REGULARITY FOR A FRACTIONAL p-LAPLACE EQUATION

ARMIN SCHIKORRA

Mathematisches Institut, Abt. für Reine Mathematik, Albert-Ludwigs-Universität,
Eckerstraße 1, 79104 Freiburg im Breisgau, Germany
armin.schikorra@math.uni-freiburg.de

TIEN-TSAN SHIEH

National Center for Theoretical Sciences, National Taiwan University,
No. 1, Sec. 4, Roosevelt Rd., Taipei City 106, Taiwan, R.O.C.
ttshieh@ncts.ntu.edu.tw

DANIEL E SPECTOR

Department of Applied Mathematics, National Chiao Tung University,
1001 Ta Hsueh Rd, Hsinchu 30010, Taiwan, R.O.C.
dspector@math.nctu.edu.tw

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In this note we consider regularity theory for a fractional p-Laplace operator which arises in the complex interpolation of the Sobolev spaces, the $H^{s,p}$-Laplacian. We obtain the natural analogue to the classical p-Laplacian situation, namely $C^{s+\alpha}_{loc}$-regularity for the homogeneous equation.

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1. Introduction and main result

In recent years equations involving what we will call the distributional $W^{s,p}$-Laplacian, defined for test functions φ as

$$(-\Delta)^s_p u[\varphi] := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+sp}}\,dy\,dx,$$

have received a lot of attention, e.g. [3,6,7,13,15,16,21]. The $W^{s,p}$-Laplacian $(-\Delta)^s_p$ appears when one computes the first variation of certain energies involving the $W^{s,p}$ semi-norm

$$[u]_{W^{s,p}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dy\,dx\right)^{\frac{1}{p}}, \tag{1.1}$$

which was introduced by Gagliardo [11] and independently by Slobodeckij [27] to describe the trace spaces of Sobolev maps. We refer to [8] for a modern introduction to the spaces $W^{s,p}$.

Regularity theory for equations involving this fractional p-Laplace operator is a very challenging open problem and only partial results are known: $C^{0,\alpha}_{loc}$-regularity for suitable right-hand-side data was obtained by Di Castro, Kuusi and Palatucci [6,7]; A generalization of the Gehring lemma was proven by Kuusi, Mingione and Sire [15, 16]; A stability theorem similar to the Iwaniec stability result for the $p-$Laplacian was established by the first-named author [23]. The current state-of-the art with respect to regularity theory is higher Sobolev-regularity by Brasco and Lindgren [3].

Aside from their origins as trace spaces, the fractional Sobolev spaces

$$W^{s,p}(\mathbb{R}^d) := \left\{u \in L^p(\mathbb{R}^d) : [u]_{W^{s,p}(\mathbb{R}^d)} < +\infty\right\}$$

also arise in the real interpolation of L^p and $\dot{W}^{1,p}$, see [28]. If one alternatively considers the complex interpolation method, one is naturally led to another kind of fractional Sobolev space $H^{s,p}(\mathbb{R}^d)$, where taking the place of the differential energy (1.1) one can utilize the semi-norm

$$[u]_{H^{s,p}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |D^s u|^p\right)^{\frac{1}{p}}. \tag{1.2}$$

Here $D^s = (\frac{\partial^s}{\partial x_1^s}, \ldots, \frac{\partial^s}{\partial x_d^s})$ is the fractional gradient for

$$\frac{\partial^s u}{\partial x_i^s}(x) := c_{d,s}\; p.v. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+s}}\frac{x_i-y_i}{|x-y|}\,dy, \quad i=1,\ldots,d.$$

Composition formulae for the fractional gradient have been studied in the classical work [12], while more recently they have been considered by a number of authors [1,4,5,22,24,26]. While it is common in the literature (for example in [18]) to see $H^{s,p}(\mathbb{R}^d)$ equipped with the L^p-norm of the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ (see Section 2 for a definition), we here utilize (1.2) because it preserves the structural properties of the spaces for $s \in (0,1)$ more appropriately. In particular, for $s = 1$ we have $D^1 = D$ (the constant $c_{d,s}$ tends to zero as s tends to one), while for $s \in (0,1)$ the fractional Sobolev spaces defined this way support a fractional Sobolev inequality in the case $p = 1$, see [25]. Let us also remark that for $p = 2$ these spaces are the same, $W^{s,2} = H^{s,2}$, but for $p \neq 2$ this is not the case.[a]

Returning to the question of a fractional p-Laplacian, in the context of $H^{s,p}(\mathbb{R}^d)$ computing the first variation of energies involving the $H^{s,p}$ semi-norms (1.2) yields an alternative fractional version of a p-Laplacian, we shall call it the $H^{s,p}$-Laplacian

$$\operatorname{div}_s(|D^s u|^{p-2}\, D^s u) = \sum_{i=1}^{d} \frac{\partial^s}{\partial x_i^s}(|D^s u|^{p-2}\, \frac{\partial^s u}{\partial x_i^s}).$$

Somewhat surprisingly while the regularity theory for the homogeneous equation of the $W^{s,p}$-Laplacian

$$(-\Delta)_p^s u = 0$$

is far from being understood, the regularity for the $H^{s,p}$-Laplacian

$$\operatorname{div}_s(|D^s u|^{p-2}\, D^s u) = 0 \tag{1.3}$$

actually follows the classical theory, which is the main result we prove in this note:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be open, $p \in (2 - \frac{1}{d}, \infty)$ and $s \in (0,1]$. Suppose $u \in H^{s,p}(\mathbb{R}^d)$ is a distributional solution to (1.3), that is*

$$\int_{\mathbb{R}^d} |D^s u|^{p-2} D^s u \cdot D^s \varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.4}$$

Then $u \in C^{s+\alpha}_{loc}(\Omega)$ for some $\alpha > 0$ only depending on p.

The key observation for Theorem 1.1 is that $v := I_{1-s}u$, where I_{1-s} denotes the Riesz potential, actually solves an inhomogeneous classical p-Laplacian equation with good right-hand side.

Proposition 1.1. *Let u be as in Theorem 1.1. Then $v := I_{1-s}u$ satisfies*

$$-\operatorname{div}(|Dv|^{p-2} Dv) \in L^\infty_{loc}(\Omega).$$

Therefore, Theorem 1.1 follows from the regularity theory of the classical p-Laplacian: By Proposition 1.1, v is a distributional solution to

$$\operatorname{div}(|Dv|^{p-2}\, Dv) = \mu$$

[a]For a complete picture of the Sobolev spaces $W^{s,p}$ and $H^{s,p}$ and the relation between them we refer to [20].

and μ is sufficiently integrable whence $v \in C^{1,\alpha}_{loc}(\Omega)$ [9,10,29] (see also the excellent survey paper by Mingione [19]). In particular, one can apply the potential estimates by Kuusi and Mingione [14, Theorem 1.4, Theorem 1.6] to deduce that $Dv \in C^{0,\alpha}_{loc}(\Omega)$, which implies that $u \in C^{s+\alpha}_{loc}(\Omega)$.

Let us also remark, that the reduction argument used for Proposition 1.1 extends to the class of fractional partial differential equations introduced in [26], which will be treated in a forthcoming work.

2. Proof of Proposition 1.1

With $(-\Delta)^{\frac{\sigma}{2}}$ we denote the fractional Laplacian

$$(-\Delta)^{\frac{\sigma}{2}} f(x) := \tilde{c}_{d,\sigma}\ p.v. \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+\sigma}}\ dy,$$

and with I_σ its inverse, the Riesz potential. Let $v := I_{1-s}u$ where u satisfies (1.4), so that

$$\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s\varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{2.1}$$

Now let $\Omega_1 \Subset \Omega$ be an arbitrary open set compactly contained in Ω, and let ϕ be a test function supported in Ω_1. Pick an open set Ω_2 so that $\Omega_1 \Subset \Omega_2 \Subset \Omega$ and a cutoff function η, supported in Ω and constantly one in Ω_2. Then in particular one can take

$$\varphi := \eta(-\Delta)^{\frac{1-s}{2}}\phi$$

as a test function in (2.1) to obtain

$$\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s(\eta(-\Delta)^{\frac{1-s}{2}}\phi) = 0.$$

That is,

$$\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D\phi = \int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s(\eta^c(-\Delta)^{\frac{1-s}{2}}\phi).$$

where $\eta^c := (1-\eta)$. We set

$$T(\phi) := D^s(\eta^c(-\Delta)^{\frac{1-s}{2}}\phi)$$

Now we show that by the disjoint support of η^c and ϕ we have

$$\|T(\phi)\|_{L^p(\mathbb{R}^d)} \le C_{\Omega_1,\Omega_2,d,s,p}\, \|\phi\|_{L^1(\mathbb{R}^d)}. \tag{2.2}$$

Once we have this, the claim is proven as Hölder's inequality and realizing the L^∞ norm via duality implies

$$-\operatorname{div}(|Dv|^{p-2}Dv) \in L^\infty_{loc}(\Omega).$$

To see (2.2), we use the disjoint support arguments as in [2, Lemma A.1] [17, Lemma 3.6.]: First we see that since $\eta^c(x)\phi(x) \equiv 0$,

$$T(\phi) = \tilde{c}_{d,1-s}\, D^s \int_{\mathbb{R}^d} \frac{\eta^c(x)\phi(y)}{|x-y|^{N+1-s}}\, dy.$$

Now taking a cutoff-function ζ supported in Ω_2, $\zeta \equiv 1$ on Ω_1 we have

$$T(\phi) = \tilde{c}_{d,1-s}\, D^s \int_{\mathbb{R}^d} \frac{\eta^c(x)\zeta(y)\phi(y)}{|x-y|^{N+1-s}}\, dy = \tilde{c}_{d,1-s} \int_{\mathbb{R}^d} k(x,y)\,\phi(y)\, dy,$$

where

$$k(x,y) := D^s_x \kappa(x,y) := D^s_x \frac{\eta^c(x)\,\zeta(y)}{|x-y|^{N+1-s}}.$$

The positive distance between the supports of η^c and ζ implies that these kernels k, κ are a smooth, bounded, integrable (both, in x and in y), and thus by a Young-type convolution argument we obtain (2.2). One can also argue by interpolation,

$$\left\| \int_{\mathbb{R}^d} \kappa(x,y)\,\phi(y) \right\|_{L^p(\mathbb{R}^d)} \lesssim \|\phi\|_{L^1(\mathbb{R}^d)},$$

as well as

$$\left\| \int_{\mathbb{R}^d} D_x\kappa(x,y)\,\phi(y)\, dy \right\|_{L^p(\mathbb{R}^d)} \lesssim \|\phi\|_{L^1(\mathbb{R}^d)}.$$

Interpolating this implies the desired result that

$$\left\| \int_{\mathbb{R}^d} D^s_x\kappa(x,y)\,\phi(y)\, dy \right\|_{L^p(\mathbb{R}^d)} \lesssim \|\phi\|_{L^1(\mathbb{R}^d)}.$$

Thus (2.2) is established and the proof of Proposition 1.1 is finished. □

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