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H(div) conforming and DG methods for incompressible Euler's equations

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H(div) conforming and discontinuous Galerkin (DG) methods are designed for incompressible Euler's equation in two and three dimension. Error estimates are proved for both the semi-discrete method and fully-discrete method using backward Euler time stepping. Numerical examples exhibiting the performance of the methods are given.

Keywords: Euler equation, discontinuous Galerkin method

1. Introduction

In this paper we study H(div) conforming and DG finite element methods for the incompressible Euler equations in both two and three dimensions. Our methods are based on the velocity-pressure formulation. Let Ω be a bounded and simply connected polygonal domain in R^d, $d \in \{2,3\}$, with boundary Γ. The velocity $\mathbf{u} \in \mathbf{H}_0^1(\Omega) := [H_0^1(\Omega)]^d$, and the pressure $p \in L_0^2(\Omega)$ satisfy

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in} \quad (0,T)\times\Omega, \tag{1.1a}$$
$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{1.1b}$$
$$\mathbf{u}\cdot\mathbf{n} = \mathbf{0} \quad \text{on} \quad (0,T)\times\Gamma, \tag{1.1c}$$
$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \text{in} \quad \Omega, \tag{1.1d}$$

where $\mathbf{u}_t = \partial_t\mathbf{u}$ is the time derivative, $\nabla\mathbf{u}$ is the tensor gradient of $\mathbf{u}$, and $T > 0$.

The goal of this paper is to define methods that are L^2 stable, and, for DG methods, are also locally conservative. The methods are inspired by the work Cockburn *et al.* (2005) where they developed locally conservative DG methods for the steady state Navier-Stokes equations. There they take Newton iterations to solve numerically the equations and in each step they postprocess the DG approximation to

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get a new approximation that belongs to H(div) and is divergence-free. Here we apply this idea to DG methods in each time step for Euler's equations. However, we first consider H(div) conforming elements as they seem natural for incompressible Euler's equations and are easier to analyze. In order to make the H(div) elements L^2 stable, one has to add numerical fluxes of the nonlinear term on the interfaces of the triangulation. We start with the semi-discrete method, using both central and upwind fluxes, and then analyze a backward Euler time stepping method. Once we have developed H(div) conforming methods, we develop DG methods using the post-processing idea used in Cockburn *et al.* (2005). In Cockburn *et al.* (2005) upwind fluxes are used, but it is important to note that central numerical fluxes can also guarantee L^2 stability for Euler's equations.

The development and study of finite element methods for incompressible flows have a long history; see for example the books of Temam (see Temam (1984)) and Girault and Raviart (see Girault & Raviart (1986)). More recently there has been an interest in using H(div) conforming methods for these problems (see, e.g., Cockburn *et al.* (2007)) since they produce divergence-free approximations. However, to the best of our knowledge, an analysis of these methods for the inviscid problem (i.e. Euler's equation) has not been considered. On the other hand, there has been recent work on proving convergence rates for other finite element methods for problems with arbitrarily low viscosity (see Burman & Fernández (2007)).

We give an error analysis for both the semi-discrete methods and the backward Euler time stepping methods. The error estimate for the velocity in the L^2 norm converges with rate $\mathscr{O}(h^k)$ if the velocity space contains the polynomials of degree k. Notice that this is sub-optimal by one order. However, numerical experiments suggest that these results are not sharp for some polynomial orders and using a central numerical flux. More specifically, when using even degree polynomial order for the velocity it seems the methods with central flux converge optimally. In particular, the error estimate will not give an error estimate for the lowest-order Raviart-Thomas element. However, on structured grids our numerical experiments show that the lowest-order Raviart-Thomas elements seem to be converging. Moreover, when using the upwind numerical flux numerical experiments suggest that the method is optimal. However, at the present time we are not able to prove this result. Our estimates assume that the velocity belongs to $W^{1,\infty}$. Of course, these a-priori estimates are not known (and might not hold) in three dimensions for general smooth initial data. However, in two dimensions the a-priori estimates were proved by Kato (see Kato (1967)) for smooth initial data.

In addition to providing numerical experiments to check the order of convergence of our methods, we give numerical experiments to show how the methods behave in high gradient flows. We see that using upwind flux the method seems to do very well and comparable to DG methods that use the vorticity-potential formulation (see Liu & Shu (2000)).

One of the advantage of the H(div) conforming methods is that it gives approximations that are pointwise divergence-free. An advantage of the DG methods is that they give locally conservative methods (see below). The other advantage for the upwind versions of both H(div) conforming and DG methods is that numerically they converge optimally on structured meshes. Indeed, it is well known that on general quasi-uniform mesh all the standard methods (Galerkin, streamline diffusion, DG) lose at least a half-order accuracy for scalar problems. On the other hand, upwind DG methods can be shown to convergence optimally (also assuming the correct regularity) for scalar problems on classes of meshes (see Cockburn *et al.* (2010a), Cheng & Shu (2010)). In fact, as far as we know these are the only methods that have been proving to have this property. To extend these results to the current setting seems non-trivial, but we will pursue this in the future. Moreover, DG methods can be stabilized by adding consistent terms (i.e. jump terms) whereas methods like the streamline diffusion method add inconsistent terms with parameter that need to be tuned to stabilize the method. The paper is organized as follows. In the next section we present the semi-discrete methods and prove error estimates. In section 3 we present the backward Euler methods. Finally, in section 4 we provide some numerical examples.

2. Semi-discrete methods

We begin by introducing some preliminary notations. Let $\mathscr{T}_h$ be a shape-regular and quasi-uniform triangulation of $\overline{\Omega}$ without the presence of hanging nodes, and let $\mathscr{E}_h$ be the set of edges/faces F of $\mathscr{T}_h$. In addition, we denote by $\mathscr{E}_h^i$ and $\mathscr{E}_h^\partial$ the set of interior and boundary faces, respectively, of $\mathscr{E}_h$, and we set $\partial\mathscr{T}_h := \cup\{\partial K \,:\, K \in \mathscr{T}_h\}$.

Next, let $(\cdot,\cdot)_U$ denote the usual L^2 and $\mathbf{L}^2 := [L^2]^d$ inner product over the domain $U \subset R^d$, and similarly let $\langle\cdot,\cdot\rangle_G$ be the L^2 and $\mathbf{L}^2$ inner product over the surface $G \subset R^{d-1}$. Then, we introduce the inner products:

$$(\cdot,\cdot)_{\mathscr{T}_h} := \sum_{K\in\mathscr{T}_h} (\cdot,\cdot)_K \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\partial\mathscr{T}_h} := \sum_{K\in\mathscr{T}_h} \langle\cdot,\cdot\rangle_{\partial K}.$$

On the other hand, let $\mathbf{n}^+$ and $\mathbf{n}^-$ be the outward unit normal vectors on the boundaries of two neighboring elements K^+ and K^-, respectively. We use $(\tau^\pm,\mathbf{v}^\pm,q^\pm)$ to denote the traces of $(\tau,\mathbf{v},q)$ on $F := \overline{K}^+ \cap \overline{K}^-$ from the interior of $K^\pm$, where τ, $\mathbf{v}$ and q are second-order tensorial, vectorial and scalar functions, respectively. Then, we define the means $\{\!\{\cdot\}\!\}$ and jumps $[\![\cdot]\!]$ for $F \in \mathscr{E}_h^i$, as follows

$$\begin{aligned}
\{\!\{\tau\}\!\} &:= \frac{1}{2}\left(\tau^+ + \tau^-\right), & \{\!\{\mathbf{v}\}\!\} &:= \frac{1}{2}\left(\mathbf{v}^+ + \mathbf{v}^-\right), & \{\!\{q\}\!\} &:= \frac{1}{2}\left(q^+ + q^-\right),\\
[\![\tau\mathbf{n}]\!] &:= \tau^+\mathbf{n}^+ + \tau^-\mathbf{n}^-, & [\![\mathbf{v}\cdot\mathbf{n}]\!] &:= \mathbf{v}^+\cdot\mathbf{n}^+ + \mathbf{v}^-\cdot\mathbf{n}^-, & [\![q\mathbf{n}]\!] &:= q^+\mathbf{n}^+ + q^-\mathbf{n}^-.
\end{aligned}$$

The method is derived using the conservative or divergence form of the equation. To this end, denoting $\otimes$ as the usual dyadic or tensor product, that is, $(\mathbf{u}\otimes\mathbf{v})_{ij} = (\mathbf{u}^{\mathtt{t}}\mathbf{v})_{ij} = u_i v_j$, we consider the formula

$$\mathbf{div}(\mathbf{u}\otimes\mathbf{v}) \;=\; \mathbf{v}\cdot\nabla\mathbf{u} \,+\, \mathrm{div}(\mathbf{v})\,\mathbf{u}, \tag{2.1}$$

together with the divergence-free condition, to write the problem (1.1) in the form

$$\begin{aligned}
\mathbf{u}_t + \mathbf{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla p \;=\; \mathbf{0} \quad \text{in} \quad (0,T)\times\Omega, \qquad \mathrm{div}(\mathbf{u}) \;=\; 0 \quad \text{in} \quad (0,T)\times\Omega\,,\\
\mathbf{u}\cdot\mathbf{n} \;=\; \mathbf{0} \quad \text{on} \quad (0,T)\times\Gamma\,, \qquad \mathbf{u}(0,\mathbf{x}) \;=\; \mathbf{u}_0(\mathbf{x}) \quad \text{in} \quad \Omega\,,
\end{aligned} \tag{2.2}$$

where $\mathbf{div}$ denotes the usual divergence operator div acting along each row of the corresponding tensor.

Finally, given an integer $\ell \geqslant 0$ and a subset U of R^d, we denote by $P_\ell(U)$ the space of polynomials defined in U of total degree at most ℓ, with $\mathbf{P}_\ell(U) := [P_\ell(U)]^d$. Furthermore, for each $K \in \mathscr{T}_h$, we define the local Raviart-Thomas space of order ℓ (see, e.g. Brezzi & Fortin (1991); Roberts & Thomas (1991))

$$\mathbf{RT}_\ell(K) \;:=\; \mathbf{P}_\ell(K) \,+\, P_\ell(K)\,\mathbf{x}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$ is a generic vector of R^d. In addition, we set

$$\mathbf{ND}_\ell(K) \;:=\; \mathbf{P}_\ell(K) \,+\, \mathbf{P}_\ell(K)\times\mathbf{x}$$

be the local Nédélec space of order ℓ on $K \in \mathscr{T}_h$.

2.1 *H(div) conforming methods*

In this section, we define H(div) conforming finite element schemes associated with the model problem (2.2). We start by introducing the method using the central flux, but in a later section we present the method using the upwind flux. For simplicity we only consider the Raviart-Thomas finite element spaces, but we note that one can use instead the BDM finite elements (see, e.g. Brezzi & Fortin (1991); Roberts & Thomas (1991)). The globally defined Raviart-Thomas spaces are given by $\mathbf{V}_h$ for the velocity and Q_h for the pressure, given by

$$\begin{aligned} \mathbf{V}_h &:= \left\{ \mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) \,:\, \mathbf{v}|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathscr{T}_h \quad \text{and} \quad \mathbf{v}\cdot\mathbf{n} = \mathbf{0} \text{ on } \Gamma \right\}, \\ Q_h &:= \left\{ q \in L^2_0(\Omega) \,:\, q|_K \in P_k(K) \quad \forall\, K \in \mathscr{T}_h \right\}. \end{aligned}$$

Now, the finite element method is defined by: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$, such that

$$\begin{aligned} (\partial_t \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} - (\mathbf{u}_h \otimes \mathbf{u}_h, \nabla_h \mathbf{v}_h)_{\mathscr{T}_h} - (p_h, \mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} + \langle \widehat{\sigma}(\mathbf{u}_h, p_h)\mathbf{n}, \mathbf{v}_h \rangle_{\partial \mathscr{T}_h} &= 0, \\ (q_h, \mathrm{div}(\mathbf{u}_h))_{\mathscr{T}_h} &= 0, \\ \mathbf{u}_h(0,\mathbf{x}) = \mathbf{u}_{h,0}(\mathbf{x}) \quad \text{in } \Omega&, \end{aligned} \tag{2.3}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$, where ∇_h is the broken gradient, $\mathbf{u}_{h,0}$ is some projection of $\mathbf{u}_0$ on $\mathbf{V}_h$, and $\widehat{\sigma}(\mathbf{u}_h, p_h)$ represents the numerical flux of $\mathbf{u} \otimes \mathbf{u} + p\,\mathbb{I}$ on $\mathscr{E}_h$. In particular, we take $\widehat{\sigma}(\mathbf{u}_h, p_h) := \mathbf{u}_h \otimes \mathbf{u}_h + p_h \mathbb{I}$ on $\mathscr{E}_h^\partial$ and for $\mathscr{E}_h^i$ we define

$$\widehat{\sigma}(\mathbf{u}_h, p_h) := \{\!\{\mathbf{u}_h\}\!\} \otimes \{\!\{\mathbf{u}_h\}\!\} + \{\!\{p_h\}\!\}\mathbb{I}. \tag{2.4}$$

This is the method using the central flux. In a later section we introduce the method using the upwind flux which seems to do better numerically.

Next, using the above definition for $\widehat{\sigma}$, together with the formula (2.1), the fact that $\mathbf{u}_h$ is divergence-free (from the second equation in (2.3)), and integration by parts, we can rewrite (2.3) as: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$, such that

$$\begin{aligned} (\partial_t \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} + (\mathbf{u}_h \cdot \nabla_h \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} - \sum_{F \in \mathscr{E}_h^i} \langle [\![(\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n}]\!], \{\!\{\mathbf{v}_h\}\!\} \rangle_F - (p_h, \mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} &= 0, \\ (q_h, \mathrm{div}(\mathbf{u}_h))_{\mathscr{T}_h} &= 0, \\ \mathbf{u}_h(0,\mathbf{x}) = \mathbf{u}_{h,0}(\mathbf{x}) \quad \text{in } \Omega&, \end{aligned} \tag{2.5}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$.

It will be useful to rewrite the $[\![(\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n}]\!]|_F$. Let $F = \overline{K}^+ \cap \overline{K}^-$. Then,

$$[\![(\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n}]\!] = [\![(\mathbf{u}_h \cdot \mathbf{n})\mathbf{u}_h]\!] = (\mathbf{u}_h^+ \cdot \mathbf{n}^+)\mathbf{u}_h^+ + (\mathbf{u}_h^- \cdot \mathbf{n}^-)\mathbf{u}_h^-.$$

In addition, from the fact that $\mathbf{u}_h^+ \cdot \mathbf{n}^+ = \mathbf{u}_h^- \cdot \mathbf{n}^+$, since $\mathbf{u}_h \in \mathbf{H}(\mathrm{div};\Omega)$, it follows that

$$[\![(\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n}]\!] = (\mathbf{u}_h^+ \cdot \mathbf{n}^+)\mathbf{u}_h^+ - (\mathbf{u}_h^- \cdot \mathbf{n}^+)\mathbf{u}_h^- = (\mathbf{u}_h^+ \cdot \mathbf{n}^+)(\mathbf{u}_h^+ - \mathbf{u}_h^-).$$

From now on we will use the notation (without loss of generality) $[\![\mathbf{v}]\!] := \mathbf{v}^+ - \mathbf{v}^-$. Also, we use the notation $(\mathbf{u}_h \cdot \mathbf{n})|_F = (\mathbf{u}_h^+ \cdot \mathbf{n}^+)|_F$. Hence, we write

$$[\![(\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n}]\!] = (\mathbf{u}_h \cdot \mathbf{n})[\![\mathbf{u}_h]\!].$$

Now from this we see that the third term in the right-hand side of first equation in (2.5) is consistent, since $[\![\mathbf{u}]\!] = \mathbf{0}$ on $\mathscr{E}_h^i$ when $\mathbf{u}$ is smooth.

LEMMA 2.1 (Conservation of energy) Given $\mathbf{u}_h \in \mathbf{V}_h$ the solution of (2.5), we have

$$\frac{d}{dt}\,\|\mathbf{u}_h\|^2_{L^2(\Omega)} \;=\; 0.$$

Proof. Taking $\mathbf{v}_h := \mathbf{u}_h$ in the first equation of (2.5) and using that $\mathbf{u}_h$ is divergence-free, it follows that

$$\frac{1}{2}\frac{d}{dt}\,\|\mathbf{u}_h\|^2_{L^2(\Omega)} + (\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h,\mathbf{u}_h)_{\mathscr{T}_h} - \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h\cdot\mathbf{n})[\![\mathbf{u}_h]\!],\{\!\{\mathbf{u}_h\}\!\}\rangle_F \quad=\quad 0. \tag{2.6}$$

Thus, note that

$$\begin{aligned}(\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h,\mathbf{u}_h)_{\mathscr{T}_h} &= \frac{1}{2}\sum_{K\in\mathscr{T}_h}\int_K \mathbf{u}_h\cdot\nabla(|\mathbf{u}_h|^2) \;=\; \frac{1}{2}\sum_{K\in\mathscr{T}_h}\left\{-\int_K \mathrm{div}(\mathbf{u}_h)|\mathbf{u}_h|^2 \;+\; \int_{\partial K}(\mathbf{u}_h\cdot\mathbf{n})|\mathbf{u}_h|^2\right\}\\ &= \frac{1}{2}\sum_{K\in\mathscr{T}_h}\int_{\partial K}(\mathbf{u}_h\cdot\mathbf{n})|\mathbf{u}_h|^2 \;=\; \frac{1}{2}\sum_{F\in\mathscr{E}_h^i}\int_F \{\!\{\mathbf{u}_h\}\!\}\cdot[\![\,|\mathbf{u}_h|^2\mathbf{n}]\!]\\ &= \sum_{F\in\mathscr{E}_h^i}\int_F(\mathbf{u}_h\cdot\mathbf{n})[\![\mathbf{u}_h]\!]\cdot\{\!\{\mathbf{u}_h\}\!\},\end{aligned} \tag{2.7}$$

which, together with (2.6) complete the proof. $\square$

We remark here that, from the previous lemma, integrating in time over $(0,t)$, we can deduce that $\|\mathbf{u}_h(t,\cdot)\|_{L^2(\Omega)} = \|\mathbf{u}_{h,0}\|_{L^2(\Omega)}$ for each $t\in(0,T)$. That is, we proved that the scheme (2.5) is stable.

2.1.1 *Error estimates.* Our next goal is to obtain error estimates for the scheme (2.5). In order to do that, we now introduce the Raviart-Thomas interpolation operator (see Brezzi & Fortin (1991); Roberts & Thomas (1991)) $\Pi_h^k : \mathbf{H}^1(\Omega)\to\mathbf{V}_h$, which satisfies the following approximation properties: for each $\mathbf{v}\in\mathbf{H}^m(\Omega)$, with $1\leqslant m\leqslant k+1$, there holds

$$\|\mathbf{v}-\Pi_h^k(\mathbf{v})\|_{L^2(K)} + h_K\|\nabla(\mathbf{v}-\Pi_h^k(\mathbf{v}))\|_{L^2(K)} \quad\leqslant\quad C h_K^m|\mathbf{v}|_{m,K} \quad \forall\, K\in\mathscr{T}_h. \tag{2.8}$$

Moreover, we also have the following bounds

$$\|\mathbf{v}-\Pi_h^k(\mathbf{v})\|_{L^\infty(K)} + h_K\|\nabla(\mathbf{v}-\Pi_h^k(\mathbf{v}))\|_{L^\infty(K)} \quad\leqslant\quad C h_K\|\nabla\mathbf{v}\|_{L^\infty(K)} \quad \forall\, K\in\mathscr{T}_h. \tag{2.9}$$

In addition, let $\mathscr{P}_h^k : L^2(\Omega)\to Q_h$ be the L^2-orthogonal projector. Hence, for each $q\in H^m(\Omega)$, with $0\leqslant m\leqslant k+1$, there holds (see, e.g. Ciarlet (1978))

$$\|q-\mathscr{P}_h^k(q)\|_{L^2(K)} \quad\leqslant\quad C h_K^m|q|_{H^m(K)} \quad \forall\, K\in\mathscr{T}_h. \tag{2.10}$$

We now aim to derive the *a priori* error estimates for the scheme (2.5). To this end, thanks to the triangle inequality, we only need to provide estimates for the approximation errors, namely, $\mathrm{E}^{\mathbf{u}} := \Pi_h^k(\mathbf{u})-\mathbf{u}_h$ and $\mathrm{E}^p := \mathscr{P}_h^k(p)-p_h$. To do this, we use the fact that the exact solution satisfies the approximation method (2.5), in order to obtain the error equations:

$$\begin{aligned}(\partial_t(\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h)_{\mathscr{T}_h} + (\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h,\mathbf{v}_h)_{\mathscr{T}_h}&\\ -\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h\cdot\mathbf{n})[\![\mathbf{u}-\mathbf{u}_h]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F - (p-p_h,\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} &= 0,\\ (q_h,\mathrm{div}(\mathbf{u}-\mathbf{u}_h))_{\mathscr{T}_h} &= 0,\end{aligned}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$. In addition, from the property $\mathrm{div}(\Pi_h^k(\mathbf{u})) = \mathscr{P}_h^k(\mathrm{div}(\mathbf{u})) = 0$, we can rewrite the error equations in the form

$$
\begin{aligned}
&(\partial_t \mathrm{E}^{\mathbf{u}}, \mathbf{v}_h)_{\mathscr{T}_h} + (\mathbf{u}\cdot\nabla_h \mathbf{u} - \mathbf{u}_h\cdot\nabla_h \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} - \sum_{F\in\mathscr{E}_h^i} \langle (\mathbf{u}_h\cdot\mathbf{n})[\![\mathrm{E}^{\mathbf{u}}]\!], \{\!\{\mathbf{v}_h\}\!\}\rangle_F - (\mathrm{E}^p, \mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} \\
&= (\partial_t(\Pi_h^k(\mathbf{u}) - \mathbf{u}), \mathbf{v}_h)_{\mathscr{T}_h} - \sum_{F\in\mathscr{E}_h^i} \langle (\mathbf{u}_h\cdot\mathbf{n})[\![\Pi_h^k(\mathbf{u}) - \mathbf{u}]\!], \{\!\{\mathbf{v}_h\}\!\}\rangle_F - (\mathscr{P}_h^k(p) - p, \mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h}, \qquad (2.11) \\
&\qquad (q_h, \mathrm{div}(\mathrm{E}^{\mathbf{u}}))_{\mathscr{T}_h} = 0,
\end{aligned}
$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$, where it is important to remark here that $\mathrm{E}^{\mathbf{u}}$ is divergence-free.

THEOREM 2.1 Assume that $\mathbf{u} \in W^{1,\infty}([0,T]\times\Omega)^d$ is uniformly bounded. In addition, given an integer $k \geqslant 1$, suppose that $\mathbf{u}_0 \in \mathbf{H}^{k+1}(\Omega)$, $\mathbf{u} \in L^2(0,T;\mathbf{H}^{k+1}(\Omega))$, and $\mathbf{u}_t \in L^2(0,T;\mathbf{H}^{k+1}(\Omega))$. Then, there exists $C > 0$, independent of h, such that

$$
\|(\mathbf{u}-\mathbf{u}_h)(T,\cdot)\|_{L^2(\Omega)} \leqslant C(u)\, h^k B(u),
$$

where

$$
C(u) \quad := \quad (1 + C(1+C_u))\exp(C(1+C_u)T),
$$

with $C_u := \|\mathbf{u}\|_{W^{1,\infty}([0,T]\times\Omega)}$. Also,

$$
B(u) \quad := \quad h\|\mathbf{u}_0\|_{H^{k+1}(\Omega)} + \|\mathbf{u}\|_{L^2(0,T;H^{k+1}(\Omega))} + h\|\mathbf{u}_t\|_{L^2(0,T;H^{k+1}(\Omega))}.
$$

Proof. We begin by choosing $\mathbf{v}_h := \mathrm{E}^{\mathbf{u}}$ in (2.11). Thus, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} \quad = \quad & \underbrace{-(\mathbf{u}\cdot\nabla_h\mathbf{u} - \mathbf{u}_h\cdot\nabla_h\mathbf{u}_h, \mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h}}_{I_1} + \underbrace{\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h\cdot\mathbf{n})[\![\mathrm{E}^{\mathbf{u}}]\!], \{\!\{\mathrm{E}_h^{\mathbf{u}}\}\!\}\rangle_F}_{I_2} \\
& + (\Pi_h^k(\mathbf{u}_t) - \mathbf{u}_t, \mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - \underbrace{\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h\cdot\mathbf{n})[\![\Pi_h^k(\mathbf{u}) - \mathbf{u}]\!], \{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F}_{I_3}, \qquad (2.12)
\end{aligned}
$$

where we have used the fact that $\partial_t\Pi_h^k(\mathbf{u}) = \Pi_h^k(\mathbf{u}_t)$. Next, note that

$$
\begin{aligned}
I_1 \quad &= \quad -(\mathbf{u}\cdot\nabla_h\{\mathbf{u} - \Pi_h^k(\mathbf{u})\}, \mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - ((\mathbf{u}-\mathbf{u}_h)\cdot\nabla_h\Pi_h^k(\mathbf{u}), \mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - (\mathbf{u}_h\cdot\nabla_h\mathrm{E}^{\mathbf{u}}, \mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h}, \\
&= \quad -(\mathbf{u}\cdot\nabla_h\{\mathbf{u} - \Pi_h^k(\mathbf{u})\}, \mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - ((\mathbf{u}-\mathbf{u}_h)\cdot\nabla_h\Pi_h^k(\mathbf{u}), \mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - I_2,
\end{aligned}
$$

where in the last term, we apply the same arguments of (2.7) by using $\mathrm{E}^{\mathbf{u}}$ instead of $\mathbf{u}_h$ in the last two functions. Furthermore, using (2.9) we deduce that

$$
\begin{aligned}
I_1 + I_2 \quad &\leqslant \quad C_u\|\nabla_h\{\Pi_h^k(\mathbf{u}) - \mathbf{u}\}\|_{L^2(\Omega)}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + CC_u\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} \\
&\leqslant \quad C_u\|\nabla_h\{\Pi_h^k(\mathbf{u}) - \mathbf{u}\}\|_{L^2(\Omega)}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + CC_u\Big\{\|\Pi_h^k(\mathbf{u}) - \mathbf{u}\|_{L^2(\Omega)} + \|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}\Big\}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} \\
&\leqslant \quad CC_u\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} + CC_u\Big\{\|\Pi_h^k(\mathbf{u}) - \mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla_h\{\Pi_h^k(\mathbf{u}) - \mathbf{u}\}\|^2_{L^2(\Omega)}\Big\}. \qquad (2.13)
\end{aligned}
$$

On the other hand, for I_3 it follows

$$\begin{aligned} I_3 &= -\sum_{F\in\mathscr{E}_h^i}\langle(\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n})[\![\Pi_h^k(\mathbf{u})-\mathbf{u}]\!],\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F+\sum_{F\in\mathscr{E}_h^i}\langle(\Pi_h^k(\mathbf{u})\cdot\mathbf{n})[\![\Pi_h^k(\mathbf{u})-\mathbf{u}]\!],\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F \\ &\leqslant Ch^{-1}\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|_{L^\infty(\Omega)}\sum_{F\in\mathscr{E}_h^i}h_F\|\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\|^2_{L^2(F)} \\ &\quad+C\|\Pi_h^k(\mathbf{u})\|_{L^\infty(\Omega)}\left(\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\|[\![\Pi_h^k(\mathbf{u})-\mathbf{u}]\!]\|^2_{L^2(F)}\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^i}h_F\|\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\|^2_{L^2(F)}\right)^{1/2}. \end{aligned} \tag{2.14}$$

In addition, given $\mathbf{v}\in\mathbf{H}^1(\mathscr{T}_h)$ and applying a discrete trace inequality, we note that there exists $\widehat{C}>0$, independent of h, such that

$$\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\|[\![\mathbf{v}]\!]\|^2_{L^2(F)}\leqslant\widehat{C}\Big\{h^{-2}\|\mathbf{v}\|^2_{L^2(\Omega)}+\|\nabla_h\mathbf{v}\|^2_{L^2(\Omega)}\Big\}, \tag{2.15}$$

and, in the same way together with an inverse inequality we obtain

$$\sum_{F\in\mathscr{E}_h^i}h_F\|\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\|^2_{L^2(F)}\leqslant\widehat{C}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}. \tag{2.16}$$

Hence, replacing (2.15) and (2.16) in (2.14) and using (2.9) we deduce that

$$I_3\leqslant CC_u\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}+CC_u\left\{h^{-2}\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|^2_{L^2(\Omega)}+\|\nabla_h\{\Pi_h^k(\mathbf{u})-\mathbf{u}\}\|^2_{L^2(\Omega)}\right\}. \tag{2.17}$$

Now, we return to (2.12), which satisfies that

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}\leqslant\frac{1}{2}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}+\frac{1}{2}\|\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t\|^2_{L^2(\Omega)}+(I_1+I_2)+I_3,$$

where, replacing (2.13) and (2.17), we obtain that

$$\begin{aligned}\frac{d}{dt}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}&\leqslant C(1+C_u)\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}+C\|\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t\|^2_{L^2(\Omega)}\\ &\quad+CC_u\left\{h^{-2}\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|^2_{L^2(\Omega)}+\|\nabla_h\{\Pi_h^k(\mathbf{u})-\mathbf{u}\}\|^2_{L^2(\Omega)}\right\}.\end{aligned} \tag{2.18}$$

Hence, applying (2.8) we get

$$\frac{d}{dt}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}\leqslant C(1+C_u)\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}+C(1+C_u)h^{2k}\left(h^2\|\mathbf{u}_t\|^2_{H^{k+1}(\Omega)}+\|\mathbf{u}\|^2_{H^{k+1}(\Omega)}\right).$$

which, applying the Gronwall's inequality (see, e.g. Evans (2010)), yields

$$\begin{aligned}\|\mathrm{E}^{\mathbf{u}}(T,\cdot)\|^2_{L^2(\Omega)}&\leqslant\exp(C(1+C_u)T)\Big\{\|\mathrm{E}^{\mathbf{u}}(0,\cdot)\|^2_{L^2(\Omega)}\\ &\quad+C(1+C_u)\left(h^2\|\mathbf{u}_t\|^2_{L^2(0,T;H^{k+1}(\Omega))}+\|\mathbf{u}\|^2_{L^2(0,T;H^{k+1}(\Omega))}\right)\Big\}.\end{aligned}$$

Finally, we use that $\|\mathrm{E}^{\mathbf{u}}(0,\cdot)\|^2_{L^2(\Omega)}\leqslant Ch^{2(k+1)}\|\mathbf{u}_0\|^2_{H^{k+1}(\Omega)}$ to complete the proof. □

The next goal is to establish error estimates for the pressure variable. To do this, we first obtain an estimate for $\partial_t(\mathbf{u}-\mathbf{u}_h)$, which is the subject of the next result.

LEMMA 2.2 Assume the same hypotheses of Theorem 2.1. Then, there exists $C > 0$, independent of h, such that

$$\begin{aligned}\|\partial_t \mathrm{E}^{\mathbf{u}}(T,\cdot)\|_{L^2(\Omega)} \;&\leqslant\; (C(u)h^{k-d/2}B(u)+C_u)h^{k-1}\Big\{C(u)\,B(u)+\|\mathbf{u}(T,\cdot)\|_{H^{k+1}(\Omega)}\Big\}\\ &\quad + Ch^{k+1}\|\mathbf{u}_t(T,\cdot)\|_{H^{k+1}(\Omega)}\,.\end{aligned}$$

Proof. First, we take $\mathbf{v}_h := \partial_t \mathrm{E}^{\mathbf{u}}$ in (2.11) and using that $\mathrm{div}(\partial_t \mathrm{E}^{\mathbf{u}}) = \partial_t \mathrm{div}(\mathrm{E}^{\mathbf{u}}) = 0$, we obtain

$$\begin{aligned}\|\partial_t \mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} \;&=\; -(\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h,\partial_t\mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} + \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h\cdot\mathbf{n})[\![\mathrm{E}^{\mathbf{u}}]\!],\{\!\{\partial_t\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F\\
&\quad + (\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t,\partial_t\mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h\cdot\mathbf{n})[\![\Pi_h^k(\mathbf{u})-\mathbf{u}]\!],\{\!\{\partial_t\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F\\
&\leqslant\; \|\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h\|_{L^2(\Omega)}\|\partial_t\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + \|\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t\|_{L^2(\Omega)}\|\partial_t\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}\\
&\quad + C\|\mathbf{u}_h\|_{L^\infty(\Omega)}\left(\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\|[\![\mathrm{E}^{\mathbf{u}}]\!]\|^2_{L^2(F)}\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^i}h_F\|\{\!\{\partial_t\mathrm{E}^{\mathbf{u}}\}\!\}\|^2_{L^2(F)}\right)^{1/2}\\
&\quad + C\|\mathbf{u}_h\|_{L^\infty(\Omega)}\left(\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\|[\![\Pi_h^k(\mathbf{u})-\mathbf{u}]\!]\|^2_{L^2(F)}\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^i}h_F\|\{\!\{\partial_t\mathrm{E}^{\mathbf{u}}\}\!\}\|^2_{L^2(F)}\right)^{1/2}.\end{aligned}$$

Next, using (2.15) and (2.16), we deduce after some algebraic manipulation that

$$\begin{aligned}\|\partial_t\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} \;&\leqslant\; C\Big\{h^{-1}\|\mathbf{u}_h\|_{L^\infty(\Omega)}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h\|_{L^2(\Omega)}\\
&\quad + \|\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t\|_{L^2(\Omega)} + \|\mathbf{u}_h\|_{L^\infty(\Omega)}(h^{-1}\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|_{L^2(\Omega)} + \|\nabla_h(\Pi_h^k(\mathbf{u})-\mathbf{u})\|_{L^2(\Omega)})\Big\}.\end{aligned}\tag{2.19}$$

To bound the nonlinear term we add and subtract terms to get

$$\begin{aligned}\|\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h\|_{L^2(\Omega)} \;&=\; \|(\mathbf{u}-\mathbf{u}_h)\cdot\nabla_h\mathbf{u}+\mathbf{u}_h\cdot\nabla_h(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)}\\
&\leqslant C_u\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} + \|\mathbf{u}_h\|_{L^\infty(\Omega)}\|\nabla_h(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)}\\
&\leqslant C_u\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} + \|\mathbf{u}_h\|_{L^\infty(\Omega)}(\|\nabla_h(\mathbf{u}-\Pi_h^k\mathbf{u})\|_{L^2(\Omega)} + Ch^{-1}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)})\\
&\leqslant C_u\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + C_u\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|_{L^2(\Omega)}\\
&\quad + \|\mathbf{u}_h\|_{L^\infty(\Omega)}(\|\nabla_h(\mathbf{u}-\Pi_h^k\mathbf{u})\|_{L^2(\Omega)} + Ch^{-1}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}),\end{aligned}$$

where we have used an inverse estimate. Therefore,

$$\begin{aligned}\|\partial_t\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} \;&\leqslant\; C\Big\{(h^{-1}\|\mathbf{u}_h\|_{L^\infty(\Omega)}+C_u)\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + \|\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t\|_{L^2(\Omega)}\\
&\quad + \|\mathbf{u}_h\|_{L^\infty(\Omega)}\|\nabla_h(\Pi_h^k(\mathbf{u})-\mathbf{u})\|_{L^2(\Omega)} + (h^{-1}\|\mathbf{u}_h\|_{L^\infty(\Omega)}+C_u)\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|_{L^2(\Omega)}\Big\}.\end{aligned}$$

We can bound $\|\mathbf{u}_h\|_{L^\infty(\Omega)}$ using an inverse estimate

$$\|\mathbf{u}_h\|_{L^\infty(\Omega)} \;\leqslant\; \|\mathrm{E}^{\mathbf{u}}\|_{L^\infty(\Omega)} + \|\Pi_h^k(\mathbf{u})\|_{L^\infty(\Omega)} \;\leqslant\; C\,h^{-d/2}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + CC_u.\tag{2.20}$$

Hence,

$$\begin{aligned}\|\partial_t \mathrm{E}^{\mathbf{u}}(t)\|_{L^2(\Omega)} \;&\leqslant\; C\Big\{ h^{-1}(h^{-d/2}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}+C_u)\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + \|\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t\|_{L^2(\Omega)} \\ &\quad + (h^{-d/2}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}+C_u)\Big(\|\nabla_h(\Pi_h^k(\mathbf{u})-\mathbf{u})\|_{L^2(\Omega)} + h^{-1}\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|_{L^2(\Omega)}\Big)\Big\}.\end{aligned}$$

Finally, using Theorem 2.1 and (2.8) establishes the result. □

Note that in the above proof we have also proved

$$\begin{aligned}\|(\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h)(T,\cdot)\|_{L^2(\Omega)} \;&\leqslant\; (C(u)h^{k-d/2}B(u)+C_u)h^{k-1}\Big\{C(u)\,B(u)+\|\mathbf{u}(T,\cdot)\|_{H^{k+1}(\Omega)}\Big\} \\ &\quad + Ch^{k+1}\|\partial_t\mathbf{u}(T,\cdot)\|_{H^{k+1}(\Omega)}\,. \end{aligned} \tag{2.21}$$

We end this section with the a-priori error estimate for the pressure, which is established next.

THEOREM 2.2 Assume the hypothesis of Theorem 2.1. Also, suppose that $p\in L^2(0,T;H^{k+1}(\Omega))$. Then, there exists $C>0$, independent of h, such that

$$\begin{aligned}\|(p-p_h)(T,\cdot)\|_{L^2(\Omega)} \;&\leqslant\; (C(u)h^{k-d/2}B(u)+C_u+C\,h)h^{k-1}\Big\{C(u)\,B(u)+\|\mathbf{u}(T,\cdot)\|_{H^{k+1}(\Omega)}\Big\} \\ &\quad + Ch^{k+1}\Big\{\|\mathbf{u}_t(T,\cdot)\|_{H^{k+1}(\Omega)} + \|p(T,\cdot)\|_{H^{k+1}(\Omega)}\Big\}.\end{aligned}$$

Proof. We begin by recalling here the discrete inf-sup given by

$$\beta\,\|q_h\|_{L^2(\Omega)} \;\leqslant\; \sup_{\substack{\mathbf{v}_h\in\mathbf{V}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{(q_h,\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h}}{\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}} \qquad \forall\, q_h\in Q_h, \tag{2.22}$$

which, in particular for $q_h := \mathrm{E}^p$, it follows

$$\|\mathrm{E}^p\|_{L^2(\Omega)} \;\leqslant\; \frac{1}{\beta}\sup_{\substack{\mathbf{v}_h\in\mathbf{V}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{(\mathrm{E}^p,\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h}}{\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)}}. \tag{2.23}$$

Now, from the error equation (2.11) and proceeding as in the proof of Lemma 2.2, we have

$$\begin{aligned}(\mathrm{E}^p,\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} \;&=\; (\partial_t\mathrm{E}^{\mathbf{u}},\mathbf{v}_h)_{\mathscr{T}_h} + (\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h,\mathbf{v}_h)_{\mathscr{T}_h} - \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h\cdot\mathbf{n})[\![\mathrm{E}^{\mathbf{u}}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F \\ &\quad - (\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t,\mathbf{v}_h)_{\mathscr{T}_h} + \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h\cdot\mathbf{n})[\![\Pi_h^k(\mathbf{u})-\mathbf{u}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F + (\mathscr{P}_h^k(p)-p,\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} \\ &\leqslant\; \|\partial_t\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)} + Ch^{-1}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)} \\ &\quad + C\Big\{h^{-1}\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|_{L^2(\Omega)} + \|\nabla_h(\Pi_h^k(\mathbf{u})-\mathbf{u})|_{L^2(\Omega)}\Big\}\|\mathbf{v}_h\|_{L^2(\Omega)} \\ &\quad + \|\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)} + \|\mathscr{P}_h^k(p)-p\|_{L^2(\Omega)}\|\mathrm{div}(\mathbf{v}_h)\|_{L^2(\Omega)}.\end{aligned}$$

The above result together with (2.23) establishes

$$\begin{aligned}\|\mathrm{E}^p\|_{L^2(\Omega)} \;&\leqslant\; C\Big\{\|\partial_t\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h\cdot\nabla_h\mathbf{u}_h\|_{L^2(\Omega)} + h^{-1}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} \\ &\quad + h^{-1}\|\Pi_h^k(\mathbf{u})-\mathbf{u}\|_{L^2(\Omega)} + \|\nabla_h(\Pi_h^k(\mathbf{u})-\mathbf{u})\|_{L^2(\Omega)} + \|\Pi_h^k(\mathbf{u}_t)-\mathbf{u}_t\|_{L^2(\Omega)} + \|\mathscr{P}_h^k(p)-p\|_{L^2(\Omega)}\Big\}.\end{aligned}$$

Therefore, thanks to $\|p - p_h\|_{L^2(\Omega)} \leqslant \|\mathrm{E}^p\|_{L^2(\Omega)} + \|\mathscr{P}_h^k(p) - p\|_{L^2(\Omega)}$, (2.21), Lemma 2.2 and the approximation properties (2.8) and (2.10), we can easily complete the proof. □

Notice the the error estimate for the pressure predicts $\mathscr{O}(h^{k-1})$ (for $k \geqslant 2$) in two and three dimensions.

2.1.2 *Using an upwind flux.* Here, we introduce an alternative version of the conforming method (2.5), analyzed in previous sections. In order to do that, we begin by redefining the numerical flux $\widehat{\sigma}$ (cf. (2.4)) in a new general form, given by:

$$\widehat{\sigma}(\mathbf{u}_h, p_h) \;:=\; \widehat{\mathbf{u}}_h^{\mathbf{w}} \otimes \{\!\{\mathbf{u}_h\}\!\} \;+\; \{\!\{p_h\}\!\}\, \mathbb{I},$$

where $\widehat{\mathbf{u}}_h^{\mathbf{w}}$ is a new numerical trace for $\mathbf{u}_h$ related with the convective term. In particular, taking $\widehat{\mathbf{u}}_h^{\mathbf{w}} := \{\!\{\mathbf{u}_h\}\!\} = \frac{1}{2}\left(\mathbf{u}_h^{\text{int}} + \mathbf{u}_h^{\text{ext}}\right)$ we arrive exactly to the scheme (2.5). That is, the method (2.5) correspond to a *central scheme*.

On the other hand, for some problems with high gradients, it is more natural to use an *upwind scheme*, in order to get better accuracy and order of convergence. In Section 4 we will present some examples of this. In fact, we see numerically that using upwind flux gives optimal convergence rates for both the velocity and pressure variables.

According to above, we consider the following upwind flux

$$\widehat{\mathbf{u}}_h^{\mathbf{w}} \;:=\; \begin{cases} \mathbf{u}_h^{\text{int}} & \text{if} \quad \mathbf{u}_h \cdot \mathbf{n} \geqslant 0, \\ \mathbf{u}_h^{\text{ext}} & \text{if} \quad \mathbf{u}_h \cdot \mathbf{n} < 0. \end{cases}$$

This definition is given in the same way of that presented in Liu & Shu (2000) for the vorticity, and it is not difficult to check that we can obtain again the method (2.5), with an extra term given by a weighted full jumps onto $\mathscr{E}_h^i$. That is, we seek $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in Q_h$, such that

$$\begin{aligned}(\partial_t \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} + (\mathbf{u}_h \cdot \nabla_h \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} - \sum_{F \in \mathscr{E}_h^i} \langle (\mathbf{u}_h \cdot \mathbf{n}) [\![\mathbf{u}_h]\!], \{\!\{\mathbf{v}_h\}\!\} \rangle_F & \\ + \sum_{F \in \mathscr{E}_h^i} \langle |\mathbf{u}_h \cdot \mathbf{n}| [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!] \rangle_F - (p_h, \mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} &= 0 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ (q_h, \mathrm{div}(\mathbf{u}_h))_{\mathscr{T}_h} &= 0 \quad \forall\, q_h \in Q_h, \\ \mathbf{u}_h(0, \mathbf{x}) \;=\; \mathbf{u}_{h,0}(\mathbf{x}) \quad \text{in } \Omega. &\end{aligned} \tag{2.24}$$

It is important to remark here, that the introduction of this new term does not pose any difficulty in order to prove stability and convergence. In fact, both follow the same arguments, using that when $\mathbf{v}_h = \mathbf{u}_h$ this term is positive. In particular, the error estimates are basically the same and the stability, see remark after the proof of Lemma 2.1, now is given by $\|\mathbf{u}_h(t, \cdot)\|_{L^2(\Omega)} \leqslant \|\mathbf{u}_{h,0}\|_{L^2(\Omega)}$ for each $t \in (0, T)$.

2.2 *DG schemes*

In this section, we introduce a discontinuous Galerkin method for the model problem (2.2). The velocity space will consist of polynomials of degree $k+1$ for the fully discontinuous subspace

$$\mathbf{V}_h^{\text{dg}} \quad := \quad \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \,:\, \mathbf{v}|_K \in \mathbf{P}_{k+1}(K) \quad \forall\, K \in \mathscr{T}_h \quad \text{and} \quad \mathbf{v} \cdot \mathbf{n} = \mathbf{0} \;\text{ on }\; \Gamma \right\},$$

whereas, the pressure space remains unchanged. That is,

$$Q_h \quad := \quad \left\{ q \in L_0^2(\Omega) \;:\; q|_K \in P_k(K) \quad \forall\, K \in \mathscr{T}_h \right\}.$$

In the previous section we only defined the jumps and averages on the interior faces/edges. Here we also define them on boundary faces. That is, for $F \in \mathscr{E}_h^\partial$, as is usual, we set

$$\{\!\{\mathbf{v}\}\!\} := \mathbf{v}, \qquad [\![\mathbf{v}\cdot\mathbf{n}]\!] := \mathbf{v}\cdot\mathbf{n} \qquad \text{and} \qquad \{\!\{q\}\!\} := q.$$

Thus, in order to define the approximation scheme, we first introduce a postprocessed flux. For each $\mathbf{v} \in \mathbf{H}^1(\mathscr{T}_h)$, we find $\mathbf{v}^\star \in \mathbf{P}_{k+1}(\mathscr{T}_h)$ such that

$$\int_F (\mathbf{v}^\star\cdot\mathbf{n})q \quad = \quad \int_F (\{\!\{\mathbf{v}\}\!\}\cdot\mathbf{n})q \quad \forall\, q \in P_{k+1}(F), \quad \forall\, F \in \partial K, \tag{2.25}$$

$$\int_K \mathbf{v}^\star\cdot\mathbf{p} \quad = \quad \int_K \mathbf{v}\cdot\mathbf{p} \quad \forall\, \mathbf{p} \in \mathbf{ND}_{k-1}(K), \tag{2.26}$$

for each $K \in \mathscr{T}_h$. Note that if $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{dg}}$ then $\mathbf{v}_h^\star \in \mathbf{BDM}_{k+1}^0(\Omega)$ where,

$$\begin{aligned}
\mathbf{BDM}_{k+1}(\Omega) \quad &:= \quad \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) \;:\; \mathbf{v}|_K \in \mathbf{P}_{k+1}(K) \quad \forall\, K \in \mathscr{T}_h\} \\
\mathbf{BDM}_{k+1}^0(\Omega) \quad &:= \quad \{\mathbf{v} \in \mathbf{BDM}_{k+1}(\Omega) \;:\; \mathbf{v}\cdot\mathbf{n} = 0 \;\text{ on }\; \Gamma\}.
\end{aligned}$$

For this postprocessed flux, we have the following result.

LEMMA 2.3 Given $K \in \mathscr{T}_h$ and $\mathbf{v}_h \in \mathbf{P}_{k+1}(K)$, there is exists a constants $C^\star > 0$, independent of K, such that

$$\|\mathbf{v}_h^\star - \mathbf{v}_h\|_{L^2(K)} \;\leqslant\; C^\star\, h_K^{1/2} \sum_{F\in\partial K} \|[\![\mathbf{v}_h\cdot\mathbf{n}]\!]\|_{L^2(F)}\,.$$

Proof. We proceed as in (Cockburn *et al.*, 2010b, Lemma 4.2). Indeed, if we set $\delta := \mathbf{v}_h^\star - \mathbf{v}_h \in \mathbf{P}_{k+1}(K)$ we have that δ satisfying the equations

$$\begin{aligned}
\int_F (\delta\cdot\mathbf{n})q \quad &= \quad \int_F (\{\!\{\mathbf{v}_h\}\!\} - \mathbf{v}_h)\cdot\mathbf{n}q \quad \forall\, q \in P_{k+1}(F), \quad \forall\, F \in \partial K, \\
\int_K \delta\cdot\mathbf{p} \quad &= \quad 0 \quad \forall\, \mathbf{p} \in \mathbf{ND}_{k-1}(K).
\end{aligned}$$

The result together with a scaling argument (see Brezzi & Fortin (1991)), imply that

$$\|\delta\|_{L^2(K)} \;\leqslant\; C h_K^{1/2} \|(\{\!\{\mathbf{v}_h\}\!\} - \mathbf{v}_h)\cdot\mathbf{n}\|_{0,\partial K}\,,$$

which, using the fact that $(\{\!\{\mathbf{v}_h\}\!\} - \mathbf{v}_h)\cdot\mathbf{n} = \pm[\![\mathbf{v}_h\cdot\mathbf{n}]\!]$, we complete the proof. □

Now, similar as in (2.3), we consider the Galerkin scheme: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^{\mathrm{dg}} \times Q_h$, such that

$$\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} - (\mathbf{u}_h \otimes \mathbf{u}_h^\star, \nabla_h \mathbf{v}_h)_{\mathscr{T}_h} - (p_h, \mathrm{div}_h(\mathbf{v}_h))_{\mathscr{T}_h} + \langle \widehat{\sigma}(\mathbf{u}_h, p_h)\mathbf{n}, \mathbf{v}_h\rangle_{\partial\mathscr{T}_h} \quad &= \quad 0, \\
-(\nabla_h q_h, \mathbf{u}_h)_{\mathscr{T}_h} + \langle \widehat{\mathbf{u}}_h\cdot\mathbf{n}, q_h\rangle_{\partial\mathscr{T}_h} \quad &= \quad 0, \\
\mathbf{u}_h(0,\mathbf{x}) \;=\; \mathbf{u}_{h,0}(\mathbf{x}) \quad \text{in } \Omega,&
\end{aligned} \tag{2.27}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h^{\mathrm{dg}} \times Q_h$, where

$$\widehat{\sigma}(\mathbf{u}_h, p_h) \quad := \quad \{\!\{\mathbf{u}_h\}\!\} \otimes \{\!\{\mathbf{u}_h^\star\}\!\} + \{\!\{p_h\}\!\}\,\mathbb{I} + \alpha\, h_F^{-1} [\![\mathbf{u}_h \cdot \mathbf{n}]\!]\,\mathbb{I}, \tag{2.28}$$

and $\alpha > 0$ is stabilization parameter. In addition, we define the numerical flux $\widehat{\mathbf{u}}_h$ as

$$\widehat{\mathbf{u}}_h \;:=\; \{\!\{\mathbf{u}_h\}\!\} \text{ on } \mathscr{E}_h.$$

Thus, from the second equation of (2.27) and the definition of $\mathbf{u}_h^\star$ (cf. (2.25) and (2.26)), we note that

$$0 \;=\; -(\nabla_h q_h, \mathbf{u}_h)_{\mathscr{T}_h} + \langle \widehat{\mathbf{u}}_h \cdot \mathbf{n}, q_h \rangle_{\partial \mathscr{T}_h} \;=\; -(\nabla_h q_h, \mathbf{u}_h^\star)_{\mathscr{T}_h} + \langle \mathbf{u}_h^\star \cdot \mathbf{n}, q_h \rangle_{\partial \mathscr{T}_h} \;=\; (q_h, \mathrm{div}(\mathbf{u}_h^\star))_{\mathscr{T}_h}$$

for all $q_h \in Q_h$. The above identity and the fact that $\mathrm{div}(\mathbf{u}_h^\star)|_K \in P_k(K)$ for each $K \in \mathscr{T}_h$, imply that $\mathbf{u}_h^\star$ is divergence-free. This conclusion and the fact that $\mathbf{u}_h^\star$ has a continuous normal component are the main reasons that while we consider $\mathbf{u}_h^\star$ instead of $\mathbf{u}_h$ in the method (2.27).

Then, using integration by parts, the fact that $\mathbf{div}(\mathbf{u}_h \otimes \mathbf{u}_h^\star) = \mathbf{u}_h^\star \cdot \nabla \mathbf{u}_h$ (cf. (2.1)), and the definition of the numerical fluxes, it is not difficult to check that the above DG scheme is as follows: Find $\mathbf{u}_h \in \mathbf{V}_h^{\mathrm{dg}}$ and $p_h \in Q_h$ such that

$$\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} + (\mathbf{u}_h^\star \cdot \nabla_h \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} + \alpha \sum_{F \in \mathscr{E}_h^i} h_F^{-1} \langle [\![\mathbf{u}_h \cdot \mathbf{n}]\!], [\![\mathbf{v}_h \cdot \mathbf{n}]\!] \rangle_F & \\
- \sum_{F \in \mathscr{E}_h^i} \langle (\mathbf{u}_h^\star \cdot \mathbf{n}) [\![\mathbf{u}_h]\!], \{\!\{\mathbf{v}_h\}\!\} \rangle_F - (p_h, \mathrm{div}_h(\mathbf{v}_h))_{\mathscr{T}_h} + \sum_{F \in \mathscr{E}_h^i} \langle [\![\mathbf{v}_h \cdot \mathbf{n}]\!], \{\!\{p_h\}\!\} \rangle_F \quad &= \quad 0, \\
(q_h, \mathrm{div}_h(\mathbf{u}_h))_{\mathscr{T}_h} - \sum_{F \in \mathscr{E}_h^i} \langle [\![\mathbf{u}_h \cdot \mathbf{n}]\!], \{\!\{q_h\}\!\} \rangle_F \quad &= \quad 0, \\
\mathbf{u}_h(0, \mathbf{x}) \;=\; \mathbf{u}_{h,0}(\mathbf{x}) \quad \text{in } \Omega, &
\end{aligned} \tag{2.29}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h^{\mathrm{dg}} \times Q_h$. It is important to note here, that $\mathbf{u}_h$ is not necessarily divergence-free as in the method of Section 2.1. In addition, unlike the methods in the previous section, the DG method (2.29) is locally conservative. Indeed, given $K \in \mathscr{T}_h$ we take $\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{dg}}$ such that $\mathbf{v}_h|_K = \mathbf{e}_i$ is a 1 in the i-th coordinate and 0's elsewhere. Also, $\mathbf{v}_h$ vanishes in the exterior of K, it means in particular that $\mathbf{v}_h$ not belong to $\mathbf{V}_h$. Then, replacing $\mathbf{v}_h$ in the first equation of (2.27) we obtain

$$\int_K \partial_t (\mathbf{u}_h)_i \;+\; \int_{\partial K} (\widehat{\sigma}(\mathbf{u}_h, p_h)\mathbf{n})_i \;=\; 0 \qquad \forall\, i \in \{1, 2, \ldots, d\}\,,$$

and hence

$$\int_K \partial_t \mathbf{u}_h \;+\; \int_{\partial K} \widehat{\sigma}(\mathbf{u}_h, p_h)\mathbf{n} \;=\; \mathbf{0}.$$

The foregoing equation establishes that the DG method (2.29) is in fact locally conservative.

On the other hand, note that the condition $\mathbf{u}_h \cdot \mathbf{n} = 0$ on Γ was imposed in the space $\mathbf{V}_h^{\mathrm{dg}}$. We emphasize, however, that the reason of this is just for theoretical purposes and by no means for the explicit computation of the solution of (2.29), which is solved as usual by imposing the boundary condition as a penalization term in (2.29).

LEMMA 2.4 (Stability) Given $\mathbf{u}_h \in \mathbf{V}_h^{\mathrm{dg}}$ the solution of (2.29). Then, we have

$$\frac{d}{dt}\, \|\mathbf{u}_h\|_{L^2(\Omega)}^2 \;\leqslant\; 0.$$

Proof. We take $\mathbf{v}_h := \mathbf{u}_h$ and $q_h := p_h$ in (2.29), and then we deduce

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2_{L^2(\Omega)} + (\mathbf{u}_h^\star\cdot\nabla_h\mathbf{u}_h,\mathbf{u}_h)_{\mathscr{T}_h} + \alpha\sum_{F\in\mathscr{E}_h^i} h_F^{-1}\|[\![\mathbf{u}_h\cdot\mathbf{n}]\!]\|^2_{L^2(F)} - \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^\star\cdot\mathbf{n})[\![\mathbf{u}_h]\!],\{\!\{\mathbf{u}_h\}\!\}\rangle_F \;=\; 0.$$

Next, with that same arguments of (2.7), we have

$$(\mathbf{u}_h^\star\cdot\nabla_h\mathbf{u}_h,\mathbf{u}_h)_{\mathscr{T}_h} - \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^\star\cdot\mathbf{n})[\![\mathbf{u}_h]\!],\{\!\{\mathbf{u}_h\}\!\}\rangle_F \;=\; 0,$$

which establish that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2_{L^2(\Omega)} + \alpha\sum_{F\in\mathscr{E}_h^i} h_F^{-1}\|[\![\mathbf{u}_h\cdot\mathbf{n}]\!]\|^2_{L^2(F)} \;=\; 0.$$

Finally, from the fact that $\alpha > 0$, we complete the proof. □

2.2.1 *Error estimates for DG method.* Now we are ready to provide error estimates for the DG scheme (2.29). We will need to define the BDM/Nédélec projection.

$$\int_F((\Pi_h^{\mathrm{BDM}}\mathbf{v}-\mathbf{v})\cdot\mathbf{n})q \;=\; 0 \quad \forall\, q\in P_{k+1}(F), \quad \forall\, F\in\partial K, \tag{2.30}$$

$$\int_K(\Pi_h^{\mathrm{BDM}}(\mathbf{v})-\mathbf{v})\cdot\mathbf{p} \;=\; 0 \quad \forall\, \mathbf{p}\in\mathbf{ND}_{k-1}(K). \tag{2.31}$$

We have the following approximation results for $1\leqslant m\leqslant k+2$.

$$\|\mathbf{v}-\Pi_h^{\mathrm{BDM}}(\mathbf{v})\|_{L^2(K)} + h_K\|\nabla(\mathbf{v}-\Pi_h^{\mathrm{BDM}}(\mathbf{v}))\|_{L^2(K)} \;\leqslant\; C h_K^m|\mathbf{v}|_{m,K} \quad \forall\, K\in\mathscr{T}_h. \tag{2.32}$$

Moreover, we also have the following bounds

$$\|\mathbf{v}-\Pi_h^{\mathrm{BDM}}(\mathbf{v})\|_{L^\infty(K)} + h_K\|\nabla(\mathbf{v}-\Pi_h^{\mathrm{BDM}}(\mathbf{v}))\|_{L^\infty(K)} \;\leqslant\; C h_K\|\nabla\mathbf{v}\|_{L^\infty(K)} \quad \forall\, K\in\mathscr{T}_h. \tag{2.33}$$

Let now $\mathrm{E}^{\mathbf{u}} = \Pi_h^{\mathrm{BDM}}(\mathbf{u}) - \mathbf{u}_h$ and $\mathrm{E}^p = \mathscr{P}_h^k(p) - p_h$. Then, we follow (2.11) and consider the error equations:

$$\begin{aligned}
&(\partial_t\mathrm{E}^{\mathbf{u}},\mathbf{v}_h)_{\mathscr{T}_h} + (\mathbf{u}\cdot\nabla_h\mathbf{u}-\mathbf{u}_h^\star\cdot\nabla_h\mathbf{u}_h,\mathbf{v}_h)_{\mathscr{T}_h} + \alpha\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\langle[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!],[\![\mathbf{v}_h\cdot\mathbf{n}]\!]\rangle_F\\
&\quad - \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^\star\cdot\mathbf{n})[\![\mathrm{E}^{\mathbf{u}}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F - (\mathrm{E}^p,\mathrm{div}_h(\mathbf{v}_h))_{\mathscr{T}_h} + \sum_{F\in\mathscr{E}_h^i}\langle[\![\mathbf{v}_h\cdot\mathbf{n}]\!],\{\!\{\mathrm{E}^p\}\!\}\rangle_F\\
&\;= (\partial_t(\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}),\mathbf{v}_h)_{\mathscr{T}_h} + \alpha\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\langle[\![(\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u})\cdot\mathbf{n}]\!],[\![\mathbf{v}_h\cdot\mathbf{n}]\!]\rangle_F\\
&\quad - \sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^\star\cdot\mathbf{n})[\![\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F - (\mathscr{P}_h^k(p)-p,\mathrm{div}_h(\mathbf{v}_h))_{\mathscr{T}_h}\\
&\quad + \sum_{F\in\mathscr{E}_h^i}\langle[\![\mathbf{v}_h\cdot\mathbf{n}]\!],\{\!\{\mathscr{P}_h^k(p)-p\}\!\}\rangle_F, \qquad\qquad (2.34)\\
&(q_h,\mathrm{div}_h(\mathrm{E}^{\mathbf{u}}))_{\mathscr{T}_h} - \sum_{F\in\mathscr{E}_h^i}\langle[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!],\{\!\{q_h\}\!\}\rangle_F = 0,
\end{aligned}$$

for all $(\mathbf{v}_h,q_h)\in\mathbf{V}_h^{\mathrm{dg}}\times Q_h$.

THEOREM 2.3 Assume that $\mathbf{u} \in W^{1,\infty}([0,T]\times\Omega)^d$. Also, given an integer $k \geqslant 1$, suppose that $\mathbf{u}_0 \in \mathbf{H}^{k+2}(\Omega)$, $\mathbf{u} \in L^2(0,T;\mathbf{H}^{k+2}(\Omega))$, $\mathbf{u}_t \in L^2(0,T;\mathbf{H}^{k+2}(\Omega))$, and $p \in L^2(0,T;H^{k+1}(\Omega))$. Then, there exists $C>0$, independent of h, such that

$$\|(\mathbf{u}-\mathbf{u}_h)(T,\cdot)\|_{L^2(\Omega)} \;\leqslant\; C(u)h^{k+1}B(u)$$

where

$$C(u) \;:=\; (1+C(1+C_u))\exp(C(1+C_u)T),$$

with $C_u := \|\mathbf{u}\|_{W^{1,\infty}([0,T]\times\Omega)}$. Also,

$$B(u) \quad:=\quad h\|\mathbf{u}_0\|_{H^{k+2}(\Omega)} + \|\mathbf{u}\|_{L^2(0,T;H^{k+2}(\Omega))} + h\|\mathbf{u}_t\|_{L^2(0,T;H^{k+2}(\Omega))} + \|p\|_{L^2(0,T;H^{k+1}(\Omega))}\,.$$

Proof. We begin by choosing $\mathbf{v}_h := \mathrm{E}^{\mathbf{u}}$ and $q_h := \mathrm{E}^p$ in the error equations (2.34). Then, we have that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} + \alpha\sum_{F\in\mathscr{E}_h^i} h_F^{-1}\|[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!]\|^2_{L^2(F)} \;=\; &\underbrace{-(\mathbf{u}\cdot\nabla_h\mathbf{u} - \mathbf{u}_h^\star\cdot\nabla_h\mathbf{u}_h, \mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h}}_{I_1}\\
&+\underbrace{\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^\star\cdot\mathbf{n})[\![\mathrm{E}^{\mathbf{u}}]\!],\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F}_{I_2} + (\Pi_h^{\mathrm{BDM}}(\mathbf{u}_t)-\mathbf{u}_t,\mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h}\\
&+\underbrace{\alpha\sum_{F\in\mathscr{E}_h^i} h_F^{-1}\langle[\![(\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u})\cdot\mathbf{n}]\!],[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!]\rangle_F}_{I_3} - \underbrace{\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^\star\cdot\mathbf{n})[\![\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}]\!],\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F}_{I_4}\\
&-\underbrace{(\mathscr{P}_h^k(p)-p,\mathrm{div}_h(\mathrm{E}^{\mathbf{u}}))_{\mathscr{T}_h}}_{I_5} + \underbrace{\sum_{F\in\mathscr{E}_h^i}\langle[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!],\{\!\{\mathscr{P}_h^k(p)-p\}\!\}\rangle_F}_{I_6}\,.
\end{aligned} \tag{2.35}$$

Next, we want to find bounds for I_i, $i=1,\ldots,6$. First since $\mathrm{div}_h(\mathrm{E}^{\mathbf{u}})$ is a piecewise polynomial of degree k we have $I_5=0$. Also, note that by (2.30) $I_3=0$. Before we bound the rest of the terms. We note that by Lemma 2.3 and $[\![\Pi_h^{\mathrm{BDM}}(\mathbf{u})\cdot\mathbf{n}]\!]=0$, we know

$$\|\mathbf{u}_h-\mathbf{u}_h^\star\|^2_{L^2(\Omega)} \;\leqslant\; C\sum_{F\in\mathscr{E}_h} h_F\|[\![\mathbf{u}_h\cdot\mathbf{n}]\!]\|^2_{L^2(F)} \;=\; C\sum_{F\in\mathscr{E}_h} h_F\|[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!]\|^2_{L^2(F)} \;\leqslant\; C\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}. \tag{2.36}$$

Now we bound I_1, using that

$$\begin{aligned}
I_1 \quad&=\quad -(\mathbf{u}\cdot\nabla_h\left\{\mathbf{u}-\Pi_h^{\mathrm{BDM}}(\mathbf{u})\right\},\mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - ((\mathbf{u}-\mathbf{u}_h^\star)\cdot\nabla_h\Pi_h^{\mathrm{BDM}}(\mathbf{u}),\mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - (\mathbf{u}_h^\star\cdot\nabla_h\mathrm{E}^{\mathbf{u}},\mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h},\\
&=\quad -(\mathbf{u}\cdot\nabla_h\left\{\mathbf{u}-\Pi_h^{\mathrm{BDM}}(\mathbf{u})\right\},\mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - ((\mathbf{u}-\mathbf{u}_h^\star)\cdot\nabla_h\Pi_h^{\mathrm{BDM}}(\mathbf{u}),\mathrm{E}^{\mathbf{u}})_{\mathscr{T}_h} - I_2,
\end{aligned}$$

where in last term, we apply the same argument of (2.7) as in the proof of Theorem 2.1. Furthermore, using that $\|\mathbf{u}\|_{L^\infty(\Omega)}\leqslant C_u$ and $\|\nabla_h\Pi_h^{\mathrm{BDM}}(\mathbf{u})\|_{L^\infty(\Omega)}\leqslant CC_u$ (see (2.33)), we deduce that

$$\begin{aligned}
I_1+I_2 \;&\leqslant\; C_u\|\nabla_h\{\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\}\|_{L^2(\Omega)}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} + CC_u\|\mathbf{u}-\mathbf{u}_h^\star\|_{L^2(\Omega)}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}\\
&\leqslant\; CC_u\Big\{\|\nabla_h\{\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\}\|_{L^2(\Omega)} + \|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|_{L^2(\Omega)} + \|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}\Big\}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}\\
&\quad + CC_u\|\mathbf{u}_h-\mathbf{u}_h^\star\|_{L^2(\Omega)}\|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)}\\
&\leqslant\; CC_u\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} + CC_u\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|^2_{L^2(\Omega)} + CC_u\|\nabla_h\{\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\}\|^2_{L^2(\Omega)},
\end{aligned} \tag{2.37}$$

where we also used (2.36).

In the case of I_4, from $\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})\|_{L^\infty(\Omega)} \leqslant C_u$, note that

$$\begin{aligned}
I_4 &= \sum_{F\in\mathscr{E}_h^i} \langle (\{\!\{\mathbf{u}_h^\star\}\!\}\cdot\mathbf{n})[\![\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}]\!], \{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F \\
&= \sum_{F\in\mathscr{E}_h^i} \langle (\{\!\{\mathbf{u}_h^\star-\mathbf{u}_h\}\!\}\cdot\mathbf{n})[\![\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}]\!], \{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F - \sum_{F\in\mathscr{E}_h^i} \langle (\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\cdot\mathbf{n})[\![\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}]\!], \{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F \\
&\quad + \sum_{F\in\mathscr{E}_h^i} \langle (\{\!\{\Pi_h^{\mathrm{BDM}}(\mathbf{u})\}\!\}\cdot\mathbf{n})[\![\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}]\!], \{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\rangle_F \\
&\leqslant \|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|_{L^\infty(\Omega)} \left(\sum_{F\in\mathscr{E}_h^i} h_F^{-1}\|\{\!\{\mathbf{u}_h^\star-\mathbf{u}_h\}\!\}\|^2_{L^2(F)}\right)^{1/2} \left(\sum_{F\in\mathscr{E}_h^i} h_F\|\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\|^2_{L^2(F)}\right)^{1/2} \\
&\quad + \|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|_{L^\infty(\Omega)} \sum_{F\in\mathscr{E}_h^i} \|\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\|^2_{L^2(F)} \\
&\quad + C_u \left(\sum_{F\in\mathscr{E}_h^i} h_F^{-1}\|[\![\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}]\!]\|^2_{L^2(F)}\right)^{1/2} \left(\sum_{F\in\mathscr{E}_h^i} h_F\|\{\!\{\mathrm{E}^{\mathbf{u}}\}\!\}\|^2_{L^2(F)}\right)^{1/2},
\end{aligned}$$

and from (2.15), (2.16), and (2.36) with an inverse inequality, we deduce that

$$\begin{aligned}
I_4 &\leqslant \|\mathbf{u}_h-\mathbf{u}_h^\star\|_{L^2(\Omega)} \left(Ch^{-1}\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|_{L^\infty(\Omega)}\right) \|\mathrm{E}^{\mathbf{u}}\|_{L^2(\Omega)} \\
&\quad + \left(Ch^{-1}\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|_{L^\infty(\Omega)}\right) \|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} + CC_u\Big\{h^{-2}\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|^2_{L^2(\Omega)} \\
&\quad + \|\nabla_h\{\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\}\|^2_{L^2(\Omega)}\Big\} + CC_u\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} \\
&\leqslant C\left(1+h^{-1}\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|_{L^\infty(\Omega)}\right)\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} \\
&\quad + Ch^{-2}\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|^2_{L^2(\Omega)} + C\|\nabla_h\{\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\}\|^2_{L^2(\Omega)}.
\end{aligned}$$

In addition, applying (2.33), we conclude that

$$I_4 \leqslant CC_u\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} + CC_u\Big\{h^{-2}\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla_h\{\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\}\|^2_{L^2(\Omega)}\Big\}. \qquad (2.38)$$

Now, in similar way to (2.15), given $q\in H^1(\mathscr{T}_h)$ we have that

$$\sum_{F\in\mathscr{E}_h^i} h_F\|\{\!\{q\}\!\}\|^2_{L^2(F)} \leqslant \widehat{C}\Big\{\|q\|^2_{L^2(\Omega)} + h^2\|\nabla_h q\|^2_{L^2(\Omega)}\Big\},$$

which allows us to deduce

$$\begin{aligned}
I_6 &= \sum_{F\in\mathscr{E}_h} \langle h_F^{-1/2}[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!], h_F^{1/2}\{\!\{\mathscr{P}_h^k(p)-p\}\!\}\rangle_F \leqslant \frac{\alpha}{2}\sum_{F\in\mathscr{E}_h} h_F^{-1}\|[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!]\|^2_{L^2(F)} \\
&\quad + C\sum_{F\in\mathscr{E}_h^i} h_F\|\{\!\{\mathscr{P}_h^k(p)-p\}\!\}\|^2_{L^2(F)} \\
&\leqslant \frac{\alpha}{2}\sum_{F\in\mathscr{E}_h} h_F^{-1}\|[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!]\|^2_{L^2(F)} + C\|\mathscr{P}_h^k(p)-p\|^2_{L^2(\Omega)} + Ch^2\|\nabla_h\{\mathscr{P}_h^k(p)-p\}\|^2_{L^2(\Omega)}. \qquad (2.39)
\end{aligned}$$

On the other hand, replacing (2.37)–(2.39) in (2.35), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\sum_{F\in\mathscr{E}_h^i} h_F^{-1}\|[\![\mathrm{E}^{\mathbf{u}}\cdot\mathbf{n}]\!]\|^2_{L^2(F)} \leqslant CC_u\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Pi_h^{\mathrm{BDM}}(\mathbf{u}_t)-\mathbf{u}_t\|^2_{L^2(\Omega)}$$
$$+ CC_u\Big\{h^{-2}\|\Pi_h^{\mathrm{BDM}}(\mathbf{u})-\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla_h\{\Pi_h^{\mathrm{BDM}}\mathbf{u}-\mathbf{u}\}\|^2_{L^2(\Omega)}\Big\} + C\|\mathscr{P}_h^k(p)-p\|^2_{L^2(\Omega)}$$
$$+ Ch^2\|\nabla_h\{\mathscr{P}_h^k(p)-p\}\|^2_{L^2(\Omega)}.$$

Hence, using (2.32) we have

$$\begin{aligned}\frac{d}{dt}\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)} \quad &\leqslant \quad C(1+C_u)\|\mathrm{E}^{\mathbf{u}}\|^2_{L^2(\Omega)}\\ &\quad + C(1+C_u)\,h^{2(k+1)}\Big\{h^2\|\mathbf{u}_t\|^2_{H^{k+2}(\Omega)} + \|\mathbf{u}\|^2_{H^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)}\Big\}.\end{aligned}$$

Finally, applying Gronwall's inequality gives the result. □

THEOREM 2.4 Assuming the hypothesis of the previous theorem we have the existence of a $C>0$, independent of h, such that

$$\begin{aligned}\|(p-p_h)(T,\cdot)\|_{L^2(\Omega)} \quad &\leqslant \quad (C(u)h^{k+1-d/2}B(u)+C_u+C)h^k\Big\{C(u)B(u)+\|\mathbf{u}(T,\cdot)\|_{H^{k+2}(\Omega)}\Big\}\\ &\quad + Ch^{k+1}\Big\{h\|\mathbf{u}_t(T,\cdot)\|_{H^{k+2}(\Omega)} + \|p(T,\cdot)\|_{H^{k+1}(\Omega)}\Big\}.\end{aligned}$$

Proof. Similar to the proof of Theorem 2.2. □

2.2.2 *Upwind flux for DG method.* Similarly as Section 2.1.2, we now introduce a DG method using an upwind flux. Indeed, as before, we redefine the numerical flux $\widehat{\sigma}$ (see (2.28)) in the form

$$\widehat{\sigma}(\mathbf{u}_h,p_h) := \widehat{\mathbf{u}}_h^{\mathbf{w}}\otimes\{\!\{\mathbf{u}_h^\star\}\!\} + \{\!\{p_h\}\!\}\,\mathbb{I} + \alpha\,h_F^{-1}[\![\mathbf{u}_h\cdot\mathbf{n}]\!]\,\mathbb{I},$$

where we take $\widehat{\mathbf{u}}_h^{\mathbf{w}}$ as

$$\widehat{\mathbf{u}}_h^{\mathbf{w}} := \begin{cases}\mathbf{u}_h^{\mathrm{int}} & \text{if} \quad \mathbf{u}_h^\star\cdot\mathbf{n}\geqslant 0,\\ \mathbf{u}_h^{\mathrm{ext}} & \text{if} \quad \mathbf{u}_h^\star\cdot\mathbf{n}<0.\end{cases}$$

Once again, with this definition we can obtain again the method (2.29), with an extra consistent term given by

$$\sum_{F\in\mathscr{E}_h^i}\langle\,|\mathbf{u}_h^\star\cdot\mathbf{n}|\,[\![\mathbf{u}_h]\!],\ [\![\mathbf{v}_h]\!]\rangle_F,$$

which, allow us to prove stability and convergence in the same way of before, using the fact that when

$\mathbf{v}_h = \mathbf{u}_h$ the above term is positive. Summarizing, we find $\mathbf{u}_h \in \mathbf{V}_h^{\mathrm{dg}}$ and $p_h \in Q_h$ such that

$$
\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} + (\mathbf{u}_h^\star \cdot \nabla_h \mathbf{u}_h, \mathbf{v}_h)_{\mathscr{T}_h} + \alpha \sum_{F \in \mathscr{E}_h^i} h_F^{-1} \langle [\![\mathbf{u}_h \cdot \mathbf{n}]\!], [\![\mathbf{v}_h \cdot \mathbf{n}]\!] \rangle_F & \\
- \sum_{F \in \mathscr{E}_h^i} \langle (\mathbf{u}_h^\star \cdot \mathbf{n}) [\![\mathbf{u}_h]\!], \{\!\{ \mathbf{v}_h \}\!\} \rangle_F + \sum_{F \in \mathscr{E}_h^i} \langle |\mathbf{u}_h^\star \cdot \mathbf{n}| [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!] \rangle_F & \\
- (p_h, \mathrm{div}_h(\mathbf{v}_h))_{\mathscr{T}_h} + \sum_{F \in \mathscr{E}_h^i} \langle [\![\mathbf{v}_h \cdot \mathbf{n}]\!], \{\!\{ p_h \}\!\} \rangle_F &= 0, \\
(q_h, \mathrm{div}_h(\mathbf{u}_h))_{\mathscr{T}_h} - \sum_{F \in \mathscr{E}_h^i} \langle [\![\mathbf{u}_h \cdot \mathbf{n}]\!], \{\!\{ q_h \}\!\} \rangle_F &= 0, \\
\mathbf{u}_h(0, \mathbf{x}) = \mathbf{u}_{h,0}(\mathbf{x}) \quad \text{in } \Omega, &
\end{aligned} \tag{2.40}
$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h^{\mathrm{dg}} \times Q_h$.

3. Fully-discrete methods

In this section we define fully-discrete versions of both approaches introduced in Section 2. In order to do that, for the time discretization we consider the backward Euler method, that is, we write

$$
\mathbf{u}_t(t_{n+1}, \cdot) = \frac{1}{\Delta t} \big\{ \mathbf{u}(t_{n+1}, \cdot) - \mathbf{u}(t_n, \cdot) \big\} + \mathbf{E}_0(t_{n+1}), \tag{3.1}
$$

where $\Delta t > 0$ is the time step, $t_n := n\Delta t$, $0 \leqslant n \leqslant N$, and $\mathbf{E}_0(t_{n+1})$ is the truncation error. We know that

$$
\|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)} \leqslant C \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}(s, \cdot)\|_{L^2(\Omega)} \, ds. \tag{3.2}
$$

For simplicity of the following analysis we denote $\mathbf{u}^n := \mathbf{u}(t_n, \cdot)$ for the exact value and $\mathbf{u}_h^n := \mathbf{u}_h(t_n, \cdot)$ for the approximation. Also, given Π_h^k the corresponding projection used before in each case, respectively, we define $\mathbf{e}_{\mathbf{u}}^n := \Pi_h^k(\mathbf{u}^n) - \mathbf{u}_h^n$ as the discrete error. Similar convention is used for the pressure variable.

On the other hand, using (3.1) we have that the exact solution of (1.1) satisfies that

$$
\begin{aligned}
(\mathbf{u}^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} + \Delta t(\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} - \Delta t(p^{n+1}, \mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} &= (\mathbf{u}^n, \mathbf{v}_h)_{\mathscr{T}_h} - \Delta t(\mathbf{E}_0(t_{n+1}), \mathbf{v}_h)_{\mathscr{T}_h}, \\
(q_h, \mathrm{div}(\mathbf{u}^{n+1}))_{\mathscr{T}_h} &= 0,
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
(\mathbf{u}^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} + \Delta t(\mathbf{u}^n \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} - \Delta t(p^{n+1}, \mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} &= (\mathbf{u}^n, \mathbf{v}_h)_{\mathscr{T}_h} \\
+ \Delta t((\mathbf{u}^n - \mathbf{u}^{n+1}) \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} - \Delta t(\mathbf{E}_0(t_{n+1}), \mathbf{v}_h)_{\mathscr{T}_h}, & \\
(q_h, \mathrm{div}(\mathbf{u}^{n+1}))_{\mathscr{T}_h} = 0, &
\end{aligned} \tag{3.3}
$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h^{\mathrm{dg}} \times Q_h$. We recall here that $\mathbf{V}_h \subset \mathbf{V}_h^{\mathrm{dg}}$.

3.1 *H(div) conforming methods*

Next, using (3.1) in the semi-discrete method (2.5), we introduce the fully-discrete approximation as: Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned}(\mathbf{u}_h^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} + \Delta t(\mathbf{u}_h^n \cdot \nabla_h \mathbf{u}_h^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} - \Delta t \sum_{F\in\mathscr{E}_h^i} \langle(\mathbf{u}_h^n\cdot\mathbf{n})[\![\mathbf{u}_h^{n+1}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F \qquad & \\ - \Delta t(p_h^{n+1},\operatorname{div}(\mathbf{v}_h))_{\mathscr{T}_h} &= (\mathbf{u}_h^n,\mathbf{v}_h)_{\mathscr{T}_h}, \\ (q_h,\operatorname{div}(\mathbf{u}_h^{n+1}))_{\mathscr{T}_h} &= 0,\end{aligned} \tag{3.4}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$. Note that we eliminated the nonlinearity of the problem using the previous approximation. Also, it follows from the proof of Lemma 2.1 that when we take $\mathbf{v}_h := \mathbf{u}_h^{n+1}$ in (3.4), we have

$$\|\mathbf{u}_h^{n+1}\|_{L^2(\Omega)}^2 = (\mathbf{u}_h^n,\mathbf{u}_h^{n+1})_{\mathscr{T}_h},$$

which establish that $\|\mathbf{u}_h^{n+1}\|_{L^2(\Omega)} \leqslant \|\mathbf{u}_h^n\|_{L^2(\Omega)}$, that is, the method (3.4) is stable.

Our next goal is establish an error estimate for the velocity.

THEOREM 3.1 Assume that $\mathbf{u} \in W^{1,\infty}([0,T]\times\Omega)^d$ is uniformly bounded. Also, given an integer $k \geqslant 1$, suppose that $\mathbf{u}_0 \in \mathbf{H}^{k+1}(\Omega)$, $\mathbf{u}_t \in L^2(0,T;\mathbf{H}^{k+1}(\Omega))$, and $\mathbf{u}_{tt} \in L^2(0,T;\mathbf{L}^2(\Omega))$. Then, there exists $C > 0$, independent of h, such that

$$\|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)} \leqslant C\exp(CC_u T)(h^k + \Delta t)A(u), \quad \text{for all } 0 \leqslant n \leqslant N,$$

with $C_u := \|\mathbf{u}\|_{W^{1,\infty}([0,T]\times\Omega)}$. Also, where

$$\begin{aligned}A(u) := &(h\sqrt{T} + C_u T^{3/2})\|\mathbf{u}_t\|_{L^2(0,T;H^{k+1}(\Omega))} + C_u\sqrt{T}\|\mathbf{u}_t\|_{L^2(0,T;L^2(\Omega))} + \sqrt{T}\|\mathbf{u}_{tt}\|_{L^2(0,T;L^2(\Omega))} \\ &+ (C_u T + h)\|\mathbf{u}_0\|_{H^{k+1}(\Omega)}.\end{aligned}$$

Proof. We begin by subtracting equation (3.3) from equation (3.4) together with the fact that $[\![\mathbf{u}^{n+1}]\!] = \mathbf{0}$ on $\mathscr{E}_h^i$, in order to obtain the error equation

$$\begin{aligned}&(\mathbf{e}_{\mathbf{u}}^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} + \Delta t(\mathbf{u}^n\cdot\nabla_h\mathbf{u}^{n+1} - \mathbf{u}_h^n\cdot\nabla_h\mathbf{u}_h^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} - \Delta t\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^n\cdot\mathbf{n})[\![\mathbf{e}_{\mathbf{u}}^{n+1}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F \\ &- \Delta t(p^{n+1} - p_h^{n+1},\operatorname{div}(\mathbf{v}_h))_{\mathscr{T}_h} = (\mathbf{u}^n - \mathbf{u}_h^n,\mathbf{v}_h)_{\mathscr{T}_h} + (\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} \\ &+ \Delta t((\mathbf{u}^n - \mathbf{u}^{n+1})\cdot\nabla\mathbf{u}^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} - \Delta t\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^n\cdot\mathbf{n})[\![\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F \\ &- \Delta t(\mathbf{E}_0(t_{n+1}),\mathbf{v}_h)_{\mathscr{T}_h}.\end{aligned} \tag{3.5}$$

Now, we take $\mathbf{v}_h := \mathbf{e}_{\mathbf{u}}^{n+1}$ and using that $\operatorname{div}(\mathbf{e}_{\mathbf{u}}^{n+1}) = 0$ in Ω, it follows that

$$
\begin{aligned}
\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2(\Omega)}^2 \;=\; & \underbrace{-\Delta t(\mathbf{u}^n \cdot \nabla_h \mathbf{u}^{n+1} - \mathbf{u}_h^n \cdot \nabla_h \mathbf{u}_h^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1})_{\mathscr{T}_h}}_{I_1} + \underbrace{\Delta t \sum_{F \in \mathscr{E}_h^i} \langle (\mathbf{u}_h^n \cdot \mathbf{n}) [\![\mathbf{e}_{\mathbf{u}}^{n+1}]\!], \{\!\{ \mathbf{e}_{\mathbf{u}}^{n+1} \}\!\} \rangle_F}_{I_2} \\
& + (\mathbf{u}^n - \mathbf{u}_h^n, \mathbf{e}_{\mathbf{u}}^{n+1})_{\mathscr{T}_h} + (\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1})_{\mathscr{T}_h} + \Delta t((\mathbf{u}^n - \mathbf{u}^{n+1}) \cdot \nabla \mathbf{u}^{n+1}, \mathbf{e}_{\mathbf{u}}^{n+1})_{\mathscr{T}_h} \\
& \underbrace{- \Delta t \sum_{F \in \mathscr{E}_h^i} \langle (\mathbf{u}_h^n \cdot \mathbf{n}) [\![\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}]\!], \{\!\{ \mathbf{e}_{\mathbf{u}}^{n+1} \}\!\} \rangle_F}_{I_3} - \Delta t(\mathbf{E}_0(t_{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1})_{\mathscr{T}_h},
\end{aligned} \tag{3.6}
$$

which, in similar way to (2.13), we note that

$$
\begin{aligned}
I_1 + I_2 \;=\; & -\Delta t(\mathbf{u}^n \cdot \nabla_h \{\mathbf{u}^{n+1} - \Pi_h^k(\mathbf{u}^{n+1})\}, \mathbf{e}_{\mathbf{u}}^{n+1})_{\mathscr{T}_h} - \Delta t((\mathbf{u}^n - \mathbf{u}_h^n) \cdot \nabla_h \Pi_h^k(\mathbf{u}^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1})_{\mathscr{T}_h} \\
\leqslant\; & \Delta t \Big\{ C_u \|\nabla_h \{\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}\}\|_{L^2(\Omega)} + C C_u \|\Pi_h^k(\mathbf{u}^n) - \mathbf{u}^n\|_{L^2(\Omega)} \\
& + C C_u \|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)} \Big\} \|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2(\Omega)},
\end{aligned} \tag{3.7}
$$

where, we used that $\|\mathbf{u}^n\|_{L^\infty(\Omega)} \leqslant C_u$ and $\|\nabla_h \Pi_h^k(\mathbf{u}^{n+1})\|_{L^\infty(\Omega)} \leqslant C C_u$. Also, follows (2.14) and using (2.15), (2.16) and (2.9), we have

$$
\begin{aligned}
I_3 \;\leqslant\; & C \Delta t \left\{ h^{-1} \|\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}\|_{L^\infty(\Omega)} \left(\sum_{F \in \mathscr{E}_h^i} h_F \|\{\!\{ \mathbf{e}_{\mathbf{u}}^n \}\!\}\|_{L^2(F)}^2 \right)^{\frac{1}{2}} \right. \\
& \left. + \|\Pi_h^k(\mathbf{u}^n)\|_{L^\infty(\Omega)} \left(\sum_{F \in \mathscr{E}_h^i} h_F^{-1} \|[\![\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}]\!]\|_{L^2(F)}^2 \right)^{\frac{1}{2}} \right\} \left(\sum_{F \in \mathscr{E}_h^i} h_F \|\{\!\{ \mathbf{e}_{\mathbf{u}}^{n+1} \}\!\}\|_{L^2(F)}^2 \right)^{\frac{1}{2}} \\
\leqslant\; & C C_u \Delta t \Big\{ \|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)} + h^{-1} \|\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}\|_{L^2(\Omega)} \\
& + \|\nabla_h \{\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}\}\|_{L^2(\Omega)} \Big\} \|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2(\Omega)}.
\end{aligned} \tag{3.8}
$$

On the other hand, we return to (3.6), and observe

$$
\begin{aligned}
\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2(\Omega)}^2 \;\leqslant\; & \Big\{ \|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)} + \|\Pi_h^k(\mathbf{u}^{n+1} - \mathbf{u}^n) - (\mathbf{u}^{n+1} - \mathbf{u}^n)\|_{L^2(\Omega)} + C_u \Delta t \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{L^2(\Omega)} \\
& + \Delta t \|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)} \Big\} \|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2(\Omega)} + (I_1 + I_2) + I_3,
\end{aligned}
$$

which, replacing (3.7) and (3.8), we deduce that

$$
\begin{aligned}
\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2(\Omega)} \;\leqslant\; & (1 + C C_u \Delta t) \|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)} + \|\Pi_h^k(\mathbf{u}^{n+1} - \mathbf{u}^n) - (\mathbf{u}^{n+1} - \mathbf{u}^n)\|_{L^2(\Omega)} \\
& + C C_u \Delta t \Big\{ \|\Pi_h^k(\mathbf{u}^n) - \mathbf{u}^n\|_{L^2(\Omega)} + h^{-1} \|\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}\|_{L^2(\Omega)}. \\
& + \|\nabla_h \{\Pi_h^k(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}\}\|_{L^2(\Omega)} \Big\} + \Delta t \Big\{ C_u \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{L^2(\Omega)} + \|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)} \Big\}.
\end{aligned}
$$

Next, using that

$$(\mathbf{u}^{n+1}-\mathbf{u}^{n})(\mathbf{x}) \quad = \quad \int_{t_n}^{t_{n+1}} \mathbf{u}_t(s,\mathbf{x})\,ds\,, \tag{3.9}$$

together with (2.8), it follows that

$$\|\Pi_h^k(\mathbf{u}^{n+1}-\mathbf{u}^{n})-(\mathbf{u}^{n+1}-\mathbf{u}^{n})\|_{L^2(\Omega)} \quad \leqslant \quad Ch^{k+1}\int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(s,\cdot)\|_{H^{k+1}(\Omega)}\,ds\,.$$

Similarly, we can show

$$\Delta t\,C_u\,\|\mathbf{u}^{n+1}-\mathbf{u}^{n}\|_{L^2(\Omega)} \quad \leqslant \quad \Delta t\,CC_u\int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(s,\cdot)\|_{L^2(\Omega)}\,ds$$

and, from (3.2),

$$\Delta t\,\|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)} \quad \leqslant \quad C\,\Delta t\int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}(s,\cdot)\|_{L^2(\Omega)}\,ds\,.$$

In addition, using that

$$\mathbf{u}^{n+1}(\mathbf{x}) \quad = \quad \mathbf{u}_0(\mathbf{x})\,+\,\int_{0}^{t_{n+1}} \mathbf{u}_t(s,\mathbf{x})\,ds\,, \tag{3.10}$$

and (2.8), we have

$$h^{-1}\|\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\|_{L^2(\Omega)} \quad \leqslant \quad Ch^k\left\{\|\mathbf{u}_0\|_{H^{k+1}(\Omega)}+\int_0^{t_{n+1}}\|\mathbf{u}_t(s,\cdot)\|_{H^{k+1}(\Omega)}\,ds\right\}.$$

Analogously, we can show

$$\begin{aligned}
&CC_u\,\Delta t\left\{\|\Pi_h^k(\mathbf{u}^{n})-\mathbf{u}^{n}\|_{L^2(\Omega)}+h^{-1}\|\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\|_{L^2(\Omega)}+\|\nabla_h\{\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\}\|_{L^2(\Omega)}\right\}\\
&\leqslant\; CC_u\,\Delta t\,h^k\left\{\|\mathbf{u}_0\|_{H^{k+1}(\Omega)}+\int_0^{t_{n+1}}\|\mathbf{u}_t(s,\cdot)\|_{H^{k+1}(\Omega)}\,ds\right\}.
\end{aligned}$$

Therefore, gathering together all the above equations, we deduce that

$$\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2(\Omega)} \quad \leqslant \quad (1+CC_u\Delta t)\,\|\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2(\Omega)}\,+\,C(\Delta t+h^k)B(u,n)\,, \tag{3.11}$$

where

$$\begin{aligned}
B(u,n) \quad := \quad & h\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(s,\cdot)\|_{H^{k+1}(\Omega)}\,ds+C_u\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(s,\cdot)\|_{L^2(\Omega)}\,ds+\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}(s,\cdot)\|_{L^2(\Omega)}\,ds\\
&+\Delta t\,C_u\left\{\|\mathbf{u}_0\|_{H^{k+1}(\Omega)}+\int_0^{t_{n+1}}\|\mathbf{u}_t(s,\cdot)\|_{H^{k+1}(\Omega)}\,ds\right\}.
\end{aligned}$$

Now, from the recurrence relation (3.11), we obtain that

$$\begin{aligned}
\|\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2(\Omega)} \quad &\leqslant \quad (1+CC_u\Delta t)^n\,\|\mathbf{e}_{\mathbf{u}}^{0}\|_{L^2(\Omega)}\,+\,C\left\{\sum_{i=0}^{n-1}(1+C_u\Delta t)^iB(u,n-1-i)\right\}(h^k+\Delta t)\\
&\leqslant \quad C\,(1+CC_u\Delta t)^n\,(h^k+\Delta t)\left\{h\|\mathbf{u}_0\|_{H^{k+1}(\Omega)}+\sum_{i=0}^{n-1}B(u,n-1-i)\right\}\\
&= \quad C\left(1+C\frac{C_u\,T}{n}\right)^n(h^k+\Delta t)\left\{h\|\mathbf{u}_0\|_{H^{k+1}(\Omega)}+\sum_{i=0}^{n-1}B(u,n-1-i)\right\}.
\end{aligned}$$

Finally, noting that

$$\sum_{i=0}^{n-1} B(u,n-1-i) \leqslant h\int_0^{t_n} \|\mathbf{u}_t(s,\cdot)\|_{H^{k+1}(\Omega)}\,ds + C_u\int_0^{t_n} \|\mathbf{u}_t(s,\cdot)\|_{L^2(\Omega)}\,ds$$
$$+ \int_0^{t_n} \|\mathbf{u}_{tt}(s,\cdot)\|_{L^2(\Omega)}\,ds + C_u t_n \left\{ \|\mathbf{u}_0\|_{H^{k+1}(\Omega)} + \int_0^{t_n} \|\mathbf{u}_t(s,\cdot)\|_{H^{k+1}(\Omega)}\,ds \right\},$$

the result now follows by using Cauchy-Schwarz inequality. □

Now, we establish the a-priori error estimate for the pressure, and for that we first consider the next result.

LEMMA 3.1 Assuming the hypothesis of the previous theorem we have the existence of a $C > 0$, independent of h, such that for all $0 \leqslant n \leqslant N$

$$\left\| \frac{\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}}{\Delta t} - \frac{\mathbf{u}^{n} - \mathbf{u}_h^{n}}{\Delta t} \right\|_{L^2(\Omega)} \leqslant C C_{h,\Delta t}(u) \exp(C C_u T) \left(h^{k-1} + \frac{\Delta t}{h}\right) A(u)$$
$$+ C\left\{1 + C_{h,\Delta t}(u)\right\} (h^k + \Delta t) D_n(u),$$

where

$$C_{h,\Delta t}(u) := \exp(C C_u T)\, h^{-d/2} (h^k + \Delta t) A(u) + C_u$$

and

$$D_n(u) := h\|\mathbf{u}_t\|_{L^\infty(t_n,t_{n+1};H^{k+1}(\Omega))} + \|\mathbf{u}_t\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))} + \|\mathbf{u}_{tt}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}$$
$$+ \|\mathbf{u}\|_{L^\infty(t_n,t_{n+1};H^{k+1}(\Omega))}.$$

Proof. From the error equation (3.5) we have

$$(\delta_h, \mathbf{v}_h)_{\mathscr{T}_h} = -(\mathbf{u}^n \cdot \nabla_h \mathbf{u}^{n+1} - \mathbf{u}_h^n \cdot \nabla_h \mathbf{u}_h^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} + \sum_{F \in \mathscr{E}_h^i} \langle (\mathbf{u}_h^n \cdot \mathbf{n}) [\![\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}]\!], \{\!\{ \mathbf{v}_h \}\!\} \rangle_F$$
$$+ (p^{n+1} - p_h^{n+1}, \operatorname{div}(\mathbf{v}_h))_{\mathscr{T}_h} + \frac{1}{\Delta t} (\Pi_h^k(\mathbf{u}^{n+1} - \mathbf{u}^n) - (\mathbf{u}^{n+1} - \mathbf{u}^n), \mathbf{v}_h)_{\mathscr{T}_h}$$
$$+ ((\mathbf{u}^n - \mathbf{u}^{n+1}) \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} - (\mathbf{E}_0(t_{n+1}), \mathbf{v}_h)_{\mathscr{T}_h} \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h,$$

where $\delta_h := \frac{1}{\Delta t}(\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n})$. Then, taking $\mathbf{v}_h := \delta_h$ and using that $\operatorname{div}(\delta_h) = 0$, we deduce that

$$\|\delta_h\|_{L^2(\Omega)}^2 \leqslant \|\mathbf{u}^n \cdot \nabla_h \mathbf{u}^{n+1} - \mathbf{u}_h^n \cdot \nabla_h \mathbf{u}_h^{n+1}\|_{L^2(\Omega)} \|\delta_h\|_{L^2(\Omega)}$$
$$+ C\|\mathbf{u}_h^n\|_{L^\infty(\Omega)} \left(\sum_{F \in \mathscr{E}_h^i} h_F^{-1} \| [\![\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}]\!] \|_{L^2(F)}^2 \right)^{1/2} \left(\sum_{F \in \mathscr{E}_h^i} h_F \| \{\!\{ \delta_h \}\!\} \|_{L^2(F)}^2 \right)^{1/2}$$
$$+ \left\| \Pi_h^k \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \right) - \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \right) \right\|_{L^2(\Omega)} \|\delta_h\|_{L^2(\Omega)}$$
$$+ C_u \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{L^2(\Omega)} \|\delta_h\|_{L^2(\Omega)} + \|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)} \|\delta_h\|_{L^2(\Omega)}.$$

Now, we follow the proof of Lemma 2.2, to obtain that

$$\|\mathbf{u}^n\cdot\nabla_h\mathbf{u}^{n+1}-\mathbf{u}_h^n\cdot\nabla_h\mathbf{u}_h^{n+1}\|_{L^2(\Omega)} \leqslant C_u\|\mathbf{u}^n-\mathbf{u}_h^n\|_{L^2(\Omega)}$$
$$+C\,(h^{-d/2}\|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)}+C_u)\Big\{h^{-1}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2(\Omega)}+\|\nabla_h\{\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\}\|_{L^2(\Omega)}\Big\}, \tag{3.12}$$

and from (2.15) and (2.20) we have

$$\|\mathbf{u}_h^n\|_{L^\infty(\Omega)}\left(\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\|[\![\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}]\!]\|_{L^2(F)}^2\right)^{\frac{1}{2}} \leqslant C\,(h^{-d/2}\|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)}+C_u)\Big\{h^{-1}\|\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}\|_{L^2(\Omega)}$$
$$+h^{-1}\|\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\|_{L^2(\Omega)}+\|\nabla_h\{\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\}\|_{L^2(\Omega)}\Big\}. \tag{3.13}$$

Next, applying (3.12) and (3.13), together with (2.16), it follows that

$$\begin{aligned}\|\delta_h\|_{L^2(\Omega)} \leqslant\;& C_u\|\mathbf{u}^n-\mathbf{u}_h^n\|_{L^2(\Omega)}+Ch^{-1}(h^{-d/2}\|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)}+C_u)\|\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}\|_{L^2(\Omega)}\\ &+C(h^{-d/2}\|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)}+C_u)\Big\{h^{-1}\|\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\|_{L^2(\Omega)}\\ &+\|\nabla_h\{\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\}\|_{L^2(\Omega)}\Big\}+\left\|\Pi_h^k\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right)-\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right)\right\|_{L^2(\Omega)}\\ &+C_u\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{L^2(\Omega)}+\|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)}.\end{aligned}$$

On the other hand, using the fact that

$$\left\|\frac{\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}}{\Delta t}-\frac{\mathbf{u}^n-\mathbf{u}_h^n}{\Delta t}\right\|_{L^2(\Omega)} \leqslant \left\|\Pi_h^k\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right)-\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right)\right\|_{L^2(\Omega)}+\|\delta_h\|_{L^2(\Omega)},$$

we have

$$\begin{aligned}\left\|\frac{\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}}{\Delta t}-\frac{\mathbf{u}^n-\mathbf{u}_h^n}{\Delta t}\right\|_{L^2(\Omega)} &\leqslant C_u\|\mathbf{u}^n-\mathbf{u}_h^n\|_{L^2(\Omega)}\\ &+Ch^{-1}(h^{-d/2}\|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)}+C_u)\|\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}\|_{L^2(\Omega)}\\ &+C(h^{-d/2}\|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)}+C_u)\Big\{h^{-1}\|\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\|_{L^2(\Omega)}\\ &+\|\nabla_h\{\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\}\|_{L^2(\Omega)}\Big\}+2\left\|\Pi_h^k\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right)-\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right)\right\|_{L^2(\Omega)}\\ &+C_u\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{L^2(\Omega)}+\|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)}.\end{aligned} \tag{3.14}$$

Next, we proceed as in the last part of the proof of Theorem 3.1. Indeed, from (2.8), we obtain that

$$h^{-1}\|\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\|_{L^2(\Omega)}+\|\nabla_h\{\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\}\|_{L^2(\Omega)}$$
$$\leqslant Ch^k\|\mathbf{u}^{n+1}\|_{H^{k+1}(\Omega)} \leqslant Ch^k\|\mathbf{u}\|_{L^\infty(t_n,t_{n+1};H^{k+1}(\Omega))}$$

Similarly, from (3.9) and (2.8), we have

$$\begin{aligned}\left\|\Pi_h^k\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right)-\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right)\right\|_{L^2(\Omega)} &\leqslant Ch^{k+1}\left\{\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(s,\cdot)\|_{H^{k+1}(\Omega)}\,ds\right\}\\ &\leqslant Ch^{k+1}\|\mathbf{u}_t\|_{L^\infty(t_n,t_{n+1};H^{k+1}(\Omega))}\,.\end{aligned}$$

In addition, using again (3.9) and (3.2), we deduce, respectively, that

$$\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{L^2(\Omega)} \leqslant \int_{t_n}^{t_{n+1}}\|\mathbf{u}_t(s,\cdot)\|_{L^2(\Omega)}\,ds \leqslant \Delta t\,\|\mathbf{u}_t\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}\,,$$

and

$$\|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)} \leqslant C\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}(s,\cdot)\|_{L^2(\Omega)}\,ds \leqslant C\,\Delta t\,\|\mathbf{u}_{tt}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}\,.$$

The result now follows after applying the previous theorem and the last four estimates into (3.14). □

THEOREM 3.2 Assume the hypothesis of Theorem 3.1. In addition, suppose that $p\in L^2(0,T;H^{k+1}(\Omega))$. Then, there exists $C>0$, independent of h, such that for all $0\leqslant n\leqslant N$ the following estimate holds

$$\begin{aligned}\|p^n-p_h^n\|_{L^2(\Omega)} &\leqslant CC_{h,\Delta t}(u)\exp(CC_u\,T)\left(h^{k-1}+\frac{\Delta t}{h}\right)A(u)\\ &\quad + C\left\{1+C_{h,\Delta t}(u)\right\}(h^k+\Delta t)\,D_n(u)\,,\\ &\quad + Ch^{k+1}\|p(t_n,\cdot)\|_{H^{k+1}(\Omega)}.\end{aligned}$$

Proof. We proceed as in the proof of Theorem 2.2. Indeed, from error equation (3.5), we deduce that

$$\begin{aligned}(\mathbf{e}_p^{n+1},\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} &= \Delta t^{-1}((\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1})-(\mathbf{u}^n-\mathbf{u}_h^n),\mathbf{v}_h)_{\mathscr{T}_h}+(\mathbf{u}^n\cdot\nabla_h\mathbf{u}^{n+1}-\mathbf{u}_h^n\cdot\nabla_h\mathbf{u}_h^{n+1},\mathbf{v}_h)_{\mathscr{T}_h}\\ &\quad -((\mathbf{u}^n-\mathbf{u}^{n+1})\cdot\nabla\mathbf{u}^{n+1},\mathbf{v}_h)_{\mathscr{T}_h}-\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^n\cdot\mathbf{n})[\![\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F\\ &\quad +(\mathscr{P}_h^k(p^{n+1})-p^{n+1},\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h}+(\mathbf{E}_0(t_{n+1}),\mathbf{v}_h)_{\mathscr{T}_h}\\ &\leqslant \left\|\frac{\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}}{\Delta t}-\frac{\mathbf{u}^n-\mathbf{u}_h^n}{\Delta t}\right\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)}+\|\mathbf{u}^n\cdot\nabla_h\mathbf{u}^{n+1}-\mathbf{u}_h^n\cdot\nabla_h\mathbf{u}_h^{n+1}\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)}\\ &\quad +C\|\mathbf{u}_h^n\|_{L^\infty(\Omega)}\left(\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\|[\![\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}]\!]\|_{L^2(F)}^2\right)^{1/2}\left(\sum_{F\in\mathscr{E}_h^i}h_F\|\{\!\{\mathbf{v}_h\}\!\}\|_{L^2(F)}^2\right)^{1/2}\\ &\quad +C_u\|\mathbf{u}^{n+1}-\mathbf{u}^n\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)}+\|\mathscr{P}_h^k(p^{n+1})-p^{n+1}\|_{L^2(\Omega)}\|\mathrm{div}(\mathbf{v}_h)\|_{L^2(\Omega)}\\ &\quad +\|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)}\,.\end{aligned}$$

Thus, using (3.12), (3.13) and (2.16), we obtain that

$$
\begin{aligned}
(\mathbf{e}_p^{n+1},\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} \quad \leqslant \quad & C\Bigg\{ \left\| \frac{\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}}{\Delta t} - \frac{\mathbf{u}^{n}-\mathbf{u}_h^{n}}{\Delta t}\right\|_{L^2(\Omega)} + C_u\|\mathbf{u}^n-\mathbf{u}_h^n\|_{L^2(\Omega)} \\
& + Ch^{-1}(h^{-d/2}\|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)}+C_u)\|\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}\|_{L^2(\Omega)} \\
& + C(h^{-d/2}\|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Omega)}+C_u)\Big\{h^{-1}\|\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\|_{L^2(\Omega)} \\
& + \|\nabla_h\{\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\}\|_{L^2(\Omega)}\Big\} \\
& + C_u\|\nabla_h\{\Pi_h^k(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\}\|_{L^2(\Omega)} + C_u\|\mathbf{u}^{n+1}-\mathbf{u}^{n}\|_{L^2(\Omega)} \\
& + \|\mathscr{P}_h^k(p^{n+1})-p^{n+1}\|_{L^2(\Omega)} + \|\mathbf{E}_0(t_{n+1})\|_{L^2(\Omega)}\Bigg\}\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega)},
\end{aligned}
$$

which, together with the inf-sup condition (2.22), Lemma 3.1, Theorem 3.1, (2.10), and the last estimates obtained in the proof of Lemma 3.1, we can complete the proof. □

We end this section by remarking that we can extend the previous analysis for the upwind version of the method (cf. (2.24)) given by: Find $(\mathbf{u}_h^{n+1},p_h^{n+1})\in\mathbf{V}_h\times Q_h$ such that

$$
\begin{aligned}
(\mathbf{u}_h^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} + \Delta t(\mathbf{u}_h^n\cdot\nabla_h\mathbf{u}_h^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} - \Delta t\sum_{F\in\mathscr{E}_h^i}\langle(\mathbf{u}_h^n\cdot\mathbf{n})[\![\mathbf{u}_h^{n+1}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F & \\
+\Delta t\sum_{F\in\mathscr{E}_h^i}\langle|\mathbf{u}_h^n\cdot\mathbf{n}|[\![\mathbf{u}_h^{n+1}]\!],[\![\mathbf{v}_h]\!]\rangle_F - \Delta t(p_h^{n+1},\mathrm{div}(\mathbf{v}_h))_{\mathscr{T}_h} &= (\mathbf{u}_h^n,\mathbf{v}_h)_{\mathscr{T}_h}, \\
(q_h,\mathrm{div}(\mathbf{u}_h^{n+1}))_{\mathscr{T}_h} &= 0,
\end{aligned}
\tag{3.15}
$$

for all $(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times Q_h$.

3.2 *DG schemes*

Here we only mention that when we combine the techniques used in sections 2.2 and 3.1 we can also obtain the same error estimates for DG schemes (2.29) and (2.40). The fully-discrete versions of both methods, using (3.1), are given by: Find $\mathbf{u}_h\in\mathbf{V}_h^{\mathrm{dg}}$ and $p_h\in Q_h$ such that

$$
\begin{aligned}
(\mathbf{u}_h^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} + \Delta t((\mathbf{u}_h^\star)^n\cdot\nabla_h\mathbf{u}_h^{n+1},\mathbf{v}_h)_{\mathscr{T}_h} + \alpha\Delta t\sum_{F\in\mathscr{E}_h^i}h_F^{-1}\langle[\![\mathbf{u}_h^{n+1}\cdot\mathbf{n}]\!],[\![\mathbf{v}_h\cdot\mathbf{n}]\!]\rangle_F & \\
-\Delta t\sum_{F\in\mathscr{E}_h^i}\langle((\mathbf{u}_h^\star)^n\cdot\mathbf{n})[\![\mathbf{u}_h^{n+1}]\!],\{\!\{\mathbf{v}_h\}\!\}\rangle_F - \Delta t(p_h^{n+1},\mathrm{div}_h(\mathbf{v}_h))_{\mathscr{T}_h} & \\
+\Delta t\sum_{F\in\mathscr{E}_h^i}\langle[\![\mathbf{v}_h\cdot\mathbf{n}]\!],\{\!\{p_h^{n+1}\}\!\}\rangle_F &= (\mathbf{u}_h^n,\mathbf{v}_h)_{\mathscr{T}_h}, \\
(q_h,\mathrm{div}_h(\mathbf{u}_h^{n+1}))_{\mathscr{T}_h} - \sum_{F\in\mathscr{E}_h^i}\langle[\![\mathbf{u}_h^{n+1}\cdot\mathbf{n}]\!],\{\!\{q_h\}\!\}\rangle_F &= 0, \\
\mathbf{u}_h(0,\mathbf{x}) = \mathbf{u}_{h,0}(\mathbf{x}) \quad \text{in } \Omega, &
\end{aligned}
\tag{3.16}
$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h^{\mathrm{dg}} \times Q_h$ for the central flux, and: Find $\mathbf{u}_h \in \mathbf{V}_h^{\mathrm{dg}}$ and $p_h \in Q_h$ such that

$$\begin{aligned}
(\mathbf{u}_h^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} + \Delta t((\mathbf{u}_h^\star)^n \cdot \nabla_h \mathbf{u}_h^{n+1}, \mathbf{v}_h)_{\mathscr{T}_h} + \alpha \Delta t \sum_{F \in \mathscr{E}_h^i} h_F^{-1} \langle [\![\mathbf{u}_h^{n+1} \cdot \mathbf{n}]\!], [\![\mathbf{v}_h \cdot \mathbf{n}]\!] \rangle_F & \\
- \Delta t \sum_{F \in \mathscr{E}_h^i} \langle ((\mathbf{u}_h^\star)^n \cdot \mathbf{n}) [\![\mathbf{u}_h^{n+1}]\!], \{\!\{\mathbf{v}_h\}\!\} \rangle_F + \Delta t \sum_{F \in \mathscr{E}_h^i} \langle |(\mathbf{u}_h^\star)^n \cdot \mathbf{n}| [\![\mathbf{u}_h^{n+1}]\!], [\![\mathbf{v}_h]\!] \rangle_F & \\
- \Delta t (p_h^{n+1}, \mathrm{div}_h(\mathbf{v}_h))_{\mathscr{T}_h} + \Delta t \sum_{F \in \mathscr{E}_h^i} \langle [\![\mathbf{v}_h \cdot \mathbf{n}]\!], \{\!\{p_h^{n+1}\}\!\} \rangle_F &= (\mathbf{u}_h^n, \mathbf{v}_h)_{\mathscr{T}_h}, \\
(q_h, \mathrm{div}_h(\mathbf{u}_h^{n+1}))_{\mathscr{T}_h} - \sum_{F \in \mathscr{E}_h^i} \langle [\![\mathbf{u}_h^{n+1} \cdot \mathbf{n}]\!], \{\!\{q_h\}\!\} \rangle_F &= 0, \\
\mathbf{u}_h(0, \mathbf{x}) \;=\; \mathbf{u}_{h,0}(\mathbf{x}) \quad \text{in } \Omega, &
\end{aligned} \tag{3.17}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h^{\mathrm{dg}} \times Q_h$ for the upwind flux.

THEOREM 3.3 Assume that $\mathbf{u} \in W^{1,\infty}([0,T] \times \Omega)^d$ is uniformly bounded. Also, given an integer $k \geqslant 1$, suppose that $\mathbf{u}_0 \in \mathbf{H}^{k+2}(\Omega)$, $\mathbf{u}_t \in L^2(0,T;\mathbf{H}^{k+2}(\Omega))$, $\mathbf{u}_{tt} \in L^2(0,T;\mathbf{L}^2(\Omega))$, and $p \in L^\infty(0,T;H^{k+1}(\Omega))$. Then, there exists $C > 0$, independent of h, such that

$$\|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)} \;\leqslant\; C \exp(C C_u T) (h^{k+1} + \Delta t) A(u,p), \qquad \text{for all } \; 0 \leqslant n \leqslant N,$$

where $C_u := \|\mathbf{u}\|_{W^{1,\infty}([0,T]\times\Omega)}$ and

$$\begin{aligned}
A(u,p) \quad := \quad & (h\sqrt{T} + C_u T^{3/2}) \|\mathbf{u}_t\|_{L^2(0,T;H^{k+2}(\Omega))} + C_u \sqrt{T} \|\mathbf{u}_t\|_{L^2(0,T;L^2(\Omega))} + \sqrt{T} \|\mathbf{u}_{tt}\|_{L^2(0,T;L^2(\Omega))} \\
& + (C_u T + h) \|\mathbf{u}_0\|_{H^{k+2}(\Omega)} + \sqrt{T} \|p\|_{L^\infty(0,T;H^{k+1}(\Omega))} .
\end{aligned}$$

Proof. It follows straightforwardly from the proof of Theorems 2.3 and 3.1. □

THEOREM 3.4 Assume the hypothesis of Theorem 3.3. In addition, assume that the parameter α lies in $(0, \alpha_0 \Delta t)$, for some $\alpha_0 > 0$ independent of h. Then, there exists $C > 0$, independent of h, such that for all $0 \leqslant n \leqslant N$ the following estimate holds

$$\begin{aligned}
\|p^n - p_h^n\|_{L^2(\Omega)} \quad \leqslant \quad & C C_{h,\Delta t}(u,p) \exp(C C_u T) \left(h^k + \frac{\Delta t}{h} \right) A(u,p) \\
& + C \left\{ 1 + C_{h,\Delta t}(u,p) \right\} (h^k + \Delta t) D_n(u,p) , \\
& + C h^{k+1} \|p(t_n, \cdot)\|_{H^{k+1}(\Omega)}
\end{aligned}$$

where

$$C_{h,\Delta t}(u,p) \;:=\; \exp(C C_u T) h^{-d/2} (h^{k+1} + \Delta t) A(u,p) \;+\; C_u$$

and

$$\begin{aligned}
D_n(u,p) \quad := \quad & h^2 \|\mathbf{u}_t\|_{L^\infty(t_n,t_{n+1};H^{k+2}(\Omega))} + \|\mathbf{u}_t\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))} + \|\mathbf{u}_{tt}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))} \\
& + \|\mathbf{u}\|_{L^\infty(t_n,t_{n+1};H^{k+2}(\Omega))} + \|p\|_{L^\infty(t_n,t_{n+1};H^{k+1}(\Omega))} .
\end{aligned}$$

Proof. Similar as the proof of Theorem 3.2. □

4. Numerical results

In this section, we present some numerical results for two dimensional problem (i.e. $d = 2$), illustrating the performance of the fully discrete schemes analyzed in Sections 3.1 and 3.2. In all the computations we consider four uniform meshes that are Cartesian refinements of a domain defined in terms of squares, and then we split each square into two congruent triangles. Also, we consider polynomial degree $k \in \{0,1,2\}$ and for the DG schemes, we use only $\alpha = 1$. In addition, the numerical results presented below were obtained using a MATLAB code, where the zero integral mean condition for the pressure is imposed via a real Lagrange multiplier.

In Example 1 we follow Liu & Shu (2000) and consider the Taylor-Green vortex (see Chorin (1968)). That is, we set $\Omega := [0,2\pi]^2$, and the exact solution is given by

$$
\begin{aligned}
\mathbf{u}(t,\mathbf{x}) &= \Big(\sin(x_1)\cos(x_2)e^{-2t/\lambda},\, -\cos(x_1)\sin(x_2)e^{-2t/\lambda}\Big)^{\mathtt{t}},\\
p(t,\mathbf{x}) &= \frac{1}{4}\big(\cos(2x_1)+\cos(2x_2)\big)e^{-4t/\lambda},
\end{aligned}
$$

for all $\mathbf{x} := (x_1,x_2)^{\mathtt{t}} \in \Omega$ and $t \in (0,1)$, where $\lambda = 100$. It is easy to check that $\mathbf{u}$ is divergence-free and $\int_\Omega p = 0$. Here, we compute the approximation of $\mathbf{u}$ at $t = 1$, where we consider $\Delta t = 1/160 = 0.00625$. In Table 1 we present the results obtained for Raviart-Thomas schemes (3.4) and (3.15), whereas in Table 2 we use DG schemes (3.16) and (3.17). For the theory presented in previous sections we assume that $k \geqslant 1$, however here we also use $k = 0$ in order to appreciate the behavior of the proposed schemes in practice.

We see that the estimates we obtained using the Raviart-Thomas spaces and the central flux are sharp for the velocity when $k = 1$. However, for $k = 0$, $k = 2$ the convergence rates are higher than predicted theoretically. In particular, we could not prove convergence for $k = 0$, however numerically the method seems to be converging with order 1. Similarly, for DG method using the central flux the estimate we gave seem to be sharp for the velocity for $k = 0$ and $k = 2$ (notice that the velocity space contains polynomials of degree $k+1$ for the DG space), but numerically the case $k = 1$ does better than the theory predicts. Finally, using the upwind flux for both the Raviart-Thomas method or the DG method one observes, as expected, numerically optimal convergence rates for both the velocity and pressure variables. Unfortunately, we cannot prove these optimal error estimates.

For Example 2 we consider the double shear layer problem taken from Bell *et al.* (1989) (see also Liu & Shu (2000)). We solve the Euler equation (1.1) in the domain $\Omega := [0,2\pi]^2$ with a periodic boundary condition and an initial data given by $\mathbf{u}_0(\mathbf{x}) = (u_1^0(\mathbf{x}), u_2^0(\mathbf{x}))^{\mathtt{t}}$, with

$$
u_1^0(\mathbf{x}) = \begin{cases} \tanh((x_2-\pi/2)/\rho) & x_2 \leqslant \pi \\ \tanh((3\pi/2-x_2)/\rho) & x_2 > \pi \end{cases}, \quad \text{and} \quad u_2^0(\mathbf{x}) = \delta\sin(x_1),
$$

for all $\mathbf{x} := (x_1,x_2)^{\mathtt{t}} \in \Omega$, where we take $\rho = \pi/15$ and $\delta = 0.05$.

In Figures 1−6, we present some contours of the vorticity $\omega_h := \mathrm{curl}(\mathbf{u}_h) = \partial_{x_1}u_2 - \partial_{x_2}u_1$ at $t = 6$ and $t = 8$ to show the resolution. We use 99 contours between -4.9 and 4.9, using the previous four meshes, where $h \in \{0.7405, 0.3702, 0.2468, 0.1851\}$. For this Figures, we use the DG scheme with the central flux (cf. (3.16)). Analogously, in Figures 7−12 we use the DG scheme with the upwind flux (cf. (3.17)). In all this figures, we take $\Delta t = 8/200 = 0.04$. In order to save space, we not report here the results using the divergence conforming methods. However, we remark that the Raviart-Thomas schemes (3.4) and (3.15) behave very similarly as the DG methods for this example.

TABLE 1. *History of convergence for Example 1, Raviart-Thomas scheme with* $t = 1$.

k	h	d.o.f	Central flux				Upwind flux			
			$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$		$\|p-p_h\|_{L^2(\Omega)}$		$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$		$\|p-p_h\|_{L^2(\Omega)}$	
			error	order	error	order	error	order	error	order
0	0.7405	745	1.14e-0	−−	4.69e-1	−−	1.50e-0	−−	8.69e-1	−−
	0.3702	2929	5.70e-1	1.00	2.08e-1	1.17	8.82e-1	0.76	5.17e-1	0.75
	0.2468	6553	3.80e-1	1.00	1.35e-1	1.07	6.29e-1	0.83	3.68e-1	0.84
	0.1851	11617	2.85e-1	1.00	1.00e-1	1.04	4.90e-1	0.87	2.86e-1	0.88
1	0.7405	2353	5.84e-1	−−	1.48e-1	−−	1.75e-1	−−	9.86e-2	−−
	0.3702	9313	2.97e-1	0.98	6.65e-2	1.15	4.40e-2	2.00	2.68e-2	1.88
	0.2468	20881	1.98e-1	1.00	4.32e-2	1.07	1.94e-2	2.01	1.22e-2	1.94
	0.1851	37057	1.49e-1	1.00	3.22e-2	1.02	1.09e-2	2.01	6.91e-3	1.96
2	0.7405	4825	2.15e-2	−−	6.53e-3	−−	1.28e-2	−−	5.19e-3	−−
	0.3702	19153	3.31e-3	2.70	9.53e-4	2.78	1.53e-3	3.06	6.80e-4	2.93
	0.2468	42985	8.61e-4	3.32	3.09e-4	2.78	4.36e-4	3.09	2.13e-4	2.87
	0.1851	76321	3.52e-4	3.11	1.39e-4	2.78	1.79e-4	3.08	9.49e-5	2.81

TABLE 2. *History of convergence for Example 1, DG scheme with* $t = 1$.

k	h	d.o.f	Central flux				Upwind flux			
			$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$		$\|p-p_h\|_{L^2(\Omega)}$		$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$		$\|p-p_h\|_{L^2(\Omega)}$	
			error	order	error	order	error	order	error	order
0	0.7405	2017	5.13e-1	−−	3.83e-1	−−	2.85e-1	−−	3.81e-1	−−
	0.3702	8065	2.39e-1	1.10	1.89e-1	1.02	8.17e-2	1.80	1.88e-1	1.02
	0.2468	18145	1.57e-1	1.03	1.25e-1	1.01	3.85e-2	1.85	1.25e-1	1.01
	0.1851	32257	1.18e-1	1.01	9.39e-2	1.01	2.24e-2	1.89	9.34e-2	1.01
1	0.7405	4321	4.48e-2	−−	4.83e-2	−−	3.41e-2	−−	4.80e-2	−−
	0.3702	17281	6.48e-3	2.79	1.20e-2	2.01	4.60e-3	2.89	1.20e-2	2.00
	0.2468	38881	2.01e-3	2.89	5.32e-3	2.00	1.37e-3	2.99	5.32e-3	2.00
	0.1851	69121	8.72e-4	2.90	3.00e-3	2.00	5.75e-4	3.01	2.99e-3	2.00
2	0.7405	7489	4.83e-3	−−	4.14e-3	−−	2.23e-3	−−	4.12e-3	−−
	0.3702	29953	5.89e-4	3.03	5.52e-4	2.91	1.49e-4	3.90	5.50e-4	2.91
	0.2468	67393	1.74e-4	3.00	1.76e-4	2.82	3.10e-5	3.88	1.71e-4	2.89
	0.1851	119809	7.36e-5	3.00	8.31e-5	2.61	1.03e-5	3.83	7.82e-5	2.71

We see that the method using the upwind flux seems to do much better than the method using the central flux. In particular, when using $k = 2$ and using the upwind flux the method seems to do quite well. In fact, the method seems to be comparable to DG methods using the vorticity-potential formulation and high-order time integrators developed by Liu and Shu in Liu & Shu (2000).

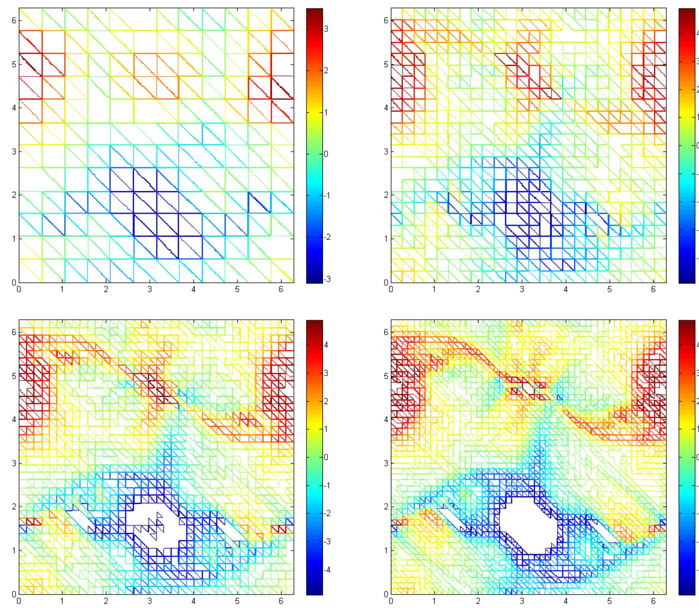

FIG. 1. Example 2 (DG + central flux), contours for the vorticity with $k = 0$ and $t = 6$.

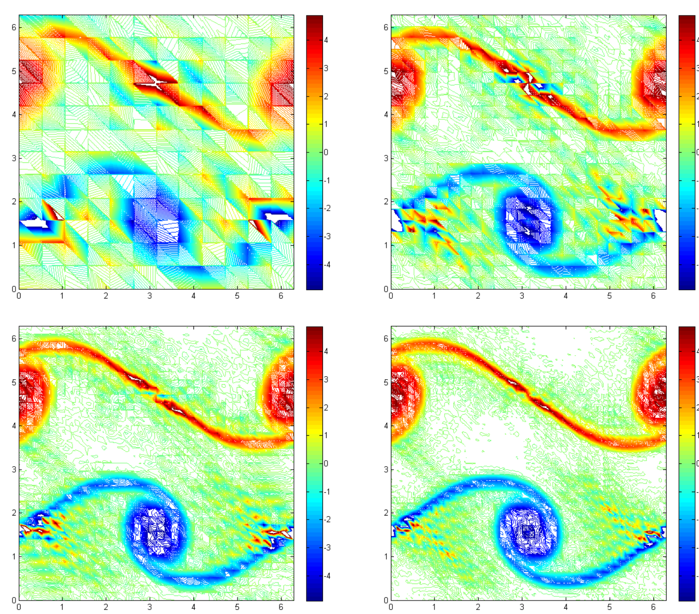

FIG. 2. Example 2 (DG + central flux), contours for the vorticity with $k = 1$ and $t = 6$.

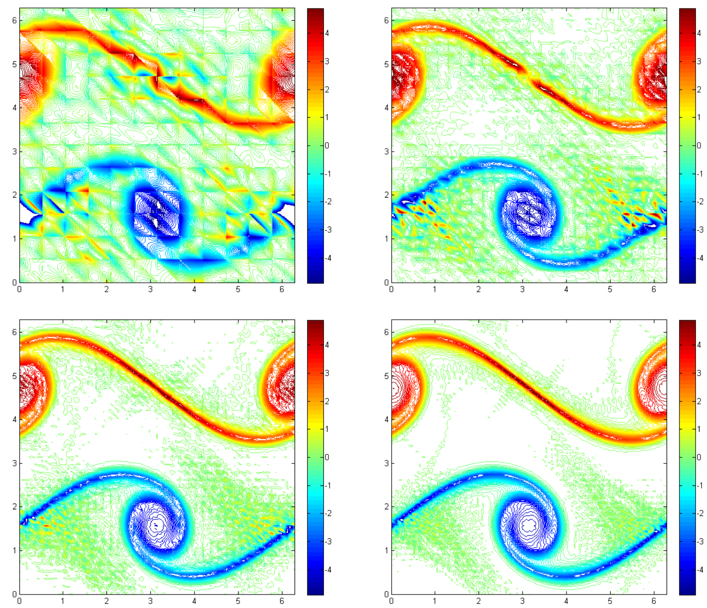

FIG. 3. Example 2 (DG + central flux), contours for the vorticity with $k = 2$ and $t = 6$.

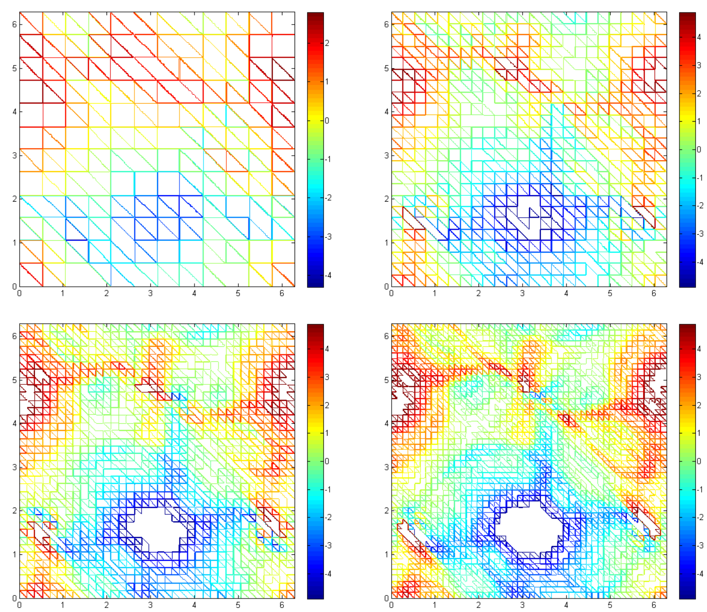

FIG. 4. Example 2 (DG + central flux), contours for the vorticity with $k = 0$ and $t = 8$.

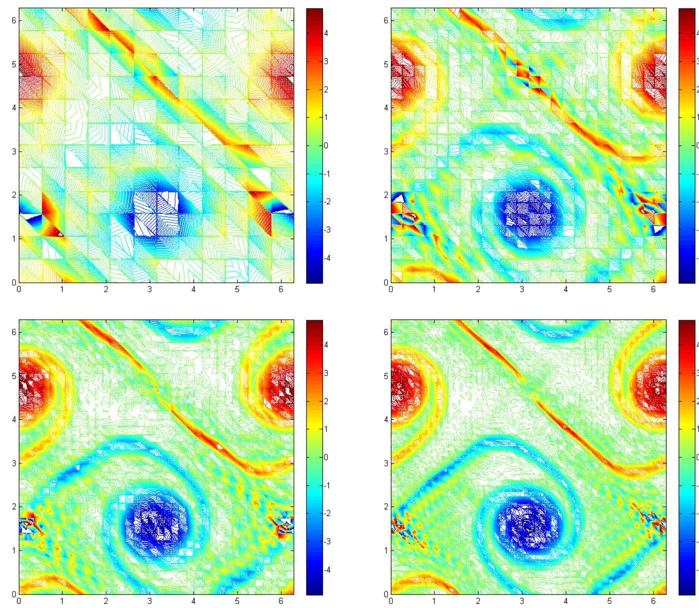

FIG. 5. Example 2 (DG + central flux), contours for the vorticity with $k = 1$ and $t = 8$.

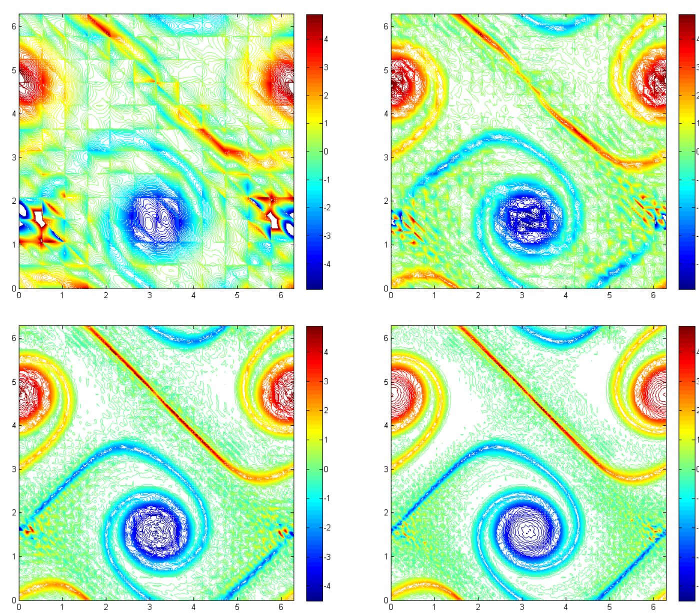

FIG. 6. Example 2 (DG + central flux), contours for the vorticity with $k = 2$ and $t = 8$.

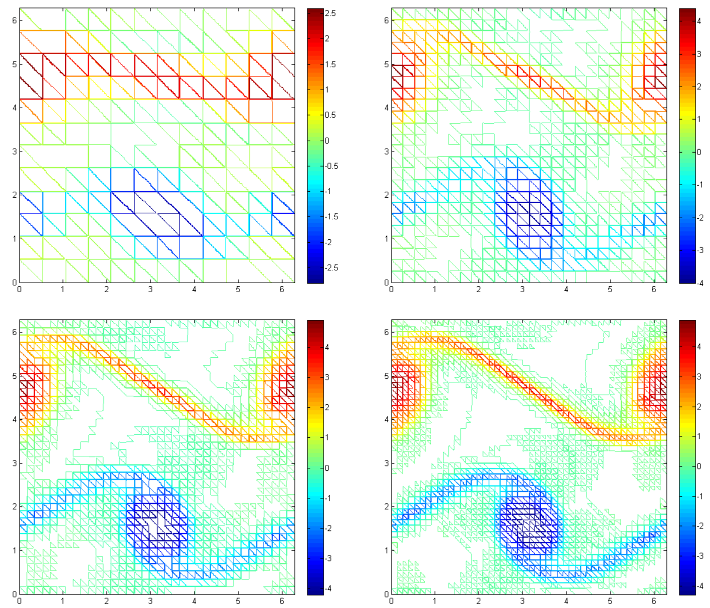

FIG. 7. Example 2 (DG + upwind flux), contours for the vorticity with $k = 0$ and $t = 6$.

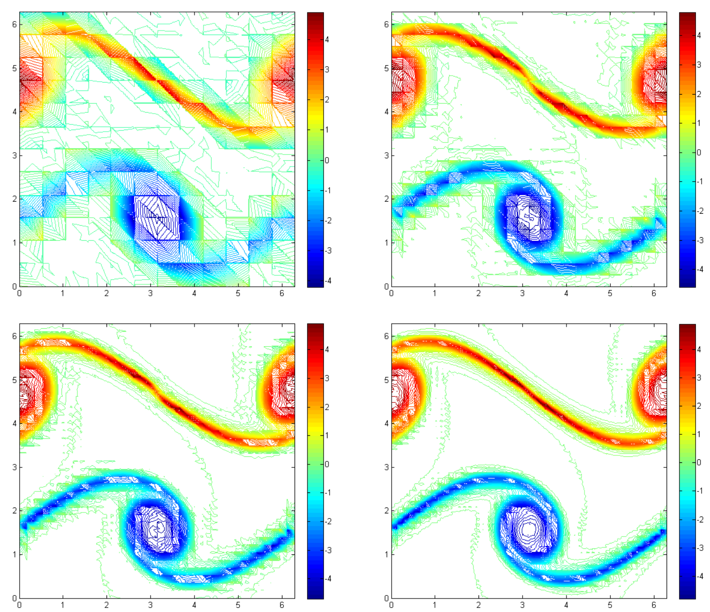

FIG. 8. Example 2 (DG + upwind flux), contours for the vorticity with $k = 1$ and $t = 6$.

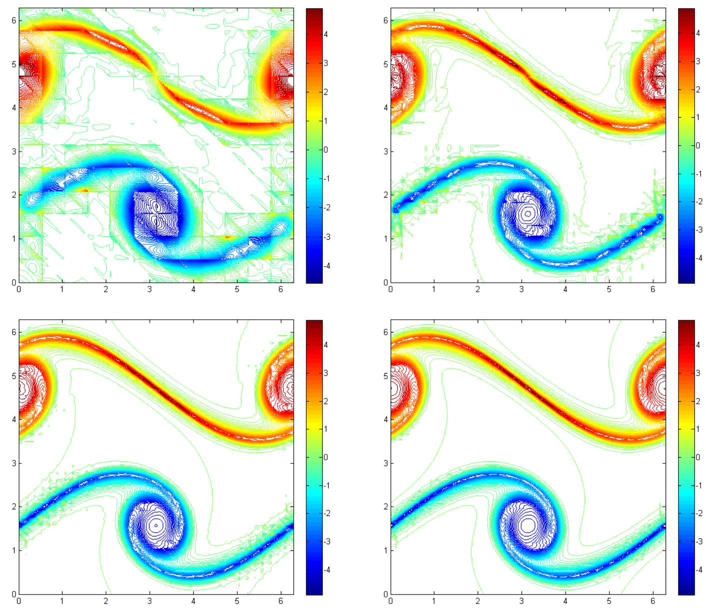

FIG. 9. Example 2 (DG + upwind flux), contours for the vorticity with $k = 2$ and $t = 6$.

5. Conclusions and future directions

In this paper we have developed finite element methods for incompressible Euler equations. We prove error estimates, however, numerical experiments suggest that our analysis is not sharp, at least for the upwind methods. It would be interesting to see if a new analysis can prove the optimal estimates for the upwind schemes. Our fully discrete methods are implicit. In the future we would like to consider numerical methods that treat the nonlinear part explicitly in order to make the method more efficient. In addition, in this work we used only zero boundary conditions in order to perform the analysis in a cleaner way. However, the results presented here can be extended to the case of nonzero velocity on the boundary.

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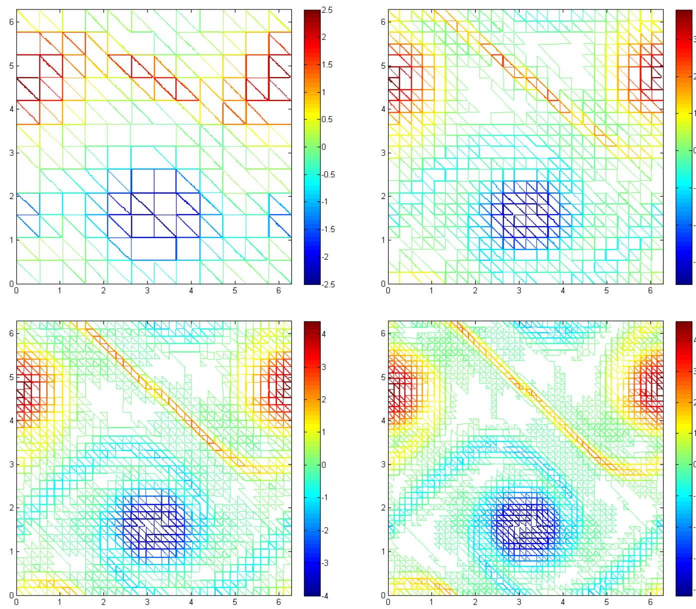

FIG. 10. Example 2 (DG + upwind flux), contours for the vorticity with $k = 0$ and $t = 8$.

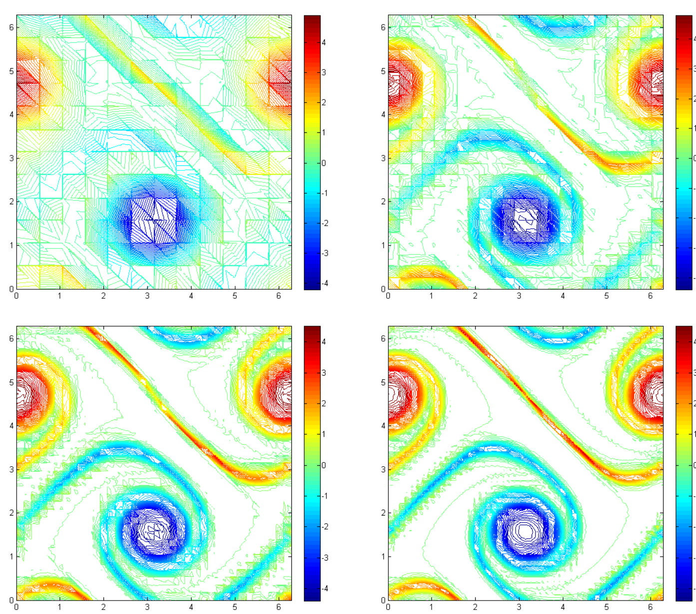

FIG. 11. Example 2 (DG + upwind flux), contours for the vorticity with $k = 1$ and $t = 8$.

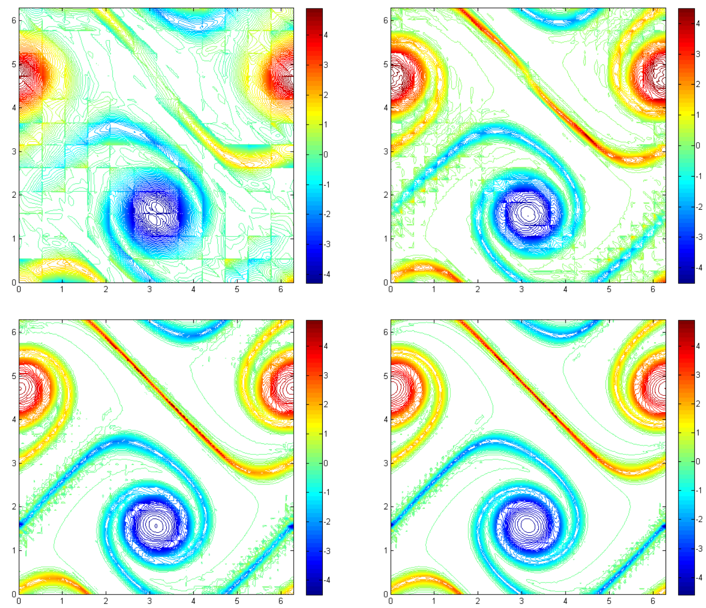

FIG. 12. Example 2 (DG + upwind flux), contours for the vorticity with $k = 2$ and $t = 8$.

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