AVERAGE VALUES OF HIGHER MOMENTS OF QUADRATIC L-FUNCTIONS OVER RATIONAL FUNCTION FIELDS

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ABSTRACT. The aim of this paper is to establish asymptotic formulas in the region $\Re(s) \geq 1$ of the ℓ-th moment of quadratic L-functions over a rational function field $\mathbb{F}_q(t)$, for arbitrary positive integer ℓ and odd prime power q. Specifically, we obtain the asymptotic formulas relating to two families. One is over all discriminants and another is over all fundamental discriminants. In addition, we give applications including asymptotic formulas for the size of class numbers, algebraic K-groups K_2, and limit distribution functions associated with the quadratic L-functions.

INTRODUCTION

Let $k = \mathbb{F}_q(t)$ be a rational function field with odd characteristic p. Let $A = \mathbb{F}_q[t]$, and A^+ be the set of all monic polynomials. The infinite place ∞ of k corresponds to the degree valuation. The letter P always denotes a monic irreducible polynomial in A, which corresponds to a finite place of k.

Let $\left[\frac{\cdot}{\cdot}\right]$ be the quadratic symbol for $\mathbb{F}_q[t]$ (cf. [9, Chapter 3]). Given $m \in A$ non-square, define the function $\chi_m(n) := \left[\frac{m}{n}\right]$ for $n \in A - \{0\}$. We are interested in the L-function associated to χ_m which is defined by

$$L(s, \chi_m) := \sum_{n \in A^+} \chi_m(n) q^{-s \deg n} = \prod_P (1 - \chi_m(P) q^{-s \deg P})^{-1}, \text{ on } \Re(s) > 1.$$

This is equal to a polynomial of degree at most $\deg m - 1$ in q^{-s}.

Suppose that $m \in A$ is square-free. Let

$$\lambda_\infty(m) := \begin{cases} 0, & \text{if } \infty \text{ is ramified in } k(\sqrt{m})/k; \\ -1, & \text{if } \infty \text{ is inert in } k(\sqrt{m})/k; \\ 1, & \text{if } \infty \text{ splits in } k(\sqrt{m})/k. \end{cases}$$

Then

$$L^*(s, \chi_m) := (1 - \lambda_\infty(m) q^{-s})^{-1} \cdot L(s, \chi_m)$$

is the Artin L-function associated to the unique non-trivial character of the Galois group $\mathrm{Gal}(k(\sqrt{m})/k)$ (cf. [9, Theorem 17.6]). Let $\mathfrak{m}(\chi_m) := \deg m - 1 - |\lambda_\infty(m)|$. Then the functional equation for the complete L-function $L^*(s, \chi_m)$ is as follows:

$$L^*(s, \chi_m) = q^{\mathfrak{m}(\chi_m)(1/2 - s)} L^*(1 - s, \chi_m),$$

which implies that $L(s, \chi_m)$ is a polynomial of degree $\deg m - 1$ in q^{-s}. We classify quadratic function fields $K := k(\sqrt{m})$ according to whether ∞ splits, is inert, or ramified in K/k. This is analogous to classifying quadratic number fields as real or imaginary. That is

- If $\deg m$ is even and $\mathrm{sgn}(m) \in (\mathbb{F}_q^\times)^2$, then ∞ splits in K/k.
- If $\deg m$ is even, and $\mathrm{sgn}(m) \notin (\mathbb{F}_q^\times)^2$, then ∞ is inert in K/k.

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- If $\deg m$ is odd, then ∞ ramifies in K/k.

Here $\mathrm{sgn}(m)$ is the leading coefficient of $m \in A$. Let B be the integral closure of A in K/k. Then the L-function $L(s, \chi_m)$ satisfies

$$\zeta_B(s) = \zeta_A(s) \cdot L(s, \chi_m),$$

where ζ_A (reps. ζ_B) is the zeta function of A (reps. B):

- $\zeta_A(s) := \sum_{I \subset A} \mathbf{N}(I)^{-s} = \prod_P (1 - q^{-s \deg P})^{-1}$ on $\Re(s) > 1$, where $\mathbf{N}(I)$ denotes the absolute norm.
- $\zeta_B(s) := \sum_{I \subset B} \mathbf{N}(I)^{-s} = \prod_{\mathfrak{P}} (1 - \mathbf{N}(\mathfrak{P})^{-s})^{-1}$ on $\Re(s) > 1$, where the sum is over all non-zero ideals in B, and the product is over all non-zero prime ideals in B.

These zeta functions both have a simple pole at $s = 1$, and are rational functions in q^{-s}.

Throughout, we use the symbol $\square$ to denote square polynomials. Let γ be a fixed generator of $\mathbb{F}_q^\times$, and ℓ be a positive integer. On the basis of classifying the quadratic fields K over k, we are interested in considering the following averaging problems:

- Summing over all non-square monic polynomials ("discriminants"):

$$\mathcal{L}(s, M, \ell)_\mathcal{R} = \sum_{\substack{m \in A^+: \\ \deg m = M}} L(s, \chi_m)^\ell, \qquad \text{if } M \text{ is an odd integer;}$$
$$\mathcal{L}(s, M, \ell)_\mathcal{S} = \sum_{\substack{m \in A^+: \\ \deg m = M, m \neq \square}} L(s, \chi_m)^\ell, \qquad \text{if } M \text{ is an even integer;}$$
$$\mathcal{L}(s, M, \ell)_\mathcal{I} = \sum_{\substack{m \in A^+: \\ \deg m = M, m \neq \square}} L(s, \chi_{\gamma \cdot m})^\ell, \quad \text{if } M \text{ is an even integer.}$$

- Summing over all square-free monic polynomials ("fundamental discriminants"):

$$\mathcal{L}^*(s, M, \ell)_\mathcal{R} := \sideset{}{^*}\sum_{\substack{m \in A^+: \\ \deg m = M}} L(s, \chi_m)^\ell, \qquad \text{if } M \text{ is an odd integer;}$$
$$\mathcal{L}^*(s, M, \ell)_\mathcal{S} := \sideset{}{^*}\sum_{\substack{m \in A^+: \\ \deg m = M}} L(s, \chi_m)^\ell, \qquad \text{if } M \text{ is an even integer;}$$
$$\mathcal{L}^*(s, M, \ell)_\mathcal{I} := \sideset{}{^*}\sum_{\substack{m \in A^+: \\ \deg m = M}} L(s, \chi_{\gamma \cdot m})^\ell, \quad \text{if } M \text{ is an even integer.}$$

Here $*$ means that the sum in question runs over all square-free monic polynomials.

We note that for any odd integer M,

$$\mathcal{L}(s, M, \ell)_\mathcal{R} = \sum_{\substack{m \in A^+: \\ \deg m = M}} L(s, \chi_{\gamma \cdot m})^\ell, \text{ and } \mathcal{L}^*(s, M, \ell)_\mathcal{R} = \sideset{}{^*}\sum_{\substack{m \in A^+: \\ \deg m = M}} L(s, \chi_{\gamma \cdot m})^\ell.$$

We divide our results into two parts.

0.1. **The non-square case:** In this case, when $\ell = 1$, J. Hoffstein and M. Rosen (cf. [6, Theorem 0.7, Theorem 1.4 and Theorem 1.5]) obtained formulas for above sums, as functions of q^{-s}, which immediately gives asymptotic formulas as M approaches infinity. One of goals of this paper is derived asymptotic formulas for these sums when M goes to infinity with ℓ arbitrary. That is

Theorem 0.1. *Let ℓ, M be positive integers and $\star$ be either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$, then we have, for $\Re(s) \geq 1$,*

$$\mathcal{L}(s, M, \ell)_\star = \zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(s) \cdot q^M + O\left(q^{(1/2+\delta)M}\right),$$

for any $\delta > 0$*, as* $M \to \infty$*. Here* $c_\ell(s) :=$

$$\prod_P \left\{ \left((1 - q^{-\deg P}) \left(\frac{(1 + q^{-s\deg P})^\ell + (1 - q^{-s\deg P})^\ell}{2} \right) + q^{-\deg P}(1 - q^{-2s\deg P})^\ell \right) (1 - q^{-2s\deg P})^{\frac{\ell(\ell-1)}{2}} \right\}$$

is absolutely convergent on $\Re(s) > 1/4$*.*

Remark 0.2.

(1). When $\ell = 1$,

$$c_1(s) = \zeta_A(2s+1)^{-1}$$

is a rational function in q^{-s}. The above theorem reduces to the well-known averaging L-values of J. Hoffstein and M. Rosen (cf. [9, p. 323-324]).

(2). Let D denote an integer congruent to 0 or 1 modulo 4 and non-square. Let $\psi_D(n) := \left(\frac{D}{n}\right)$ denote the Kronecker symbol. The Dirichlet L-function associated with ψ_D is given by

$$L(s, \psi_D) := \sum_{n=1}^{\infty} \psi_D(n)/n^s, \text{ on } \Re(s) > 1.$$

Concerning higher moments, M. B. Barban [2, Lemma 5.6] established the following asymptotic formula, for any fixed positive integer ℓ,

$$\sum_{-N \le D \le -1} L^\ell(1, \psi_D) = r_\ell N + O(N \exp(-c\sqrt{N})), \text{ as } N \to \infty,$$

where $c > 0$ is a constant independent of ℓ and

$$r_\ell = \sum_{\substack{n=1 \\ n \equiv 1 \mod 2}}^{\infty} \frac{\varphi(n) d_\ell(n^2)}{n^3}.$$

Here $d_\ell(n)$ is the number of ways of expressing n as the product ℓ positive integers, expressions in which only the order of the factors begin different is regarded as distinct, and $\varphi(n)$ is the Euler totient function. In the function field case, the constant of our main term is (cf. (2.1)):

$$\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(s) = \sum_{n \in A^+} \frac{d_\ell(n^2) \cdot \varphi(n)}{|n|^{2s+1}} := r_\ell(s).$$

Here $|n| := q^{\deg n}$ for any $n \in A - \{0\}$. In particular, if $s = 1$, then

$$\zeta_A(2)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(1) = \sum_{n \in A^+} \frac{d_\ell(n^2) \cdot \varphi(n)}{|n|^3},$$

which is to be compared with the classical result. Here $d_\ell(n)$ is the number of ways of expressing n as the product ℓ monic polynomials, expressions in which only the order of the factors begin different is regarded as distinct, and $\varphi(n)$ is the Euler totient function for A.

(3). Although $\zeta_A^\ell(s) = \sum_{n \in A^+} \frac{d_\ell(n)}{|n|^s}$ and $\frac{\zeta_A(s-1)}{\zeta_A(s)} = \sum_{n \in A^+} \frac{\varphi(n)}{|n|^s}$ are both rational functions in q^{-s}, we do not know whether $\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(s) = \sum_{n \in A^+} \frac{d_\ell(n^2) \cdot \varphi(n)}{|n|^{2s+1}}$ is a rational function in q^{-s} for $\ell > 1$.

(4). For each local factor at prime P, we have

$$\left((1-q^{-\deg P})\left(\frac{(1+q^{-s\deg P})^\ell+(1-q^{-s\deg P})^\ell}{2}\right)+q^{-\deg P}(1-q^{-2s\deg P})^\ell\right)$$

$$=1+\sum_{\substack{t:2|t\\ 2\le t\le \ell}}\binom{\ell}{t}q^{-st\deg P}-q^{-\deg P}\sum_{\substack{t:2|t\\ 2\le t\le \ell}}\binom{\ell}{t}q^{-st\deg P}+q^{-\deg P}\sum_{t=1}^{\ell}\binom{\ell}{t}(-1)^t q^{-2st\deg P},$$

which is a polynomial of degree 2ℓ in $q^{-s\deg P}$. For each $\ell>1$, the infinite product

$$\prod_P\left((1-q^{-\deg P})\left(\frac{(1+q^{-s\deg P})^\ell+(1-q^{-s\deg P})^\ell}{2}\right)+q^{-\deg P}(1-q^{-2s\deg P})^\ell\right)$$

is holomorphic on $\Re(s)>1/2$ and has a pole at $s=1/2$ of order $\frac{\ell(\ell-1)}{2}$. Since $c_\ell(s)$ is equal to $\zeta_A(2s)^{-\frac{\ell(\ell-1)}{2}}$ multiplying this infinite product, the function $c_\ell(s)$ is holomorphic on $\Re(s)>1/4$.

In M. B. Barban's paper, he not only obtains the asymptotic formula of higher moment described in the above remark (2), but also achieves a result of limit distributions with its corresponding characteristic function (cf. [2, Theorem 5.2]). In the function field context, we also have an analogous result and prove it in Subsection 2.1:

Corollary 0.3. *For real $s_0\ge 1$, as $M\to\infty$, the quantity*

$$f_M(x,s_0)=q^{-M}\#\{m\in A^+,\deg m=M:\ L(s_0,\chi_m)\le x\},\ x\in\mathbb{R}$$

converges to a distribution function $f(x):=f(x,s_0)$ at each point of continuity of the latter, and the corresponding characteristic function has the form

$$\phi_{f,s_0}(x)=1+\sum_{\ell\ge1}\frac{r_\ell(s_0)}{\ell!}(ix)^\ell,\ x\in\mathbb{R}.$$

Write $m=m_0m_1^2$, where m_0 is square-free. The polynomial m_0 is well defined up to the square of a constant. Define B_m to be the ring $A+Am_1\sqrt{m_0}\subset K=k(\sqrt{m})$. It is an A-order in K, (i.e. it is a ring, finitely generated as an A-module, and its quotient field is K). Meanwhile, B_m is the unique subring of B_{m_0} such that m_1 is the annihilator of the A module B_{m_0}/B_m (cf. [9, Theorem 17.6]). The Picard group $\operatorname{Pic}(B_m)$ is the group of invertible fractional ideals of B_m modulo the subgroup of principal fractional ideals. We set the class number $h_m:=\#\operatorname{Pic}(B_m)$.

Since $L(1,\chi_m)$ gives the size of $\operatorname{Pic}(B_m)$, setting $s=1$ in Theorem 0.1, we obtain the following average value results for the ℓ-th moment of h_m.

Corollary 0.4. *Let ℓ be a positive integer.*

(1). *If $\deg m=M$ is an odd integer, then*

$$\sum_{\substack{m\in A^+:\\ \deg m=M, m\neq\square}}h_m^\ell=q^{(1+\frac{\ell}{2})M}\cdot q^{-\ell/2}\cdot\zeta_A(2)^{\frac{\ell(\ell+1)}{2}}\cdot c_\ell(1)+O(q^{(1/2+\delta+\frac{\ell}{2})M}).$$

(2). *If $\deg m=M$ is an even integer and γ is a generator of $\mathbb{F}_q^\times$, then*

$$\sum_{\substack{m\in A^+:\\ \deg m=\gamma M, m\neq\square}}h_{\gamma m}^\ell=q^{(1+\frac{\ell}{2})M}\cdot\left(\frac{2}{q+1}\right)^{\ell/2}\cdot\zeta_A(2)^{\frac{\ell(\ell+1)}{2}}\cdot c_\ell(1)+O(q^{(1/2+\delta+\frac{\ell}{2})M}).$$

(3). *If* $\deg m = M$ *is an even integer, then*

$$\sum_{\substack{m\in A^+:\\ \deg m=M, m\neq\square}} (h_m \cdot R_m)^\ell = q^{\left(1+\frac{\ell}{2}\right)M} \cdot (q-1)^{-\ell/2} \cdot \zeta_A(2)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell(1) + O(q^{\left(1/2+\delta+\frac{\ell}{2}\right)M}).$$

Here R_m *is the regulator of the ring* B_m.

0.2. **The square-free case:** When $\ell = 1$ and $q \equiv 1 \mod 4$, J. Hoffstein and M. Rosen established that (cf. [6, Theorem 0.8]):

Theorem 0.5. *Let*

$$P(s) := \prod_P \left(1 - |P|^{-2} - |P|^{-(2s+1)} + |P|^{-(2s+2)}\right), \text{ on } \Re(s) \geq 1/2.$$

Choose any $\epsilon > 0$. *If* $\Re(s) \geq 1$, *then*

(1). *If* M *is odd, then*

$$(q-1)^{-1}(q^M - q^{M-1})^{-1} \sum L(s, \chi_m) = \zeta_A(2)\zeta_A(2s)P(s) + O(q^{-(M/2)(1-\epsilon)}),$$

where the sum is over all square-free m *such that* $\deg(m) = M$.

(2). *If* M *is even, then*

$$2(q-1)^{-1}(q^M - q^{M-1})^{-1} \sum L(s, \chi_m) = \zeta_A(2)\zeta_A(2s)P(s) + O(q^{-(M/2)(1-\epsilon)}),$$

where the sum is over all square-free m *such that* $\deg(m) = M$ *and* $\operatorname{sgn}(m) \in (\mathbb{F}_q^\times)^2$ *or over all square-free* m *such that* $\deg(m) = M$ *and* $\operatorname{sgn}(m) \notin (\mathbb{F}_q^\times)^2$.

Hoffstein-Rosen uses the fact that the Fourier coefficients of Eisenstein series for the metaplectic group involve the values $L(s, \chi_m)$. In this paper, we are able to derive asymptotic formulas for these averaging values, with q, ℓ arbitrary, by more elementary direct approach. We have

Theorem 0.6. *Let* ℓ, M *be positive integers. Suppose that* $\star$ *is either* $\mathcal{S}$, $\mathcal{I}$, *or* $\mathcal{R}$. *Then, for* $\Re(s) \geq 1$,

$$\mathcal{L}^*(s, M, \ell)_\star = \zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell^*(s) \cdot (q^M \cdot \zeta_A(2)^{-1}) + O\left(q^{(1/2+\delta)M}\right)$$

for any $\delta > 0$, *as* $M \to \infty$. *Here*

$$c_\ell^*(s) := \prod_P \left\{ \frac{\left(1 - q^{-2s\deg P}\right)^{\frac{\ell(\ell+1)}{2}}}{1 + q^{-\deg P}} \left(\left(\frac{(1+q^{-s\deg P})^{-\ell} + (1-q^{-s\deg P})^{-\ell}}{2} \right) + q^{-\deg P} \right) \right\},$$

which is absolutely convergent in $\Re(s) > 1/4$. *Here*

$(q^M \cdot \zeta_A(2)^{-1}) = q^M - q^{M-1} = \#\{m \in A^+ : \deg m = M$ *and* m *is square-free*$\}$, *for* $M > 1$.

Remark 0.7.

(1). When $\ell = 1$, we have (cf. (2.3))

$$\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}} \cdot c_1^*(s) = \zeta_A(2) \cdot \zeta_A(2s) \cdot \prod_P (1 - |P|^{-2} - |P|^{-2s-1} + |P|^{-2s-2}) := r_\ell^*(s).$$

The above theorem reduces to the averaging L-values of J. Hoffstein and M. Rosen (cf. [6, Theorem 5.2]) in the region $\Re(s) \geq 1$.

(2). For each local factor at prime P, we have

$$\left(\frac{\left(1-q^{-2s\deg P}\right)^{\frac{\ell(\ell+1)}{2}}}{1+q^{-\deg P}}\left(\left(\frac{(1+q^{-s\deg P})^{-\ell}+(1-q^{-s\deg P})^{-\ell}}{2}\right)+q^{-\deg P}\right)\right)(1-q^{-2\deg P})$$

$$=\Big[1+\frac{\ell(\ell-1)}{2}q^{-2s\deg P}+\sum_{\substack{t=4\\2|t}}^{\ell}\binom{\ell}{t}q^{-st\deg P}-\sum_{\substack{t=2\\2|t}}^{\ell}\binom{\ell}{t}q^{-(st+1)\deg P}$$

$$+\sum_{t=1}^{\ell}(-1)^t\binom{\ell}{t}q^{-(2st+1)\deg P}-\sum_{t=0}^{\ell}(-1)^t\binom{\ell}{t}q^{-(2st+2)\deg P}\Big]\cdot\left(1-q^{-2s\deg P}\right)^{\frac{\ell(\ell-1)}{2}}$$

which is a polynomial of degree $\ell(\ell+1)$ in $q^{-s\deg P}$. Thus, for each $\ell\geq 1$, the infinite product gives that $c^*_\ell(s)$ is holomorphic on $\Re(s)>1/4$.

Similarly, Theorem 0.5 also gives a limit distribution function of the square-free case.

Corollary 0.8. *For real $s_0\geq 1$, as $M\to\infty$, the quantity*

$$f^*_M(x,s_0)=(q^M-q^{M-1})^{-1}\#\{m\in A^+,\deg m=M \text{ and } m \text{ is square-free}: L(s_0,\chi_m)\leq x\},\ x\in\mathbb{R}$$

converges to a distribution function $f^*(x):=f^*(x,s_0)$ at each point of continuity of the latter, and the corresponding characteristic function has the form

$$\phi^*_{f,s_0}(x)=1+\sum_{\ell\geq 1}\frac{r^*_\ell(s_0)}{\ell!}(ix)^\ell,\ x\in\mathbb{R}.$$

Use the same argument as Corollary 0.4. Setting $s=1$ in Theorem 0.6, we obtain the following average value results for the ℓ-th moment of h_m.

Corollary 0.9. *Let ℓ be a positive integer. Then, for any $\delta>0$,*

(1). *If $\deg m=M$ is an odd integer, then*

$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} h^\ell_m=q^{\left(1+\frac{\ell}{2}\right)M}\cdot q^{-\ell/2}\cdot\zeta_A(2)^{-1}\cdot\zeta_A(2)^{\frac{\ell(\ell+1)}{2}}\cdot c^*_\ell(1)+O(q^{\left(1/2+\delta+\frac{\ell}{2}\right)M}).$$

(2). *If $\deg m=M$ is an even integer, and γ is a generator of $\mathbb{F}_q^\times$, then*

$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} h^\ell_{\gamma m}=q^{\left(1+\frac{\ell}{2}\right)M}\cdot\left(\frac{2}{q+1}\right)^{\ell/2}\cdot\zeta_A(2)^{-1}\cdot\zeta_A(2)^{\frac{\ell(\ell+1)}{2}}\cdot c^*_\ell(1)+O(q^{\left(1/2+\delta+\frac{\ell}{2}\right)M}).$$

(3). *If $\deg m=M$ is an even integer, then*

$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} (h_m\cdot R_m)^\ell=q^{\left(1+\frac{\ell}{2}\right)M}\cdot(q-1)^{-\ell/2}\cdot\zeta_A(2)^{-1}\cdot\zeta_A(2)^{\frac{\ell(\ell+1)}{2}}\cdot c^*_\ell(1)+O(q^{\left(1/2+\delta+\frac{\ell}{2}\right)M}).$$

Here R_m is the regulator of the ring B_m.

Remark 0.10. Let F be a quadratic extension over $\mathbb{Q}$. Let $\Delta_{F/\mathbb{Q}}$, h_F, and R_F be the discriminant of $F/\mathbb{Q}$, the class number, and the regulator of F, respectively. In 2008, T. Taniguchi conjectured that (cf.[11, Theorem 1 and Conjecture 10.14]):

$$\lim_{X\to\infty}\frac{1}{X^2}\cdot\sum_{\substack{[F:\mathbb{Q}]=2\\ 0<\left|\Delta_{F/\mathbb{Q}}\right|\leq X}} h^2_F\cdot R^2_F=\frac{\zeta(2)^2}{2^4}\cdot\prod_p\left(1-\frac{3}{p^3}+\frac{2}{p^4}+\frac{1}{p^5}-\frac{1}{p^6}\right).$$

When $\ell = 2$, simplifying our cases (1), (2), and (3) in the above corollary, we have proved

$$\begin{aligned}
&(1) + (2) + (3)\\
=&q^{(1+\frac{\ell}{2})M} \cdot \left(\frac{1}{q-1} + \frac{2}{q+1} + \frac{1}{q}\right) \cdot \zeta_A(2)^{-1} \cdot \zeta_A(2)^{\frac{\ell(\ell+1)}{2}} \cdot c_2^*(1)\\
=&\left(\frac{1}{q-1} + \frac{2}{q+1} + \frac{1}{q}\right) \cdot \zeta_A(2)^2 \cdot \prod_P \left(1 - \frac{3}{|P|^3} + \frac{2}{|P|^4} + \frac{1}{|P|^5} - \frac{1}{|P|^6}\right) \cdot q^{2M},
\end{aligned}$$

which is to be compared with T. Taniguchi's conjecture.

Since $L(2, \chi_m)$ gives the size of K-groups $K_2(B_m)$ (after Tate and Quillen, cf. [8]), setting $s = 2$ in Theorem 0.6, we also obtain the following average value results for the ℓ-th moment of $\#(K_2(B_m))$ for arbitrary ℓ.

Corollary 0.11. *Let ℓ be a positive integer, $m \in A$ be square-free. Then, for any $\delta > 0$,*

(1). *If $\deg m = M$ is an odd integer, then*
$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} (\#K_2(B_m))^\ell = q^{-\frac{3\ell}{2}} \cdot \zeta_A(2)^{-1} \cdot \zeta_A(4)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell^*(2) \cdot q^{(1+\frac{3\ell}{2})M} + O(q^{(1/2+\delta+3\ell/2)M}).$$

(2). *If $\deg m = M$ is an even integer, and γ is a generator of $\mathbb{F}_q^\times$, then*
$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} (\#K_2(B_{\gamma\cdot m}))^\ell = \left(\frac{(1+q^{-1})}{q^2+1}\right)^\ell \cdot \zeta_A(2)^{-1} \cdot \zeta_A(4)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell^*(2) \cdot q^{(1+\frac{3\ell}{2})M} + O(q^{(1/2+\delta+3\ell/2)M}).$$

(3). *If $\deg m = M$ is an even integer, then*
$$\sum_{\substack{m\in A^+:\\ \deg m=M}}^{*} (\#K_2(B_m))^\ell = (q^2+q)^{-\ell} \cdot \zeta_A(2)^{-1} \cdot \zeta_A(4)^{\frac{\ell(\ell+1)}{2}} \cdot c_\ell^*(2) \cdot q^{(1+\frac{3\ell}{2})M} + O(q^{(1/2+\delta+3\ell/2)M}).$$

The strategy of proving Theorem 0.1 and Theorem 0.6 is similar. We rewrite $\mathcal{L}(s, M, \ell)_\star$ (or $\mathcal{L}^*(s, M, \ell)_\star$) as a polynomial in q^{-s} whose coefficients involve general divisor function d_ℓ as well as quadratic Gauss sums (cf. Lemma 3.1 and Lemma 4.1). For the sake of applying a function field version of the Tauberian theorem, we divide the above sums into parts according to the locations and orders of poles of L-functions associated to d_ℓ and quadratic Gauss sums (cf. Corollary 2.6). Finally, we mention that the main contribution comes form the quadratic Gauss sums degenerating to the character sums (cf. Proposition 3.2 and Proposition 4.2).

The contents of this paper are as follows. In subsection 1.1, we introduce the quadratic Gauss sums in our context and derive their exact values. The ℓ-th moment of L-functions associated with quadratic character χ_m for each positive integer ℓ is studied in subsection 1.2. We then establish asymptotic formulas in section 2 by a function field version of the Tauberian theorem. Proofs of Corollary 0.3 and Corollary 0.8 are given in subsection 2.1. We finally prove Theorem 0.1 in section 3 and Theorem 0.6 in section 4.

1. Preliminaries

Let k_∞ be the completion field of k at ∞, O_∞ be the valuation ring of k_∞, and $\pi_\infty = t^{-1}$ be a fixed uniformizer. For $y := \sum_{i=N}^{\infty} a_i \pi_\infty^i \in k_\infty$ with $a_N \neq 0$, we define $\mathrm{ord}_\infty(y) := N$. We fix an additive character ψ_∞ of k_∞ as the following, for $y := \sum_{i=N}^{\infty} a_i \pi_\infty^i \in k_\infty$,

$$\psi_\infty(y) := \exp(\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-a_1)) \in \mathbb{C}^\times.$$

For a locally constant function with compact support $f : k_\infty \to \mathbb{C}$, the Fourier transform f^* of f is defined to be

$$\hat{f}(y) := \int_{k_\infty} f(x)\psi_\infty(xy)dx,$$

where dx is the Haar measure of k_∞ such that $\mathrm{Vol}(O_\infty) = q$. Then $\hat{\hat{f}}(x) = f(-x)$ holds. It is straightforward to prove that:

Proposition 1.1. (Poisson summation formula) *Let f be a locally constant $\mathbb{C}$-valued function with compact support. Then for any $x \in k_\infty^\times$ and $y \in k_\infty$, we have*

$$\sum_{m\in A} f(xm+y) = q^{\mathrm{ord}_\infty(x)} \sum_{m'\in A} \hat{f}\left(\frac{m'}{x}\right)\cdot \psi_\infty\left(-\frac{ym'}{x}\right).$$

1.1. **Quadratic symbol and Gauss sums.** Let $\left[\frac{a}{b}\right]$ be the Kronecker symbol for A with $a \in A$ and $b \in A^+$. For $a \in A$, $a \neq 0$, we define $\mathrm{sgn}_2(a)$ to be the leading coefficient of a raised to the $\frac{q-1}{2}$ power. The reciprocity law of this symbol is stated as follows:

Proposition 1.2. (The quadratic reciprocity law) *Let $a, b \in A$ be relatively prime, nonzero elements. Then*

$$\left[\frac{a}{b}\right]\left[\frac{b}{a}\right] = (-1)^{\frac{q-1}{2}\deg a \deg b}\,\mathrm{sgn}_2(a)^{\deg b}\,\mathrm{sgn}_2(b)^{-\deg a}.$$

Let n be a monic polynomial. For all polynomials $e \in A$, we define an analogue of Gauss sum as follows:

$$G_e(n) := \sum_{a \mod n} \left[\frac{a}{n}\right]\psi_\infty\left(-\frac{ae}{n}\right) \in \mathbb{C},$$

and put

$$\tilde{G}_e(n) := \left(\frac{1+i}{2} + \left[\frac{-1}{n}\right]\frac{1-i}{2}\right)G_e(n). \tag{1.1}$$

Here $i := \sqrt{-1}$. Before we compute the exact values of $\tilde{G}_e(n)$ for all $n \in A^+$, we note that

Lemma 1.3. *Let P be a fixed monic irreducible polynomial. For all integers N satisfying $\left\lceil\frac{\deg P}{2}\right\rceil \leq N \leq \deg P - 1$, we have*

$$\sum_{\substack{n\in A\\ \deg n = N}} \psi_\infty\left(-\frac{n^2}{P}\right) = 0.$$

Proof. Let $n = \sum_{i=-N}^{0} a_i\pi_\infty^i$ and $P^{-1} = \sum_{j=\deg P}^{\infty} b_j\pi_\infty^j$, where $a_{-N}, b_{\deg P} \in \mathbb{F}_q^\times$, and $a_i, b_j \in \mathbb{F}_q$ for all $-N+1 \leq i \leq 0$ and $j \geq \deg P + 1$. Then

$$\frac{n^2}{P} = \sum_{u=-2N+\deg P}^{\infty}\left(\sum_{\substack{j\geq \deg P\\ -N\leq i_1,i_2\leq 0\\ i_1+i_2+j=u}} a_{i_1}a_{i_2}b_j\right)\pi_\infty^u.$$

Therefore, we have

$$\sum_{\substack{n\in A\\ \deg n=N}} \psi_\infty\left(-\frac{n^2}{P}\right) = \sum_{\substack{a_{-N}\in\mathbb{F}_q^\times\\ a_l\in\mathbb{F}_q,-N<l\le 0}} \psi_\infty\left(-\left(\sum_{\substack{j\ge\deg P\\ -N\le i_1,i_2\le 0\\ i_1+i_2+j=1}} a_{i_1}a_{i_2}b_j\right)\pi_\infty\right)$$
$$= \sum_{\substack{a_{-N}\in\mathbb{F}_q^\times\\ a_l\in\mathbb{F}_q,-N<l\le 0}} \prod_{\substack{j\ge\deg P\\ -N\le i_1,i_2\le 0\\ i_1+i_2+j=1}} \psi_\infty\left(\left(-a_{i_1}a_{i_2}b_j\right)\pi_\infty\right).$$

The definition of ψ_∞ implies that we only focus on $i_1+i_2+j=1$. Taking $i_1=-N$ and $j=\deg P$ in the above equality, we have $i_2=1-\deg P+N$, which implies that

$$\left\lceil\frac{-\deg P}{2}\right\rceil+1\le i_2\le 0$$

by the assumption of N. Thus, there always exists $i_2\neq -N$ such that $-N+i_2+\deg P=1$. Meanwhile, for this $i_2=1-\deg P+N$, the solution $i_1=-N$ and $j=\deg P$ is the unique solution satisfying $i_1+i_2+j=1$ for $-N\le i_1\le 0$ and $\deg P\le j$. Thus, we derive

$$\sum_{\substack{n\in A\\ \deg n=N}} \psi_\infty\left(-\frac{n^2}{P}\right) = \sum_{\substack{a_{-N}\in\mathbb{F}_q^\times\\ a_l\in\mathbb{F}_q,-N<l\le 0}} \left(\prod_{\substack{j\ge\deg P\\ -N\le i_1,i_2\le 0\\ i_1+i_2+j=1\\ (i_1,i_2,j)\neq(-N,1-\deg P+N,\deg P)\\ (i_1,i_2,j)\neq(1-\deg P+N,-N,\deg P)}} \psi_\infty\left(\left(-a_{i_1}a_{i_2}b_j\right)\pi_\infty\right)\right)$$
$$\cdot\psi_\infty\left(-2\cdot a_{-N}\cdot b_{\deg P}\cdot a_{1-\deg P+N}\cdot\pi_\infty\right)$$
$$= \sum_{\substack{a_{-N}\in\mathbb{F}_q^\times\\ a_l\in\mathbb{F}_q,-N<l\neq 1-\deg P+N\le 0}} \left(\prod_{\substack{j\ge\deg P\\ -N\le i_1,i_2\le 0\\ i_1+i_2+j=1\\ (i_1,i_2,j)\neq(-N,1-\deg P+N,\deg P)\\ (i_1,i_2,j)\neq(1-\deg P+N,-N,\deg P)}} \psi_\infty\left(\left(-a_{i_1}a_{i_2}b_j\right)\pi_\infty\right)\right)$$
$$\cdot\left(\sum_{a_{1-\deg P+N}\in\mathbb{F}_q} \psi_\infty\left(a_{1-\deg P+N}\cdot\pi_\infty\right)\right) = 0.$$

□

Now, we compute $\tilde{G}_e(n)$ for all $n\in A^+$.

Lemma 1.4. (1). *Suppose m and n are co-prime monic polynomials. Then*

$$\tilde{G}_e(mn)=\tilde{G}_e(m)\tilde{G}_e(n).$$

(2). *Suppose that $d\in A$, and α is the largest power of irreducible polynomial P dividing e (If $e=0$ then set $\alpha=\infty$). Then for $\beta\ge 1$*

$$\tilde{G}_e(P^\beta):=\begin{cases} 0, & \text{if } \beta\le\alpha \text{ is odd;}\\ \varphi(P^\beta), & \text{if } \beta\le\alpha \text{ is even;}\\ -q^{\alpha\deg P}, & \text{if } \beta=\alpha+1 \text{ is even;}\\ (\gamma_{p,q})^{\deg P}\cdot\left[\frac{eP^{-\alpha}}{P}\right]\cdot q^{(\alpha+1/2)\deg P}, & \text{if } \beta=\alpha+1 \text{ is odd;}\\ 0, & \text{if } \beta\ge\alpha+2.\end{cases}$$

Here,

$$\gamma_{p,q} := -\left(-\sqrt{\left(\frac{-1}{p}\right)}\right)^{[\mathbb{F}_q:\mathbb{F}_p]} \cdot \left(\frac{1+i}{2} + (-1)^{\frac{q-1}{2}}\frac{1-i}{2}\right) \in \{\pm 1\}. \tag{1.2}$$

where $\left(\frac{\cdot}{p}\right)$ *is the Legendre symbol modulo* p, $[\mathbb{F}_q : \mathbb{F}_p]$ *is the dimension of* $\mathbb{F}_q$ *over* $\mathbb{F}_p$*, and for* $f \in A$, $\varphi(f) := \#(A/fA)^\times$ *is the Euler totient function for* A.

Proof. The statement (1). comes from the Chinese Remainder theorem and the quadratic reciprocity law.
(2). We only show the crucial case $\beta = \alpha + 1$. Others are straightforward to verify. If $\beta = \alpha + 1$, then

$$\begin{aligned} G_e(P^\beta) &= \sum_{a \mod P^\beta} \left[\frac{a}{P^\beta}\right] \psi_\infty\left(-\frac{ae}{P^\beta}\right) \\ &= \sum_{l \mod P} \left[\frac{l}{P^\beta}\right] \sum_{b \mod P^{\beta-1}} \psi_\infty\left(-\frac{(bp+l)e}{P^\beta}\right) = q^{(\beta-1)\deg P} \sum_{l \mod P} \left[\frac{l}{P^\beta}\right] \psi_\infty\left(-\frac{le}{P^\beta}\right). \end{aligned}$$

If β is even, then $\sum_{l \mod P} \left[\frac{l}{P^\beta}\right] \psi_\infty\left(-\frac{le}{P^\beta}\right) = -1$. Set $\deg 0 = -\infty$. If β is odd, and $\deg P$ is odd, then

$$\begin{aligned} &\sum_{l \mod P} \left[\frac{l}{P}\right] \psi_\infty\left(-\frac{l(eP^{-\alpha})}{P}\right) = \left[\frac{eP^{-\alpha}}{P}\right] \sum_{l \mod P} \psi_\infty\left(-\frac{l^2}{P}\right) \\ &= \left[\frac{eP^{-\alpha}}{P}\right]\left[\sum_{\substack{l \mod P \\ 2\deg l - \deg P < -1}} \psi_\infty\left(-\frac{l^2}{P}\right) + \sum_{\substack{l \mod P \\ 2\deg l - \deg P = -1}} \psi_\infty\left(-\frac{l^2}{P}\right) + \sum_{\substack{l \mod P \\ 2\deg l - \deg P > -1}} \psi_\infty\left(-\frac{l^2}{P}\right)\right], \text{ by Lemma 1.3} \\ &= \left[\frac{eP^{-\alpha}}{P}\right]\left[q^{(\deg P-1)/2} + q^{(\deg P-1)/2} \sum_{a \in \mathbb{F}_q^\times} \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a^2)}{p}\right)\right] \\ &= \left[\frac{eP^{-\alpha}}{P}\right] q^{(\deg P-1)/2} \sum_{a \in \mathbb{F}_q} \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a^2)}{p}\right) \\ &= -\left(-\sqrt{\left(\frac{-1}{p}\right)}\right)^{[\mathbb{F}_q:\mathbb{F}_p]} \left[\frac{eP^{-\alpha}}{P}\right] q^{\deg P/2}, \text{ by Davenport-Hasse relation [7, p. 158-162].} \end{aligned}$$

Meanwhile, in this case, we have $\frac{1+i}{2} + \left[\frac{-1}{P^\beta}\right]\frac{1-i}{2} = \frac{1+i}{2} + (-1)^{\frac{q-1}{2}}\frac{1-i}{2}$. Combining the above equalities, we obtain the desired result.
If $\beta = \alpha + 1$ is odd and $\deg P$ is even, then we have

$$\sum_{l \mod P} \left[\frac{l}{P}\right] \psi_\infty\left(-\frac{l(eP^{-\alpha})}{P}\right) = \left[\frac{eP^{-\alpha}}{P}\right] q^{(\beta-1/2)\deg P}$$

by the same method as the above case.

□

1.2. **Quadratic L-functions.** Let $K := k(\sqrt{m})$ be a quadratic field over k, where m is non-square with $\deg m \geq 1$. One of our goals is to investigate the mean value of ℓ-th moment of the class numbers h_m. If ∞ doesn't split in K/k, then $B_m^\times = \mathbb{F}_q^\times$, and if ∞ splits in K/k, then $B_m^\times = \mathbb{F}_q^\times \times <\epsilon_m>$, where $<\epsilon_m>$ is infinite cyclic. In this case, we set R_m equal to the absolute value of $\log_q q^{\operatorname{ord}_\infty(\epsilon_m)}$.

Suppose m is square-free, the connection between $L(1,\chi_m)$ and class numbers is proven by E. Artin. This result also can be generalized to the case of non-square polynomials (cf. [9, Theorem 17.8B]) .

Theorem 1.5. *Let $m \in A$ be a non-square polynomial of degree $M \geq 1$.*

(1). $L(1,\chi_m) = q^{\frac{1-M}{2}} \cdot h_m$, *if M is odd.*
(2). $L(1,\chi_m) = \frac{q+1}{2} \cdot q^{-M/2} \cdot h_m$, *if M is even and* $\mathrm{sgn}_2(m) = -1$.
(3). $L(1,\chi_m) = (q-1) \cdot q^{-M/2} \cdot h_m \cdot R_m$, *if M is even and* $\mathrm{sgn}_2(m) = 1$. *Here R_m is the regulator of the ring B_m.*

Suppose that m is square-free. We are able to investigate the mean value of ℓ-th moment of $\#(K_2(B_m))$, since the connection between $L(2,\chi_m)$ and $\#(K_2(B_m))$ is already known by Tate and Quillen. That is (cf. [8, Proposition 2]):

Theorem 1.6. *Let $m \in A$ be a square-free polynomial of degree $M \geq 1$. Then*

(1). $\#(K_2(B_m)) = q^{(3/2)M} \cdot q^{-3/2} \cdot L(2,\chi_m)$, *if M is odd.*
(2). $\#(K_2(B_m)) = q^{(3/2)M} \cdot (1+q^{-1}) \cdot (q^2+1)^{-1} \cdot L(2,\chi_m)$, *if M is even and* $\mathrm{sgn}_2(m) = -1$.
(3). $\#(K_2(B_m)) = q^{(3/2)M} \cdot (q^2+q)^{-1} \cdot L(2,\chi_m)$, *if M is even and* $\mathrm{sgn}_2(m) = 1$.

For each positive integer ℓ, the ℓ-th moment of L-function $L(s,\chi_m)$ is:

$$\big(L(s,\chi_m)\big)^\ell = \left[\sum_{N=0}^{M-1}\left(\sum_{\substack{n\in A^+:\\ \deg n = N}} \chi_m(n)\right) q^{-sN}\right]^\ell$$
$$= \sum_{N=0}^{\ell(M-1)}\left(\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n)\chi_m(n)\right) q^{-sN},$$

where

$$d_\ell(n) := \sum_{\substack{n_1,n_2,\ldots,n_\ell\in A^+:\\ n_1 n_2\cdots n_\ell = n}} 1 \tag{1.3}$$

is the number of ways of expressing n as the product of k monic polynomials, expressions in which only the order of the factors being different is regarded as distinct. The main purpose of this paper is to study the mean values of these ℓ-th moments.

2. Asymptotic formulas for arithmetic functions

The following Tauberian Theorem is used to study asymptotic formulas for arithmetic functions (cf. [3, Theorem 7]):

Theorem 2.1. *Let $f(u) := \sum_{N\geq 0} a_N u^N$ with the numbers $a_N \in \mathbb{C}$ for all N, be convergent in*

$$\{u \in \mathbb{C} : |u| < q^{-a}\}$$

for a fixed real number $a > 0$. Assume that in the above domain

$$f(u) = g(u)(u-q^{-a})^{-w} + h(u)$$

holds, where $h(u)$, $g(u)$ are analytic functions in $\{u \in \mathbb{C} : |u| \leq q^{-a}\}$, $g(q^{-a}) \neq 0$, and $w > 0$ is a positive integer. Then

$$a_N = (-1)^{-w}\frac{g(q^{-a})q^{aw}}{\Gamma(w)} \cdot q^{aN} N^{w-1} + O\left(q^{aN}N^{w-2}\right), \quad \text{as } N \to \infty.$$

Let $a = 1$, then Theorem 2.1 reduces to the case [9, Theorem 17.4].

Corollary 2.2. *Let* $f(u) := \sum_{N\geq 0} a_N u^N$ *with the numbers* $a_N \in \mathbb{C}$ *for all* N, *be convergent in*

$$\{u \in \mathbb{C} : |u| < q^{-a}\}$$

for a fixed real number $a > 0$. *Assume that in the above domain*

$$f(u) = g(u)(u + q^{-a})^{-w} + h(u)$$

holds, where $h(u)$, $g(u)$ *are analytic functions in* $\{u \in \mathbb{C} : |u| \leq q^{-a}\}$, $g(-q^{-a}) \neq 0$ *and* w *is a positive integer. Then*

$$a_N = (-1)^N \frac{g(-q^{-a})q^{aw}}{\Gamma(w)} q^{aN} N^{w-1} + O\left(q^{aN} N^{w-2}\right), \text{ as } N \to \infty.$$

Proof. Set $\tilde{f}(u) = f(-u)$, $\tilde{g}(u) = g(-u)$ and $\tilde{h}(u) = h(-u)$. Thus, $\tilde{f}(u) := \sum_{N\geq 0} (-1)^N a_N u^N$ The condition $f(u) = g(u)(u + q^{-a})^{-w} + h(u)$ is equivalent to

$$\tilde{f}(u) = (-1)^{-w} \tilde{g}(u)(u - q^{-a})^{-w} + \tilde{h}(u).$$

Applying Theorem 2.1, we obtain that

$$(-1)^N a_N = \frac{g(-q^{-a})q^{aw}}{\Gamma(w)} q^{aN} N^{w-1} + O\left(q^{aN} N^{w-2}\right).$$

□

Let ℓ be a positive integer. The arithmetic function $d_\ell(n^2)$ will also play a key role in our asymptotic studies. Here $d_\ell(n)$ is the general divisor function defined in (1.3). Let $\varphi(n)$ be the Euler totient function for A. Applying Theorem 2.1 with $a = 3$ and $w = \frac{\ell(\ell+1)}{2}$, it is not difficult to drive the estimation below

Lemma 2.3. *Let* ℓ *be a positive integer, then*

$$\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n^2)\varphi(n^2) = c_\ell(1/2) \cdot \frac{q^{3N} \cdot N^{\frac{\ell(\ell+1)}{2}-1}}{\Gamma\left(\frac{\ell(\ell+1)}{2}\right)} + O\left(\frac{q^{3N} \cdot N^{\frac{\ell(\ell+1)}{2}-1}}{N}\right),$$

as $N \to \infty$.

Proof. For $\ell \geq 1$, we note that

$$\sum_{t=0}^{\infty} \frac{(2t+1)(2t+2)\cdot \ldots \cdot (2t+\ell-1)}{(\ell-1)\,!} x^t = \left[\frac{(1+\sqrt{x})^\ell + (1-\sqrt{x})^\ell}{2}\right](1-x)^{-\ell}.$$

Suppose $\ell = 1$. $d_\ell(n) = 1$ for all $n \in A^+$. Thus,

$$\sum_{\substack{n\in A^+\\ \deg n = N}} d_\ell(n^2)\varphi(n^2) = q^{3N}(1 - q^{-1})$$

by [9, Proposition 2.7], which satisfies the statement of our lemma. For $\ell > 1$, the generating function of $d_\ell(n^2)\varphi(n^2)$ is:

$$\begin{aligned}\zeta_{d_\ell,\varphi}(s) :=& \sum_{n\in A^+} d_\ell(n^2)\varphi(n^2)q^{-s\deg n}\\ =& \prod_P \left\{(1-q^{-\deg P})\left[\frac{(1+q^{\frac{(2-s)\deg P}{2}})^\ell + (1-q^{\frac{(2-s)\deg P}{2}})^\ell}{2(1-q^{(2-s)\deg P})^\ell}\right] + q^{-\deg P}\right\} \qquad (2.1)\\ =& \zeta_A^{\frac{\ell(\ell+1)}{2}}(s-2)\cdot c_\ell\left(\frac{s-2}{2}\right),\end{aligned}$$

which has a pole of order $\frac{\ell(\ell+1)}{2}$ at $s=3$. Set $u = q^{-s}$, $\tilde{\zeta}_{d_\ell,\varphi}(u) := \zeta_{d_\ell,\varphi}(s)$, and $\tilde{c}_\ell(u) := c_\ell\left(\frac{s-2}{2}\right)$. Then Theorem 2.1 (or [9, Theorem 17.4]) with $a=3$ and $w = \frac{\ell(\ell+1)}{2}$ gives us the desired result, since the leading coefficient of Laurent series of $\tilde{\zeta}_{d_\ell,\varphi}(u)$ at $u = q^{-3}$ is equal to

$$(-q^{-3})^{\frac{\ell(\ell+1)}{2}}\cdot\tilde{c}_\ell(q^{-3}).$$

□

For each $n\in A^+$, we define

$$\nu(n) := \prod_{P|n}\left(1-q^{-2\deg P}\right)^{-1}. \qquad (2.2)$$

Applying Theorem 2.1 with $a=3$ and $w=\frac{\ell(\ell+1)}{2}$ to $d_\ell(n^2)\varphi(n^2)\nu(n)$, we have

Lemma 2.4. *Let ℓ be a positive integer, then*

$$\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n^2)\varphi(n^2)\nu(n) = c_\ell^*(1/2)\cdot\frac{q^{3N}\cdot N^{\frac{\ell(\ell+1)}{2}-1}}{\Gamma\left(\frac{\ell(\ell+1)}{2}\right)} + O\left(\frac{q^{3N}\cdot N^{\frac{\ell(\ell+1)}{2}-1}}{N}\right),$$

as $N\to\infty$.

Proof. The generating function of $d_\ell(n^2)\varphi(n^2)\nu(n)$ is:

$$\begin{aligned}\zeta_{d_\ell,\varphi,\nu}(s) :=& \sum_{n\in A^+} d_\ell(n^2)\varphi(n^2)\nu(n)q^{-s\deg n}\\ =& \prod_P\left\{\frac{1}{1+q^{-\deg P}}\left[\left[\frac{(1+q^{\frac{(2-s)\deg P}{2}})^{-\ell} + (1-q^{\frac{(2-s)\deg P}{2}})^{-\ell}}{2}\right] + q^{-\deg P}\right]\right\} \qquad (2.3)\\ =& \zeta_A^{\frac{\ell(\ell+1)}{2}}(s-2)\cdot c_\ell^*\left(\frac{s-2}{2}\right),\end{aligned}$$

which has a pole of order $\frac{\ell(\ell+1)}{2}$ at $s=3$.

Set $u=q^{-s}$, $\zeta_{d_\ell,\varphi,\nu}(s) = \tilde{\zeta}_{d_\ell,\varphi,\nu}(u)$, and $\tilde{c}_\ell^*(u) := c_\ell^*\left(\frac{s-2}{2}\right)$. Then Theorem 2.1 (or [9, Theorem 17.4]) with $a=3$ and $w=\frac{\ell(\ell+1)}{2}$ gives us the desired result, since the leading coefficient of Laurent series of $\tilde{\zeta}_{d_\ell,\varphi,\nu}(u)$ at $u=q^{-3}$ is equal to

$$(-q^{-3})^{\frac{\ell(\ell+1)}{2}}\cdot\tilde{c}_\ell^*(q^{-3}).$$

□

We finally study the asymptotic formulas of the general divisor function d_ℓ and the Gauss sum $\tilde{G}_e$. Let $g\in A^+$ and

$$L_g(s,\gamma_{p,q}\cdot\chi_e) := \prod_{P\nmid g}\left(1-\gamma_{p,q}^{\deg P}\chi_e(P)q^{-s\deg P}\right)^{-1}, \text{ on } \Re(s) > 1$$

for each monic polynomial e. We have

Lemma 2.5. *Write $e = e_1 e_2^2$, where $e_1 \in A^+$ is a square-free polynomial, and e_2 is a monic polynomial. In the region $\Re(s) > 3/2$*

$$\sum_{\substack{n\in A^+:\\(n,g)=1}} d_\ell(n)\tilde{G}_e(n)q^{-s\deg n} = L_g(s-1/2, \gamma_{p,q}\cdot\chi_{e_1})^\ell \prod_{P\nmid g} \mathcal{G}_{P,d_\ell,\tilde{G}_e}(s) := L_g(s-1/2, \gamma_{p,q}\cdot\chi_{e_1})^\ell \mathcal{G}_{d_\ell,\tilde{G}_e,g}(s),$$

where $\mathcal{G}_{P,d_\ell,\tilde{G}_e}(s)$ is defined as follows:

$$\mathcal{G}_{P,d_\ell,\tilde{G}_e}(s) := \left(1-\gamma_{p,q}^{\deg P}\cdot\chi_{e_1}(P)q^{(1/2-s)\deg P}\right)^\ell \left(\sum_{t\geq 0}^{\infty} d_\ell(P^t)\tilde{G}_e(P^t)q^{-st\deg P}\right).$$

Then $\mathcal{G}_{d_\ell,\tilde{G}_e,g}(s)$ is holomorphic on $\Re(s) > 1$. Here $\gamma_{p,q}$ is defined in (1.2).

Proof. The Euler product of $\mathcal{G}_{d_\ell,\tilde{G}_e,g}(s)$ follows from the multiplicativity of $\tilde{G}_e$ and d_ℓ.

By Lemma 1.4, we see that for $P \nmid e$ and $\ell \geq 3$,

$$\begin{aligned}\mathcal{G}_{P,d_\ell,\tilde{G}_e}(s) =& \left(1-\gamma_{p,q}^{\deg P}\chi_{e_1}(P)q^{(1/2-s)\deg P}\right)^\ell \left(1+\gamma_{p,q}^{\deg P}\cdot\ell\cdot\chi_{e_1}(P)q^{(1/2-s)\deg P}\right)\\ =&1-\frac{\ell(\ell+1)}{2}q^{(1-2s)\deg P}+O(q^{(3/2-3s)\deg P})\end{aligned}$$

which implies that $\mathcal{G}_{d_\ell,\tilde{G}_e,g}(s)$ is holomorphic on $\Re(s) > 1$. When $\ell = 1$ or 2, we use the same method to prove that $\mathcal{G}_{d_\ell,\tilde{G}_e,g}(s)$ is holomorphic on $\Re(s) > 1$.

□

Corollary 2.6. *If $e \in A$ and $g \in A^+$, then we have, for any $\delta > 0$,*

$$\sum_{\substack{n\in A^+:\deg n=N\\(n,g)=1}} d_\ell(n)\tilde{G}_e(n) \ll \begin{cases} q^{(1+\delta)N}, & \text{if } e \neq \square;\\ q^{(\frac{3}{2}+\delta)N}, & \text{if } e = \square,\end{cases}$$

as $N \to \infty$.

Proof. Let $L_g(s,\chi_e) := \prod_{P\nmid g}(1-\chi_e(P)q^{-s\deg P})^{-1}$ for $\Re(s) > 1$. Set $L_g(s,\gamma_{p,q}\cdot\chi_e) = \tilde{L}_g(u,\gamma_{p,q}\cdot\chi_e)$ and $\tilde{L}_g(u,\chi_e) = L_g(s,\chi_e)$ with $u = q^{-s}$. Then

$$\tilde{L}_g(u,\gamma_{p,q}\cdot\chi_e) = \begin{cases} \tilde{L}_g(u,\chi_e), & \text{if } \gamma_{p,q} = 1;\\ \tilde{L}_g(-u,\chi_e), & \text{if } \gamma_{p,q} = -1.\end{cases}$$

If e is not square, then the function $\tilde{L}_g(u,\gamma_{p,q}\cdot\chi_e)$ is holomorphic on $\mathbb{C}$. If e is square, then $\tilde{L}_g(u,\gamma_{p,q}\cdot\chi_e)$ is holomorphic on $\{u : |u| < q^{-3/2}\}$ and has a pole at the circle $\{u : |u| = q^{-3/2}\}$.

Thus, for all $g \in A^+$, the function $\mathcal{G}_{d_\ell,\tilde{G}_e,g}(s)$ is holomorphic on $\Re(s) > 1$ by the above lemma, so Theorem 2.1 and Corollary 2.2 with $a = 3/2$ for the case $e = \square$ and $a = 1$ for the case $e \neq \square$ imply that, for any $g \in A^+$,

$$\sum_{\substack{n\in A^+:\deg n=N\\(n,g)=1}} d_\ell(n)\tilde{G}_e(n) \ll \begin{cases} q^{(1+\delta)N}, & \text{if } e \neq \square;\\ q^{(\frac{3}{2}+\delta)N}, & \text{if } e = \square\end{cases}$$

for any $\delta > 0$.

□

When $\deg m = M$ is an even number, we will encounter extra contribution which is

Lemma 2.7. *Let ℓ be a positive integer. Then*

$$\sum_{n\in A^+} d_\ell(n) = \frac{q^N \cdot N^{\ell-1}}{\Gamma(\ell)} + O\left(q^N N^{\ell-2}\right) \text{ as } N\to\infty.$$

This proof is simpler than the above cases, so we omit it.

2.1. **Limit distributions.** A distribution function is a non-decreasing function $f:\mathbb{R}\to[0,1]$ which is right continuous and satisfies $f(-\infty)=0$ and $f(\infty)=1$. In 1931, M. Fréchet's and J. Shohat's proved that (cf. [5, Lemma 1.43])

Lemma 2.8. *If all the $f_N(x)$ from a sequence of distribution functions $f_N(x)$ have finite moments $\alpha_T(N)=\int_{\mathbb{R}} x^T df_N(x)$ of every order and if $\alpha_T(N)\to\beta_T$ as $N\to\infty$ for each $T\in\mathbb{N}$, then the β_T are the moments of some distribution function $f(x)$. If, moreover, $f(x)$ is uniquely determined by its moments, then as $N\to\infty$ the sequence $f_N(x)$ converges to $f(x)$ at each point of continuity of $f(x)$.*

Now to justify the application of the above lemma, we still need one lemma (cf. [5, Lemma 1.44]).

Lemma 2.9. *Let $\alpha_0=1$, $\alpha_1,\dots\alpha_T,\dots$ be the moments of some distribution function $f(x)$, each being assumed finite, and suppose that the series*

$$\sum_{T=0}^{\infty}\frac{\alpha_T}{T!}\tau_0^T$$

is absolutely convergent for some $\tau_0>0$. Then $f(x)$ is the unique distribution function with moments α_0, α_1, $\alpha_2,\dots$. Moreover the characteristic function $\phi_f(y):\mathbb{R}\to\mathbb{C}$ of the distribution f has the representation

$$\phi_f(y)=\sum_{T=0}^{\infty}\frac{\alpha_T}{T!}(iy)^T$$

for $|y|<\tau_0$.

The proof of Corollary 0.3 is similar to Corollary 0.8. We only prove one of them.

Proof of Corollary 0.3.

For a fixed $s_0\in\mathbb{R}$ with $s_0\geq 1$, the real value function

$$f_M(x,s_0):=\frac{1}{q^M}\#\{m\in A^+:\deg m=M \text{ and } L(s_0,\chi_m)\leq x\},\ x\in\mathbb{R}$$

are distribution functions for all $M\in\mathbb{N}$. Theorem 0.1 says that

$$\frac{1}{q^M}\mathcal{L}(s_0,M,\ell)_\star=\zeta_A(2s_0)^{\frac{\ell(\ell+1)}{2}}\cdot c_\ell(s_0)=\sum_{n\in A^+}\frac{d_\ell(n^2)\cdot\varphi(n)}{|n|^{2s_0+1}}=r_\ell(s_0),\text{ as } M\to\infty,$$

where $\star$ is either $\mathcal{S}$, $\mathcal{I}$ or $\mathcal{R}$. According to Lemma 2.3,

$$r_\ell(s_0)=\sum_{N\geq 0}\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n^2)\varphi(n^2)\right)q^{-(2s_0+2)N}\ll\frac{1}{1-q^{(2s_0-1-\delta)}},\text{ for any }\delta>0,$$

we obtain that

$$1+\sum_{\ell=1} r_\ell(s_0)x^\ell/\ell!$$

has the infinite radius of convergence. Hence Lemma 2.8 and Lemma 2.9 imply that for a fixed $s_0 \in \mathbb{R}$ with $s_0 \geq 1$, there exits a distribution function f such that

$$\lim_{x\to\infty} \frac{1}{q^M} \#\{m \in A^+ : \deg m = M \text{ and } L(s, \chi_m) \leq x\} = f(x, s)$$

holds for $s = s_0$ and $f(\cdot, s_0) = f$ at all points of continuity x of f. Moreover, the ℓ-th moment of f is equal to $r_\ell(s_0)$, so f has a characteristic function given by

$$\phi_{f,s_0}(y) = 1 + \sum_{\ell\geq 1} \frac{r_\ell(s_0)}{\ell!}(iy)^\ell, \ y \in \mathbb{R}.$$

□

3. Average values of ℓ-th moments (the non-square case).

In this section, we prove Theorem 0.1. Basing on Lemma 3.1, we divide $\mathcal{L}(s, M, \ell)_\star$ into four parts, where $\star$ is either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$. Proposition 3.2 is the source of the main term. The others (cf. Proposition 3.3, Proposition 3.4, and Proposition 3.5) give error terms.

3.1. **Dividing averaging sums into parts.** For convenience, we set $\deg 0 = -\infty$ and a function j from $\{\mathcal{R}, \mathcal{I}, \mathcal{S}\}$ to $\{0, 1\}$ defined by

$$j(\mathcal{R}) := 0,\ j(\mathcal{I}) := 1, \text{and } j(\mathcal{S}) := 0. \tag{3.1}$$

The following form can be regarded as a generalization of Hoffstein-Rosen's closed form in [6, Theorem 0.7, Theorem 1.4 and Theorem 1.5] (cf. Remark 3.6).

Lemma 3.1. *Let ℓ be a positive integer, and $\star$ can be either $\mathcal{S}, \mathcal{I}$ or $\mathcal{R}$. Then*

$$\begin{aligned}\mathcal{L}(s, M, \ell)_\star = &\sum_{\substack{N=0\\2\mid N}}^{\ell(M-1)} q^{M-N} \sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n) \left(\sum_{\substack{e\in A:\\ \deg e \leq N-M-2}} \tilde{G}_e(n) - \sum_{\substack{e\in A^+:\\ \deg e = N-M-1}} \tilde{G}_e(n) \right) q^{-sN} \\ &+ \sqrt{q} \sum_{\substack{N=0\\2\nmid N}}^{\ell(M-1)} q^{M-N} \left(\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n) \sum_{\substack{e\in A^+:\\ \deg e = N-M-1}} \gamma_{p,q} \cdot \tilde{G}_e(n) \right) (-1)^{j(\star)} \cdot q^{-sN} \\ &- \sum_{\substack{m\in A^+:\\ \deg m = M\\ m=\square}} \sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n)\chi_m(n) \right) (-1)^{j(\star)N} q^{-sN}.\end{aligned}$$

where $\tilde{G}_e(n)$ is the Gauss sum defined in (1.1), $\gamma_{p,q}$ is defined in (1.2), $d_\ell(n)$ is the general divisor function defined in (1.3), and j is the function defined in (3.1).

Proof. We have

$$\begin{aligned}\mathcal{L}(s, M, \ell)_\star = &\sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n) \sum_{m\in A} f(m)\chi_m(n) \right) (-1)^{j(\star)N} \cdot q^{-sN} \\ &- \sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n) \sum_{\substack{m\in A^+:\\ \deg m = M\\ m=\square}} \chi_m(n) \right) (-1)^{j(\star)N} q^{-sN} \\ := &\ \mathrm{I} - \mathrm{II},\end{aligned}$$

where $f(x) = \mathbf{1}_{\pi_\infty^{-M}(1+\pi_\infty O_\infty)}(x)$ and $\left[\frac{\gamma}{n}\right] = (-1)^N$. Splitting the sum over m below according to the residue classes mod n and using Proposition 1.1, we have

$$(3.2)\quad \begin{cases} \mathrm{I} &= \displaystyle\sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n) \sum_{b \mod n} \left[\frac{b}{n}\right] \sum_{e\in A} f(b+ne) \right) (-1)^{j(\star)N} \cdot q^{-sN} \\ &= \displaystyle\sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n) q^{-N} \sum_{e\in A} \hat{f}(d/n) G_e(n) \right) (-1)^{j(\star)N} \cdot q^{-sN}, \end{cases}$$

where $\hat{f}(x) = q^M \cdot \psi_\infty(\pi_\infty^{-M}x)\cdot \mathbf{1}_{\pi_\infty^{M+1}O_\infty}(x)$. Observe that

$$(3.3)\quad \begin{cases} \displaystyle\sum_{e\in A} \hat{f}(e/n) G_e(n) \\ = \displaystyle\sum_{e\in A} q^M \cdot \psi_\infty(\pi_\infty^{-M} e/n)\cdot \mathbf{1}_{\pi_\infty^{M+1}O_\infty}(e/n) G_e(n) \\ = \displaystyle\sum_{\substack{e\in A-\{0\}:\\ \deg e = \deg n - M - 1}} q^M \cdot \psi_\infty(\pi_\infty^{-M} e/n) G_e(n) + \sum_{\substack{e\in A:\\ \deg e \le \deg n - M-2}} q^M \cdot G_e(n), \\ := \mathrm{III} + \mathrm{IV}. \end{cases}$$

Now, we simplify III and IV. We have

$$\begin{aligned} \mathrm{III} &= \sum_{\substack{e\in A^+:\\ \deg e = \deg n - M-1}} q^M \cdot G_e(n) \left\{ \sum_{\epsilon\in\mathbb{F}_q^\times} \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-\epsilon)}{p} \right) \cdot \left[\frac{\epsilon}{n}\right] \right\} \\ &= q^M \cdot \begin{cases} -\displaystyle\sum_{\substack{e\in A^+:\\ \deg e = \deg n - M-1}} \tilde{G}_e(n), & \text{if } \deg n \text{ is even;} \\ \gamma_{p,q}\cdot\sqrt{q}\cdot \displaystyle\sum_{\substack{e\in A^+:\\ \deg e = \deg n - M-1}} \tilde{G}_e(n), & \text{if } \deg n \text{ is odd,} \end{cases} \end{aligned}$$

because of

$$\sum_{\epsilon\in\mathbb{F}_q^\times} \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-\epsilon)}{p} \right) \cdot \left[\frac{\epsilon}{n}\right] = \begin{cases} -1, & \text{if } \deg n \text{ is even;} \\ -\sqrt{q}\left[\frac{-1}{n}\right] \cdot \left(-\sqrt{\left(\frac{-1}{p}\right)} \right)^{[\mathbb{F}_q:\mathbb{F}_p]}, & \text{if } \deg n \text{ is odd.} \end{cases}$$

For IV, we note that

$$\sum_{\substack{e\in A-\{0\}:\\ \deg e\le N-M-2}} G_e(n) = \sum_{\substack{e\in A-\{0\}:\\ \deg e\le N-M-2}} G_{\gamma\cdot e}(n) = \sum_{\substack{e\in A-\{0\}:\\ \deg e\le N-M-2}} \left[\frac{\gamma}{n}\right] \cdot G_e(n),$$

and $G_0(n) = 0$, if $\deg n$ is odd. Thus, the above equality implies that

$$\mathrm{IV} = \frac{q^M}{2} \sum_{\substack{e\in A:\\ \deg e\le N-M-2}} \left(1 + (-1)^{\deg n}\right) G_e(n) = \frac{q^M}{2} \sum_{\substack{e\in A:\\ \deg e\le N-M-2}} \left(1 + (-1)^{\deg n}\right) \tilde{G}_e(n).$$

The last equality comes from $\tilde{G}_e(n) = G_e(n)$, if $\deg n$ is even. Inserting (3.3)= III+ IV into (3.2), we complete the proof.

□

On the basis of the above lemma, we divide $\mathcal{L}(s,M,\ell)_\star$, where $\star$ is either $\mathcal{S}$, $\mathcal{I}$ or $\mathcal{R}$, into four parts which are

$$\mathcal{P}_0(s)_\star := \sum_{\substack{N=0\\2|N}}^{\ell(M-1)} q^{M-N}\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n)\tilde{G}_0(n)\right)q^{-sN},$$

$$\mathcal{P}_\diamond(s)_\star := \sum_{\substack{N=0\\2|N}}^{\ell(M-1)} q^{M-N}\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n)\left(\sum_{\substack{e\in A-\{0\}:e\diamond\square\\ \deg e\le N-M-2}}\tilde{G}_e(n)-\sum_{\substack{e\in A^+:e\diamond\square\\ \deg e=N-M-1}}\tilde{G}_e(n)\right)q^{-sN}$$
$$+\gamma_{p,q}\cdot\sqrt{q}\cdot\sum_{\substack{N=0\\2\nmid N}}^{\ell(M-1)} q^{M-N}\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n)\sum_{\substack{e\in A^+:e\diamond\square\\ \deg e=N-M-1}}\tilde{G}_e(n)\right)(-1)^{j(\star)}q^{-sN},$$

where $\diamond$ is $=$ or $\neq$, and the term

$$\mathcal{P}_1(s)_\star := \sum_{\substack{m\in A^+:\\ \deg m=M\\ m=\square}}\sum_{N=0}^{\ell(M-1)}\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n)\chi_m(n)\right)(-1)^{j(\star)N}q^{-sN},$$

such that

$$\mathcal{L}(s,M,\ell)_\star = \mathcal{P}_0(s)_\star+\mathcal{P}_=(s)_\star-\mathcal{P}_1(s)_\star+\mathcal{P}_{\neq}(s)_\star.$$

3.2. **The contributions of $\mathcal{P}_0$, $\mathcal{P}_=$, $\mathcal{P}_1$, and $\mathcal{P}_{\neq}$.** For $\mathcal{P}_0(s)_\star$, we establish the following asymptotic formula:

Proposition 3.2. *Let ℓ, M be positive integers, and $\star$ be either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$. then, for any $\delta>0$,*

$$\mathcal{P}_0(s)_\star = \zeta_A(2s)^{\frac{\ell(\ell+1)}{2}}\cdot c_\ell(s)\cdot q^M + O(q^{(1+(-\Re(s)+1/2+\delta))M}),\ \textit{if}\ \Re(s)\ge 1,$$

as $M\to\infty$. Here $c_\ell(s)$ is introduced in Theorem 0.1.

Proof. Suppose that $\ell\ge 1$ and $\Re(s)>1/2$. Then we have

$$\sum_{\substack{N=0\\2|N}}^{\ell(M-1)} q^{M-N}\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n)\cdot\tilde{G}_0(n)\right)q^{-sN}$$
$$=\sum_{N=0}^{\infty} q^M\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n^2)\cdot\varphi(n^2)\right)q^{-2(1+s)N} - \sum_{N=\lfloor\frac{\ell(M-1)}{2}\rfloor+1}^{\infty} q^M\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n^2)\cdot\varphi(n^2)\right)q^{-2(1+s)N}$$
$$:= \mathrm{I}-\mathrm{II}.$$

By Lemma 2.3, we have

$$\mathrm{I} = q^M\cdot\sum_{N=0}^{\infty}\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n^2)\cdot\varphi(n^2)\right)q^{-2(1+s)N} = q^M\cdot\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}}\cdot c_\ell(s),\ \text{on}\ \Re(s)>\frac{1}{2},$$

and for any $\delta>0$,

$$\mathrm{II}\ll q^M\frac{q^{-(2\Re(s)-1-\delta)\frac{\ell(M-1)}{2}}}{1-q^{-(2\Re(s)-1-\delta)}}.$$

Combining the above estimations, we complete this proof. $\square$

As for $\mathcal{P}_{=}(s)_\star$, it appears on the case $\ell > 1$. If $\ell = 1$, then $\mathcal{P}_{=}(s)_\star$ always equals to 0.

Proposition 3.3. *Let $\ell \geq 2, M$ be positive integers, and $\star$ be either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$. Then we have, for any $\delta > 0$,*

$$\mathcal{P}_{=}(s)_\star = q^{(1/2+\delta)M}, \ \textit{if } \Re(s) \geq 1,$$

as $M \to \infty$.

Proof. We have Suppose $\Re(s) \geq 1$. We have, by Corollary 2.6,

$$\begin{aligned}
\mathcal{P}_{=}(s)_\star \ll \sum_{\substack{N \geq M+1 \\ 2 \mid N}}^{\ell(M-1)} q^{M-N} \sum_{\substack{d \in A - \{0\}: \\ \deg d \leq \left\lfloor \frac{N-M-1}{2} \right\rfloor}} q^{(\frac{3}{2}+\delta)N} q^{-\Re(s)N}, \text{ for any } \delta > 0 \\
\ll \sum_{\substack{N \geq M+1 \\ 2 \mid N}}^{\ell(M-1)} q^{M/2} q^{(1+\delta-\Re(s))N} + \sum_{\substack{N \geq M+1 \\ 2 \mid N}}^{\ell(M-1)} q^{M} q^{(1/2+\delta-\Re(s))N} \\
\ll q^{\left(\frac{1}{2}+(1+\delta-\Re(s))\right)M} \ll q^{(1/2+\delta)M}.
\end{aligned}$$

□

The third contribution only occurs to the case of $2 \mid M$.

Proposition 3.4. *Let ℓ be a positive integer and $\star$ be either $\mathcal{S}$ or $\mathcal{I}$. Then we have, for $\delta > 0$,*

$$\mathcal{P}_1(s)_\star = O\left(q^{\left(\frac{1}{2}+\delta\right)M}\right), \ \textit{if } \Re(s) \geq 1,$$

as $M \to \infty$.

Proof. Suppose that $\Re(s) \geq 1$.

$$\begin{aligned}
\mathcal{P}_1(s)_\star = & \sum_{\substack{m \in A^+: \\ \deg m = \frac{M}{2}}} \sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n \in A^+: \\ \deg n = N \\ (n,m)=1}} d_\ell(n) \right) (-1)^{j(\star)N} q^{-sN} \\
\ll & \sum_{\substack{m \in A^+: \\ \deg m = \frac{M}{2}}} \sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n \in A^+: \\ \deg n = N}} d_\ell(n) \right) q^{-\Re(s)N} \\
\ll & q^{M/2} \sum_{N=0}^{\ell(M-1)} q^{(1-\Re(s)+\delta)N}, \text{ by By Lemma 2.7} \\
\ll & q^{\left(\frac{1}{2}+\ell\delta\right)M}, \text{ for any } \delta > 0.
\end{aligned}$$

□

The estimation of $\mathcal{P}_{\neq}(s)_\star$ is stated as follows:

Proposition 3.5. *Let ℓ and M be positive integers and $\star$ be either $\mathcal{S}$, $\mathcal{I}$ or $\mathcal{R}$, then we have, for any $\delta > 0$,*

$$\mathcal{P}_{\neq}(s)_\star = O\left(q^{\delta M}\right), \ \textit{if } \Re(s) \geq 1,$$

as $M \to \infty$.

Proof. Suppose $\Re(s) \geq 1$.

$$\mathcal{P}_{\neq}(s)_\star \ll \sum_{\substack{N=0\\2|N}}^{\ell(M-1)} q^{M-N}\left(\sum_{\substack{d\in A^+:d\neq\square\\ \deg d\leq N-M-1}} q^{(1+\delta)N}\right) q^{-\Re(s)N}, \text{ by Corollary 2.6}$$

$$\ll \sum_{\substack{N\geq M+1\\2|N}}^{\ell(M-1)} q^{(1+\delta)N}\cdot q^{-\Re(s)N} + \sum_{\substack{N\geq M+1\\2|N}}^{\ell(M-1)} q^M q^{(\delta-\Re(s))N} \ll q^{\ell\delta M}.$$

The proof is finished. □

Remark 3.6. When $\ell = 1$, we have, for $\star = \mathcal{I}$, $\mathcal{S}$ or $\mathcal{R}$,

$$\mathcal{L}(s,M,1)_\star = q^M + \sum_{\substack{N=2\\2|N}}^{M-1} q^{M-N}\left(\sum_{\substack{n\in A^+:\\ \deg n=N\\ a=\square}} \sum_{a \mod n}\left[\frac{a}{n}\right] + \sum_{\substack{n\in A^+:\\ \deg n=N\\ a\neq\square}} \sum_{a \mod n}\left[\frac{a}{n}\right]\right) q^{-sN}$$

$$= q^M + \sum_{\substack{N=2\\2|N}}^{M-1} q^{M-N}\left(\sum_{\substack{n\in A^+:\\ \deg n=N/2}} \varphi(n^2)\right) q^{-sN},$$

$$= q^M + q^M(1-q^{-1})\sum_{\substack{N=2\\2|N}}^{M-1} q^{(1/2-s)N}, \text{ by [9, Proposition 2.7]}$$

which also leads to the result [6, Theorem 0.7]. Similarly, [6, Theorem 1.4 and Theorem 1.5] also can be obtained by computing extra term $\mathcal{P}_1(s)_\star$, where $\star$ is $\mathcal{S}$ or $\mathcal{I}$.

4. Average values of ℓ-th moment of quadratic L-funcitons (the square-free case).

The idea of Theorem 0.6 is similarly to Theorem 0.1, but it is more complex. Let $\mu(f)$ be Möbius function for A and $n \in A^+$. Then $n \mapsto \sum_{g^2|n} \mu(g)$ is the characteristic function for square-free polynomials n. Using this fact, we have an analogue closed form Lemma 4.1 as Lemma 3.1. The function $\mathcal{L}(s,M,\ell)_\star$ can be divided into three parts, where $\star$ is either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$. Proposition 4.2 is the source of the main term. The others (cf. Proposition 4.3, and Proposition 4.4) give error terms.

4.1. Dividing averaging sums into parts.

Similarly, the sums in question can be rewritten as the following form:

Lemma 4.1. *Let ℓ be a positive integer, and $\star$ can be either $\mathcal{S},\mathcal{I}$ or $\mathcal{R}$. Then*

$$\mathcal{L}^*(s,M,\ell)_\star$$

$$= \sum_{\substack{N=0\\2|N}}^{\ell(M-1)} \sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n) \sum_{0\leq G\leq\lfloor\frac{M}{2}\rfloor} q^{M-2G} \sum_{\substack{g\in A^+:\\ \deg g=G,\\ (g,n)=1}} \mu(g)\left(\sum_{\substack{e\in A:\\ \deg e\leq N-M+2G-2}} \tilde{G}_e(n) - \sum_{\substack{e\in A^+:\\ \deg e=N-M+2G-1}} \tilde{G}_e(n)\right) q^{-(1+s)N}$$

$$+\gamma_{p,q}\cdot\sqrt{q}\sum_{\substack{N=0\\2\nmid N}}^{\ell(M-1)}\left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n)\sum_{0\leq G\leq\lfloor\frac{M}{2}\rfloor} q^{M-2G-N}\sum_{\substack{g\in A^+:\\ \deg g=G,(g,n)=1}} \mu(g)\sum_{\substack{e\in A^+:\\ \deg e=N-M+2G-1}} \tilde{G}_e(n)\right)(-1)^{j(\star)}\cdot q^{-sN}.$$

where $\tilde{G}_e(n)$ is the Gauss sum defined in (1.1), $\gamma_{p,q}$ is defined in (1.2), $d_\ell(n)$ is the general divisor function defined in (1.3), and j is the function defined in (3.1).

Proof. We have

$$\mathcal{L}^*(s,M,\ell)_\star = \sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n) \sum_{\substack{g\in A^+:\\ \deg g\le \lfloor \frac{M}{2}\rfloor}} \mu(g) \sum_{\substack{m\in A^+:\\ g^2|m}} \chi_m(n)\right) (-1)^{j(\star)N}\cdot q^{-sN}.$$

Write $m=g^2m_1$, where $m_1\in A^+$. Then the above equality is equal to

$$\begin{aligned}
\mathrm{I} :=& \sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n) \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} \sum_{\substack{g\in A^+:\\ \deg g=G}} \mu(g) \sum_{\substack{m_1\in A^+:\\ \deg m_1=M-2G}} \chi_{g^2m_1}(n)\right) (-1)^{j(\star)N}\cdot q^{-sN}\\
=& \sum_{N=0}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n) \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} \sum_{\substack{g\in A^+:\\ \deg g=G\\ (g,n)=1}} \mu(g) \sum_{\substack{m_1\in A^+:\\ \deg m_1=M-2G}} \left[\frac{m_1}{n}\right]\right) (-1)^{j(\star)N}\cdot q^{-sN}.
\end{aligned}$$

Let $f(x)=\mathbf{1}_{\pi_\infty^{-M+2G}(1+\pi_\infty O_\infty)}(x)$. Then

$$\sum_{\substack{m_1\in A^+:\\ \deg m_1=M-2G}} \left[\frac{m_1}{n}\right] = \sum_{m_1\in A} f(m_1)\left[\frac{m_1}{n}\right].$$

Using Proposition 1.1, the above equality is equal to

$$\begin{aligned}
\mathrm{II} :=& \sum_{b \mod n} \left[\frac{b}{n}\right] \sum_{e\in A} f(b+ne) = \sum_{b \mod n} \left[\frac{b}{n}\right] q^{-N} \sum_{e\in A} \hat{f}(e/n)\psi_\infty\left(-\frac{be}{n}\right)\\
=& q^{-N}\sum_{e\in A} \hat{f}(e/n) G_e(n).
\end{aligned}$$

where $\hat{f}(x)=q^{M-2G}\cdot\psi_\infty(\pi_\infty^{-M+2G}x)\cdot \mathbf{1}_{\pi_\infty^{M-2G+1}O_\infty}(x)$.

Observe that

$$\begin{aligned}
\mathrm{II} =& \sum_{\substack{e\in A-\{0\}:\\ \deg e=\deg n-M+2G-1}} q^{M-2G-N}\cdot \psi_\infty(\pi_\infty^{-M+2G}e/n)G_e(n) + \sum_{\substack{e\in A:\\ \deg e\le \deg n-M+2G-2}} q^{M-2G-N}\cdot G_e(n)\\
:=& \mathrm{III}+\mathrm{IV}.
\end{aligned}$$

Using the same argument as Lemma 3.1, we have

$$\mathrm{III} = q^{M-2G-N}\cdot \begin{cases} -\displaystyle\sum_{\substack{e\in A^+:\\ \deg e=\deg n-M+2G-1}} \tilde{G}_e(n), & \text{if } \deg n \text{ is even;}\\ \gamma_{p,q}\cdot\sqrt{q}\cdot \displaystyle\sum_{\substack{e\in A^+:\\ \deg e=\deg n-M+2G-1}} \tilde{G}_e(n), & \text{if } \deg n \text{ is odd,}\end{cases}$$

and

$$\mathrm{IV} = q^{M-2G-N}\cdot\frac{1}{2}\sum_{\substack{e\in A:\\ \deg e\le N-M+2G-2}} (1+(-1)^{\deg n})\cdot \tilde{G}_e(n).$$

Inserting II= III+ IV into I, the proof is complete. □

On the basis of the above lemma, we divide $\mathcal{L}^*(s,M,\ell)_\star$, where $\star$ is either $\mathcal{S}$, $\mathcal{I}$ or $\mathcal{R}$, into three parts which are

$$\mathcal{P}_0^*(s)_\star := \sum_{\substack{N=0\\2|N}}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n) \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} q^{M-2G-N} \sum_{\substack{g\in A^+:\\ \deg g=G\\ (g,n)=1}} \mu(g)\tilde{G}_0(n)\right) q^{-sN},$$

and

$$\mathcal{P}_\diamond^*(s)_\star$$
$$:= \sum_{\substack{N=0\\2|N}}^{\ell(M-1)} \sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n) \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} q^{M-2G-N} \sum_{\substack{g\in A^+:\\ \deg g=G\\ (g,n)=1}} \mu(g) \left(\sum_{\substack{e\in A-\{0\}:e\diamond\Box\\ \deg e\le N-M+2G-2}} \tilde{G}_e(n) - \sum_{\substack{e\in A^+:e\diamond\Box\\ \deg e=N-M+2G-1}} \tilde{G}_e(n)\right) q^{-sN}$$

$$+\gamma_{p,q}\cdot\sqrt{q}\cdot \sum_{\substack{N=0\\2\nmid N}}^{\ell(M-1)} \left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n) \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} q^{M-2G-N} \sum_{\substack{g\in A^+:\\ \deg g=G\\ (g,n)=1}} \mu(g) \sum_{\substack{e\in A^+:e=\Box\\ \deg e\diamond N-M+2G-1}} \tilde{G}_e(n)\right) (-1)^{j(\star)} q^{-sN},$$

where $\diamond$ is $=$ or $\neq$, such that

$$\mathcal{L}^*(s,M,\ell)_\star = \mathcal{P}_0^*(s)_\star + \mathcal{P}_=^*(s)_\star + \mathcal{P}_{\neq}^*(s)_\star.$$

4.2. **The contributions of $\mathcal{P}_0^*$, $\mathcal{P}_=^*$, and $\mathcal{P}_{\neq}^*$.** For $\mathcal{P}_1^*(s)_\star$, we establish the following asymptotic formula:

Proposition 4.2. *Let ℓ, M be positive integers, and $\star$ be either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$. Then we have, for any $\delta>0$,*

$$\mathcal{P}_0^*(s)_\star = \zeta_A(2)^{-1}\zeta_A(2s)^{\frac{\ell(\ell+1)}{2}}\cdot c_\ell^*(s)\cdot q^M + O\left(q^{(1/2+\delta)M}\right),\ \textit{if}\ \Re(s)\ge 1,$$

as $M\to\infty$. Here $c_\ell^*(s)$ is introduced in Theorem 0.6.

Proof. Suppose that $\ell\ge 1$. We have

$$\mathcal{P}_0^*(s)_\star = \sum_{\substack{N=0\\2|N}}^{\ell(M-1)} \left(\sum_{\substack{\deg n=N/2\\ n\in A^+}} d_\ell(n^2)\varphi(n^2) \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} q^{M-2G-N} \sum_{\substack{g\in A^+:\\ \deg g=G\\ (g,n)=1}} \mu(g)\right) q^{-sN}. \tag{4.1}$$

Since

$$\sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} q^{-2G} \sum_{\substack{g\in A^+:\\ \deg g=G\\ (g,n)=1}} \mu(g) = \sum_{\substack{g\in A^+:\\ (g,n)=1}} \frac{\mu(g)}{q^{2\deg g}} - \sum_{\lfloor \frac{M}{2}\rfloor<G} q^{-2G} \sum_{\substack{g\in A^+:\\ \deg g=G\\ (g,n)=1}} \mu(g)$$
$$=\zeta_A(2)^{-1}\cdot\nu(n) + O(q^{-\frac{M}{2}}),\ \text{where}\ \nu(n)=\prod_{P|n}\left(1-q^{-2\deg P}\right)^{-1}\ \text{is defined in (2.2)},$$

we have (4.1) which is equal to

$$\frac{q^M}{\zeta_A(2)}\cdot \sum_{N=0}^{\lfloor\frac{\ell(M-1)}{2}\rfloor} \left(\sum_{\substack{n\in A^+:\\ \deg n=N}} d_\ell(n^2)\cdot\varphi(n^2)\cdot\nu(n)\right) q^{-2(1+s)N} + O\left(q^{M/2} \sum_{N=0}^{\lfloor\frac{\ell(M-1)}{2}\rfloor} \sum_{\substack{n\in A^+:\\ \deg n=N}} \frac{d_\ell(n^2)\cdot\varphi(n^2)}{q^{2(1+s)N}}\right)$$

$:= \mathrm{I}+\mathrm{II}.$

Suppose that $\Re(s) > 1/2$. We have

$$\mathrm{II} = O(q^{M/2}), \text{ by Lemma 2.3,}$$

and for any $\delta > 0$,

$$\mathrm{I} = \sum_{n\in A^+} \frac{d_\ell(n^2)\cdot\varphi(n^2)\cdot\nu(n)}{q^{(2s+2)\deg n}} - q^M \cdot \sum_{N=\lfloor \frac{\ell(M-1)}{2}\rfloor+1}^{\infty} \left(\sum_{\substack{n\in A^+:\\ \deg n = N}} d_\ell(n^2)\cdot\varphi(n^2)\cdot\nu(n) \right) q^{-2(1+s)N}$$

$$= \sum_{n\in A^+} \frac{d_\ell(n^2)\cdot\varphi(n^2)\cdot\nu(n)}{q^{(2s+2)\deg n}} + O\left(q^M q^{(-\ell/2+\ell\delta/2)M}\right), \text{ if } \Re(s)\ge 1.$$

Combining the above estimations, we complete the proof. □

As for $\mathcal{P}^*_{=}(s)_\star$, we have

Proposition 4.3. *Let ℓ, M be positive integers, and $\star$ be either $\mathcal{S}$, $\mathcal{I}$, or $\mathcal{R}$. Then we have, for any $\delta > 0$,*

$$\mathcal{P}^*_{=}(s)_\star = O\left(q^{\left(\frac{1}{2}+\delta\right)M}\right), \text{ if } \Re(s)\ge 1,$$

as $M\to\infty$.

Proof. The equalities $\displaystyle\sum_{\substack{g\in A^+:\\ \deg g = G\\ (g,n)=1}} \mu(g) = O(q^G)$,

$$\sum_{\substack{e\in A-\{0\}:\\ \deg e\le \lfloor \frac{N-M+2G-2}{2}\rfloor}} \tilde{G}_{e^2}(n) = (q-1) \sum_{\substack{e\in A^+:\\ \deg e\le \lfloor \frac{N-M+2G-2}{2}\rfloor}} \tilde{G}_{e^2}(n)$$

and Corollary 2.6 say that, for any $\delta > 0$,

$$\mathcal{P}^*_{=}(s)_\star \ll \sum_{\substack{N=0\\ 2|N}}^{\ell(M-1)} \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} q^{M-2G-N} \sum_{\substack{g\in A^+:\\ \deg g = G}} \mu(g) \sum_{\substack{e\in A^+:\\ \deg e\le \lfloor \frac{N-M+2G-1}{2}\rfloor}} \left(\sum_{\substack{n\in A^+:\\ \deg n = N\\ (n,g)=1}} d_\ell(n)\tilde{G}_{e^2}(n) \right) q^{-sN}$$

$$\ll \sum_{\substack{N=0\\ 2|N}}^{\ell(M-1)} \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} q^{M-G-N} \sum_{\substack{e\in A^+:\\ \deg e\le \lfloor \frac{N-M+2G-2}{2}\rfloor}} q^{(3/2+\delta-\Re(s))N}.$$

Note that if $\deg d \ge 0$, then $N \ge M-2G+2$. Thus we have

$$\ll \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} \sum_{\substack{N\ge M-2G+2\\ 2|N}}^{\ell(M-1)} q^{M-G}\left(q^{\lfloor \frac{N-M+2G-2}{2}\rfloor+1}-1\right) q^{(1/2+\delta-\Re(s))N}$$

$$\ll M\cdot q^{M/2}\cdot q^{(1+\delta-\Re(s))N} + \sum_{0\le G\le \lfloor \frac{M}{2}\rfloor} q^{M-G}\left(q^{(1/2+\delta-\Re(s))\ell M} + q^{(1/2+\delta-\Re(s))(M-2G+2)}\right)$$

$$\ll q^{(1/2+\ell\delta)M}, \text{ if } \Re(s)\ge 1.$$

□

The estimation of $\mathcal{P}^*_{\neq}(s)_\star$ is stated as follows:

Proposition 4.4. *Let ℓ, M be positive integers and $\star$ be either $\mathcal{S}$, $\mathcal{I}$ or $\mathcal{R}$, then we have, for any $\delta > 0$,*

$$\mathcal{P}^*_{\neq}(s)_\star = O\left(q^{\left(\frac{1}{2}+\delta\right)M}\right), \text{ if } \Re(s)\ge 1,$$

as $M \to \infty$.

Proof.

$$\mathcal{P}^{*}_{\neq}(s)_{\star} \ll \sum_{0 \le G \le \lfloor \frac{M}{2} \rfloor} \sum_{\substack{N=0 \\ 2|N}}^{\ell(M-1)} q^{M-G-N}(q-1) \left(\sum_{\substack{e \in A^{+}: e \neq \square \\ \deg e \le N-M+2G-1}} q^{(1+\delta)N} \right) q^{-\Re(s)N}, \text{ by Corollary 2.6}$$

Note that if $\deg d \geq 0$, then $N \geq M - 2G + 1$. Thus we have

$$\begin{aligned} &\ll \sum_{0 \le G \le \lfloor \frac{M}{2} \rfloor} \sum_{\substack{N \ge M-2G+1 \\ 2|N}}^{\ell(M-1)} q^{M-G}(q^{N-M+2G+1}-1) q^{(\delta-\Re(s))N} \\ &\ll q^{M/2} \cdot q^{(1+\delta-\Re(s))\ell M} + \sum_{0 \le G \le \lfloor \frac{M}{2} \rfloor} q^{M-G} \left(q^{(\delta-\Re(s))\ell M} + q^{(\delta-\Re(s))(M-2G)} \right) \\ &\ll q^{\left(\frac{1}{2}+\ell\delta\right)M}, \text{ if } \Re(s) \geq 1. \end{aligned}$$

□

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