 1$ such that $g(u)$ and $h(u)$ are analytic on the disc : $\{s \in \mathbb{C} : |u| \leq q^{-a(1-\delta)}\}$. Let $C$ be the boundary of this disc oriented counterclockwise and $C_\epsilon$ a small circle about $u = 0$ oriented clockwise. We have

$$\frac{1}{2\pi i} \oint_{C_\epsilon + C} \frac{f(u)}{u^{n+1}} du = -a_n + \frac{1}{2\pi i} \oint_C \frac{f(u)}{u^{n+1}} du.$$

Observe that

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{C_\epsilon + C} \frac{f(u)}{u^{n+1}} du =& \operatorname{Res}\left( \frac{g(u)}{(u - q^{-a})^w u^{n+1}}, q^{-a} \right) \\
=& \operatorname{Res}\left( \sum_{j=1}^{\infty} \frac{g^{(j)}(q^{-a})}{j!(u - q^{-a})^{w-j} u^{n+1}}, q^{-a} \right) \\
=& \frac{g^{(w-1)}(q^{-a})}{(w-1)!} q^{a(n+1)} + \sum_{j=0}^{w-2} \frac{g^{(j)}(q^{-a})(-1)^{w-j-1}}{j!(w-j-1)!} \left[ \prod_{\ell=1}^{w-j-1} (n+\ell) \right] \cdot q^{a(n+w-j)}.
\end{aligned}$$

and for any $\epsilon > 0$,

$$\frac{1}{2\pi i} \oint_C \frac{f(u)}{u^{n+1}} du \ll q^{(a-\delta+\epsilon)N}.$$

The proof is therefore complete for arbitrary positive integer $w > 1$. ∎

**Step** 2:

We use the contour which consists of a small circle $C_0$ with the center at the origin and a keyhole contour (cf. [4, Section 3] for more details). The keyhole contour consists of a small circle $C_\epsilon$ about the $q^{-a}$ of radius $\epsilon$, extending to a line segment $\gamma_1$ parallel and close to the positive real axis but not touching it, to an almost full circle $C_1$, returning to a line segment parallel $\gamma_2$, close, and below the positive real axis in the negative sense, returning to the small circle. Therefore, for each $n \in \mathbb{N}$, one has

(a) $a_n = \frac{1}{2\pi i}\oint_{C_1+\gamma_1+C_\epsilon+\gamma_2} \frac{f(u)}{u^{n+1}} du.$
(b) $\int_{C_1} \frac{f(u)}{u^{n+1}} du = o\left(\frac{q^{an}}{n^{2-w}}\right)$, as $n \to \infty$.
(c) $\lim_{\epsilon\to 0^+} \displaystyle\int_{C_\epsilon} \frac{f(u)}{u^{n+1}} du = 0.$
(d) For all $0 < w < 1$, we have

$$\begin{aligned}\lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \cdot \int_{\gamma_1+\gamma_2} \frac{f(u)}{u^{n+1}} du =& \frac{g(q^{-a})(-a)^{1-w} q^{aw}}{\Gamma(w)} \left(\frac{q^{an}}{(an)^{1-w}}\right) \\ & - \frac{(-1)^{-w}}{\Gamma(w-1)} q^{aw} \cdot \left[ g(q^{-a})(\frac{w}{2}) - \frac{g'(q^{-a})}{q^a}\right] \left(\frac{q^{an}}{n^{2-w}}\right) \\ & + O(\frac{q^{an}}{n^{3-w}}), \text{ as } n \to \infty.\end{aligned}$$

Combining (a)∼(d), we arrive at this theorem for case $0 < w < 1$.

**Step** 3:

*Proof.* We proceed by induction on $\lfloor w \rfloor$. The case $\lfloor w \rfloor = 0$ has been proved. Assume that Theorem 7 holds for all $\lfloor w \rfloor \le N$. Suppose $\lfloor w \rfloor = N + 1$, we consider the following function

$$\begin{aligned}\hat{f}(u) :=& f(u) \cdot (u - q^{-a}) \\ =& g(u) \cdot (u - q^{-a})^{-(w-1)} + h(u) \cdot (u - q^{-a}),\end{aligned}$$

We now write $\hat{f}(u) = \sum_{n\ge 0} b_n u^n$, where $b_n = -q^{-a} a_n + a_{n-1}$ for all $n \ge 0$ and set $a_{-1} = 0$. Note that $\hat{f}(u)$ is still a meromorphic function on

$$\{u \in \mathbb{C} - [q^{-a}, \infty) : |u| < q^{-a(1-\delta')}\}$$

for some $\delta' > 0$ and $\lim_{u \to q^{-a}} (u - q^{-a})^{w-1} \cdot \hat{f}(u) = g(q^{-a}) \neq 0$. Solving $a_n$ from $b_n$, we derive that

$$a_n = -\sum_{k=0}^{n} q^{a(k+1)} \cdot b_{n-k} \text{ for } n \geq 0. \tag{6}$$

Since $\lim_{r \to q^{-a}} (u - q^{-a})^{w-1} \cdot \hat{f}(u) \neq 0$ and $\lfloor w-1 \rfloor \leq N$, we have

$$b_n = (-1)^{1-w} \cdot \frac{g(q^{-a}) q^{a(w-1)}}{\Gamma(w-1)} \left( \frac{q^{an}}{n^{2-w}} \right) + o\left( \frac{q^{an}}{n^{2-w}} \right)$$

by induction hypothesis. Combining (6) and the above formula of $b_n$ implies that

$$a_n = (-1)^{-w} \frac{g(q^{-a}) q^{a(w-1)}}{\Gamma(w-1)} \cdot \left( q^{a(n+1)} \right) \cdot \left( \sum_{k=1}^{n} k^{w-2} \right) + o\left( \frac{q^{an}}{n^{1-w}} \right).$$

Note that for $w > 1$, we have

$$\sum_{k=1}^{n} k^{w-2} = \left( \frac{n^{w-1}}{w-1} \right) + o(n^{w-1}).$$

Hence

$$\begin{aligned} a_n =& (-1)^{-w} \frac{g(q^{-a}) q^{a(w-1)}}{\Gamma(w-1)} \cdot \left( q^{a(n+1)} \right) \cdot \left[ \left( \frac{n^{w-1}}{w-1} \right) + o(n^{w-1}) \right] + o\left( \frac{q^{an}}{n^{1-w}} \right) \\ =& (-1)^{-w} \frac{g(q^{-a}) q^{aw}}{\Gamma(w)} \cdot \left( \frac{q^{an}}{n^{1-w}} \right) + o\left( \frac{q^{an}}{n^{1-w}} \right). \end{aligned}$$

The proof is complete. ∎

## References


[1] M. Bhargava, *The density of discriminants of quartic rings and fields*, Ann. of Math. 162 (2005), 1031-1061.

[2] M. Bhargava, *The density of discriminants of quintic rings and fields*, Ann. of Math. 172 (2010), 1559-1591.

[3] H. Cohn, *The density of abelian cubic fields*, Proc. Amer. Math. Soc. 5, (1954), 476-477.

[4] C.-Y. Chuang, Y.-L. Kuan and J. Yu, *On counting polynomials over finite fields*, To appear in Proc. Amer. Math. Soc..

[5] H. Davenport and H. Heilbronn, *On the density of discriminants of cubic fields II*, Proc. Roy. Soc. London Ser. A 322 (1971), no. 1551, 405-420.

[6] D.R. Hayes, *Explicit class field theory for rational function fields*, Trans. Amer. Math. Soc. 189 (1974), 74-91.

[7] S. Mäki, *On the density of abelian number fields*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 54 (1985), 104 pp.

[8] M. Rosen, *Number theory in function fields*, Graduate Texts in Math. 2010, Springer (2002).

[9] David J. Wright, *Distribution of discriminants of abelian extensions*, Proc. London Math. Soc. (3) 58 (1989), no. 1, 17-50.

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