HALL ALGEBRAS OF CYCLIC QUIVERS AND q-DEFORMED FOCK SPACES

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Abstract. Based on the work of Ringel and Green, one can define the (Drinfeld) double Ringel–Hall algebra $\boldsymbol{\mathcal{D}}(Q)$ of a quiver Q as well as its highest weight modules. The main purpose of the present paper is to show that the basic representation $L(\Lambda_0)$ of $\boldsymbol{\mathcal{D}}(\Delta_n)$ of the cyclic quiver Δ_n provides a realization of the q-deformed Fock space $\bigwedge^\infty$ defined by Hayashi. This is worked out by extending a construction of Varagnolo and Vasserot. By analysing the structure of nilpotent representations of Δ_n, we obtain a decomposition of the basic representation $L(\Lambda_0)$ which induces the Kashiwara–Miwa–Stern decomposition of $\bigwedge^\infty$ and a construction of the canonical basis of $\bigwedge^\infty$ defined by Leclerc and Thibon in terms of certain monomial basis elements in $\boldsymbol{\mathcal{D}}(\Delta_n)$.

1. Introduction

In [40], Ringel introduced the Hall algebra $\boldsymbol{\mathcal{H}}(\Delta_n)$ of the cyclic quiver Δ_n with n vertices and showed that its subalgebra generated by simple representations, called the composition algebra, is isomorphic to the positive part $\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n)$ of the quantized enveloping algebra $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$. Schiffmann [41] further showed that $\boldsymbol{\mathcal{H}}(\Delta_n)$ is the tensor product of $\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n)$ with a central subalgebra which is the polynomial ring in infinitely many indeterminates. Following the approach in [46], the double Ringel–Hall algebra $\boldsymbol{\mathcal{D}}(\Delta_n)$ was defined in [6]. Based on [12, 21] and an explicit description of central elements of $\boldsymbol{\mathcal{H}}(\Delta_n)$ in [19], it was shown in [6, Th. 2.3.3] that $\boldsymbol{\mathcal{D}}(\Delta_n)$ is isomorphic to the quantum affine algebra $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$ defined by Drinfeld's new presentation [10].

The q-deformed Fock space representation $\bigwedge^\infty$ of the quantized enveloping algebra $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ has been constructed by Hayashi [17], and its crystal basis was described by Misra and Miwa [36]. Further, by work of Kashiwara, Miwa, and Stern [27], the action of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ on $\bigwedge^\infty$ is centralized by a Heisenberg algebra which arises from affine Hecke algebras. This yields a bimodule isomorphism from $\bigwedge^\infty$ to the tensor product of the basic representation of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ and the Fock space representation of the Heisenberg algebra.

By defining a natural semilinear involution on $\bigwedge^\infty$, Leclerc and Thibon [29] obtained in an elementary way a canonical basis of $\bigwedge^\infty$. It was conjectured in [28, 29] that for $q = 1$, the coefficients of the transition matrix of the canonical basis on the natural basis of $\bigwedge^\infty$ are equal to the decomposition numbers for Hecke algebras and quantum Schur algebras at roots of unity. These conjecture have been proved, respectively, by Ariki [1] and Varagnolo and Vasserot [47]. For the categorification of the Fock space, see, for example, [43, 18, 45].

In [47], Varagnolo and Vasserot extended the $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$-action on the Fock space $\bigwedge^\infty$ to that of the extended Ringel–Hall algebra $\boldsymbol{\mathcal{D}}(\Delta_n)^{\leqslant 0}$ of the cyclic quiver Δ_n. They also showed that the canonical basis of the Ringel–Hall algebra $\boldsymbol{\mathcal{H}}(\Delta_n)$ in the sense of Lusztig induces a basis of $\bigwedge^\infty$ which conjecturally coincides with the canonical basis constructed by Leclerc and Thibon [29]. This conjecture was proved by Schiffmann [41] by identifying the central subalgebra of $\boldsymbol{\mathcal{H}}(\Delta_n)$ with the ring of symmetric functions.

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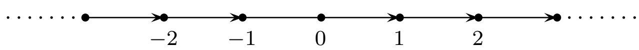
The main purpose of the present paper is to extend Varagnolo–Vasserot's construction to obtain a $\boldsymbol{\mathcal{D}}(\Delta_n)$-module structure on the Fock space $\bigwedge^\infty$ which is shown to be isomorphic to the basic representation $L(\Lambda_0)$ of $\boldsymbol{\mathcal{D}}(\Delta_n)$. Moreover, the central elements in the positive and negative parts of $\boldsymbol{\mathcal{D}}(\Delta_n)$ constructed by Hubery [19] give rise naturally to the operators introduced in [27] which generate the Heisenberg algebra. Furthermore, the structure of $\boldsymbol{\mathcal{D}}(\Delta_n)$ yields a decomposition of $L(\Lambda_0)$ which induces the Kashiwara–Miwa–Stern decomposition of $\bigwedge^\infty$. This also provides a way to construct the canonical basis of $\bigwedge^\infty$ in [29] in terms of certain monomial basis elements of $\boldsymbol{\mathcal{D}}(\Delta_n)$.

The paper is organized as follows. In Section 2 we review the classification of (nilpotent) representations of both infinite linear quiver Δ_∞ and the cyclic quiver Δ_n with n vertices and discuss their generic extensions. Section 3 recalls the definition of Ringel–Hall algebras $\boldsymbol{\mathcal{H}}(\Delta_\infty)$ and $\boldsymbol{\mathcal{H}}(\Delta_n)$ of Δ_∞ and Δ_n as well as the maps from the homogeneous spaces of $\boldsymbol{\mathcal{H}}(\Delta_n)$ to those of $\boldsymbol{\mathcal{H}}(\Delta_\infty)$ introduced in [47]. The images of basis elements of $\boldsymbol{\mathcal{H}}(\Delta_n)$ under these maps are described. In Section 4 we first follow the approach in [46] to present the construction of double Ringel–Hall algebras of both Δ_∞ and Δ_n and then study the irreducible highest weight $\boldsymbol{\mathcal{D}}(\Delta_n)$-modules based on the results in [23]. Section 5 recalls from [17, 36, 47] the Fock space representation $\bigwedge^\infty$ over $\mathbf{U}_v(\widehat{\mathfrak{sl}}_\infty)$ ($\cong \boldsymbol{\mathcal{D}}(\Delta_\infty)$) as well as over $\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n)$. In Section 6 we define the $\boldsymbol{\mathcal{D}}(\Delta_n)$-module structure on $\bigwedge^\infty$ based on [27, 47]. It is shown in Section 7 that $\bigwedge^\infty$ is isomorphic to the basic representation of $\boldsymbol{\mathcal{D}}(\Delta_n)$. In the final section, we present a way to construct the canonical basis of $\bigwedge^\infty$ and interpret the "ladder method" construction of certain basis elements in $\bigwedge^\infty$ in terms of generic extensions of nilpotent representations of Δ_n.

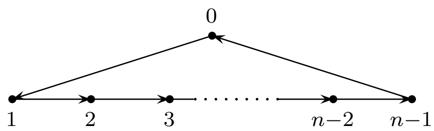
2. Nilpotent representations and generic extensions

In this section we consider nilpotent representations of both a cyclic quiver $\Delta = \Delta_n$ with n vertices ($n \geqslant 2$) and the infinite quiver $\Delta = \Delta_\infty$ of type A_∞^∞ and study their generic extensions. We show that the degeneration order of nilpotent representations of Δ_n induces the dominant order of partitions.

Let Δ_∞ denote the infinite quiver of type A_∞^∞



with vertex set $I = I_\infty = \mathbb{Z}$, and for $n \geqslant 2$, let Δ_n denote the cyclic quiver



with vertex set $I = I_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$. For each $i \in I_\infty = \mathbb{Z}$, let $\bar{i}$ denote its residue class in $I_n = \mathbb{Z}/n\mathbb{Z}$. We also simply write $\bar{i} \pm 1$ to denote the residue class of $i \pm 1$ in $\mathbb{Z}/n\mathbb{Z}$.

Given a field k, we denote by $\mathrm{Rep}^0\Delta$ the category of finite dimensional nilpotent representations of Δ ($= \Delta_\infty$ or Δ_n) over k. (Note that each finite dimensional representation of Δ_∞ is automatically nilpotent.) Given a representation $V = (V_i, V_\rho) \in \mathrm{Rep}^0\Delta$, the vector $\mathbf{dim}\, V = (\dim_k V_i)_{i\in I}$ is called the *dimension vector* of V. The Grothendieck group of $\mathrm{Rep}^0\Delta$ is identified with the free abelian group $\mathbb{Z}I$ with basis I. Let $\{\varepsilon_i \mid i \in I\}$ denote the standard basis of $\mathbb{Z}I$. Thus, elements in $\mathbb{Z}I$ will be written as $\mathbf{d} = (d_i)_{i\in I}$ or $\mathbf{d} = \sum_{i\in I} d_i\varepsilon_i$. In case $I = \mathbb{Z}/n\mathbb{Z}$, we sometimes write $\mathbb{Z}^n$ for $\mathbb{Z}I$.

The Euler form $\langle -, - \rangle : \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ is defined by

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle = \dim_k \mathrm{Hom}_{k\Delta}(M, N) - \dim_k \mathrm{Ext}^1_{k\Delta}(M, N).$$

Its symmetrization

$$(\mathbf{dim}\, M, \mathbf{dim}\, N) = \langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle + \langle \mathbf{dim}\, N, \mathbf{dim}\, M\rangle$$

is called the symmetric Euler form.

It is well known that the isoclasses (isomorphism classes) of representations in $\mathrm{Rep}^0\Delta$ are parametrized by the set $\mathfrak{M}$ consisting of all multisegments

$$\mathfrak{m} = \sum_{i\in I,\, l\geqslant 1} m_{i,l}[i,l),$$

where all $m_{i,l}\in\mathbb{N}$ but finitely many are zero. More precisely, the representation $M(\mathfrak{m}) = M_k(\mathfrak{m})$ associated with $\mathfrak{m}$ is defined by

$$M(\mathfrak{m}) = \bigoplus_{i\in I, l\geqslant 1} m_{i,l} S_i[l],$$

where $S_i[l]$ denotes the indecomposable representation of Δ with the simple top S_i and length l. For each $\mathbf{d}\in\mathbb{N}I$, put

$$\mathfrak{M}^{\mathbf{d}} = \{\mathfrak{m}\in\mathfrak{M} \mid \mathbf{dim}\, M(\mathfrak{m}) = \mathbf{d}\}.$$

Furthermore, we will write $\mathfrak{M} = \mathfrak{M}_\infty$ (resp., $\mathfrak{M} = \mathfrak{M}_n$) if $I = \mathbb{Z}$ (resp., $I = \mathbb{Z}/n\mathbb{Z}$).

It is also known that there exist Auslander–Reiten sequences in $\mathrm{Rep}^0\Delta$, that is, for each $M\in\mathrm{Rep}^0\Delta$, there is an Auslander–Reiten sequence

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0,$$

where τM denotes the Auslander–Reiten translation of M. It is clear that τ induces an isomorphism $\tau : \mathbb{Z}I \to \mathbb{Z}I$ such that $\tau(\mathbf{dim}\, M) = \mathbf{dim}\, \tau M$. In particular, $\tau(\varepsilon_i) = \varepsilon_{i+1}$, $\forall\, i\in I$. If $\Delta = \Delta_n$, then $\tau^{sn} = \mathrm{id}$ for all $s\in\mathbb{Z}$. For $\mathfrak{m}\in\mathfrak{M}$, let $\tau\mathfrak{m}$ be defined by $M(\tau\mathfrak{m}) \cong \tau M(\mathfrak{m})$.

Given $\mathbf{d}\in\mathbb{N}I$, let $V = \oplus_{i\in I} V_i$ be an I-graded vector space with dimension vector $\mathbf{d}$. Consider

$$E_V = \{(x_i)\in \bigoplus_{i\in I}\mathrm{Hom}_k(V_i, V_{i+1}) \mid x_{n-1}\cdots x_0 \text{ is nilpotent if } \Delta = \Delta_n.\}.$$

Then each element $x\in E_V$ defines a representation (V,x) of dimension vector $\mathbf{d}$ in $\mathrm{Rep}^0\Delta$. Moreover, the group

$$G_V = \prod_{i\in I}\mathrm{GL}(V_i)$$

acts on E_V by conjugation, and there is a bijection between the G_V-orbits and the isoclasses of representations in $\mathrm{Rep}^0\Delta$ of dimension vector $\mathbf{d}$. For each $x\in E_V$, by $\mathcal{O}_x$ we denote the G_V-orbit of x. In case k is algebraically closed, we have the equalities

$$\dim\mathcal{O}_x = \dim G_V - \dim\mathrm{End}_{k\Delta}(V,x) = \sum_{i\in I} d_i^2 - \dim\mathrm{End}_{k\Delta}(V,x). \tag{2.0.1}$$

By abuse of notation, for each $M\in\mathrm{Rep}^0\Delta$, we denote by $\mathcal{O}_M$ the orbit of M.

Following [3, 37, 5], given two representations M, N in $\mathrm{Rep}^0\Delta$, there exists a unique (up to isomorphism) extension G of M by N such that $\dim\mathrm{End}_{k\Delta}(G)$ is minimal. The extension G is called the *generic extension* of M by N, denoted by $M * N$. Moreover, generic extensions satisfy the associativity, i.e., for $L, M, N\in\mathrm{Rep}^0\Delta$,

$$L * (M * N) \cong (L * M) * N.$$

Let $\mathcal{M}(\Delta)$ denote the the set of isoclasses of representations in $\mathrm{Rep}^0\Delta$. Define a multiplication on $\mathcal{M}(\Delta)$ by setting

$$[M] * [N] = [M * N].$$

Then $\mathcal{M}(\Delta)$ is a monoid with identity $[0]$, the isoclass of zero representation of Δ.

By [37, 5], the generic extension $M * N$ can be also characterized as the unique maximal element among all the extensions of M by N with respect to the degeneration order $\leqslant_{\rm deg}$ which is defined by setting $M \leqslant_{\rm deg} N$ if $\mathbf{dim}\, M = \mathbf{dim}\, N$ and

$$\text{(2.0.2)} \qquad \dim{}_k \mathrm{Hom}_{k\Delta}(M, X) \geqslant \dim{}_k \mathrm{Hom}_{k\Delta}(N, X), \quad \text{for all } X \in \mathrm{Rep}^{\,0}\Delta.$$

If k is algebraically closed, then $M \leqslant_{\rm deg} N$ if and only if $\overline{\mathcal{O}}_M \subseteq \mathcal{O}_N$, where $\overline{\mathcal{O}}_M$ is the closure of $\mathcal{O}_M$. This defines a partial order relation on the set $\mathcal{M}(\Delta)$ of isoclasses of representations in $\mathrm{Rep}^{\,0}\Delta$; see [48, Th. 2] or [5, Lem. 3.2]. By [37, 2.4], for $M, N, M', N' \in \mathrm{Rep}^{\,0}\Delta$,

$$M' \leqslant_{\rm deg} M, N' \leqslant_{\rm deg} N \Longrightarrow M' * N' \leqslant_{\rm deg} M * N.$$

For $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{M}_n$ (resp., $\mathfrak{M}_\infty$), we write $\mathfrak{m} \leqslant_{\rm deg} \mathfrak{m}'$ (resp., $\mathfrak{m} \leqslant_{\rm deg}^{\infty} \mathfrak{m}'$) if $M(\mathfrak{m}) \leqslant_{\rm deg} M(\mathfrak{m}')$ in $\mathrm{Rep}^{\,0}\Delta_n$ (resp., $\mathrm{Rep}\,\Delta_\infty$).

By [4, 13], there is a covering functor

$$\mathscr{F} : \mathrm{Rep}\,\Delta_\infty \longrightarrow \mathrm{Rep}^{\,0}\Delta_n$$

sending $S_i[l]$ to $S_{\bar{i}}[l]$ for $i \in \mathbb{Z}$ and $l \geqslant 1$. Moreover, $\mathscr{F}$ is dense and exact, and the Galois group of $\mathscr{F}$ is the infinite cyclic group G generated by τ^n, i.e., $\tau^n(S_i[l] = S_{i+n}[l])$. For $\mathfrak{m} \in \mathfrak{M}_\infty$, let $\mathscr{F}(\mathfrak{m}) \in \mathfrak{M}_n$ be such that $M(\mathscr{F}(\mathfrak{m})) \cong \mathscr{F}(M(\mathfrak{m})) \in \mathrm{Rep}^{\,0}\Delta_n$. From (2.0.2) we easily deduce that for $M, N \in \mathrm{Rep}\,\Delta_\infty$,

$$\text{(2.0.3)} \qquad M \leqslant_{\rm deg} N \Longrightarrow \mathscr{F}(M) \leqslant_{\rm deg} \mathscr{F}(N).$$

The following two classes of representations will play an important role later on. For each $\mathbf{d} = (d_i) \in \mathbb{N}I$, we set

$$S_{\mathbf{d}} = \bigoplus_{i \in I} d_i S_i[1] \in \mathrm{Rep}^{\,0}\Delta.$$

In other words, $S_{\mathbf{d}}$ is the unique semisimple representation of dimension vector $\mathbf{d}$.

Let Π be the set of all partitions $\lambda = (\lambda_1, \dots, \lambda_t)$ (i.e., $\lambda_1 \geqslant \cdots \geqslant \lambda_t \geqslant 1$). For each $\lambda \in \Pi$, define

$$\mathfrak{m}_\lambda = \sum_{s=1}^{t} [1 - s, \lambda_s) \in \mathfrak{M}.$$

Then

$$M(\mathfrak{m}_\lambda) = S_0[\lambda_1] \oplus S_{-1}[\lambda_2] \oplus \cdots \oplus S_{1-t}[\lambda_t] \in \mathrm{Rep}^{\,0}\Delta.$$

If $\Delta = \Delta_\infty$, then we sometimes write $\mathfrak{m}_\lambda = \mathfrak{m}_\lambda^\infty \in \mathfrak{M}_\infty$ to make a distinction. It follows from the definition that $\mathscr{F}(\mathfrak{m}_\lambda^\infty) = \mathfrak{m}_\lambda$ for all $\lambda \in \Pi$.

Proposition 2.1. *Let $\lambda, \mu \in \Pi$.*

(1) *If $\Delta = \Delta_\infty$, then*

$$\mathbf{dim}\, M(\mathfrak{m}_\mu^\infty) = \mathbf{dim}\, M(\mathfrak{m}_\lambda^\infty) \Longleftrightarrow \mu = \lambda.$$

In particular, for each $\mathfrak{m} \in \mathfrak{M}_\infty$, there exists at most one $\nu \in \Pi$ such that $\mathfrak{m} = \mathfrak{m}_\nu^\infty$.

(2) *If $\Delta = \Delta_n$, then*

$$M(\mathfrak{m}_\mu) \leqslant_{\rm deg} M(\mathfrak{m}_\lambda) \Longrightarrow \mu \trianglelefteq \lambda,$$

where $\trianglelefteq$ is the dominance order on Π, i.e., $\mu \trianglelefteq \lambda \Longleftrightarrow \sum_{j=1}^{i} \mu_j \leqslant \sum_{j=1}^{i} \lambda_j$, $\forall\, i \geqslant 1$.

Proof. (1) By definition, both the socles of $M(\mathfrak{m}_\lambda^\infty)$ and $M(\mathfrak{m}_\mu^\infty)$ are multiplicity-free. Thus, comparing the socles of $S_0[\lambda_1]$ and $S_0[\mu_1]$ gives $\lambda_1 = \mu_1$. The lemma then follows from an inductive argument.

(2) Suppose $M(\mathfrak{m}_\mu) \leqslant_{\deg} M(\mathfrak{m}_\lambda)$. By viewing $\mathfrak{m}_\lambda$ and $\mathfrak{m}_\mu$ as multipartitions in $\mathfrak{M}_n$, we obtain by [7, Prop. 2.7] that for each $l \geqslant 1$,

$$\sum_{s=1}^{l} \widetilde{\mu}_s \geqslant \sum_{s=1}^{l} \widetilde{\lambda}_s,$$

where $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots)$ and $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \dots)$ are the dual partition of λ and μ, respectively, that is, $\widetilde{\mu} \trianglerighteq \widetilde{\lambda}$. By [35, 1.1], $\mu \trianglelefteq \lambda$. □

3. Ringel–Hall algebra of the quiver Δ

In this section we introduce the Ringel–Hall algebra $\mathcal{H}(\Delta)$ of Δ ($= \Delta_n$ or Δ_∞) and the maps from homogeneous subspaces of $\mathcal{H}(\Delta_n)$ to those of $\mathcal{H}(\Delta_\infty)$ defined in [47, 6.1]. We also describe the images of basis elements of $\mathcal{H}(\Delta_n)$ under these maps.

The cyclic quiver Δ_n gives the $n\times n$ Cartan matrix $C_n = (a_{ij})_{i,j\in I}$ of type $\widehat{A}_{n-1}$, while Δ_∞ defines the infinite Cartan matrix $C_\infty = (a_{ij})_{i,j\in\mathbb{Z}}$. Thus, we have the associated quantum enveloping algebras $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ and $\mathbf{U}_v(\mathfrak{sl}_\infty)$ which are $\mathbb{Q}(v)$-algebras with generators $K_i^{\pm1}, E_i, F_i, D^{\pm1}$ ($i \in I = \mathbb{Z}/n\mathbb{Z}$) and $K_i^{\pm1}, E_i, F_i$ ($i \in \mathbb{Z}$), respectively, and the quantum Serre relations. In particular, the relations involving the generator $D^{\pm1}$ in $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ are

$$DD^{-1} = 1 = D^{-1}D,\ K_iD = DK_i,\ DE_i = v^{\delta_{0,i}}E_iD,\ DF_i = v^{-\delta_{0,i}}F_iD,\ \ \forall\, i \in I;$$

see [2, Def. 3.16]. The subalgebra of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ generated by $K_i^{\pm1}, E_i, F_i$ ($i \in I = \mathbb{Z}/n\mathbb{Z}$) is denoted by $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$; see [27, 1.1].

By [38, 40, 16], for $\mathfrak{p}, \mathfrak{m}_1, \dots, \mathfrak{m}_t \in \mathfrak{M}$, there is a polynomial $\varphi^{\mathfrak{p}}_{\mathfrak{m}_1,\dots,\mathfrak{m}_t}(q) \in \mathbb{Z}[q]$ (called Hall polynomial) such that for each finite field k,

$$\varphi^{\mathfrak{p}}_{\mathfrak{m}_1,\dots,\mathfrak{m}_t}(|k|) = F^{M_k(\mathfrak{p})}_{M_k(\mathfrak{m}_1),\dots,M_k(\mathfrak{m}_t)},$$

which is by definition the number of the filtrations

$$M_k(\mathfrak{p}) = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{t-1} \supseteq M_t = 0$$

such that $M_{s-1}/M_s \cong M_k(\mathfrak{m}_s)$ for all $1 \leq s \leq t$. By [39, Sect. 2], for each $\mathfrak{m} \in \mathfrak{M}$, there is a polynomial $a_\mathfrak{m}(q) \in \mathbb{Z}[q]$ such that for each finite field k,

$$a_\mathfrak{m}(|k|) = |\mathrm{Aut}_{k\Delta}(M_k(\mathfrak{m}))|.$$

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ be the Laurent polynomial ring over $\mathbb{Z}$ in indeterminate v. By definition, the (twisted generic) *Ringel–Hall algebra* $\mathcal{H}(\Delta)$ of Δ is the free $\mathcal{Z}$-module with basis $\{u_\mathfrak{m} \mid \mathfrak{m} \in \mathfrak{M}\}$ and multiplication given by

$$u_\mathfrak{m}u_{\mathfrak{m}'} = v^{\langle \mathbf{dim}\, M(\mathfrak{m}), \mathbf{dim}\, M(\mathfrak{m}')\rangle} \sum_{\mathfrak{p}\in\mathfrak{M}} \varphi^{\mathfrak{p}}_{\mathfrak{m},\mathfrak{m}'}(v^2)u_\mathfrak{p}. \tag{3.0.1}$$

In practice, we also write $u_\mathfrak{m} = u_{[M(\mathfrak{m})]}$ in order to make certain calculations in terms of modules. Furthermore, for each $\mathbf{d} \in \mathbb{N}I$, we simply write $u_\mathbf{d} = u_{[S_\mathbf{d}]}$.

For each $i \in I$, set $u_i = u_{[S_i]}$. We then denote by $\mathcal{C}(\Delta)$ the subalgebra of $\mathcal{H}(\Delta)$ generated by the divided power $u_i^{(t)} = u_i^t/[t]^!$, $i \in I$ and $t \geqslant 1$, called the *composition algebra* of Δ, where

$$[t]^! = [t][t-1]\cdots[1] \ \text{ with } \ [m] = (v^m - v^{-m})/(v - v^{-1}). \tag{3.0.2}$$

Moreover, both $\mathcal{H}(\Delta)$ and $\mathcal{C}(\Delta)$ are $\mathbb{N}I$-graded:

$$\mathcal{H}(\Delta) = \bigoplus_{\mathbf{d}\in\mathbb{N}I} \mathcal{H}(\Delta)_\mathbf{d} \ \text{ and } \ \mathcal{C}(\Delta) = \bigoplus_{\mathbf{d}\in\mathbb{N}I} \mathcal{C}(\Delta)_\mathbf{d}, \tag{3.0.3}$$

where $\mathcal{H}(\Delta)_{\mathbf{d}}$ is spanned by all $u_{\mathfrak{m}}$ with $\mathfrak{m} \in \mathfrak{M}^{\mathbf{d}}$ and $\mathcal{C}(\Delta)_{\mathbf{d}} = \mathcal{C}(\Delta) \cap \mathcal{H}(\Delta)_{\mathbf{d}}$. Since the Auslander–Reiten translate $\tau : \operatorname{Rep}^0 \Delta \to \operatorname{Rep}^0 \Delta$ is an auto-equivalence, it induces an automorphism $\tau : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta), u_{\mathfrak{m}} \mapsto u_{\tau\mathfrak{m}}$. We also consider the $\mathbb{Q}(v)$-algebras

$$\boldsymbol{\mathcal{H}}(\Delta) = \mathcal{H}(\Delta) \otimes_{\mathcal{Z}} \mathbb{Q}(v) \text{ and } \boldsymbol{\mathcal{C}}(\Delta_n) = \mathcal{C}(\Delta_n) \otimes_{\mathcal{Z}} \mathbb{Q}(v).$$

Remark 3.1. We remark that the Hall algebra of Δ defined in [47] is the opposite algebra of $\mathcal{H}(\Delta)$ given here with v being replaced by v^{-1}. Thus, v and v^{-1} should be swaped when comparing with the formulas in [47].

Following [38], $\mathcal{C}(\Delta_\infty) = \mathcal{H}(\Delta_\infty)$, and there is an isomorphism $\mathbf{U}_v^+(\mathfrak{sl}_\infty) \cong \boldsymbol{\mathcal{H}}(\Delta_\infty)$ taking $E_i \mapsto u_i$, $\forall\, i \in I_\infty = \mathbb{Z}$. But, for $n \geqslant 2$, $\mathcal{C}(\Delta_n)$ is a proper subalgebra of $\mathcal{H}(\Delta_n)$. By [40],

$$\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n) \cong \boldsymbol{\mathcal{C}}(\Delta_n),\ E_i \longmapsto u_i,\ \forall\, i \in I_n.$$

By [41, Th. 2.2], $\boldsymbol{\mathcal{H}}(\Delta_n)$ is decomposed into the tensor product of $\boldsymbol{\mathcal{C}}(\Delta_n)$ and a polynomial ring in infinitely many indeterminates which are central elements in $\boldsymbol{\mathcal{H}}(\Delta_n)$. Such central elements have been explicitly constructed in [19]. More precisely, for each $t \geqslant 1$, let

$$\boldsymbol{c}_t = (-1)^t v^{-2nt} \sum_{\mathfrak{m}} (-1)^{\dim \operatorname{End}(M(\mathfrak{m}))} a_{\mathfrak{m}}(v^2) u_{\mathfrak{m}} \in \mathcal{H}(\Delta_n), \tag{3.1.1}$$

where the sum is taken over all $\mathfrak{m} \in \mathfrak{M}_n$ such that $\mathbf{dim}\, M(\mathfrak{m}) = t\delta$ with $\delta = (1, \ldots, 1) \in \mathbb{N}I_n$, and $\operatorname{soc} M(\mathfrak{m})$ is square-free, i.e., $\mathbf{dim} \operatorname{soc} M(\mathfrak{m}) \leqslant \delta$. The following result is proved in [19].

Theorem 3.2. *The elements c_m are central in $\mathcal{H}(\Delta_n)$. Moreover, there is a decomposition*

$$\boldsymbol{\mathcal{H}}(\Delta_n) = \boldsymbol{\mathcal{C}}(\Delta_n) \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v)[\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots],$$

where $\mathbb{Q}(v)[\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots]$ is the polynomial algebra in $\boldsymbol{c}_t$ for $t \geqslant 1$. In particular, $\boldsymbol{\mathcal{H}}(\Delta_n)$ is generated by u_i and $\boldsymbol{c}_t$ for $i \in I_n$ and $t \geqslant 1$.

For each $\mathfrak{m} \in \mathfrak{M}$, set $d(\mathfrak{m}) = \dim M(\mathfrak{m})$, $\mathbf{d}(\mathfrak{m}) = \mathbf{dim}\, M(\mathfrak{m})$ and define

$$\widetilde{u}_{\mathfrak{m}} = v^{\dim \operatorname{End}_{k\Delta}(M(\mathfrak{m})) - d(\mathfrak{m})} u_{\mathfrak{m}}. \tag{3.2.1}$$

Then $\{\widetilde{u}_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}\}$ is also a $\mathcal{Z}$-basis of $\mathcal{H}(\Delta)$ which plays a role in the construction of the canonical basis. In particular,

$$\widetilde{u}_i = u_i \text{ for each } i \in I \text{ and } \widetilde{u}_{\mathbf{d}} = v^{\sum_i (d_i^2 - d_i)} u_{\mathbf{d}} \text{ for each } \mathbf{d} \in \mathbb{N}I.$$

Consider the map $\pi : \mathbb{Z}I_\infty \to \mathbb{Z}I_n, \mathbf{d} \mapsto \bar{\mathbf{d}}$, where $\pi(\mathbf{d}) = \bar{\mathbf{d}} = (d_{\bar{i}})$ is defined by

$$d_{\bar{i}} = \sum_{j \in \bar{i}} d_j,\ \forall\, \bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}.$$

In particular, for each representation $M \in \operatorname{Rep} \Delta_\infty$, $\mathbf{dim}\, \mathscr{F}(M) = \pi(\mathbf{dim}\, M)$.

In the following we briefly recall from [47, 6.1] the $\mathcal{Z}$-linear map

$$\gamma_{\mathbf{d}} : \mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}} \longrightarrow \mathcal{H}(\Delta_\infty)_{\mathbf{d}}$$

for each $\mathbf{d} \in \mathbb{N}I_\infty$. These maps play a crucial role in defining an action of $\mathcal{H}(\Delta_n)$ on the Fock space later on.

Let $k = \mathbb{F}_q$ be a finite filed with q elements and let $V = \oplus_{i \in I} V_i$ be an I-graded $\mathbb{F}_q$-vector space with dimension vector $\mathbf{d}$. Then we define $\mathbb{C}_{G_V}(E_V)$ to be the set of G_V-invariant functions $E_V \to \mathbb{C}$, which is a vector space over $\mathbb{C}$. Then $\mathcal{H}(\Delta)_{\mathbf{d}} \otimes_{\mathcal{Z}} \mathbb{C}$ (at $v = \sqrt{q}$) can be identified with $\mathbb{C}_{G_V}(E_V)$ via taking $u_{[(V,x)]}$ to the characteristic function of the G_V-orbit of x in E_V.

Now take $\mathbf{d} \in \mathbb{N}I_\infty$ and let $V = \oplus_{i\in\mathbb{Z}} V_i$ be an I_∞-graded $\mathbb{F}_q$-vector space of dimension vector $\mathbf{d}$. This gives an I_n-graded space $\overline{V} = \oplus_{\bar{i}\in I_n} V_{\bar{i}}$ of dimension vector $\bar{\mathbf{d}}$ with $\overline{V}_{\bar{i}} = \oplus_{j\in\bar{i}} V_j$, $\forall\, \bar{i} \in I_n$. Moreover, $\overline{V}$ admits a filtration by the subspaces

$$\overline{V}_{\geqslant i} = \bigoplus_{j\geqslant i} V_j,\ \forall\, i \in \mathbb{Z}.$$

Then the associated graded space $\oplus_{i\in\mathbb{Z}} \overline{V}_{\geqslant i}/\overline{V}_{\geqslant i-1}$ is naturally identified with the $\mathbb{Z}$-graded space V. Set

$$E_{\overline{V},V} = \{x \in E_{\overline{V}} \mid x(\overline{V}_{\geqslant i}) \subseteq \overline{V}_{i+1}\} \subset E_{\overline{V}}.$$

This gives a map $p : E_{\overline{V},V} \to E_V$, which takes a representation of Δ_n in $E_{\overline{V}}$ to the induced representation of Δ_∞ in E_V, and the embedding $\iota : E_{\overline{V},V} \to E_{\overline{V}}$. By specializing v to $\sqrt{q}$, the map $\gamma_{\mathbf{d}}$ is then given by

$$(\gamma_{\mathbf{d}} \otimes_{\mathcal{Z}} \mathbb{C}) \mid_{v=\sqrt{q}}: \mathbb{C}_{G_{\overline{V}}}(E_{\overline{V}}) \longrightarrow \mathbb{C}_{G_V}(E_V),\ f \longmapsto \sqrt{q}^{h(\mathbf{d})} p_! \iota^*(f),$$

where $h(\mathbf{d}) = \sum_{i<j,\bar{i}=\bar{j}} d_i(d_{j+1}-d_j)$. Here we identify $\mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}} \otimes_{\mathcal{Z}} \mathbb{C}$ with $\mathbb{C}_{G_{\overline{V}}}(E_{\overline{V}})$ and $\mathcal{H}(\Delta_\infty)_{\mathbf{d}} \otimes_{\mathcal{Z}} \mathbb{C}$ with $\mathbb{C}_{G_V}(E_V)$.

The first two statements in the following lemma are taken from [47, Sect. 6.1], and the third one follows from the isomorphism $\tau : \mathcal{H}(\Delta_\infty) \to \mathcal{H}(\Delta_\infty)$.

Lemma 3.3. (1) *For each* $\mathbf{d} \in \mathbb{N}I_\infty$, $\gamma_{\mathbf{d}}(\widetilde{u}_{\bar{\mathbf{d}}}) = v^{-h(\mathbf{d})}\widetilde{u}_{\mathbf{d}}$.

(2) *Fix* $\alpha, \beta \in \mathbb{N}I_n$ *with* $\bar{\mathbf{d}} = \alpha + \beta$. *Then for* $x \in \mathcal{H}(\Delta_n)_\alpha$ *and* $y \in \mathcal{H}(\Delta_n)_\beta$,

$$\sum_{\mathbf{a},\mathbf{b}} v^{\kappa(\mathbf{a},\mathbf{b})} \gamma_{\mathbf{a}}(x)\gamma_{\mathbf{b}}(y) = \gamma_{\mathbf{d}}(xy), \tag{3.3.1}$$

where the sum is taken over all pairs $\mathbf{a}, \mathbf{b} \in \mathbb{N}I_\infty$ *satisfying* $\mathbf{a} + \mathbf{b} = \mathbf{d}$, $\bar{\mathbf{a}} = \alpha$, *and* $\bar{\mathbf{b}} = \beta$, *and* $\kappa(\mathbf{a}, \mathbf{b}) = \sum_{i>j,\bar{i}=\bar{j}} a_i(2b_j - b_{j-1} - b_{j+1})$.

(3) *For each* $\mathbf{d} \in \mathbb{N}I_\infty$ *and* $\mathfrak{m} \in \mathfrak{M}_n^{\bar{\mathbf{d}}}$, $\gamma_{\tau^n(\mathbf{d})}(\widetilde{u}_{\mathfrak{m}}) = \tau^n(\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}}))$.

We now describe the images of the basis elements $\widetilde{u}_{\mathfrak{m}}$ of $\mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}}$ under $\gamma_{\mathbf{d}}$.

Proposition 3.4. *Let* $\mathbf{d} \in \mathbb{N}I_\infty$ *and* $\mathfrak{m} \in \mathfrak{M}_n$ *be such that* $\alpha := \mathbf{dim}\, M(\mathfrak{m}) = \bar{\mathbf{d}}$. *Then*

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}}) \in \sum_{\mathfrak{z}\in\mathfrak{M}_\infty,\, \mathscr{F}(\mathfrak{z})\leqslant_{\deg}\mathfrak{m}} \mathcal{Z}\widetilde{u}_{\mathfrak{z}}.$$

Proof. Consider the radical filtration of $M = M(\mathfrak{m})$

$$M = \mathrm{rad}^{\,0} M \supseteq \mathrm{rad}\, M (= \mathrm{rad}^{\,1} M) \supseteq \cdots \supseteq \mathrm{rad}^{\,\ell-1} M \supseteq \mathrm{rad}^{\,\ell} M = 0$$

with $\mathrm{rad}^{\,s-1} M/\mathrm{rad}^{\,s} M \cong S_{\alpha_s}$, where ℓ is the Loewy length of M and $\alpha_s \in \mathbb{N}I_n$ for $1 \leqslant s \leqslant \ell$. Then $M = S_{\alpha_1} * \cdots * S_{\alpha_\ell}$. Moreover, by [8, Sect. 9],

$$\widetilde{u}_{\alpha_1} \cdots \widetilde{u}_{\alpha_\ell} = \widetilde{u}_{\mathfrak{m}} + \sum_{\mathfrak{p}<_{\deg}\mathfrak{m}} f_{\mathfrak{m},\mathfrak{p}} \widetilde{u}_{\mathfrak{p}},\quad \text{where } f_{\mathfrak{m},\mathfrak{p}} \in \mathcal{Z}.$$

On the one hand, by induction with respect to the order $\leqslant_{\deg}$, we may assume that for each $\mathfrak{p} \in \mathfrak{M}_n^{\mathbf{d}}$ with $\mathfrak{p} <_{\deg} \mathfrak{m}$, $\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{p}})$ is a $\mathcal{Z}$-linear combination of $\widetilde{u}_{\mathfrak{y}}$ with $\mathfrak{y} \in \mathfrak{M}_\infty$ satisfying $\mathscr{F}(\mathfrak{y}) \leqslant_{\deg} \mathfrak{p}$. Therefore,

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}}) = \gamma_{\mathbf{d}}(\widetilde{u}_{\alpha_1} \cdots \widetilde{u}_{\alpha_\ell}) + x, \tag{3.4.1}$$

where $x = -\sum_{\mathfrak{p}<_{\deg}\mathfrak{m}} f_{\mathfrak{m},\mathfrak{p}} \gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{p}})$ is a $\mathcal{Z}$-linear combination of $\widetilde{u}_{\mathfrak{z}}$ with $\mathscr{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}$.

On the other hand, by applying (3.3.1) inductively, we obtain

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\alpha_1}\cdots\widetilde{u}_{\alpha_\ell})=\sum_{\mathbf{a}_1,\dots,\mathbf{a}_\ell} v^{\sum_{s<t}\kappa(\mathbf{a}_s,\mathbf{a}_t)-\sum_s h(\mathbf{a}_s)}\widetilde{u}_{\mathbf{a}_1}\cdots\widetilde{u}_{\mathbf{a}_\ell}, \tag{3.4.2}$$

where the sum is taken over all sequences $\mathbf{a}_1,\dots,\mathbf{a}_\ell\in\mathbb{N}I_\infty$ satisfying

$$\mathbf{a}_1+\cdots+\mathbf{a}_\ell=\mathbf{d} \text{ and } \overline{\mathbf{a}_s}=\alpha_s,\ \forall\, 1\leqslant s\leqslant \ell.$$

By the definition, each term $\widetilde{u}_{\mathbf{a}_1}\cdots\widetilde{u}_{\mathbf{a}_\ell}$ is a $\mathcal{Z}$-linear combination of $\widetilde{u}_{\mathfrak{y}}$ such that $M(\mathfrak{y})$ admits a filtration

$$M(\mathfrak{y})=X_0\supset X_1\supset\cdots\supset X_{\ell-1}\supset X_\ell=0$$

satisfying $X_{s-1}/X_s\cong S_{\mathbf{a}_s}$ for all $1\leqslant s\leqslant\ell$. Applying the exact functor $\mathscr{F}$ gives a filtration of $\mathscr{F}(M(\mathfrak{y}))$

$$\mathscr{F}(M(\mathfrak{y}))=\mathscr{F}(X_0)\supset\mathscr{F}(X_1)\supset\cdots\supset\mathscr{F}(X_{\ell-1})\supset\mathscr{F}(X_\ell)=0$$

such that

$$\mathscr{F}(X_{s-1})/\mathscr{F}(X_s)\cong\mathscr{F}(X_{s-1}/X_s)\cong S_{\alpha_s},\ \forall\, 1\leqslant s\leqslant\ell.$$

Therefore,

$$\mathscr{F}(M(\mathfrak{y}))=M(\mathscr{F}(\pi))\leqslant_{\deg} S_{\alpha_1}*\cdots*S_{\alpha_\ell}=M(\mathfrak{m}),$$

that is, $\mathscr{F}(\mathfrak{y})\leqslant_{\deg}\mathfrak{m}$.

In conclusion, we obtain that

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}})\in\sum_{\mathfrak{z}\in\mathfrak{M}_\infty,\,\mathscr{F}(\mathfrak{z})\leqslant_{\deg}\mathfrak{m}}\mathcal{Z}\widetilde{u}_{\mathfrak{z}}.$$

□

Fix $\lambda\in\Pi$ and write

$$\mathbf{d}(\lambda)=\mathbf{dim}\,M(\mathfrak{m}_\lambda^\infty)\in\mathbb{N}I_\infty \text{ and } \alpha(\lambda)=\mathbf{dim}\,M(\mathfrak{m}_\lambda)\in\mathbb{N}I_n.$$

By the definition of $M(\mathfrak{m}_\lambda^\infty)$ and $M(\mathfrak{m}_\lambda)$, the radical filtration of $\widetilde{M}=M(\mathfrak{m}_\lambda^\infty)$

$$\widetilde{M}=\mathrm{rad}^0\widetilde{M}\supseteq\mathrm{rad}\,\widetilde{M}\supseteq\cdots\supseteq\mathrm{rad}^{\ell-1}\widetilde{M}\supseteq\mathrm{rad}^{\ell}\widetilde{M}=0$$

gives rise to the radical filtration of $M(\mathfrak{m}_\lambda)=\mathscr{F}(\widetilde{M})$

$$M(\mathfrak{m}_\lambda)=\mathscr{F}(\mathrm{rad}^0\widetilde{M})\supseteq\mathscr{F}(\mathrm{rad}\,\widetilde{M})\supseteq\cdots\supseteq\mathscr{F}(\mathrm{rad}^{\ell-1}\widetilde{M})\supseteq\mathscr{F}(\mathrm{rad}^{\ell}\widetilde{M})=0,$$

that is, $\mathscr{F}(\mathrm{rad}^s\widetilde{M})=\mathrm{rad}^s(M(\mathfrak{m}_\lambda))$ for $1\leqslant s\leqslant\ell$. Let $\mathbf{d}(\lambda)_s\in\mathbb{N}I_\infty$ and $\alpha(\lambda)_s\in\mathbb{N}I_n$, $1\leqslant s\leqslant\ell$, be such that

$$\mathrm{rad}^{s-1}\widetilde{M}/\mathrm{rad}^s\widetilde{M}\cong S_{\mathbf{d}(\lambda)_s} \text{ and } \mathrm{rad}^{s-1}M(\mathfrak{m}_\lambda)/\mathrm{rad}^sM(\mathfrak{m}_\lambda)\cong S_{\alpha(\lambda)_s}.$$

Then $\overline{\mathbf{d}(\lambda)_s}=\alpha(\lambda)_s$ for $1\leqslant s\leqslant\ell$. Applying (3.4.1) and (3.4.2) to $\mathfrak{m}_\lambda$ gives the following result.

Corollary 3.5. (1) *Let $\lambda\in\Pi$ and keep the notation above. Then*

$$\gamma_{\mathbf{d}(\lambda)}(\widetilde{u}_{\mathfrak{m}_\lambda})\in v^{\theta(\lambda)}\widetilde{u}_{\mathfrak{m}_\lambda^\infty}+\sum_{\mathfrak{z}\in\mathfrak{M}_\infty,\,\mathscr{F}(\mathfrak{z})<_{\deg}\mathfrak{m}_\lambda}\mathcal{Z}\widetilde{u}_{\mathfrak{z}},$$

where $\theta(\lambda)=\sum_{s<t}\kappa(\mathbf{d}(\lambda)_s,\mathbf{d}(\lambda)_t)-\sum_{s=1}^{\ell}h(\mathbf{d}(\lambda)_s)$.

(2) *Let $\mathbf{d}\in\mathbb{N}I_\infty$ with $\overline{\mathbf{d}}=\alpha(\lambda)$. If $\mathbf{d}=\tau^{rm}(\mathbf{d}(\lambda))$ for some $r\in\mathbb{Z}$, then*

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}_\lambda})\in v^{\theta(\lambda)}\widetilde{u}_{\tau^{rm}(\mathfrak{m}_\lambda^\infty)}+\sum_{\mathfrak{z}\in\mathfrak{M}_\infty,\,\mathscr{F}(\mathfrak{z})<_{\deg}\mathfrak{m}_\lambda}\mathcal{Z}\widetilde{u}_{\mathfrak{z}}.$$

Otherwise,

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}_\lambda}) \in \sum_{\mathfrak{z}\in\mathfrak{M}^{\mathbf{d}}_\infty,\, \mathscr{F}(\mathfrak{z})<_{\deg}\mathfrak{m}_\lambda} \mathcal{Z}\widetilde{u}_{\mathfrak{z}}.$$

In the following we briefly recall the canonical basis of $\mathcal{H}(\Delta)$ for $\Delta = \Delta_n$ or Δ_∞. By [31] and [47, Prop. 7.5], there is a semilinear ring involution $\iota : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta)$ taking $v \mapsto v^{-1}$ and $\widetilde{u}_{\mathbf{d}} \mapsto \widetilde{u}_{\mathbf{d}}$ for all $\mathbf{d} \in \mathbb{Z}I$. It is often called the bar-involution, usually written as $\bar{x} = \iota(x)$. The canonical basis (or the global crystal basis in the sense of Kashiwara) $\mathbf{B} := \{b_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}\}$ for $\mathcal{H}(\Delta)$ (at $v = \infty$) can be characterized as follows:

$$\overline{b}_{\mathfrak{m}} = b_{\mathfrak{m}}, \quad b_{\mathfrak{m}} \in \widetilde{u}_{\mathfrak{m}} + \sum_{\mathfrak{p}<_{\deg}\mathfrak{m}} v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}_{\mathfrak{p}}; \tag{3.5.1}$$

see [31]. The canonical basis elements $b_{\mathfrak{m}}$ also admit a geometric characterization given in [32, 47]. Let $H^i_{\mathcal{O}_{\mathfrak{p}}}(IC_{\mathcal{O}_{\mathfrak{m}}})$ be the stalk at a point of $\mathcal{O}_{\mathfrak{p}}$ of the i-th intersection cohomology sheaf of the closure $\overline{\mathcal{O}_{\mathfrak{m}}}$ of $\mathcal{O}_{\mathfrak{m}}$. Then

$$b_{\mathfrak{m}} = \sum_{\substack{i\in\mathbb{N}\\ \mathfrak{p}\leqslant_{\deg}\mathfrak{m}}} v^{i-\dim\mathcal{O}_{\mathfrak{m}}+\dim\mathcal{O}_{\mathfrak{p}}} \dim H^i_{\mathcal{O}_{\mathfrak{p}}}(IC_{\mathcal{O}_{\mathfrak{m}}})\widetilde{u}_{\mathfrak{p}}.$$

For the cyclic quiver case, by [33], the subset of $\mathbf{B}$

$$\mathbf{B}^{\mathrm{ap}} := \{b_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}^{\mathrm{ap}}_n\}$$

is the canonical basis of $\mathcal{C}(\Delta_n)$, where $\mathfrak{M}^{\mathrm{ap}}_n$ denotes the set of aperiodic multisegments, that is, those multisegments $\mathfrak{m} = \sum_{i\in I_n,\, l\geqslant 1} m_{i,l}[i,l)$ satisfying that for each $l \geqslant 1$, there is some $i \in I_n$ such that $m_{i,l} = 0$. In other words, $\mathbf{B}^{\mathrm{ap}}$ is the canonical basis of $\mathbf{U}^{\pm}_v(\widehat{\mathfrak{sl}}_n)$. Note that for each $\lambda = (\lambda_1, \dots, \lambda_m) \in \Pi$, the corresponding multisegment $\mathfrak{m}_\lambda$ is aperiodic if and only if λ is n-regular which, by definition, satisfies $\lambda_s > \lambda_{s+n-1}$ for $1 \leqslant s \leqslant s+n-1 \leqslant m$.

4. Double Ringel–Hall algebras and highest weight modules

In this section we follows [46, 6] to define the double Ringel–Hall algebra $\boldsymbol{\mathcal{D}}(\Delta)$ of the quiver $\Delta = \Delta_n$ or Δ_∞ and study the irreducible highest weight modules of $\boldsymbol{\mathcal{D}}(\Delta_n)$ associated with integral dominant weights in terms of a quantized generalized Kac–Moody algebra.

The Ringel–Hall algebra $\boldsymbol{\mathcal{H}}(\Delta)$ of Δ can be extended to a Hopf algebra $\boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0}$ which is a $\mathbb{Q}(v)$-vector space with a basis $\{u^+_{\mathfrak{m}}K_\alpha \mid \alpha \in \mathbb{Z}I, \mathfrak{m} \in \mathfrak{M}\}$; see [38, 15, 46] or [6, Prop. 1.5.3]. Its algebra structure is given by

$$\begin{aligned} &K_\alpha K_\beta = K_{\alpha+\beta}, \quad K_\alpha u^+_{\mathfrak{m}} = v^{(\mathbf{d}(\mathfrak{m}),\alpha)}u^+_{\mathfrak{m}}K_\alpha, \\ &u^+_{\mathfrak{m}}u^+_{\mathfrak{m}'} = \sum_{\mathfrak{p}\in\mathfrak{M}} v^{\langle\mathbf{d}(\mathfrak{m}),\mathbf{d}(\mathfrak{m}')\rangle}\varphi^{\mathfrak{p}}_{\mathfrak{m},\mathfrak{m}'}(v^2)u^+_{\mathfrak{p}}, \end{aligned} \tag{4.0.2}$$

where $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{M}$ and $\alpha, \beta \in \mathbb{Z}I$, and its coalgebra structure is given by

(4.0.3)

$$\begin{aligned} &\Delta(u^+_{\mathfrak{m}}) = \sum_{\mathfrak{m}',\mathfrak{m}''\in\mathfrak{M}} v^{\langle\mathbf{d}(\mathfrak{m}'),\mathbf{d}(\mathfrak{m}'')\rangle}\frac{\mathfrak{a}_{\mathfrak{m}'}(v^2)\mathfrak{a}_{\mathfrak{m}''}(v^2)}{\mathfrak{a}_{\mathfrak{m}}(v^2)}\varphi^{\mathfrak{m}}_{\mathfrak{m}',\mathfrak{m}''}(v^2)u^+_{\mathfrak{m}''}\otimes u^+_{\mathfrak{m}'}K_{\mathbf{d}(\mathfrak{m}'')}, \\ &\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \varepsilon(u^+_{\mathfrak{m}}) = 0 \ (\mathfrak{m} \neq 0), \quad \varepsilon(K_\alpha) = 1, \end{aligned}$$

where $\mathfrak{m} \in \mathfrak{M}$ and $\alpha \in \mathbb{Z}I$. We refer to [46] or [6] for the definition of the antipode.

Dually, there is a Hopf algebra $\boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0}$ with basis $\{K_\alpha u_{\mathfrak{m}}^- \mid \alpha \in \mathbb{Z}I, \mathfrak{m} \in \mathfrak{M}\}$. In particular, the multiplication is given by

$$
\begin{aligned}
&K_\alpha K_\beta = K_{\alpha+\beta}, \quad K_\alpha u_{\mathfrak{m}}^- = v^{-(\mathbf{d}(\mathfrak{m}),\alpha)} u_{\mathfrak{m}}^- K_\alpha, \\
&u_{\mathfrak{m}}^- u_{\mathfrak{m}'}^- = \sum_{\mathfrak{p}\in\mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}'),\mathbf{d}(\mathfrak{m})\rangle} \varphi_{\mathfrak{m}',\mathfrak{m}}^{\mathfrak{p}}(v^2) u_{\mathfrak{p}}^-,
\end{aligned}
\tag{4.0.4}
$$

where $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{M}$ and $\alpha, \beta \in \mathbb{Z}I$. The comultiplication and the counit are given by

$$
\begin{aligned}
&\Delta(u_{\mathfrak{m}}^-) = \sum_{\mathfrak{m}',\mathfrak{m}''\in\mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}'),\mathbf{d}(\mathfrak{m}'')\rangle} \frac{\mathfrak{a}_{\mathfrak{m}'}\mathfrak{a}_{\mathfrak{m}''}}{\mathfrak{a}_{\mathfrak{m}}} \varphi_{\mathfrak{m}',\mathfrak{m}''}^{\mathfrak{m}}(v^2) u_{\mathfrak{m}''}^- K_{-\mathbf{d}(\mathfrak{m}')} \otimes u_{\mathfrak{m}'}^-, \\
&\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \varepsilon(u_{\mathfrak{m}}^-) = 0 \ (\mathfrak{m} \neq 0), \quad \varepsilon(K_\alpha) = 1,
\end{aligned}
\tag{4.0.5}
$$

where $\alpha \in \mathbb{Z}I$ and $\mathfrak{m} \in \mathfrak{M}$.

It is routine to check that the bilinear form $\psi : \boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0} \times \boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0} \to \mathbb{Q}(v)$ defined by

$$
\psi(K_\alpha u_{\mathfrak{m}}^+, K_\beta u_{\mathfrak{m}'}^-) = v^{(\alpha,\beta)-\langle \mathbf{d}(\mathfrak{m}),\mathbf{d}(\mathfrak{m})\rangle + 2d(\mathfrak{m})} \frac{\delta_{\mathfrak{m},\mathfrak{m}'}}{\mathfrak{a}_{\mathfrak{m}}(v^2)}
\tag{4.0.6}
$$

is a skew-Hopf pairing in the sense of [24]; see, for example, [6, Prop. 2.1.3].

Following [46] or [6, §2.1], with the triple $(\boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0}, \boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0}, \psi)$ we obtain the associated reduced *double Ringel–Hall algebra* $\boldsymbol{\mathcal{D}}(\Delta)$ which inherits a Hopf algebra structure from those of $\boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0}$ and $\boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0}$. In particular, for all elements $x \in \boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0}$ and $y \in \boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0}$, we have in $\boldsymbol{\mathcal{D}}(\Delta)$ the following relations

$$
\sum \psi(x_1, y_1) y_2 x_2 = \sum \psi(x_2, y_2) x_1 y_1,
\tag{4.0.7}
$$

where $\Delta(x) = \sum x_1 \otimes x_2$ and $\Delta(y) = \sum y_1 \otimes y_2$ (Here we use the Sweedler notation). Moreover, $\boldsymbol{\mathcal{D}}(\Delta)$ admits a triangular decomposition

$$
\boldsymbol{\mathcal{D}}(\Delta) = \boldsymbol{\mathcal{D}}(\Delta)^+ \otimes \boldsymbol{\mathcal{D}}(\Delta)^0 \otimes \boldsymbol{\mathcal{D}}(\Delta)^-,
\tag{4.0.8}
$$

where $\boldsymbol{\mathcal{D}}(\Delta)^{\pm}$ are subalgebras generated by $u_{\mathfrak{m}}^{\pm}$ ($\mathfrak{m} \in \mathfrak{M}$), and $\boldsymbol{\mathcal{D}}(\Delta)^0$ is generated by K_α ($\alpha \in \mathbb{Z}I$). Thus, $\boldsymbol{\mathcal{D}}(\Delta)^0$ is identified with the Laurent polynomial ring $\mathbb{Q}(v)[K_i^{\pm 1} : i \in I]$,

$$
\begin{aligned}
\boldsymbol{\mathcal{H}}(\Delta) = \mathcal{H}(\Delta) \otimes_{\mathcal{Z}} \mathbb{Q}(v) &\xrightarrow{\sim} \boldsymbol{\mathcal{D}}(\Delta)^+, \quad u_{\mathfrak{m}} \longmapsto u_{\mathfrak{m}}^+, \\
\boldsymbol{\mathcal{H}}(\Delta)^{\mathrm{op}} = \mathcal{H}(\Delta)^{\mathrm{op}} \otimes_{\mathcal{Z}} \mathbb{Q}(v) &\xrightarrow{\sim} \boldsymbol{\mathcal{D}}(\Delta)^-, \quad u_{\mathfrak{m}} \longmapsto u_{\mathfrak{m}}^-.
\end{aligned}
$$

For $i \in I$, $\alpha \in \mathbb{N}I$ and $\mathfrak{m} \in \mathfrak{M}$, we write

$$
u_i^{\pm} = u_{[S_i]}^{\pm}, \quad u_\alpha^{\pm} = u_{[S_\alpha]}^{\pm}, \quad \text{and} \quad \widetilde{u}_{\mathfrak{m}}^{\pm} = v^{\dim \mathrm{End}_\Delta(M(\mathfrak{m})) - \dim M(\mathfrak{m})} u_{\mathfrak{m}}^{\pm}.
$$

The canonical basis of $\mathcal{H}(\Delta)$ in (3.5.1) gives the canonical bases $\mathbf{B}^{\pm} := \{b_{\mathfrak{m}}^{\pm} \mid \mathfrak{m} \in \mathfrak{M}\}$ of $\boldsymbol{\mathcal{D}}(\Delta)^{\pm}$ satisfying

$$
b_{\mathfrak{m}}^{\pm} \in \widetilde{u}_{\mathfrak{m}}^{\pm} + \sum_{\mathfrak{p} <_{\deg} \mathfrak{m}} v^{-1}\mathbb{Z}[v^{-1}] \widetilde{u}_{\mathfrak{p}}^{\pm}.
\tag{4.0.9}
$$

It is known that $\boldsymbol{\mathcal{D}}(\Delta_\infty)$ is generated by $u_i^{\pm}, K_i^{\pm 1}$ ($i \in \mathbb{Z}$) and is isomorphic to $\mathbf{U}_v(\mathfrak{sl}_\infty)$. By [40], the $\mathbb{Q}(v)$-subalgebra of $\boldsymbol{\mathcal{D}}(\Delta_n)$ generated by $u_i^{\pm}, K_i^{\pm 1}$ ($i \in I_n = \mathbb{Z}/n\mathbb{Z}$) is isomorphic to $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$, while $\boldsymbol{\mathcal{D}}(\Delta_n)$ is isomorphic to $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$; see [42, 21, 6]. From now on, we write for notational simplicity,

$$
\boldsymbol{\mathcal{D}}(\infty) = \boldsymbol{\mathcal{D}}(\Delta_\infty) \quad \text{and} \quad \boldsymbol{\mathcal{D}}(n) = \boldsymbol{\mathcal{D}}(\Delta_n).
$$

Remarks 4.1. (1) The construction of $\boldsymbol{\mathcal{D}}(n)$ is slightly different from that in [6, §2.1]. In particular, the K_i here play a role as $\widetilde{K}_i = K_iK_{i+1}^{-1}$ there. In particular, they do not satisfy the equality $K_0K_1\cdots K_{n-1}=1$.

(2) We can extend $\boldsymbol{\mathcal{D}}(n)$ to the $\mathbb{Q}(v)$-algebra $\widehat{\boldsymbol{\mathcal{D}}}(n)$ by adding new generators $D^{\pm 1}$ with relations

$$DD^{-1}=1=D^{-1}D,\ K_iD=DK_i,\ DE_i=v^{\delta_{0,i}}E_iD,\ DF_i=v^{-\delta_{0,i}}F_iD,\ Du^{\pm}_{\mathfrak{m}}=v^{\pm a_0}u^{\pm}_{\mathfrak{m}}D$$

for all $i\in I_n$ and $\mathfrak{m}\in\mathfrak{M}$, where $\mathbf{d}(\mathfrak{m})=(a_i)_{i\in I_n}$. Then $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ clearly becomes a subalgebra of $\widehat{\boldsymbol{\mathcal{D}}}(n)$.

As in (3.1.1), define for each $t\geqslant 1$,

$$\boldsymbol{c}_t^{\pm}=(-1)^tv^{-2tn}\sum_{\mathfrak{m}}(-1)^{\dim\operatorname{End}(M(\mathfrak{m}))}\mathfrak{a}_{\mathfrak{m}}(v^2)u^{\pm}_{\mathfrak{m}}\in\boldsymbol{\mathcal{D}}(n)^{\pm},$$

By Theorem 3.2, the elements $\boldsymbol{c}_t^+$ and $\boldsymbol{c}_t^-$ are central in $\boldsymbol{\mathcal{D}}(n)^+$ and $\boldsymbol{\mathcal{D}}(n)^-$, respectively. Following [21, Sect. 4], define recursively for $t\geqslant 1$,

$$\boldsymbol{x}_t^{\pm}=t\boldsymbol{c}_t^{\pm}-\sum_{s=1}^{t-1}\boldsymbol{x}_s^{\pm}\boldsymbol{c}_{t-s}^{\pm}\in\boldsymbol{\mathcal{D}}(n)^{\pm}.$$

Clearly, $\boldsymbol{x}_t^+$ and $\boldsymbol{x}_t^-$ are again central elements in $\boldsymbol{\mathcal{D}}(n)^+$ and $\boldsymbol{\mathcal{D}}(n)^-$, respectively. By applying [19, Cor. 10 & 12], the $\boldsymbol{x}_t^{\pm}$ are primitive, i.e.,

$$\Delta(\boldsymbol{x}_t^+)=\boldsymbol{x}_t^+\otimes K_{t\delta}+1\otimes\boldsymbol{x}_t^+\quad\text{and}\quad\Delta(\boldsymbol{x}_t^-)=\boldsymbol{x}_t^-\otimes 1+K_{-t\delta}\otimes\boldsymbol{x}_t^-,$$

and they satisfy

$$\psi(\boldsymbol{x}_t^+,\boldsymbol{x}_s^-)=v^{2tn}\{\boldsymbol{x}_t,\boldsymbol{x}_s\}=\delta_{t,s}tv^{2tn}v^{-2tn}(1-v^{-2tn})=\delta_{t,s}t(1-v^{-2tn}).$$

Finally, as in [6, § 2.2], we scale the elements $\boldsymbol{x}_t^{\pm}$ by setting

$$\boldsymbol{z}_t^{\pm}=\frac{v^{tn}}{v^t-v^{-t}}\boldsymbol{x}_t^{\pm}\in\boldsymbol{\mathcal{D}}(n)^{\pm}\quad\text{for } t\geqslant 1.$$

Then

$$\Delta(\boldsymbol{z}_t^+)=\boldsymbol{z}_t^+\otimes K_{t\delta}+1\otimes\boldsymbol{z}_t^+,\ \Delta(\boldsymbol{z}_t^-)=\boldsymbol{z}_t^-\otimes 1+K_{-t\delta}\otimes\boldsymbol{z}_t^-,\tag{4.1.1}$$

and

$$\psi(\boldsymbol{z}_t^+,\boldsymbol{z}_s^-)=\delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t-v^{-t})^2}.$$

Lemma 4.2. (1) *For each $i\in I_n$,*

$$[u_i^+,u_i^-]=\frac{K_i-K_i^{-1}}{v-v^{-1}}.$$

(2) *For $\alpha\in\mathbb{N}I_n$ and $t,s\geqslant 1$, $K_\alpha\boldsymbol{z}_t^{\pm}=\boldsymbol{z}_t^{\pm}K_\alpha$ and*

$$[\boldsymbol{z}_t^+,\boldsymbol{z}_s^-]=\delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t-v^{-t})^2}(K_{t\delta}-K_{-t\delta}).\tag{4.2.1}$$

Moreover, for each $i\in I_n$ and $t\geqslant 1$,

$$[u_i^+,\boldsymbol{z}_t^-]=0=[u_i^-,\boldsymbol{z}_t^+].$$

Proof. We only prove the formula (4.2.1). The remaining ones are easy calculations. Since $\Delta(\boldsymbol{z}_t^+) = \boldsymbol{z}_t^+ \otimes K_{t\delta} + 1 \otimes \boldsymbol{z}_t^+$ and $\Delta(\boldsymbol{z}_s^-) = \boldsymbol{z}_s^- \otimes 1 + K_{-s\delta} \otimes \boldsymbol{z}_s^-$, we have by (4.0.7) that

$$\begin{aligned} & K_{t\delta}\psi(\boldsymbol{z}_t^+, \boldsymbol{z}_s^-) + \boldsymbol{z}_t^+\psi(1, \boldsymbol{z}_s^-) + \boldsymbol{z}_s^- K_{t\delta}\psi(\boldsymbol{z}_t^+, K_{-s\delta}) + \boldsymbol{z}_s^-\boldsymbol{z}_t^+\psi(1, K_{-s\delta}) \\ =& \boldsymbol{z}_t^+\boldsymbol{z}_s^-\psi(K_{t\delta}, 1) + \boldsymbol{z}_s^-\psi(\boldsymbol{z}_t^+, 1) + \boldsymbol{z}_t^+ K_{-s\delta}\psi(K_{t\delta}, \boldsymbol{z}_s^-) + K_{-s\delta}\psi(\boldsymbol{z}_t^+, \boldsymbol{z}_s^-). \end{aligned}$$

This implies that

$$[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] = \psi(\boldsymbol{z}_t^+, \boldsymbol{z}_s^-)(K_{t\delta} - K_{-s\delta}) = \delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t - v^{-t})^2}(K_{t\delta} - K_{-t\delta})$$

since $\psi(1, \boldsymbol{z}_s^-) = \psi(\boldsymbol{z}_t^+, K_{s\delta}) = \psi(\boldsymbol{z}_t^+, 1) = \psi(K_{t\delta}, \boldsymbol{z}_s^-) = 0$ and $\psi(1, K_{s\delta}) = \psi(K_{-t\delta}, 1) = 1$. $\square$

Using arguments similar to those in the proof of [6, Th. 2.3.1], we obtain a presentation of $\boldsymbol{\mathcal{D}}(n)$. More precisely, $\boldsymbol{\mathcal{D}}(n)$ is the $\mathbb{Q}(v)$-algebra generated by $K_i^{\pm 1}$, $u_i^+ = E_i$, $u_i^- = F_i$, and $\boldsymbol{z}_t^\pm$ for $i \in I_n$ and $t \geqslant 1$ with defining relations:

(DH1) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1 = K_i^{-1}K_i$;

(DH2) $K_iE_j = v^{a_{ij}}E_jK_i,\ K_iF_j = v^{-a_{ij}}F_jK_i,\ K_i\boldsymbol{z}_t^\pm = \boldsymbol{z}_t^\pm K_i$;

(DH3) $[E_i, F_j] = \delta_{i,j}\frac{K_i - K_i^{-1}}{v - v^{-1}},\ [E_i, \boldsymbol{z}_t^-] = 0,\ [\boldsymbol{z}_t^+, F_i] = 0,$
$[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] = \delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t - v^{-t})^2}(K_{t\delta} - K_{-t\delta})$;

(DH4) $\displaystyle\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix} E_i^a E_j E_i^b = 0$ for $i \neq j$,
$\boldsymbol{z}_t^+\boldsymbol{z}_s^+ = \boldsymbol{z}_s^+\boldsymbol{z}_t^+,\ E_i\boldsymbol{z}_t^+ = \boldsymbol{z}_t^+E_i$;

(DH5) $\displaystyle\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix} F_i^a F_j F_i^b = 0$ for $i \neq j$,
$\boldsymbol{z}_t^-\boldsymbol{z}_s^- = \boldsymbol{z}_s^-\boldsymbol{z}_t^-,\ F_i\boldsymbol{z}_t^- = \boldsymbol{z}_t^-F_i$,

where $i, j \in I_n$ and $t, s \geqslant 1$.

In the following we simply identify $I_n = \mathbb{Z}/n\mathbb{Z}$ with the subset $\{0, 1, \ldots, n-1\}$ of $\mathbb{Z}$. Let $P^\vee = (\oplus_{i \in I_n}\mathbb{Z}h_i) \oplus \mathbb{Z}d$ be the free abelian group with basis $\{h_i \mid i \in I_n\} \cup \{d\}$. Set $\mathfrak{h} = P^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ and define

$$P = \{\Lambda \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q}) \mid \Lambda(P^\vee) \subset \mathbb{Z}\}.$$

Then $P = (\oplus_{i \in I_n}\mathbb{Z}\Lambda_i) \oplus \mathbb{Z}\omega$, where $\{\Lambda_i \mid i \in I_n\} \cup \{\omega\}$ is the dual basis of $\{h_i \mid i \in I_n\} \cup \{d\}$. This gives rise to the Cartan datum $(P^\vee, P, \Pi^\vee, \Pi)$ associated with the Cartan matrix $C_n = (a_{ij})$, where $\Pi^\vee = \{h_i \mid i \in I_n\}$ is set of simple coroots and $\Pi = \{\alpha_i \mid i \in I_n\}$ is the set of simple roots defined by

$$\alpha_i(h_j) = a_{ji},\ \alpha_i(d) = \delta_{0,i} \ \text{ for all } i, j \in I_n.$$

Finally, let

$$P^+ = \{\Lambda \in P \mid \Lambda(h_i) \geqslant 0,\ \forall\, i \in I_n\} = \big(\bigoplus_{i \in I_n}\mathbb{N}\Lambda_i\big) \oplus \mathbb{Z}\omega$$

denote the set of dominant weights.

For each $\Lambda \in P$, consider the left ideal J_Λ of $\boldsymbol{\mathcal{D}}(n)$ defined by

$$\begin{aligned} J_\Lambda &= \sum_{\mathfrak{m} \in \mathfrak{M}_n \setminus \{0\}} \boldsymbol{\mathcal{D}}(n)u_{\mathfrak{m}}^+ + \sum_{\alpha \in \mathbb{Z}I_n} \boldsymbol{\mathcal{D}}(n)(K_\alpha - v^{\Lambda(\alpha)}) \\ &= \sum_{\mathfrak{m} \in \mathfrak{M}_n \setminus \{0\}} \boldsymbol{\mathcal{D}}(n)u_{\mathfrak{m}}^+ + \sum_{i \in I_n} \boldsymbol{\mathcal{D}}(n)(K_i - v^{\Lambda(h_i)}), \end{aligned}$$

where $\Lambda(\alpha)=\sum_{i\in I_n} a_i\Lambda(h_i)$ if $\alpha=\sum_{i\in I_n} a_i\varepsilon_i\in\mathbb{Z}I_n$. The quotient module

$$M(\Lambda):=\boldsymbol{\mathcal{D}}(n)/J_\Lambda$$

is called the Verma module which is a highest weight module with highest vector $\eta_\Lambda:=1+J_\Lambda$. Applying the triangular decomposition (4.0.8) shows that

$$\boldsymbol{\mathcal{D}}(n)^- \longrightarrow M(\Lambda),\ x^-\longmapsto x^-+J_\Lambda$$

is an isomorphism of $\mathbb{Q}(v)$-vector spaces. Via this isomorphism, $\boldsymbol{\mathcal{D}}(n)^-$ becomes a $\boldsymbol{\mathcal{D}}(n)$-module. It is clear that $M(\Lambda)$ contains a unique maximal submodule M' which gives rise to an irreducible $\boldsymbol{\mathcal{D}}(n)$-module $L(\Lambda)=M(\Lambda)/M'$.

Remark 4.3. By the construction, if $\Lambda,\Lambda'\in P^+$ satisfy $\Lambda-\Lambda'\in\mathbb{Z}\omega$, then $L(\Lambda)=L(\Lambda')$. Therefore, it might be more appropriate to work with the algebra $\widehat{\boldsymbol{\mathcal{D}}}(n)$ defined in Remark 4.1(2).

Theorem 4.4. *Let $\Lambda=\sum_{i\in I_n} a_i\Lambda_i+b\omega\in P^+$ be a dominant weight with $\sum_{i\in I_n} a_i>0$. Then*

$$L(\Lambda)\cong\boldsymbol{\mathcal{D}}(n)^-/\big(\sum_{i\in I_n}\boldsymbol{\mathcal{D}}(n)^-(u_i^-)^{a_i+1}\big).$$

Proof. As in [9, Sect. 3], we extend the Cartan matrix $C=(a_{ij})_{i,j\in I_n}$ to a Borcherds–Cartan matrix $\widetilde{C}=(\widetilde{a}_{ij})_{i,j\in\mathbb{N}}$ by setting $\widetilde{a}_{ij}=a_{ij}$ for $0\leqslant i,j<n$ and $\widetilde{a}_{ij}=0$ otherwise. Consider the free abelian group $\widetilde{P}^\vee=(\oplus_{i\in\mathbb{N}}\mathbb{Z}h_i)\oplus(\oplus_{i\in\mathbb{N}}\mathbb{Z}d_i)$ and define

$$\widetilde{P}=\{\theta\in(\widetilde{P}^\vee\otimes\mathbb{Q})^*\mid\theta(\widetilde{P}^\vee)\subset\mathbb{Z}\}.$$

We then obtain a Cartan datum of type $\widetilde{C}$

$$(\widetilde{P}^\vee,\widetilde{P},\widetilde{\Pi}^\vee=\{h_i\mid i\in\mathbb{N}\},\widetilde{\Pi}=\{\widetilde{\alpha}_i\mid i\in\mathbb{N}\})$$

where the $\widetilde{\alpha}_i$ are defined by

$$\widetilde{\alpha}_i(h_j)=\widetilde{a}_{ji}\ \text{ and }\ \widetilde{\alpha}_i(d_j)=\delta_{i,j},\ \ \forall\, i,j\in\mathbb{N}.$$

Following [25, Def. 2.1] or [23, Def. 1.3], with the above Cartan datum we have the associated quantum generalized Kac–Moody algebra $\mathbf{U}_v(\widetilde{C})$ which is by definition a $\mathbb{Q}(v)$-algebra generated by $K_i^{\pm1},D_i^{\pm1},E_i,F_i$ for $i\in\mathbb{N}$ with relations; see [23, (1.4)] for the details. Clearly, the subalgebra of $\mathbf{U}_v(\widetilde{C})$ generated by $K_i^{\pm1},D_0^{\pm1},E_i,F_i$ for $0\leqslant i<n$ is isomorphic to $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$.

In order to make a comparison with $\boldsymbol{\mathcal{D}}(n)$, we consider the subalgebra $\widetilde{\mathbf{U}}$ of $\mathbf{U}_v(\widetilde{C})$ generated by $K_i^{\pm1},E_i,F_i$ for $i\in\mathbb{N}$. Then $\widetilde{\mathbf{U}}$ admits a triangular decomposition

$$\widetilde{\mathbf{U}}=\widetilde{\mathbf{U}}^-\otimes\widetilde{\mathbf{U}}^0\otimes\widetilde{\mathbf{U}}^+,$$

where $\widetilde{\mathbf{U}}^-$, $\widetilde{\mathbf{U}}^+$, and $\widetilde{\mathbf{U}}^0$ are subalgebras generated by F_i, E_i, and $K_i^{\pm1}$ for $i\in\mathbb{N}$, respectively. In particular, $\widetilde{\mathbf{U}}^0=\mathbb{Q}(v)[K_i^{\pm1}:i\in\mathbb{N}]$. It follows from the definition that there is a surjective algebra homomorphism $\Psi:\widetilde{\mathbf{U}}\to\boldsymbol{\mathcal{D}}(n)$ given by

$$\Psi(E_i)=\begin{cases}u_i^+, & \text{if } 0\leqslant i<n;\\ y_{i-n+1}z_{i-n+1}^+, & \text{if } i\geqslant n,\end{cases}\quad \Psi(F_i)=\begin{cases}u_i^-, & \text{if } 0\leqslant i<n;\\ z_{i-n+1}^-, & \text{if } i\geqslant n\end{cases},\quad\text{and}$$

$$\Psi(K_i^{\pm1})=\begin{cases}K_i^{\pm1}, & \text{if } 0\leqslant i<n;\\ K_{(i-n+1)\delta}^{\pm1}, & \text{if } i\geqslant n,\end{cases}$$

where $y_t=t(v^{2tn}-1)(v-v^{-1})/(v^t-v^{-t})^2$ for $t\geqslant1$; see (4.2.1). Hence, each $\boldsymbol{\mathcal{D}}(n)$-module can be viewed as a $\widetilde{\mathbf{U}}$-module via the homomorphism Ψ. By the definition, Ψ induces isomorphisms $\widetilde{\mathbf{U}}^\pm\cong\boldsymbol{\mathcal{D}}(n)^\pm$. Thus, in what follows, we will identify $\widetilde{\mathbf{U}}^\pm$ with $\boldsymbol{\mathcal{D}}(n)^\pm$ via Ψ.

As defined in [23, Sect. 2.1], for each $\theta \in \widetilde{P}$, there is an associated irreducible $\widetilde{\mathbf{U}}$-module $L(\theta)$. By [23, Prop. 3.3], $L(\theta)$ is integrable if and only if θ is dominant, that is,

$$\theta \in \widetilde{P}^+ = \{\rho \in (\widetilde{P}^\vee \otimes \mathbb{Q})^* \mid \rho(\widetilde{P}^\vee) \subset \mathbb{N}\}.$$

Moreover, by [25, Cor. 4.7], for $\theta \in \widetilde{P}^+$,

$$L(\theta) \cong \widetilde{\mathbf{U}}^-/\big(\sum_{i\in I_n} \widetilde{\mathbf{U}}^- F_i^{\theta(h_i)+1} + \sum_{i\geqslant n, \theta(h_i)=0} \widetilde{\mathbf{U}}^- F_i\big).$$

Viewing the irreducible $\boldsymbol{\mathcal{D}}(n)$-module $L(\Lambda)$ as a $\widetilde{\mathbf{U}}$-module, it is then isomorphic to $L(\widetilde{\Lambda})$, where $\widetilde{\Lambda} \in \widetilde{P}$ is defined by

$$\widetilde{\Lambda}(h_i) = \begin{cases} \Lambda(h_i) = a_i, & \text{if } 0 \leqslant i < n; \\ (i-n+1)\sum_{0\leqslant j<n} a_j, & \text{if } i \geqslant n \end{cases} \quad \text{and} \quad \widetilde{\Lambda}(d_i) = \delta_{i,0} b.$$

From the assumption $\sum_{i\in I} a_i > 0$ it follows that $\widetilde{\Lambda}(h_i) > 0$ for all $i \geqslant n$. Consequently,

$$L(\Lambda) \cong L(\widetilde{\Lambda}) \cong \widetilde{\mathbf{U}}^-/\big(\sum_{i\in I_n} \widetilde{\mathbf{U}}^- F_i^{a_i+1}\big) = \boldsymbol{\mathcal{D}}(n)^-/\big(\sum_{i\in I_n} \boldsymbol{\mathcal{D}}(n)^-(u_i^-)^{a_i+1}\big).$$

□

For each $\Lambda \in P$, let $L_0(\Lambda)$ denote the irreducible $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module of highest weight Λ. Applying Theorem 3.2 gives the following result.

Corollary 4.5. *Let $\Lambda = \sum_{i\in I_n} a_i\Lambda_i + b\omega \in P^+$ with $\sum_{i\in I_n} a_i > 0$. Then $L_0(\Lambda)$ is the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-submodule of $L(\Lambda)$ generated by the highest weight vector η_Λ and there is a vector space decomposition*

$$L(\Lambda) = L_0(\Lambda) \otimes \mathbb{Q}(v)[\boldsymbol{z}_1^-, \boldsymbol{z}_2^-, \dots].$$

In particular, if $L(\Lambda)|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)}$ denotes the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module via restriction, then

$$L(\Lambda)|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m\geqslant 0} L_0(\Lambda - m\delta^*)^{\oplus p(m)}, \tag{4.5.1}$$

where $\delta^* = \sum_{i\in I_n} \alpha_i$ and $p(m)$ is the number of partitions of m.

Proof. By Theorem 3.2,

$$\boldsymbol{\mathcal{D}}(n)^- = \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) \otimes \mathbb{Q}(v)[\boldsymbol{z}_1^-, \boldsymbol{z}_2^-, \dots].$$

This implies that

$$L(\Lambda) \cong \boldsymbol{\mathcal{D}}(n)^-/\big(\sum_{i\in I_n} \boldsymbol{\mathcal{D}}(n)^-(u_i^-)^{a_i+1}\big) \cong \big(\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)/\big(\sum_{i\in I_n} \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) F_i^{a_i+1}\big)\big) \otimes \mathbb{Q}(v)[\boldsymbol{z}_1^-, \boldsymbol{z}_2^-, \dots].$$

By [34, Cor. 6.2.3], $L_0(\Lambda) \cong \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)/\big(\sum_{i\in I_n} \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) F_i^{a_i+1}\big)$. Hence, $L_0(\Lambda)$ is the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-submodule of $L(\Lambda)$ generated by η_Λ and the desired decomposition is obtained.

For each family of nonnegative integers $\{m_t \mid t \geqslant 1\}$ satisfying all but finitely many m_t are zero, $L_0(\Lambda) \otimes \prod_{t\geqslant 1} (\boldsymbol{z}_t^-)^{m_t}$ is a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-submodule of $L(\Lambda)$ since $[u_i^\pm, \boldsymbol{z}_t^-] = 0$ for all $i \in I_n$ and $t \geqslant 1$. It is easy to see that

$$L_0(\Lambda) \otimes \prod_{t\geqslant 1} (\boldsymbol{z}_t^-)^{m_t} \cong L_0(\Lambda - (\sum_{t\geqslant 1} m_t)\delta^*).$$

We conclude that

$$L(\Lambda)|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m\geqslant 0} L_0(\Lambda - m\delta^*)^{\oplus p(m)}.$$

□

By [34, Th. 14.4.11], for each $\Lambda \in P^+$, the canonical basis $\{b^-_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}^{\rm ap}_n\}$ of $\mathbf{U}^-_v(\widehat{\mathfrak{sl}}_n)$ gives rise to the canonical basis

$$\{b^-_{\mathfrak{m}}\eta_\Lambda \neq 0 \mid \mathfrak{m} \in \mathfrak{M}^{\rm ap}_n\}$$

of $L_0(\Lambda)$. On the other hand, the crystal basis theory for the quantum generalized Kac–Moody algebra $\mathbf{U}(\widetilde{C})$ has been developed in [23]. Since all the F_i for $i \geqslant n$ correspond to imaginary simple roots and are central in $\widetilde{\mathbf{U}}^- = \boldsymbol{\mathcal{D}}(n)^-$, applying the construction in [23, Sect. 6] shows that the set

$$\mathbf{B}' := \Big\{\big(\prod_{i\geqslant n} F_i^{m_i}\big)b^-_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}^{\rm ap}_n \text{ and all } m_i \in \mathbb{N} \text{ but finitely many are zero}\Big\}$$

forms the global crystal basis of $\widetilde{\mathbf{U}}^- = \boldsymbol{\mathcal{D}}(n)^-$. We remark that $\mathbf{B}'$ does not coincide with the canonical basis $\mathbf{B}^-$ of $\boldsymbol{\mathcal{D}}(n)^-$.

5. The q-deformed Fock space I: $\boldsymbol{\mathcal{D}}(\infty)$-module

In this section we introduce the q-deformed Fock space Λ^∞ from [17] and review its module structure over $\boldsymbol{\mathcal{D}}(\infty) = \mathbf{U}_v(\mathfrak{sl}_\infty)$ defined in [36, 47], as well as its $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$-module structure. We also provide a proof of [47, Prop. 5.1] by using the properties of representations of Δ_∞. Throughout this section, we identify $\boldsymbol{\mathcal{D}}(\infty)$ with $\mathbf{U}_v(\widehat{\mathfrak{sl}}_\infty)$ via taking $u_i^+ \mapsto E_i$, $u_i^- \mapsto F_i$ for all $i \in I_\infty = \mathbb{Z}$.

For each partition $\lambda \in \Pi$, let $T(\lambda)$ denote the tableau of shape λ whose box in the intersection of the i-th row and the j-th column is labelled with $j - i$ (The box is then said to be with color $j - i$). For example, if $\lambda = (4, 2, 2, 1)$, then $T(\lambda)$ has the form

$$\begin{array}{|c|c|c|c|}\cline{1-1} -3 \\ \cline{1-2} -2 & -1 \\ \cline{1-2} -1 & 0 \\ \hline 0 & 1 & 2 & 3 \\ \hline\end{array}$$

For given $i \in \mathbb{Z}$, a removable i-box of $T(\lambda)$ is by definition a box with color i which can be removed in such a way that the new tableau has the form $T(\mu)$ for some $\mu \in \Pi$. On the contrary, an indent i-box of $T(\lambda)$ is a box with color i which can be added to $T(\lambda)$. For $i \in \mathbb{Z}$ and $\lambda \in \Pi$, define

$$n_i(\lambda) = |\{\text{indent } i\text{-boxes of } T(\lambda)\}| - |\{\text{removable } i\text{-boxes of } T(\lambda)\}|.$$

Let $\bigwedge^\infty$ be the $\mathbb{Q}(v)$-vector space with basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$. Following [47, 4.2], there is a left $\mathbf{U}_v(\mathfrak{sl}_\infty)$-module structure on $\bigwedge^\infty$ defined by

$$K_i \cdot |\lambda\rangle = v^{n_i(\lambda)}|\lambda\rangle,\ E_i \cdot |\lambda\rangle = |\nu\rangle,\ F_i \cdot |\lambda\rangle = |\mu\rangle,\ \forall\, i \in \mathbb{Z}, \lambda \in \Pi, \tag{5.0.2}$$

where $\mu, \nu \in \Pi$ are such that $T(\mu) - T(\lambda)$ and $T(\lambda) - T(\nu)$ are a box with color i. As remarked in [36, Sect. 2] and [47, 4.2], $\bigwedge^\infty$ is isomorphic to the basic representation of $\mathbf{U}_v(\mathfrak{sl}_\infty)$ with the canonical basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$.

Lemma 5.1. (1) *For $i \in \mathbb{Z}$ and $\lambda, \mu \in \Pi$, if $u_i^- \cdot |\mu\rangle = |\lambda\rangle$, then there is an exact sequence*

$$0 \longrightarrow S_i \longrightarrow M(\mathfrak{m}_\lambda) \longrightarrow M(\mathfrak{m}_\mu) \longrightarrow 0.$$

(2) *Let $\mathfrak{m} = [i, l)$ for some $i \in \mathbb{Z}$ and $l \geqslant 1$. Then $\widetilde{u}^-_{\mathfrak{m}} \cdot |\emptyset\rangle \in \mathcal{Z}|\lambda\rangle$ if $i \leqslant 0$ and $i + l - 1 \geqslant 0$ and 0 otherwise, where $\lambda = (i + l, 1^{(-i)})$. In particular, if $i = 0$, then $\widetilde{u}^-_{\mathfrak{m}} \cdot |\emptyset\rangle = |\lambda\rangle$.*

Proof. (1) This follows directly from the definition.

(2) We proceed induction on l. The statement is trivial if $l = 1$. Suppose now $l > 1$. By the definition, $M(\mathfrak{m}) = S_i[l]$ with $\mathbf{dim}\, M(\mathfrak{m}) = \sum_{j=i}^{i+l-1} \varepsilon_j$. Then

$$u^-_{i+l-1} \cdots u^-_{i+1} u^-_i = v^{1-l} u^-_{\mathfrak{m}} + \sum_{\mathfrak{z} <^{\infty}_{\deg} \mathfrak{m}} v^{1-l} u^-_{\mathfrak{z}}.$$

For each $\mathfrak{z}$ with $\mathfrak{z} <^{\infty}_{\deg} \mathfrak{m}$, $M(\mathfrak{z})$ is decomposable. Thus, we may write

$$M(\mathfrak{z}) = M(\mathfrak{y}) \oplus M(\mathfrak{z}_1),$$

where $\mathfrak{y} \in \mathfrak{M}_\infty$ and $\mathfrak{z}_1 = [j, i+l-j)$ for some $i < j \leqslant i+l-1$. This implies that

$$u^-_{\mathfrak{y}} u^-_{\mathfrak{z}_1} = u^-_{\mathfrak{z}}.$$

By the induction hypothesis,

$$u^-_{\mathfrak{z}_1} \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle \ \text{ if } j \leqslant 0 \text{ and } i+l-1 \geqslant 0,$$

and 0 otherwise, where $\mu = (i+l, 1^{(-j)})$. Let now $j \leqslant 0$ and $i+l-1 \geqslant 0$ and let $k_1, \ldots, k_{j-i}$ be a permutation of $i, i+1, \ldots, j-1$. Then

$$(u^-_{k_1} u^-_{k_2} \cdots u^-_{k_{j-i}}) \cdot |\mu\rangle = 0$$

unless $k_1 = i, k_2 = i+1 \ldots, k_{j-i} = j-1$, and moreover

$$(u^-_i u^-_{i+1} \cdots u^-_{j-1}) \cdot |\mu\rangle = |\lambda\rangle.$$

Since $u^-_{\mathfrak{y}}$ is a $\mathcal{Z}$-linear combination of the monomials $u^-_{k_1} u^-_{k_2} \cdots u^-_{k_{j-i}}$, we have $\widetilde{u}^-_{\mathfrak{m}} \cdot |\emptyset\rangle \in \mathcal{Z}|\lambda\rangle$.

Now let $i = 0$. Then $u^-_{\mathfrak{z}_1} \cdot |\emptyset\rangle = 0$ for each $\mathfrak{z}_1 = [j, i+l-j)$ with $0 < j \leqslant i+l-1$. Hence,

$$\widetilde{u}^-_{\mathfrak{m}} \cdot |\emptyset\rangle = v^{1-l} u^-_{\mathfrak{m}} \cdot |\emptyset\rangle = (u^-_{l-1} \cdots u^-_1 u^-_0) \cdot |\emptyset\rangle + \sum_{\mathfrak{z} <^{\infty}_{\deg} \mathfrak{m}} u^-_{\mathfrak{z}} \cdot |\emptyset\rangle = |\lambda\rangle.$$

□

Lemma 5.2. *Let $\mathfrak{m} = \sum_{l \geqslant 1} m_{i,l}[i,l) \in \mathfrak{M}_\infty$ and $\lambda \in \Pi$.*

1. *If there is $j \in \mathbb{Z}$ such that $\sum_{l \geqslant 1} m_{j,l} \geqslant 2$, then $\widetilde{u}^-_{\mathfrak{m}} \cdot |\lambda\rangle = 0$. In particular, for each $i \in \mathbb{Z}$ and $t \geqslant 2$, $(u^-_i)^{(t)} \cdot |\lambda\rangle = 0$, where $(u^-_i)^{(t)} = (u^-_i)^t/[t]!$; see (3.0.2).*
2. *The element $\widetilde{u}^-_{\mathfrak{m}} \cdot |\lambda\rangle$ is a $\mathcal{Z}$-linear combination of $|\mu\rangle$ with $\mu \in \Pi$.*

Proof. (1) For each $i \in \mathbb{Z}$, we put

$$m_i = \sum_{l \geqslant 1} m_{i,l} \ \text{ and } \ M_i = \bigoplus_{l \geqslant 1} m_{i,l} S_i[l].$$

Then $M = M(\mathfrak{m}) = \oplus_{i \in \mathbb{Z}} M_i$, where all but finitely many M_i are zero and

$$u^-_{\mathfrak{m}} = v^{-\sum_{i>j} \langle \mathbf{dim}\, M_i, \mathbf{dim}\, M_j \rangle} (\cdots u^-_{[M_{-1}]} u^-_{[M_0]} u^-_{[M_1]} \cdots).$$

Suppose there is $j \in \mathbb{Z}$ with $m = m_j \geqslant 2$. Then M_j admits a decomposition

$$M_j = S_j[a_1] \oplus \cdots \oplus S_j[a_m] \ \text{ with } a_1 \geqslant \cdots \geqslant a_m \geqslant 1.$$

This implies that

$$u^-_{[S_j[a_m]]} \cdots u^-_{[S_j[a_1]]} = v^{b_j} u^-_{[M_j]},$$

where $b_j = \sum_{1 \leqslant p < q \leqslant m} \langle \mathbf{dim}\, S_j[m_p], \mathbf{dim}\, S_j[m_q] \rangle$. Hence, it suffices to show that for each $\mu \in \Pi$,

$$u^-_{[M_j]} \cdot |\mu\rangle = v^{-b_j} (u^-_{[S_j[a_m]]} \cdots u^-_{[S_j[a_1]]}) \cdot |\mu\rangle = 0.$$

By the definition, $u^-_{[S_j[a_1]]}\cdot|\mu\rangle$ is a $\mathbb{Q}(v)$-linear combination of ν which are obtained from μ by adding a $(j+r)$-box for each $0\leqslant r<a_1$. Thus, each such ν does not admit an indent j-box. Thus, $u^-_{[S_j[a_1]]}\cdot|\nu\rangle=0$ and, hence, $(u^-_{[S_j[a_m]]}\cdots u^-_{[S_j[a_1]]})\cdot|\mu\rangle=0$. We conclude that $\widetilde{u}^-_{\mathfrak{m}}\cdot|\lambda\rangle=0$.

(2) It is known that $\widetilde{u}^-_{\mathfrak{m}}$ is a $\mathcal{Z}$-linear combination of monomials of divided powers $(u_i^-)^{(t)}$ for $i\in\mathbb{Z}$ and $t\geqslant 1$. Since by (1), $(u_i^-)^{(t)}\cdot|\mu\rangle=0$ for all $i\in\mathbb{Z}$, $\mu\in\Pi$ and $t\geqslant 2$, it follows that $\widetilde{u}^-_{\mathfrak{m}}\cdot|\lambda\rangle$ is a $\mathcal{Z}$-linear combination of $(u^-_{i_1}\cdots u^-_{i_m})\cdot|\lambda\rangle$, where $m=\dim M(\mathfrak{m})$ and $i_1,\ldots,i_m\in\mathbb{Z}$. By the definition, $(u^-_{i_1}\cdots u^-_{i_m})\cdot|\lambda\rangle$ either is zero or equal to $|\mu\rangle$ for some $\mu\in\Pi$. Therefore, $\widetilde{u}^-_{\mathfrak{m}}\cdot|\lambda\rangle$ is a $\mathcal{Z}$-linear combination of $|\mu\rangle$ with $\mu\in\Pi$. □

Proposition 5.3. (1) *For each* $\mathfrak{m}\in\mathfrak{M}_\infty$,
$$\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle\in\mathcal{Z}|\lambda\rangle\ \ \textit{for some } \lambda\in\Pi \textit{ with } \mathfrak{m}_\lambda\leqslant^\infty_{\deg}\mathfrak{m}.$$

(2) *For each* $\lambda\in\Pi$,
$$\widetilde{u}^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle=|\lambda\rangle\ \ \textit{and}\ \ \widetilde{u}^-_{\mathfrak{p}}\cdot|\emptyset\rangle=0\ \ \textit{for all } \mathfrak{p}\in\mathfrak{M} \textit{ with } \mathfrak{p}<^\infty_{\deg}\mathfrak{m}_\lambda.$$
In particular, $b^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle=|\lambda\rangle$.

Proof. (1) If $\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle=0$, there is nothing to prove. Now suppose $\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle\neq 0$. By Lemma 5.2(2), we write
$$\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle=\sum_{\lambda\in\Pi}f_\lambda(v)|\lambda\rangle,$$
where all $f_\lambda(v)\in\mathcal{Z}$ but finitely many are zero. If $f_\lambda(v)\neq 0$, then $\mathbf{dim}\,M(\mathfrak{m}_\lambda)=\mathbf{dim}\,M(\mathfrak{m})$. By Lemma 2.1(1), such a $\lambda\in\Pi$ is unique. Hence, we may suppose $\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle=f(v)|\lambda\rangle$ for some $0\neq f(v)\in\mathcal{Z}$ and $\lambda\in\Pi$. It remains to show that $\mathfrak{m}_\lambda\leqslant^\infty_{\deg}\mathfrak{m}$.

Applying Lemma 5.2(1) implies that
$$M=M(\mathfrak{m})=S_{i_1}[a_1]\oplus\cdots\oplus S_{i_t}[a_t],$$
where $i_1<\cdots<i_t$ and $a_1,\ldots,a_t\geqslant 1$. Then
$$u^-_{[S_{i_1}[a_1]]}\cdots u^-_{[S_{i_t}[a_t]]}=v^a u^-_{\mathfrak{m}},$$
where $a=\sum_{1\leqslant p<q\leqslant t}\langle\mathbf{dim}\,S_{i_q}[a_q],\mathbf{dim}\,S_{i_p}[a_p]\rangle$.

We proceed induction on t to show that $M(\mathfrak{m}_\lambda)\leqslant^\infty_{\deg}M=M(\mathfrak{m})$. If $t=1$, this follows from Lemma 5.1(2). Let now $t>1$ and let $\mu\in\Pi$ be such that
$$(u^-_{[S_{i_2}[a_2]]}\cdots u^-_{[S_{i_t}[a_t]]})\cdot|\emptyset\rangle=g(v)|\mu\rangle\ \ \text{for some } 0\neq g(v)\in\mathcal{Z}.$$
Then $u^-_{[S_{i_1}[a_1]]}\cdot|\mu\rangle=v^af(v)g(v)^{-1}|\lambda\rangle$. By the induction hypothesis,
$$M(\mathfrak{m}_\mu)\leqslant^\infty_{\deg}S_{i_2}[a_2]\oplus\cdots\oplus S_{i_t}[a_t].$$
By writing $u^-_{[S_{i_1}[a_1]]}$ as a $\mathcal{Z}$-linear combination of monomials of u_i^-'s and applying Lemma 5.1(1), there exists $X\in\operatorname{Rep}\Delta_\infty$ satisfying $\mathbf{dim}\,X=\mathbf{dim}\,S_{i_1}[a_1]$ with an exact sequence
$$0\longrightarrow X\longrightarrow M(\mathfrak{m}_\lambda)\longrightarrow M(\mathfrak{m}_\mu)\longrightarrow 0.$$
Since $S_{i_1}[a_1]$ is indecomposable, it follows that $X\leqslant^\infty_{\deg}S_{i_1}[a_1]$. Therefore,
$$\begin{aligned}M(\mathfrak{m}_\lambda)\leqslant^\infty_{\deg}M(\mathfrak{m}_\mu)*X\leqslant^\infty_{\deg}&(S_{i_2}[a_2]\oplus\cdots\oplus S_{i_t}[a_t])*S_{i_1}[a_1]\\ =&(S_{i_2}[a_2]\oplus\cdots\oplus S_{i_t}[a_t])\oplus S_{i_1}[a_1]=M(\mathfrak{m}),\end{aligned}$$
that is, $\mathfrak{m}_\lambda\leqslant^\infty_{\deg}\mathfrak{m}$.

(2) Write $\lambda=(\lambda_1,\ldots,\lambda_t)$ with $\lambda_1\geqslant\cdots\geqslant\lambda_t\geqslant 1$. Since
$$M(\mathfrak{m}_\lambda)=S_0[\lambda_1]\oplus S_{-1}[\lambda_2]\oplus\cdots\oplus S_{1-t}[\lambda_t],$$

we have that

$$u^-_{[S_{1-t}[\lambda_t]]}\cdots u^-_{[S_{-1}[\lambda_2]]}u^-_{[S_0[\lambda_1]]}=v^c u^-_{\mathfrak{m}_\lambda},$$

where

$$c=\sum_{1\leqslant r<s\leqslant t}\langle \mathbf{dim}\, S_{1-r}[\lambda_r],\mathbf{dim}\, S_{1-s}[\lambda_s]\rangle=\sum_{1\leqslant r<s\leqslant t}\dim \mathrm{Hom}_{\Delta_\infty}(S_{1-r}[\lambda_r],S_{1-s}[\lambda_s]).$$

By using an argument similar to that in the proof of Lemma 5.1(2), we obtain that

$$\begin{aligned}v^c u^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle&=(u^-_{[S_{1-t}[\lambda_t]]}\cdots u^-_{[S_{-1}[\lambda_2]]}u^-_{[S_0[\lambda_1]]})\cdot|\emptyset\rangle\\&=v^{\lambda_1-1}(u^-_{[S_{1-t}[\lambda_t]]}\cdots u^-_{[S_{-1}[\lambda_2]]})\cdot|(\lambda_1)\rangle\\&=c^{\lambda_1+\lambda_2-2}(u^-_{[S_{1-t}[\lambda_t]]}\cdots u^-_{[S_{-2}[\lambda_3]]})\cdot|(\lambda_1,\lambda_2)\rangle\\&=v^{\lambda_1+\cdots+\lambda_t-t}|(\lambda_1,\ldots,\lambda_t)\rangle=v^{\lambda_1+\cdots+\lambda_t-t}|\lambda\rangle.\end{aligned}$$

Since

$$\dim \mathrm{End}_{\Delta_\infty}(M(\mathfrak{m}_\lambda))=\sum_{1\leqslant r\leqslant s\leqslant t}\dim \mathrm{Hom}_{\Delta_\infty}(S_{1-r}[\lambda_r],S_{1-s}[\lambda_s])=c+t$$

and $\dim M(\mathfrak{m}_\lambda)=\lambda_1+\cdots+\lambda_t$, it follows that

$$\widetilde{u}^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle=v^{c+t-(\lambda_1+\cdots+\lambda_t)}u^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle=|\lambda\rangle.$$

Now let $\mathfrak{p}<^\infty_{\deg}\mathfrak{m}_\lambda$ and suppose $\widetilde{u}^-_{\mathfrak{p}}\cdot|\emptyset\rangle\neq 0$. By (1), there exists $\mu\in\Pi$ with $\mathfrak{m}_\mu\leqslant^\infty_{\deg}\mathfrak{p}$ such that $\widetilde{u}^-_{\mathfrak{p}}\cdot|\emptyset\rangle=f(v)|\mu\rangle$ for some $f(v)\in\mathcal{Z}$. Thus, $\mathfrak{m}_\mu<^\infty_{\deg}\mathfrak{m}_\lambda$. By Lemma 2.1(1), $\mu=\lambda$ since $\mathbf{dim}\, M(\mathfrak{m}_\mu)=\mathbf{dim}\, M(\mathfrak{m}_\lambda)$. This is a contradiction. Hence, $\widetilde{u}^-_{\mathfrak{p}}\cdot|\emptyset\rangle=0$.

By (4.0.9),

$$b^-_{\mathfrak{m}_\lambda}\in\widetilde{u}^-_{\mathfrak{m}_\lambda}+\sum_{\mathfrak{p}<^\infty_{\deg}\mathfrak{m}_\lambda}v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}^-_{\mathfrak{p}}.$$

We conclude that $b^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle=\widetilde{u}^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle=|\lambda\rangle$. □

As a consequence of the proposition above, we obtain [47, Prop. 5.1] as follows.

Corollary 5.4. *The subspace $\mathcal{I}$ of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$ spanned by $b^-_{\mathfrak{m}}$ with $\mathfrak{m}\in\mathfrak{M}-\{\mathfrak{m}_\lambda\mid\lambda\in\Pi\}$ is a left ideal of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$. Moreover, the map*

$$\mathbf{U}_v^-(\mathfrak{sl}_\infty)/\mathcal{I}\longrightarrow\textstyle\bigwedge^\infty,\ \ \mathfrak{b}^-_{\mathfrak{m}_\lambda}+\mathcal{I}\longmapsto|\lambda\rangle,\ \forall\,\lambda\in\Pi$$

is an isomorphism of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$-modules.

Proof. On the one hand, by [34, Th. 14.4.11], the set

$$\{b^-_{\mathfrak{m}}\cdot|\emptyset\rangle\neq 0\mid\mathfrak{m}\in\mathfrak{M}_\infty\}$$

is a basis of $\bigwedge^\infty$. On the other hand, there is a $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$-module homomorphism

$$\phi:\mathbf{U}_v^-(\mathfrak{sl}_\infty)\longrightarrow\textstyle\bigwedge^\infty,\ \ x\longmapsto x\cdot|\emptyset\rangle.$$

It follows from Proposition 5.3(2) that $\mathcal{I}=\mathrm{Ker}\,\phi$ is a left ideal of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$ and that ϕ induces the desired isomorphism. □

Finally, for $i\in\mathbb{Z}$ and $\lambda\in\Pi$, put

$$n_i^-(\lambda)=\sum_{j<i,\,j\in\bar{i}}n_j(\lambda),\ \ n_i^+(\lambda)=\sum_{j>i,\,j\in\bar{i}}n_j(\lambda),\quad\text{and}\ \ n_{\bar{i}}(\lambda)=\sum_{j\in\bar{i}}n_j(\lambda).$$

By [17, 36], there is a $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$-module structure on $\bigwedge^\infty$ defined by

(5.4.1)
$$K_{\bar i}\cdot|\lambda\rangle=v^{n_{\bar i}(\lambda)}|\lambda\rangle,\ \ E_{\bar i}\cdot|\lambda\rangle=\sum_{j\in\bar i}v^{n_j^-(\lambda)}E_j\cdot|\lambda\rangle,\ \ F_{\bar i}\cdot|\lambda\rangle=\sum_{j\in\bar i}v^{-n_j^+(\lambda)}F_j\cdot|\lambda\rangle,$$

where $\bar i\in I_n=\mathbb{Z}/n\mathbb{Z}$.

6. The q-deformed Fock space II: $\boldsymbol{\mathcal{D}}(n)$-module

In this section we first recall the left $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module structure on the Fock space $\bigwedge^\infty$ defined by Varagnolo and Vasserot in [47] and then extend their construction to obtain a $\boldsymbol{\mathcal{D}}(n)$-module structure on $\bigwedge^\infty$.

For each $x=\sum_{\mathfrak{m}}x_{\mathfrak{m}}u_{\mathfrak{m}}\in\mathcal{H}(\Delta)$ with $\Delta=\Delta_n$ or Δ_∞, we write
$$x^\pm=\sum_{\mathfrak{m}}x_{\mathfrak{m}}u_{\mathfrak{m}}^\pm\in\boldsymbol{\mathcal{D}}(\Delta)^\pm.$$

Then for each $\mathbf{d}\in\mathbb{N}I_\infty$, the map $\gamma_{\mathbf{d}}:\mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}}\to\mathcal{H}(\Delta_\infty)_{\mathbf{d}}$ defined in Section 3 induces $\mathbb{Q}(v)$-linear maps
$$\gamma_{\mathbf{d}}^\pm:\boldsymbol{\mathcal{D}}(n)_{\bar{\mathbf{d}}}^\pm\longrightarrow\boldsymbol{\mathcal{D}}(\infty)_{\mathbf{d}}^\pm$$
such that $\gamma_{\mathbf{d}}^\pm(x^\pm)=(\gamma_{\mathbf{d}}(x))^\pm$ for each $x\in\mathcal{H}(\Delta_\infty)$.

Following [47, 6.2], for each $\bar i\in I_n=\mathbb{Z}/n\mathbb{Z}$, $\lambda\in\mathfrak{M}_n$ and $x\in\boldsymbol{\mathcal{D}}(n)_\alpha^-$, define

$$K_{\bar i}\cdot|\lambda\rangle=v^{n_{\bar i}(\lambda)}|\lambda\rangle\ \text{ and }\ x\cdot|\lambda\rangle=\sum_{\mathbf{d}}\big(\gamma_{\mathbf{d}}^-(x)K_{-\mathbf{d}'}\big)\cdot|\lambda\rangle,\tag{6.0.2}$$

where the sum is taken over all $\mathbf{d}\in\mathbb{N}I_\infty$ such that $\bar{\mathbf{d}}=\alpha$ and $\mathbf{d}'=\sum_{i>j,\bar i=\bar j}d_j\varepsilon_i$. By [47, Cor. 6.2], this defines a left $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module structure on $\bigwedge^\infty$ which extends the Hayashi action of $\mathbf{U}_v^{\leqslant 0}(\widehat{\mathfrak{sl}}_n)$ on $\bigwedge^\infty$ defined in (5.4.1).

Dually, for each $\lambda\in\Pi$ and $x\in\boldsymbol{\mathcal{D}}(n)_\alpha^+$, define

$$x\cdot|\lambda\rangle=\sum_{\mathbf{d}}\big(\gamma_{\mathbf{d}}^+(x)K_{\mathbf{d}''}\big)\cdot|\lambda\rangle,\tag{6.0.3}$$

where the sum is taken over all $\mathbf{d}\in\mathbb{N}I_\infty$ such that $\bar{\mathbf{d}}=\alpha$ and $\mathbf{d}''=\sum_{i<j,\bar i=\bar j}d_j\varepsilon_i$.

Proposition 6.1. *The formula* (6.0.3) *defines a left $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module structure on $\bigwedge^\infty$ which extends the Hayashi action of $\mathbf{U}_v^{\geqslant 0}(\widehat{\mathfrak{sl}}_n)$ on $\bigwedge^\infty$.*

Proof. Let $x\in\boldsymbol{\mathcal{D}}(n)_\alpha^+$ and $y\in\boldsymbol{\mathcal{D}}(n)_\beta^+$, where $\alpha,\beta\in\mathbb{N}I_n$. By the definition, we have, on the one hand, that
$$(xy)\cdot|\lambda\rangle=\sum_{\mathbf{d}}\big(\gamma_{\mathbf{d}}^+(xy)K_{\mathbf{d}''}\big)\cdot|\lambda\rangle$$
and, on the other hand, that
$$x\cdot(y\cdot|\lambda\rangle)=\sum_{\mathbf{a},\mathbf{b}}\big(\gamma_{\mathbf{a}}^+(x)K_{\mathbf{a}''}\gamma_{\mathbf{b}}^+(y)K_{\mathbf{b}''}\big)\cdot|\lambda\rangle,$$
where the sum is taken over all $\mathbf{a},\mathbf{b}\in\mathbb{N}I_\infty$ such that $\bar{\mathbf{a}}=\alpha$ and $\bar{\mathbf{b}}=\beta$.

Since $K_{\mathbf{a}''}\gamma_{\mathbf{b}}^+(y)=v^{(\mathbf{a}'',\mathbf{b})}\gamma_{\mathbf{b}}^+(y)K_{\mathbf{a}''}$, we obtain that
$$x\cdot(y\cdot|\lambda\rangle)=\sum_{\mathbf{d}}\sum_{\mathbf{a}+\mathbf{b}=\mathbf{d}}v^{(\mathbf{a}'',\mathbf{b})}\big(\gamma_{\mathbf{a}}^+(x)\gamma_{\mathbf{b}}^+(y)K_{\mathbf{d}''}\big)\cdot|\lambda\rangle.$$

By the definition,

$$(\mathbf{a}'', \mathbf{b}) = \Big(\sum_{i<j,\, \bar{i}=\bar{j}} a_j \varepsilon_i, \sum_i b_i \varepsilon_i \Big) = \sum_{i<j,\, \bar{i}=\bar{j}} b_i(2a_j - a_{j-1} - a_{j+1}) = \kappa(\mathbf{a}, \mathbf{b}).$$

Applying Lemma 3.3(2) gives that

$$(xy) \cdot |\lambda\rangle = x \cdot (y \cdot |\lambda\rangle).$$

Hence, $\bigwedge^\infty$ becomes a left $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module.

For each $\bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}$ and $\lambda \in \Pi$, we have

$$u_{\bar{i}}^+ \cdot |\lambda\rangle = \sum_{j\in\bar{i}} (u_j^+ K_{-\varepsilon_j''}) \cdot |\lambda\rangle.$$

Since $\varepsilon_j'' = \sum_{l<j,\, \bar{l}=\bar{j}} \varepsilon_l$ for each $j \in \bar{i}$, it follows that

$$K_{\varepsilon_j''} \cdot |\lambda\rangle = \prod_{l<j,\, \bar{l}=\bar{j}} K_{\varepsilon_l} \cdot |\lambda\rangle = v^{\sum_{l<j,\bar{l}=\bar{j}} n_l(\lambda)} |\lambda\rangle = v^{n_j^-(\lambda)} |\lambda\rangle.$$

This implies that

$$u_{\bar{i}}^+ \cdot |\lambda\rangle = \sum_{j\in\bar{i}} v^{n_j^-(\lambda)} u_j^+ \cdot |\lambda\rangle,$$

which coincides with the formula for $E_{\bar{i}} \cdot |\lambda\rangle$ in (5.4.1), as required. $\square$

Remark 6.2. The proof of Proposition 6.1 is analogous to that of [47, Cor. 6.2]. However, it seems to us that the $\boldsymbol{\mathcal{D}}(n)^+$-module structure on $\bigwedge^\infty$ can not be directly obtained form its $\boldsymbol{\mathcal{D}}(n)^-$-module structure via certain duality between $\boldsymbol{\mathcal{D}}(n)^-$ and $\boldsymbol{\mathcal{D}}(n)^+$.

The main purpose of this section is to prove that formulas (6.0.2) and (6.0.3) indeed define a $\boldsymbol{\mathcal{D}}(n)$-module structure on $\bigwedge^\infty$. The strategy is to pass to the semi-infinite v-wedge spaces studied in [44, 27].

Let Ω denote the $\mathbb{Q}(v)$-vector space with basis $\{\omega_i \mid i \in \mathbb{Z}\}$. By [6, Prop. 3.5], Ω admits a $\boldsymbol{\mathcal{D}}(n)$-module structure defined by

$$\begin{aligned} u_i^+ \cdot \omega_s &= \delta_{i+1,\bar{s}} \omega_{s-1}, \quad u_i^- \cdot \omega_s = \delta_{i,\bar{s}} \omega_{s+1} \\ K_i^{\pm 1} \cdot \omega_s &= v^{\pm\delta_{i,\bar{s}} \mp \delta_{i+1,\bar{s}}} \omega_s, \quad \boldsymbol{z}_m^{\pm} \cdot \omega_s = \omega_{s\mp mn} \end{aligned} \tag{6.2.1}$$

for all $i \in I_n$ and $s, m \in \mathbb{Z}$ with $m \geqslant 1$. In particular, $K_\delta^{\pm 1} \cdot \omega_s = \omega_s$ for each $s \in \mathbb{Z}$. This is an extension of the $\mathbf{U}_v'(\widehat{\mathfrak{sl}}_n)$-action on Ω defined in [27, 1.1] as well as an extension of the $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-action on Ω defined in [47, 8.1]; see [6, 3.5].

For a fixed positive integer r, consider the r-fold tensor product $\Omega^{\otimes r}$ which has a basis

$$\{\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \mid \mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r\}.$$

The Hopf algebra structure of $\boldsymbol{\mathcal{D}}(n)$ induces a $\boldsymbol{\mathcal{D}}(n)$-module structure on the r-fold tensor product $\Omega^{\otimes r}$. By (4.1.1), we have for each $t \geqslant 1$,

$$\begin{aligned} \Delta^{(r-1)}(\boldsymbol{z}_t^+) &= \sum_{s=0}^{r-1} \underbrace{1 \otimes \cdots \otimes 1}_{s} \otimes \boldsymbol{z}_t^+ \otimes \underbrace{K_{t\delta} \otimes \cdots \otimes K_{t\delta}}_{r-s-1} \quad \text{and} \\ \Delta^{(r-1)}(\boldsymbol{z}_t^-) &= \sum_{s=0}^{r-1} \underbrace{K_{-t\delta} \otimes \cdots \otimes K_{-t\delta}}_{s} \otimes \boldsymbol{z}_t^- \otimes \underbrace{1 \otimes \cdots \otimes 1}_{r-s-1}. \end{aligned} \tag{6.2.2}$$

This implies particularly that for each $t \geqslant 1$ and $\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \in \Omega^{\otimes r}$,

(6.2.3)
$$\boldsymbol{z}_t^{\pm} \cdot \omega_{\mathbf{i}} = \sum_{s=1}^{r} \omega_{i_1} \otimes \cdots \otimes \omega_{i_{s-1}} \otimes \omega_{i_s \mp tn} \otimes \omega_{i_{s+1}} \otimes \cdots \otimes \omega_{i_r}.$$

By (4.0.3) and (4.0.5), for each $\alpha \in \mathbb{N}I_n$, we have

(6.2.4)
$$\begin{aligned}\Delta^{(r-1)}(\widetilde{u}_\alpha^+) = & \sum_{\alpha=\alpha^{(1)}+\cdots+\alpha^{(r)}} v^{\sum_{s>t}\langle \alpha^{(s)},\alpha^{(t)}\rangle} \times \\ & \qquad \widetilde{u}^+_{\alpha^{(1)}} \otimes \widetilde{u}^+_{\alpha^{(2)}} K_{\alpha^{(1)}} \otimes \cdots \otimes \widetilde{u}^+_{\alpha^{(r)}} K_{(\alpha^{(1)}+\alpha^{(2)}+\cdots+\alpha^{(r-1)})}, \\ \Delta^{(r-1)}(\widetilde{u}_\alpha^-) = & \sum_{\alpha=\alpha^{(1)}+\cdots+\alpha^{(r)}} v^{\sum_{s>t}\langle \alpha^{(s)},\alpha^{(t)}\rangle} \times \\ & \qquad \widetilde{u}^-_{\alpha^{(1)}} K_{-(\alpha^{(2)}+\cdots+\alpha^{(r)})} \otimes \cdots \otimes u^-_{\alpha^{(r-1)}} K_{-\alpha^{(r)}} \otimes \widetilde{u}^-_{\alpha^{(r)}}.\end{aligned}$$

This gives the the following lemma; see [47, Lem. 8.3] and [6, Cor. 3.5.8].

Lemma 6.3. *Let $\alpha \in \mathbb{N}I_n$ and $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$. Then*

(6.3.1)
$$\widetilde{u}_\alpha^+ \cdot \omega_{\mathbf{i}} = \sum_{\mathbf{n}} v^{c^+(\mathbf{i},\mathbf{i}-\mathbf{n})} \omega_{\mathbf{i}-\mathbf{n}},$$

where the sum is taken over the sequences $\mathbf{n} = (n_1, \dots, n_r) \in \{0,1\}^r$ satisfying $\alpha = \sum_{s=1}^r n_s \varepsilon_{\overline{i_s - 1}}$ and

$$c^+(\mathbf{i}, \mathbf{i}-\mathbf{n}) = \sum_{1 \leqslant s < t \leqslant r} n_s(n_t - 1)\langle \varepsilon_{\bar{i}_t}, \varepsilon_{\bar{i}_s}\rangle;$$

(6.3.2)
$$\widetilde{u}_\alpha^- \cdot \omega_{\mathbf{i}} = \sum_{\mathbf{n}} v^{c^-(\mathbf{i},\mathbf{i}+\mathbf{n})} \omega_{\mathbf{i}+\mathbf{n}},$$

where the sum is taken over the sequences $\mathbf{n} = (n_1, \dots, n_r) \in \{0,1\}^r$ satisfying $\alpha = \sum_{s=1}^r n_s \varepsilon_{\bar{i}_s}$ and

$$c^-(\mathbf{i}, \mathbf{i}+\mathbf{n}) = \sum_{1 \leqslant s < t \leqslant r} n_t(n_s - 1)\langle \varepsilon_{\bar{i}_t}, \varepsilon_{\bar{i}_s}\rangle.$$

On the other hand, let $\widehat{\mathbf{H}}(r)$ be the Hecke algebra of affine symmetric group of type A which is by definition a $\mathbb{Q}(v)$-algebra with generators T_i and $X_j^{\pm 1}$ for $i = 1, \dots, r-1$, $j = 1, \dots, r$ and relations:

$$\begin{aligned}&(T_i + 1)(T_i - v^2) = 0, \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \ \ T_i T_j = T_j T_i \ (|i-j| > 1), \\ &X_i X_i^{-1} = 1 = X_i^{-1} X_i, \ \ X_i X_j = X_j X_i, \\ &T_i X_i T_i = v^2 X_{i+1}, \ \ X_j T_i = T_i X_j \ (j \neq i, i+1).\end{aligned}$$

This is the so-called *Bernstein presentation* of $\widehat{\mathbf{H}}(r)$.

By [47, Sect. 8.2], there is a right $\widehat{\mathbf{H}}(r)$-module structure on $\Omega^{\otimes r}$ defined by

(6.3.3)
$$\begin{aligned}&\omega_{\mathbf{i}} \cdot X_t = \omega_{i_1} \cdots \omega_{i_{t-1}} \omega_{i_t - n} \omega_{i_{t+1}} \cdots \omega_{i_r}, \\ &\omega_{\mathbf{i}} \cdot T_k = \begin{cases} v^2 \omega_{\mathbf{i}}, & \text{if } i_k = i_{k+1}; \\ v\omega_{\mathbf{i}s_k}, & \text{if } -n < i_k < i_{k+1} \leqslant 0; \\ v\omega_{\mathbf{i}s_k} + (v^2-1)\omega_{\mathbf{i}}, & \text{if } -n < i_{k+1} < i_k \leqslant 0,\end{cases}\end{aligned}$$

where $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$, $\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r}$ and

$$\omega_{\mathbf{i}s_k} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_{k+1}} \otimes \omega_{i_k} \otimes \cdots \otimes \omega_{i_r}.$$

Following [6, Prop. 3.5.5], the tensor space $\Omega^{\otimes r}$ is indeed a $\boldsymbol{\mathcal{D}}(n)$-$\widehat{\mathbf{H}}(r)$-bimodule. Set

$$\Xi^r = \sum_{i=1}^{r-1} \mathrm{Im}\,(1+T_i) \subseteq \Omega^{\otimes r},$$

which is clearly a $\boldsymbol{\mathcal{D}}(n)$-submodule of $\Omega^{\otimes r}$. Thus, the quotient space $\Omega^{\otimes r}/\Xi^r$ becomes a $\boldsymbol{\mathcal{D}}(n)$-module. For each $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$, write

$$\wedge\omega_{\mathbf{i}} = \omega_{i_1} \wedge \ldots \wedge \omega_{i_r} = \omega_{\mathbf{i}} + \Xi^r \in \Omega^{\otimes r}/\Xi^r.$$

By [27, Prop. 1.3], the set

$$\{\wedge\omega_{\mathbf{i}} \mid i_1 > \cdots > i_r\}$$

forms a basis of $\Omega^{\otimes r}/\Xi^r$.

For each $m \in \mathbb{Z}$, let $\mathscr{B}_m$ denote the set of sequences $\mathbf{i} = (i_1, i_2, \dots) \in \mathbb{Z}^\infty$ satisfying that $i_s = m - s + 1$ for $s \gg 0$, and set $\mathscr{B}_\infty = \cup_{m\in\mathbb{Z}}\mathscr{B}_m$. As in [47, Sect. 10.1], let Ω^∞ denote the space spanned by semi-infinite monomials

$$\omega_{\mathbf{i}} = \omega_{i_1} \otimes \omega_{i_2} \otimes \cdots, \quad \text{where } \mathbf{i} = (i_1, i_2, \dots) \in \mathscr{B}_\infty.$$

Then the affine Hecke algebra $\widehat{\mathbf{H}}(\infty)$ acts on Ω^∞ via the formulas in (6.3.3). Set

$$\Xi^\infty = \sum_{i=1}^{\infty} \mathrm{Im}\,(1+T_i) \subseteq \Omega^\infty.$$

For each $\mathbf{i} = (i_1, i_2, \dots) \in \mathscr{B}_\infty$ as above, write

$$\wedge\omega_{\mathbf{i}} = \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots = \omega_{\mathbf{i}} + \Xi^\infty \in \Omega^\infty/\Xi^\infty.$$

For each $m \in \mathbb{Z}$, let $\mathcal{F}_{(m)}$ be the subspace of Ω^∞/Ξ^∞ spanned by $\wedge\omega_{\mathbf{i}}$ with $\mathbf{i} \in \mathscr{B}_m$. Then

$$\Omega^\infty/\Xi^\infty = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_{(m)}.$$

By [27, 1.4], the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module structure on $\Omega^{\otimes r}/\Xi^r$ induces a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module structure on $\mathcal{F}_{(m)}$ for each $m \in \mathbb{Z}$ and, hence, a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module structure on Ω^∞/Ξ^∞ as well. Moreover, by [27, Prop. 1.4], the injective map

$$\kappa : \textstyle\bigwedge^\infty \longrightarrow \Omega^\infty/\Xi^\infty, \ |\lambda\rangle \longmapsto \wedge\omega_{\mathbf{i}_\lambda}$$

is a homomorphism of $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-modules which induces an isomorphism $\bigwedge^\infty \cong \mathcal{F}_{(0)}$, where $\mathbf{i}_\lambda = (i_1, i_2, \dots)$ with $i_s = \lambda_s + 1 - s$, $\forall\, s \geqslant 1$.

As in [27, (49)], for each $m \in \mathbb{Z}$, write

$$|m\rangle = \omega_m \wedge \omega_{m-1} \wedge \omega_{m-2} \wedge \cdots .$$

Clearly, for each $\mathbf{i} = (i_1, i_2, \dots) \in \mathscr{B}_m$, there exists a sufficiently large N such that

$$\wedge\omega_{\mathbf{i}} = (\omega_{i_1} \wedge \cdots \wedge \omega_{i_N}) \wedge |m - N\rangle.$$

By [27, Lem. 2.2] and (6.2.4), for given $\alpha \in \mathbb{N}I$ and $\mathbf{i} \in \mathscr{B}_m$, there is $t \gg 0$ such that

$$u_\alpha^- \cdot (\wedge\omega_{\mathbf{i}}) = \big(u_\alpha^- \cdot (\omega_{i_1} \wedge \cdots \wedge \omega_{i_t})\big) \wedge |m - t\rangle.$$

Hence, the $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module structure on $\Omega^{\otimes r}/\Xi^r$ induces a $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module structure on Ω^∞/Ξ^∞; see [47, Sect. 10.1]. Moreover, by [47, Lem. 10.1], the map $\kappa : \bigwedge^\infty \to \Omega^\infty/\Xi^\infty$ is a $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module homomorphism.

Dually, for each given $\mathbf{i} \in \mathscr{B}_m$, there is $t \gg 0$ such that

$$u_\alpha^+ \cdot (\wedge\omega_{\mathbf{i}}) = \big(u_\alpha^+ \cdot (\omega_{i_1} \wedge \cdots \wedge \omega_{i_t})\big) \wedge \big(K_\alpha \cdot |m - t\rangle\big).$$

Thus, Ω^∞/Ξ^∞ becomes a left $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module, too. We have the following result whose proof is similar to that of [47, Lem. 10.1].

Proposition 6.4. *The map κ is a $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module homomorphism.*

Proof. We need to show that for each $\lambda \in \Pi$ and $\alpha \in \mathbb{N}I_n$,

$$\kappa(\widetilde{u}_\alpha^+ \cdot |\lambda\rangle) = \widetilde{u}_\alpha^+(\kappa(|\lambda\rangle)).$$

For simplicity, write $\mathbf{i} := \mathbf{i}_\lambda = (i_1, i_2, \dots)$. By (6.0.3),

$$\widetilde{u}_\alpha^+ \cdot |\lambda\rangle = \sum_{\mathbf{d}} (\gamma_{\mathbf{d}}^+(\widetilde{u}_\alpha^+) K_{\mathbf{d}''}) \cdot |\lambda\rangle = \sum_{\mathbf{d}} v^{-h(\mathbf{d})} (\widetilde{u}_{\mathbf{d}}^+ K_{\mathbf{d}''}) \cdot |\lambda\rangle,$$

where the sum is taken over all $\mathbf{d} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{d}} = \alpha$ and $h(\mathbf{d}) = \sum_{i<j, \bar{i}=\bar{j}} d_i(d_{j+1} - d_j)$.

For each fixed $\mathbf{d} = (d_i) \in \mathbb{N}I_\infty$ with $\bar{\mathbf{d}} = \alpha$, we have

$$\widetilde{u}_{\mathbf{d}}^+ = \cdots \widetilde{u}_{d_1\varepsilon_1}^+ \widetilde{u}_{d_0\varepsilon_0}^+ \widetilde{u}_{d_{-1}\varepsilon_{-1}}^+ \cdots = \prod_{i\in\mathbb{Z}} \widetilde{u}_{d_i\varepsilon_i}^+.$$

By the definition, $\widetilde{u}_{\mathbf{d}}^+ \cdot |\lambda\rangle \neq 0$ implies that

$$\mathbf{d} = \sum_{s\geqslant 1} n_s \varepsilon_{i_s - 1},$$

where $n_s \in \{0, 1\}$ for all $s \geqslant 1$. Moreover, if this is the case, then

$$\widetilde{u}_{\mathbf{d}}^+ \cdot |\lambda\rangle = |\mu_{\mathbf{n}}\rangle,$$

where $\mathbf{n} = (n_1, n_2, \dots)$ and $\mu_{\mathbf{n}} = \mu \in \Pi$ is determined by $\mathbf{i}_\mu = \mathbf{i} - \mathbf{n}$. Therefore, for $\mathbf{d} \in \mathbb{N}I_\infty$ with $\mathbf{d} = \sum_{s\geqslant 1} n_s \varepsilon_{i_s-1}$,

$$K_{\mathbf{d}''} = \prod_{\bar{i}_s=\bar{i}_t,\, i_s>i_t} K_{i_t-1}^{n_s} \quad \text{and} \quad h(\mathbf{d}) = \sum_{i_s>i_t} -n_s n_t(\delta_{\bar{i}_s,\bar{i}_t} - \delta_{\bar{i}_s,\overline{i_t+1}}) = -\sum_{i_s>i_t} n_s n_t \langle \varepsilon_{\bar{i}_t}, \varepsilon_{\bar{i}_s}\rangle.$$

A calculation together with (6.3.1) implies that

$$\kappa(\widetilde{u}_\alpha^+ \cdot |\lambda\rangle) = \widetilde{u}_\alpha^+(\wedge\omega_{\mathbf{i}}) = \widetilde{u}_\alpha^+(\kappa(|\lambda\rangle)).$$

□

As a consequence of the results above, to prove that the formulas (6.0.2) and (6.0.3) define a $\boldsymbol{\mathcal{D}}(n)$-module structure on $\bigwedge^\infty$, it suffices to show that the $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module and $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module structures on Ω^∞/Ξ^∞ define a $\boldsymbol{\mathcal{D}}(n)$-module structure. In other words, we need to show that the actions of $K_i^{\pm 1}, u_i^+, u_i^-$ $(i \in I_n)$ and $\boldsymbol{z}_s^+, \boldsymbol{z}_s^-$ $(s \geqslant 1)$ on Ω^∞/Ξ^∞ satisfy the relations (DH1)–(DH5) in Section 4.

Since, as discussed above, Ω^∞/Ξ^∞ is a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module, all the relations in (DH1)–(DH5) in which the $\boldsymbol{z}_s^\pm$ are not involved are satisfied. In the following we are going to check the relations

$$[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2}(K_{t\delta} - K_{-t\delta}), \ \forall\, s, t \geqslant 1.$$

By [27, §2], for each $t \geqslant 1$, there are Heisenberg operators

$$B_t^\pm : \Omega^\infty/\Xi^\infty \to \Omega^\infty/\Xi^\infty, \ \wedge\omega_{\mathbf{i}} \longmapsto \sum_{s=1}^{\infty} \wedge\omega_{\mathbf{i}\mp tn\mathbf{e}_s},$$

where $\mathbf{i} \in \mathscr{B}_\infty$ and $\mathbf{e}_s = (\delta_{i,s})_{i\geqslant 1} \in \mathbb{Z}^\infty$. Note that for each $\mathbf{i} \in \mathscr{B}_\infty$, $\wedge\omega_{\mathbf{i}\mp tn\mathbf{e}_s} = 0$ for $s \gg 0$.

Proposition 6.5. *For each $t \geqslant 1$ and $\mathbf{i} \in \mathscr{B}_\infty$,*

$$B_t^+(\wedge\omega_{\mathbf{i}}) = v^t \boldsymbol{z}_t^+ \cdot (\wedge\omega_{\mathbf{i}}) \quad \textit{and} \quad B_t^-(\wedge\omega_{\mathbf{i}}) = \boldsymbol{z}_t^- \cdot (\wedge\omega_{\mathbf{i}}).$$

Proof. For each $m\in\mathbb{Z}$, recall the element

$$|m\rangle=\omega_m\wedge\omega_{m-1}\wedge\omega_{m-2}\wedge\cdots\in\Omega^\infty/\Xi^\infty.$$

Then $\boldsymbol{z}_t^+\cdot|m\rangle=0$ and $K_\delta\cdot|m\rangle=q|m\rangle$. Write

$$\wedge\omega_{\mathbf{i}}=\omega_{i_1}\wedge\cdots\wedge\omega_{i_N}\wedge|N-m\rangle.$$

Applying (6.2.2) gives that

$$\begin{aligned}&\boldsymbol{z}_t^+\cdot(\wedge\omega_{\mathbf{i}})\\ =&\sum_{s=0}^{N}\underbrace{\omega_{i_1}\wedge\cdots\wedge\omega_{i_s}}_{s}\wedge\boldsymbol{z}_t^+\cdot\omega_{i_{s+1}}\wedge\underbrace{K_{t\delta}\cdot\omega_{i_{s+2}}\wedge\cdots\wedge K_{t\delta}\cdot\omega_{i_N}}_{N-s-1}\wedge(K_{t\delta}\cdot|N-m\rangle)\\ =&\sum_{s=0}^{N}v^t\underbrace{\omega_{i_1}\wedge\cdots\wedge\omega_{i_s}}_{s}\wedge\omega_{i_{s+1}+tn}\wedge\underbrace{\omega_{i_{s+2}}\wedge\cdots\wedge\omega_{i_N}}_{N-s-1}\wedge|N-m\rangle\\ =&v^tB_t^+(\wedge\omega_{\mathbf{i}})\quad(\text{since } B_t^+(|N-m\rangle)=0),\end{aligned}$$

that is, $B_t^+(\wedge\omega_{\mathbf{i}})=v^t\boldsymbol{z}_t^+\cdot(\wedge\omega_{\mathbf{i}})$. The second equality can be proved similarly. □

Corollary 6.6. *Let $t,s\geqslant 1$. Then for each $\mathbf{i}\in\mathscr{B}_\infty$,*

$$[\boldsymbol{z}_t^+,\boldsymbol{z}_s^-]\cdot(\wedge\omega_{\mathbf{i}})=\delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t-v^{-t})^2}(K_{t\delta}-K_{-t\delta})\cdot(\wedge\omega_{\mathbf{i}}).$$

Proof. By [27, Prop. 2.2 & 2.6] (with $q=v$),

$$[B_t^+,B_s^-]=\delta_{t,s}\frac{t(1-v^{2tn})}{1-v^{2n}}.$$

This together with Proposition 6.5 implies that for each $\mathbf{i}\in\mathscr{B}_\infty$,

$$[\boldsymbol{z}_t^+,\boldsymbol{z}_s^-]\cdot(\wedge\omega_{\mathbf{i}})=v^t[B_t^+,B_s^-]\delta_{t,s}\cdot(\wedge\omega_{\mathbf{i}})=\delta_{t,s}\frac{tv^t(1-v^{2tn})}{1-v^{2n}}(\wedge\omega_{\mathbf{i}}).$$

On the other hand,

$$\begin{aligned}\delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t-v^{-t})^2}(K_{t\delta}-K_{-t\delta})\cdot(\wedge\omega_{\mathbf{i}})&=\delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t-v^{-t})^2}(v^t-v^{-t})(\wedge\omega_{\mathbf{i}})\\&=\delta_{t,s}\frac{tv^t(1-v^{2tn})}{1-v^{2n}}(\wedge\omega_{\mathbf{i}}).\end{aligned}$$

This gives the desired equality. □

By [27, Prop. 2.1] (or direct calculations), the actions of $\boldsymbol{z}_t^\pm$ on Ω^∞/Ξ^∞ commutes with that of $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$. In conclusion, the actions of $K_i^{\pm1},u_i^+,u_i^-$ $(i\in I_n)$ and $\boldsymbol{z}_s^+,\boldsymbol{z}_s^-$ $(s\geqslant 1)$ on Ω^∞/Ξ^∞ satisfy the relations (DH1)–(DH5). Therefore, the formulas (6.0.2) and (6.0.3) define a $\boldsymbol{\mathcal{D}}(n)$-module structure on $\bigwedge^\infty$.

7. An isomorphism from $L(\Lambda_0)$ to $\bigwedge^\infty$

In this section we show that the Fock space $\bigwedge^\infty$ as a $\boldsymbol{\mathcal{D}}(n)$-module is isomorphic to the basic representation $L(\Lambda_0)$ defined in Section 4. As an application, the decomposition of $L(\Lambda_0)$ in Corollary 4.5 induces the Kashiwara–Miwa–Stern decomposition of $\bigwedge^\infty$ in [27].

Proposition 7.1. *For each $\mathfrak{m}\in\mathfrak{M}_n$, $\widetilde{u}_{\mathfrak{m}}^-\cdot|\emptyset\rangle$ is a $\mathcal{Z}$-linear combination of those $|\mu\rangle$ satisfying $\mathfrak{m}_\mu\leqslant_{\deg}\mathfrak{m}$.*

Proof. By (6.0.2),

$$\widetilde{u}^-_{\mathfrak{m}} \cdot |\emptyset\rangle = \sum_{\mathbf{d}} \big(\gamma^-_{\mathbf{d}}(\widetilde{u}^-_{\mathfrak{m}}) K_{-\mathbf{d}'}\big) \cdot |\emptyset\rangle, \quad \text{where } \mathbf{d}' = \sum_i \Big(\sum_{j<i,\, \bar{j}=\bar{i}} d_j \Big) \varepsilon_i.$$

Since $K_i \cdot |\emptyset\rangle = v^{\delta_{i,0}} |\emptyset\rangle$ for $i \in \mathbb{Z}$, it follows that $K_{-\mathbf{d}'} \cdot |\emptyset\rangle = v^{-\sum_{j<0,\, \bar{j}=\bar{0}} d_j} |\emptyset\rangle$. By Proposition 3.4,

$$\gamma^-_{\mathbf{d}}(\widetilde{u}^-_{\mathfrak{m}}) \in \sum_{\mathfrak{z}} \mathcal{Z} \widetilde{u}^-_{\mathfrak{z}},$$

where the sum is taken over $\mathfrak{z} \in \mathfrak{M}_\infty$ with $\mathscr{F}(\mathfrak{z}) \leqslant^\infty_{\deg} \mathfrak{m}$. Further, by Proposition 5.3(1),

$$\widetilde{u}^-_{\mathfrak{z}} \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle$$

for some $\mu \in \Pi$ with $\mathfrak{m}^\infty_\mu \leqslant^\infty_{\deg} \mathfrak{z}$. This implies that

$$\mathfrak{m}_\mu = \mathscr{F}(\mathfrak{m}^\infty_\mu) \leqslant_{\deg} \mathscr{F}(\mathfrak{z}) \leqslant_{\deg} \mathfrak{m}.$$

This finishes the proof. □

For each $\mathbf{d} = (d_i) \in \mathbb{N}I_\infty$, set

$$\sigma(\mathbf{d}) = -\sum_{i<0,\, \bar{i}=\bar{0}} d_i.$$

For $\lambda \in \Pi$, we write $\sigma(\lambda) = \sigma(\mathbf{dim}\, M(\mathfrak{m}^\infty_\lambda))$. The following result was proved in [47, 9.2 & 10.1]. We provide here a direct proof for completeness.

Corollary 7.2. *For each $\lambda \in \Pi$,*

$$\widetilde{u}^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle \in |\lambda\rangle + \sum_{\mu \lhd \lambda} \mathcal{Z}|\mu\rangle.$$

In particular, the $\boldsymbol{\mathcal{D}}(n)$-module $\bigwedge^\infty$ is generated by $|\emptyset\rangle$ and the set

$$\{b^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle \mid \lambda \in \Pi\}$$

is a basis of $\bigwedge^\infty$.

Proof. Applying Corollary 3.5 gives that

$$\begin{aligned}\widetilde{u}^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle &= \sum_{\mathbf{d}} \big(\gamma^-_{\mathbf{d}}(\widetilde{u}^-_{\mathfrak{m}_\lambda}) K_{-\mathbf{d}'}\big) \cdot |\emptyset\rangle = \sum_{\mathbf{d}} v^{\sigma(\mathbf{d})} \gamma^-_{\mathbf{d}}(\widetilde{u}^-_{\mathfrak{m}_\lambda}) \cdot |\emptyset\rangle \\ &= \sum_{r \in \mathbb{Z}} v^{\theta(\lambda)+\sigma(\lambda)} \widetilde{u}^-_{\tau^{rm}(\mathfrak{m}^\infty_\lambda)} \cdot |\emptyset\rangle + \sum_{\mathfrak{z} \in \mathfrak{M}_\infty,\, \mathscr{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}_\lambda} f_{\lambda,\mathfrak{z}} \widetilde{u}^-_{\mathfrak{z}} \cdot |\emptyset\rangle,\end{aligned}$$

where $f_{\lambda,\mathfrak{z}} \in \mathcal{Z}$. By Proposition 5.3 and its proof,

$$\widetilde{u}^-_{\mathfrak{m}^\infty_\lambda} \cdot |\emptyset\rangle = |\lambda\rangle \ \text{ and } \ \widetilde{u}^-_{\tau^{rm}(\mathfrak{m}^\infty_\lambda)} \cdot |\emptyset\rangle = 0 \text{ for } r > 0.$$

Furthermore, for each $r < 0$, $\widetilde{u}^-_{\tau^{rm}(\mathfrak{m}^\infty_\lambda)} \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle$ such that $\mathfrak{m}^\infty_\mu \leqslant^\infty_{\deg} \tau^{rm}(\mathfrak{m}^\infty_\lambda)$. Then $\mathfrak{m}_\mu = \mathscr{F}(\mathfrak{m}^\infty_\mu) \leqslant_{\deg} \mathscr{F}(\tau^{rm}(\mathfrak{m}^\infty_\lambda)) = \mathfrak{m}_\lambda$, which implies that $\mu \unlhd \lambda$. Since $M(\tau^{rm}(\mathfrak{m}^\infty_\lambda))$ does not have a composition factor isomorphic to $S_{\lambda_1 - 1}$, μ does not contain a box with color $\lambda_1 - 1$. Thus, $\mu \neq \lambda$ and $\mu \lhd \lambda$.

Finally, by Proposition 7.1, for each $\mathfrak{z} \in \mathfrak{M}_\infty$ with $\mathscr{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}_\lambda$, $\widetilde{u}^-_{\mathfrak{z}} \cdot |\emptyset\rangle$ is a $\mathcal{Z}$-linear combination of $|\mu\rangle$ satisfying $\mathfrak{m}_\mu \leqslant_{\deg} \mathscr{F}(\mathfrak{z})$. Thus, $\mathfrak{m}_\mu \leqslant_{\deg} \mathscr{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}_\lambda$, which by Lemma 2.1 implies that $\mu \lhd \lambda$. Hence, each $\widetilde{u}^-_{\mathfrak{z}} \cdot |\emptyset\rangle$ is a $\mathcal{Z}$-linear combination of $|\mu\rangle$ with $\mu \lhd \lambda$. Consequently,

$$\widetilde{u}^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle \in v^{\theta(\lambda)+\sigma(\lambda)} |\lambda\rangle + \sum_{\mu \lhd \lambda} \mathcal{Z}|\mu\rangle.$$

Therefore, it remains to show that

$$\theta(\lambda)+\sigma(\lambda)=0.$$

Write $\lambda=(\lambda_1,\ldots,\lambda_m)$ with $\lambda_1\geqslant\cdots\geqslant\lambda_m\geqslant 1$ and set $|\lambda|=\sum_{s=1}^m\lambda_s$. We proceed induction on $|\lambda|$ to show that $\theta(\lambda)+\sigma(\lambda)=0$. By the definition,

$$\theta(\lambda)=\sum_{s<t}\kappa(\mathbf{d}_s,\mathbf{d}_t)-\sum_{s=1}^{\ell}h(\mathbf{d}_s),$$

where $\ell=\lambda_1$ is the Loewy length of $M=M(\mathfrak{m}_\lambda^\infty)$ and $S_{\mathbf{d}_s}\cong\mathrm{rad}^{\,s-1}M/\mathrm{rad}^{\,s}M$ for $1\leqslant s\leqslant\ell$. Let $1\leqslant t\leqslant m$ be such that $\lambda_1=\cdots=\lambda_t>\lambda_{t+1}$ and define

$$\lambda'=(\lambda_1,\cdots,\lambda_{t-1},\lambda_t-1,\lambda_{t+1},\lambda_m).$$

Then $|\lambda'|=|\lambda|-1$. By the induction hypothesis, we have $\theta(\lambda')+\sigma(\lambda')=0$.

For each $1\leqslant s\leqslant\ell$, let $\mathbf{d}'_s\in\mathbb{N}I_\infty$ be defined by setting $S_{\mathbf{d}'_i}\cong\mathrm{rad}^{\,s-1}M'/\mathrm{rad}^{\,s}M'$, where $M'=M(\mathfrak{m}_{\lambda'}^\infty)$. Then

$$\mathbf{d}'_\ell=\mathbf{d}_\ell-\varepsilon_{\ell-t}\ \text{ and }\ \mathbf{d}'_s=\mathbf{d}_s\ \text{ for } 1\leqslant s<\ell.$$

This implies that

$$\sum_{s=1}^{\ell}h(\mathbf{d}_s)-\sum_{s=1}^{\ell}h(\mathbf{d}'_s)=h(\mathbf{d}_\ell)-h(\mathbf{d}'_\ell)=-\delta_{\bar t,\bar 1}\ \text{ and}$$
$$\sum_{s<t}\kappa(\mathbf{d}_s,\mathbf{d}_t)-\sum_{s<t}\kappa(\mathbf{d}'_s,\mathbf{d}'_t)=\sum_{1\leqslant s<\ell}\kappa(\mathbf{d}_s,\varepsilon_{\ell-t}).$$

Hence,

$$\theta(\lambda)-\theta(\lambda')=\sum_{1\leqslant s<\ell}\kappa(\mathbf{d}_s,\varepsilon_{\ell-t})+\delta_{\bar t,\bar 1}.$$

On the other hand, $\sigma(\lambda)=\sigma(\lambda')-1$ if $\ell-t<0$ and $\bar\ell=\bar t$, and $\sigma(\lambda)=\sigma(\lambda')$ otherwise. A direct calculation shows that if $\ell-t\geqslant 0$, then

$$\sum_{1\leqslant s<\ell}\kappa(\mathbf{d}_s,\varepsilon_{\ell-t})=-\delta_{\bar t,\bar 1},$$

and if $\ell-t<0$, then

$$\sum_{1\leqslant s<\ell}\kappa(\mathbf{d}_s,\varepsilon_{\ell-t})=\begin{cases}\delta_{\bar\ell,\bar t}-1, & \text{if } \bar t=\bar 1;\\ \delta_{\bar\ell,\bar t}, & \text{if } \bar t\neq\bar 1.\end{cases}$$

We conclude that in all cases,

$$\theta(\lambda)+\sigma(\lambda)=\theta(\lambda')+\sigma(\lambda')=0.$$

□

By the definition, for each $i\in I_n=\mathbb{Z}/n\mathbb{Z}$,

$$K_i|\emptyset\rangle=v^{\delta_{i,0}}|\emptyset\rangle.$$

This together with the corollary above implies that $\bigwedge^\infty$ is a highest weight $\boldsymbol{\mathcal{D}}(n)$-module of highest weight Λ_0. Consequently, there is a unique surjective $\boldsymbol{\mathcal{D}}(n)$-module homomorphism

$$\varphi:\boldsymbol{\mathcal{D}}(n)^-=M(\Lambda_0)\longrightarrow\textstyle\bigwedge^\infty,\ \eta_{\Lambda_0}\longmapsto|\emptyset\rangle.$$

Theorem 7.3. *The homomorphism φ induces an isomorphism of $\boldsymbol{\mathcal{D}}(n)$-modules*

$$\bar\varphi:L(\Lambda_0)\longrightarrow\textstyle\bigwedge^\infty.$$

Proof. By definition, we have

$$F_i \cdot |\emptyset\rangle = 0 \ \text{ for } i \in I_n \backslash \{0\} \text{ and } \ F_0^2 \cdot |\emptyset\rangle = 0.$$

This together with Theorem 4.4 implies that φ induces a surjective homomorphism

$$\bar{\varphi} : L(\Lambda_0) = \boldsymbol{\mathcal{D}}(n)^- / \big(\sum_{i \in I_n} \boldsymbol{\mathcal{D}}(n)^- F_i^{\Lambda_0(h_i)+1} \big) \longrightarrow \bigwedge^\infty.$$

Since $L(\Lambda_0)$ is simple, we conclude that $\bar{\varphi}$ is an isomorphism. □

Combining the theorem with Corollary 4.5 gives the decomposition of $\bigwedge^\infty$ obtained by Kashiwara, Miwa and Stern in [27, Prop. 2.3].

Corollary 7.4. *As a* $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$*-module,* $\bigwedge^\infty$ *has a decomposition*

$$\bigwedge^\infty|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m \geqslant 0} L_0(\Lambda_0 - m\delta^*)^{\oplus p(m)}.$$

8. The canonical basis for $\bigwedge^\infty$

In this section we show that the canonical basis of $\bigwedge^\infty$ defined in [29] can be constructed by using the monomial basis of the Ringel–Hall algebra of Δ_n given in [8]. We also interpret the "ladder method" in [28] in terms of generic extensions defined in Section 2.

Recall that there is a bar-involution $a \mapsto \iota(a) = \overline{a}$ on $\boldsymbol{\mathcal{D}}(n)^-$ which takes $\overline{v} \mapsto v^{-1}$ and fixes all $\widetilde{u}_\alpha^-$ for $\alpha \in \mathbb{N}I_n$. Then it induces a semilinear involution on the basic representation $L(\Lambda_0)$ by setting

$$\overline{a\eta_{\Lambda_0}} = \overline{a}\eta_{\Lambda_0} \ \text{ for all } a \in \boldsymbol{\mathcal{D}}(n)^-.$$

On the other hand, by [29], there is a semilinear involution $x \mapsto \overline{x}$ on $\bigwedge^\infty$ which, by [47], satisfies

(i) $\overline{|\emptyset\rangle} = |\emptyset\rangle$,

(ii) $\overline{ax} = \overline{a}\,\overline{x}$ for all $a \in \boldsymbol{\mathcal{D}}(n)^-$ and $x \in \bigwedge^\infty$.

Therefore, the isomorphism $L(\Lambda_0) \to \bigwedge^\infty$ given in Theorem 7.3 is compatible with the bar-involutions.

It is proved in [29, Th. 3.3] that for each $\lambda \in \Pi$,

$$\overline{|\lambda\rangle} = |\lambda\rangle + \sum_{\mu \lhd \lambda} a_{\mu,\lambda} |\mu\rangle, \ \text{ where } a_{\mu,\lambda} \in \mathcal{Z}. \tag{8.0.1}$$

Then applying the standard linear algebra method to the basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$ in [31] (or see [11] for more details) gives rise to an "IC basis" $\{b_\lambda \mid \lambda \in \Pi\}$ which is characterized by

$$\overline{b_\lambda} = b_\lambda \ \text{ and } \ b_\lambda \in |\lambda\rangle + \sum_{\mu \lhd \lambda} v^{-1}\mathbb{Z}[v^{-1}]|\mu\rangle,$$

The basis $\{b_\lambda \mid \lambda \in \Pi\}$ is called the *canonical basis* of $\bigwedge^\infty$. In other words, the basis elements b_λ are uniquely determined by the polynomials $a_{\mu,\lambda}$.

Remark 8.1. Varagnolo and Vasserot [47] have conjectured that

$$b^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle = b_\lambda \ \text{ for each } \lambda \in \Pi.$$

This conjecture was proved by Schiffmann [41].

In the following we provide a way to deduce (8.0.1) by using the monomial basis of the Ringel–Hall algebra of Δ_n given in [8]. As in [8, Sect. 3], set

$$I^e = I_n \cup \{\text{all sincere vectors in } \mathbb{N}I_n\}$$

and consider the set Σ of all words on the alphabet I^e. Recall that a vector $\mathbf{a} = (a_i) \in \mathbb{N}I_n$ is called sincere if $a_i \neq 0$ for all $i \in I_n$. Since $\boldsymbol{\mathcal{D}}(n)^-$ is isomorphic to the opposite Ringel–Hall algebra of Δ_n, we define

$$M *' N = N * M.$$

This gives the map

$$\wp^{\rm op} : \Sigma \longrightarrow \mathfrak{M},\ w = \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_t \longmapsto S_{\mathbf{a}_1} *' S_{\mathbf{a}_2} *' \cdots *' S_{\mathbf{a}_t}.$$

By [8, Sect. 9], for each $\mathfrak{m} \in \mathfrak{M}$, there is a distinguished word $w_\mathfrak{m} \in (\wp^{\rm op})^{-1}(\mathfrak{m})$ which defines a monomial $m^{(w_\mathfrak{m})}$ on $\widetilde{u}_\mathbf{a}^-$ with $\mathbf{a} \in \widetilde{I}$ such that

$$m^{(w_\mathfrak{m})} = \widetilde{u}_\mathfrak{m}^- + \sum_{\mathfrak{p} <_{\rm deg} \mathfrak{m}} \theta_{\mathfrak{p},\mathfrak{m}} \widetilde{u}_\mathfrak{p}^- \ \text{ for some } \theta_{\mathfrak{m},\mathfrak{p}} \in \mathcal{Z};$$

see [8, (9.1.1)]. If $\mathfrak{m} = \mathfrak{m}_\lambda$ for some $\lambda \in \Pi$, we simply write $w_{\mathfrak{m}_\lambda} = w_\lambda$. Thus,

(8.1.1) $$m^{(w_\lambda)} = \widetilde{u}_{\mathfrak{m}_\lambda}^- + \sum_{\mathfrak{p} <_{\rm deg} \mathfrak{m}_\lambda} \theta_{\mathfrak{p},\mathfrak{m}_\lambda} \widetilde{u}_\mathfrak{p}^-.$$

This together with Proposition 7.1 and Corollary 7.2 implies that

(8.1.2) $$m^{(w_\lambda)}|\emptyset\rangle = |\lambda\rangle + \sum_{\mu \lhd \lambda} \tau_{\mu,\lambda} |\mu\rangle,$$

where $\tau_{\mu,\lambda} \in \mathcal{Z}$. Since the monomials $m^{(w_\lambda)}$ are bar-invariant, we deduce that for each $\lambda \in \Pi$,

$$\overline{|\lambda\rangle} = |\lambda\rangle + \sum_{\mu \lhd \lambda} a'_{\mu,\lambda} |\mu\rangle \ \text{ for some } a'_{\mu,\lambda} \in \mathcal{Z}.$$

Comparing with (8.0.1) gives that

$$a_{\mu,\lambda} = a'_{\mu,\lambda} \ \text{ for all } \mu \lhd \lambda.$$

In case λ is n-regular, then $\mathfrak{m}_\lambda$ is aperiodic and the word w_λ can be chosen in Ω, the subset of all words on the alphabet $I_n = \mathbb{Z}/n\mathbb{Z}$; see [8, Sect. 4]. In other words, $m^{(w_\lambda)}$ is a monomial of the divided powers $(u_i^-)^{(t)} = F_i^{(t)}$ for $i \in I_n$ and $t \geqslant 1$. We now interpret the "ladder method" in [28, Sect. 6] in terms of the generic extension map. Let $\lambda = (\lambda_1, \ldots, \lambda_t) \in \Pi$ be n-regular. Recall the corresponding nilpotent representation

$$M(\mathfrak{m}_\lambda) = \bigoplus_{a=1}^t S_{1-a}[\lambda_a],$$

where $1 - a$ is viewed as an element in I_n. Take $1 \leqslant s \leqslant t$ with $\lambda_1 = \cdots = \lambda_s > \lambda_{s+1}$ ($\lambda_{t+1} = 0$ by convention) and let $k \geqslant 0$ be maximal such that

$$\lambda_{s+l(n-1)+1} = \cdots = \lambda_{s+(l+1)(n-1)} \text{ and } \lambda_{s+l(n-1)} = \lambda_{s+l(n-1)+1} + 1 \text{ for } 0 \leqslant l \leqslant k-1.$$

Let $i_1 \in I$ be such that $\mathrm{soc}\,(S_{1-s}[\lambda_s]) = S_{i_1}$. Then for each $a = s + l(n-1)$ with $0 \leqslant l \leqslant k$,

$$\mathrm{soc}\,(S_{1-a}[\lambda_a]) = S_{i_1}.$$

Define $\mu = (\mu_1, \ldots, \mu_t) \in \Pi$ by setting

$$\mu_a = \begin{cases} \lambda_a - 1, & \text{if } a = s + l(n-1) \text{ for some } 0 \leqslant l \leqslant k; \\ \lambda_a, & \text{otherwise.} \end{cases}$$

It is easy to see from the construction that μ is again n-regular. Moreover, by applying an argument similar to that in the proof of [5, Prop. 3.7],

$$(k+1)S_{i_1} *' M(\mathfrak{m}_\mu) = M(\mathfrak{m}_\mu) * (k+1)S_{i_1} = M(\mathfrak{m}_\lambda).$$

Repeating the above process, we finally obtain a sequence $i_1, \dots, i_d$ in I_n and positive integers $k_1 = k+1, \dots, k_d$ such that

$$(k_1 S_{i_1}) *' \cdots *' (k_d S_{i_d}) = M(\mathfrak{m}_\lambda).$$

In other word, the word $w_\lambda := i_1^{k_1} \cdots i_d^{k_d}$ lies in $(\wp^{\mathrm{op}})^{-1}(\mathfrak{m}_\lambda)$. It can be also checked that the word w_λ is distinguished. Thus, the corresponding monomial

$$m^{(w_\lambda)} = (u_{i_1}^-)^{(k_1)} \cdots (u_{i_d}^-)^{(k_d)} = F_{i_1}^{(k_1)} \cdots F_{i_d}^{(k_d)}$$

gives rise to the equality (8.1.2) for the element $m^{(w_\lambda)}|\emptyset\rangle$. We remark that $m^{(w_\lambda)}|\emptyset\rangle$ coincides with the element $A(\lambda)$ constructed in [28, (8)] by using the "ladder method" of James and Kerber [22].

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