PICARD NUMBER, HOLOMORPHIC SECTIONAL CURVATURE, AND AMPLENESS

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1. Introduction

Let M be compact Kähler manifold. Then, it is well known that M is hyperbolic if and only if any holomorphic map $f : \mathbb{C} \to M$ is a constant. A conjecture of Kobayashi states that *if a compact Kähler manifold M is hyperbolic, then its canonical bundle K_M is ample* (see, for example, [3, p. 370], [2], and [4]). This conjecture clearly holds when M is a compact Riemann surface. For M being a Kähler surface, the conjecture follows from Enriques–Kodaira classification [2]. Based on the results of Wilson [6], Peternell [4] proved that a 3-dimensional projective hyperbolic manifold has ample canonical bundle, possibly except for certain Calabi–Yau threefolds whose Picard number are not greater than 19.

On the other hand, if a compact Kähler manifold M has strictly negative holomorphic sectional curvature everywhere, then M is hyperbolic. Thus in this note, we would like to study, under what condition would the negativity of holomorphic sectional curvature imply the ampleness. As a first step, we consider the manifolds with Picard number equal to 1.

For a Kähler manifold M, we say that the holomorphic sectional curvature of M is *quasi-negative*, if the holomorphic sectional curvature is non-positive everywhere and is strictly negative at one point of M. We denote by $\rho(M)$ the Picard number of M. Our result is as follows:

Theorem 1. *Let M be an n-dimensional projective manifold with $\rho(M) = 1$. If M admits a Kähler metric ω whose holomorphic sectional curvature is quasi-negative, then K_M is ample.*

We remark that the curvature condition in Theorem 1 is sharp; namely, the quasi-negativity *cannot* be replaced by *non-positivity*. Indeed, there are 2-dimensional abelian varieites with Picard number equal to 1 ([1, p. 58–59]).

Our technique is essentially the third author's Schwarz lemma [8] (see also [7]). We incorporate here a trick of Royden [5], which converts the bound of holomorphic sectional curvature to the bound of holomorphic bisectional curvature.

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2. Proof of the Theorem

Let us first prove the following lemma.

Lemma 2.1 (Royden). *Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{C}$, and $R(\mu, \bar\nu, \eta, \bar\xi)$ be a symmetric bihermitian form on V, i.e., for all μ, ν, η, ξ in V,*

$$R(\mu, \bar\nu, \eta, \bar\xi) = R(\eta, \bar\nu, \mu, \bar\xi), \qquad R(\nu, \bar\mu, \xi, \bar\eta) = \bar R(\mu, \bar\nu, \eta, \bar\xi).$$

Assume, in addition, that R satisfies that

$$R(\eta, \bar\eta, \eta, \bar\eta) \le b\|\eta\|^4, \qquad \text{for all } \eta \in V,$$

where b is a constant. Then, for any m orthogonal vectors $\alpha_1, \dots, \alpha_m$,

$$\sum_{i,j=1}^m R(\alpha_i, \bar\alpha_i, \alpha_j, \bar\alpha_j) \le \frac{b}{2}\left[\Big(\sum_{i=1}^m \|\alpha_i\|^2\Big)^2 + \sum_{i=1}^m \|\alpha_i\|^4\right].$$

Proof. We denote $I = \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$. Let

$$\eta_\epsilon = \epsilon_1\alpha_1 + \cdots + \epsilon_m\alpha_m,$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in I^m$. Clearly, for each $\epsilon \in I^m$,

$$R(\eta_\epsilon, \bar\eta_\epsilon, \eta_\epsilon, \bar\eta_\epsilon) \le b\Big(\sum_{i=1}^m \|\alpha_i\|^2\Big)^2.$$

Thus,

$$\frac{1}{4^m}\sum_{\epsilon\in I^m} R(\eta_\epsilon, \bar\eta_\epsilon, \eta_\epsilon, \bar\eta_\epsilon) \le b\Big(\sum_{i=1}^m \|\alpha_i\|^2\Big)^2.$$

On the other hand, we have, by the symmetry of R and $\sum_{\delta\in I}\delta = 0$ and $\sum_{\delta\in I}\delta^2 = 0$, that

$$\begin{aligned}
&\frac{1}{4^m}\sum_{\epsilon\in I^m} R(\eta_\epsilon,\bar\eta_\epsilon,\eta_\epsilon,\bar\eta_\epsilon)\\
&=\frac{1}{4^m}\sum_{\epsilon\in I^m}\sum_{i,j,k,l=1}^m \epsilon_i\bar\epsilon_j\epsilon_k\bar\epsilon_l R(\alpha_i,\bar\alpha_j,\alpha_k,\bar\alpha_l)\\
&=\sum_{i=1}^m R(\alpha_i,\bar\alpha_i,\alpha_i,\bar\alpha_i)+\sum_{i\neq j}\left[R(\alpha_i,\bar\alpha_i,\alpha_j,\bar\alpha_j)+R(\alpha_i,\bar\alpha_j,\alpha_j,\bar\alpha_i)\right]\\
&=\sum_{i=1}^m R(\alpha_i,\bar\alpha_i,\alpha_i,\bar\alpha_i)+2\sum_{i\neq j} R(\alpha_i,\bar\alpha_i,\alpha_j,\bar\alpha_j).
\end{aligned}$$

It follows that

$$2\sum_{i,j=1}^m R(\alpha_i,\bar\alpha_i,\alpha_j,\bar\alpha_j)\le b\left[\Big(\sum_{i=1}^m\|\alpha_i\|^2\Big)^2+\sum_{i=1}^m\|\alpha_i\|^4\right].$$

□

Proof of Theorem 1. Let D be a smooth, ample divisor in M. Then, there exists an integer α such that

$$c_1(K_M)=\alpha c_1([D]).$$

If K_M is not ample, then $\alpha\le 0$. It then follows from the third author's solution of Calabi conjecture that, there exists a Kähler metric ω' on M whose Ricci curvature is nonnegative. We shall prove that ω' is not compatible with ω.

Let $R_{i\bar j}$ and $R_{i\bar jk\bar l}$ denote, respectively, the Ricci curvature tensor and curvature tensor of ω. Similarly, we denote by $R'_{i\bar j}$ and $R'_{i\bar jk\bar l}$ for ω'. Let

$$S=\frac{n(\omega')^{n-1}\wedge\omega}{(\omega')^n}=\sum_{i,j=1}^n g'^{i\bar j}g_{i\bar j}.$$

Here we locally write

$$\omega=\frac{\sqrt{-1}}{2}\sum_{i,j}g_{i\bar j}dz^i\wedge d\bar z^j,\qquad \omega'=\frac{\sqrt{-1}}{2}\sum_{i,j}g'_{i\bar j}dz^i\wedge d\bar z^j,$$

and $(g^{i\bar j})$ denotes the transposed inverse of $(g_{i\bar j})$, similarly for $(g'^{i\bar j})$.

Let us compute $\Delta'S$, where Δ' denotes the Laplacian associated with ω'. For convenience, we choose a normal coordinate system $\{z^1,\dots,z^n\}$ near a point $x\in M$ such that

$$g_{i\bar j}(x)=\delta_{ij},\qquad \frac{\partial g_{i\bar j}}{\partial z^k}(x)=0, \tag{2.1}$$

and that

$$g'_{i\bar{j}}(x) = \delta_{ij} g'_{i\bar{i}}(x). \tag{2.2}$$

Then, as in [7, p. 371] , we *assert* that

$$\Delta' S = \sum_i \frac{R'_{i\bar{i}}}{(g'_{i\bar{i}})^2} + \sum_{i,j,k} \frac{|\partial g'_{i\bar{j}}/\partial z^k|^2}{g'_{i\bar{i}}(g'_{j\bar{j}})^2 g'_{k\bar{k}}} - \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}} g'_{k\bar{k}}} \qquad \text{at } x. \tag{2.3}$$

For completeness, this assertion will be proved at the end. The assertion implies that

$$\Delta' S \geq - \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}} g'_{k\bar{k}}}.$$

Now we are in a position to apply Lemma 2.1. Let

$$\alpha_i = (g'_{i\bar{i}})^{-1/2} \frac{\partial}{\partial z^i}, \qquad i = 1, \ldots, n.$$

Then, $\alpha_1, \ldots, \alpha_n$ are orthogonal tangent vectors in $T_x M$. It follows that

$$\begin{aligned}
\sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}} g'_{k\bar{k}}} &= \sum_{i,k} R(\alpha_i, \bar{\alpha}_i, \alpha_k, \bar{\alpha}_k) \\
&\leq -\frac{\kappa}{2} \left[\Big(\sum_{i=1}^m \|\alpha_i\|^2 \Big)^2 + \sum_{i=1}^m \|\alpha_i\|^4 \right] \\
&\leq -\frac{\kappa}{2} (1 + \frac{1}{n}) \Big(\sum_{i=1}^m \|\alpha_i\|^2 \Big)^2 \\
&= -\frac{\kappa(n+1)}{2n} S^2,
\end{aligned}$$

where $\kappa = \kappa(x) \geq 0$ is a constant depending only on the upper bound of the holomorphic sectional curvature at x. Therefore, we obtain that

$$\Delta' S \geq \frac{\kappa(n+1)}{2n} S^2 \geq 0. \tag{2.4}$$

By the maximum principle, the function S must be identically equal to a (positive) constant. In particular, $\Delta' S \equiv 0$ on M. Now suppose that the holomorphic sectional curvature is strictly negative at a point x_0. That is,

$$-\kappa(x_0) \equiv \sup_{\eta \in T_{x_0} M \setminus \{0\}} \frac{R(\eta, \bar{\eta}, \eta, \bar{\eta})}{\|\eta\|_g^4} < 0.$$

Apply (2.3) to x_0 and then combine Lemma 2.1 to obtain that

$$0 = \Delta' S(x_0) \geq \frac{\kappa(x_0)(n+1)}{2n} S^2(x_0) \geq 0.$$

This implies that $S \equiv S(x_0) = 0$, which is a contradiction. This proves Theorem 1, except for verifying the assertion (2.3).

Let us now prove the assertion:

$$\Delta' S = \sum_i \frac{R'_{i\bar{i}}}{(g'_{i\bar{i}})^2} + \sum_{i,j,k} \frac{|\partial g'_{i\bar{j}}/\partial z^k|^2}{g'_{i\bar{i}}(g'_{j\bar{j}})^2 g'_{k\bar{k}}} - \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}} g'_{k\bar{k}}},$$

in which we use the normal coordinate chart satisfying (2.1) and (2.2). For simplicity, we denote

$$\partial_i = \frac{\partial}{\partial z^i}, \qquad \partial_{\bar{j}} = \frac{\partial}{\partial \bar{z}^j}, \qquad \text{for all } 1 \leq i, j \leq n;$$

and we shall use the summation convention, unless otherwise indicated. Note that at the point x,

$$\begin{aligned} \Delta' S &= g'^{k\bar{l}} \partial_k \partial_{\bar{l}} (g'^{i\bar{j}} g_{i\bar{j}}) \\ &= g'^{k\bar{l}} g_{i\bar{j}} \partial_k \partial_{\bar{l}} g'^{i\bar{j}} + g'^{k\bar{l}} g'^{i\bar{j}} \partial_k \partial_{\bar{l}} g_{i\bar{j}}. \end{aligned} \tag{2.5}$$

Observe that, by using (2.1) we have

$$R_{i\bar{j}k\bar{l}}(x) = -\partial_k \partial_{\bar{l}} g_{i\bar{j}}(x).$$

Then, the second term on the right of (2.5) is equal to

$$-\sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{g'_{i\bar{i}} g'_{k\bar{k}}},$$

in which (2.2) is used. It remains to show that

$$g'^{k\bar{l}} g_{i\bar{j}} \partial_k \partial_{\bar{l}} g'^{i\bar{j}} = \sum_i \frac{R'_{i\bar{i}}}{(g'_{i\bar{i}})^2} + \sum_{i,j,k} \frac{|\partial g'_{i\bar{j}}/\partial z^k|^2}{g'_{i\bar{i}}(g'_{j\bar{j}})^2 g'_{k\bar{k}}}. \tag{2.6}$$

Notice that

$$\begin{aligned} g'^{k\bar{l}} g_{i\bar{j}} \partial_k \partial_{\bar{l}} g'^{i\bar{j}} &= -g'^{k\bar{l}} g_{i\bar{j}} \partial_k (g'^{i\bar{q}} g'^{p\bar{j}} \partial_{\bar{l}} g'_{p\bar{q}}) \\ &= g'^{k\bar{l}} g_{i\bar{j}} (g'^{i\bar{b}} g'^{a\bar{q}} g'^{p\bar{j}} \partial_k g'_{a\bar{b}} + g'^{i\bar{q}} g'^{p\bar{b}} g'^{a\bar{j}} \partial_k g'_{a\bar{b}}) \partial_{\bar{l}} g'_{p\bar{q}} \\ &\quad - g'^{k\bar{l}} g_{i\bar{j}} g'^{i\bar{q}} g'^{p\bar{j}} \partial_k \partial_{\bar{l}} g'_{p\bar{q}}. \end{aligned} \tag{2.7}$$

Let us first handle the last term in (2.7). Recall that

$$R'_{k\bar{l}p\bar{q}} = -\partial_k \partial_{\bar{l}} g'_{p\bar{q}} + g'^{a\bar{b}} \partial_k g'_{p\bar{b}} \partial_{\bar{l}} g'_{a\bar{q}}.$$

It follows that

$$-g'^{k\bar{l}} \partial_k \partial_{\bar{l}} g'_{p\bar{q}} = R'_{p\bar{q}} - g'^{k\bar{l}} g'^{a\bar{b}} \partial_k g'_{p\bar{b}} \partial_{\bar{l}} g'_{a\bar{q}}.$$

Substituting this into the last term in (2.7) yields that

$$\begin{aligned}
&g'^{k\bar{l}} g_{i\bar{j}} \partial_k \partial_{\bar{l}} g'^{i\bar{j}} \\
&= g_{i\bar{j}} g'^{k\bar{l}} g'^{i\bar{b}} g'^{a\bar{q}} g'^{p\bar{j}} \partial_k g'_{a\bar{b}} \partial_{\bar{l}} g'_{p\bar{q}} + g_{i\bar{j}} g'^{k\bar{l}} g'^{i\bar{q}} g'^{p\bar{b}} g'^{a\bar{j}} \partial_k g'_{a\bar{b}} \partial_{\bar{l}} g'_{p\bar{q}} \\
&\quad + g_{i\bar{j}} g'^{i\bar{q}} g'^{p\bar{j}} R'_{p\bar{q}} - g_{i\bar{j}} g'^{i\bar{q}} g'^{p\bar{j}} g'^{k\bar{l}} g'^{a\bar{b}} \partial_k g'_{p\bar{b}} \partial_{\bar{l}} g'_{a\bar{q}} \\
&= g_{i\bar{j}} g'^{k\bar{l}} g'^{i\bar{b}} g'^{a\bar{q}} g'^{p\bar{j}} \partial_k g'_{a\bar{b}} \partial_{\bar{l}} g'_{p\bar{q}} + g_{i\bar{j}} g'^{i\bar{q}} g'^{p\bar{j}} R'_{p\bar{q}} \\
&= \sum_{i,k,a} \frac{|\partial_k g'_{a\bar{i}}|^2}{(g'_{i\bar{i}})^2 g'_{k\bar{k}} g'_{a\bar{a}}} + \sum_i \frac{R'_{i\bar{i}}}{(g'_{i\bar{i}})^2}.
\end{aligned}$$

This verifies (2.6). Hence, the assertion is proved. This finishes the proof of Theorem 1.

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