Error estimates to smooth solutions of semi-discrete discontinuous Galerkin methods with quadrature rules for scalar conservation laws

Juntao Huang[1] and Chi-Wang Shu[2]

Abstract

In this paper, we focus on error estimates to smooth solutions of semi-discrete discontinuous Galerkin (DG) methods with quadrature rules for scalar conservation laws. The main techniques we use are energy estimate and Taylor expansion first introduced by Zhang and Shu in [25]. We show that, with P^k (piecewise polynomials of degree k) finite elements in 1D problems, if the quadrature over elements is exact for polynomials of degree $(2k)$, error estimates of $O(h^{k+1/2})$ are obtained for general monotone fluxes, and optimal estimates of $O(h^{k+1})$ are obtained for upwind fluxes. For multidimensional problems, if in addition quadrature over edges is exact for polynomials of degree $(2k+1)$, error estimates of $O(h^k)$ are obtained for general monotone fluxes, and $O(h^{k+1/2})$ are obtained for monotone and sufficiently smooth numerical fluxes. Numerical results validate our analysis.

Keywords: Discontinuous Galerkin; error estimate; quadrature rules; conservation laws; semi-discrete

[1]Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University, Beijing 100084, China. E-mail: huangjt13@mails.tsinghua.edu.cn

[2]Division of Applied Mathematics, Brown University, Providence, RI 02912, USA. E-mail: shu@dam.brown.edu. Research supported by DOE grant DE-FG02-08ER25863 and NSF grant DMS-1418750.

1 Introduction

In this paper, we present error estimates for the semi-discrete discontinuous Galerkin (DG) methods with quadrature rules for scalar conservation laws in multidimensional case:

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T], \tag{1.1a}$$

$$u(t = 0) = u_0, \quad \mathbf{x} \in \Omega; \tag{1.1b}$$

here $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ and $\mathbf{f}(u) = (f_1(u), f_2(u), \cdots, f_d(u))$. We do not pay attention to boundary conditions in this paper: thus the solution is considered to be either periodic or compactly supported. The analysis in this paper is for *smooth* solutions of (1.1). Discontinuous solutions with shocks are not considered.

The first discontinuous Galerkin method was proposed by Reed and Hill [17] in the framework of neutron transport. It was then developed for conservation laws by Cockburn et al. in a series of papers [6, 5, 4, 3, 8], which use DG discretization in space and explicit total variation diminishing (TVD) Runge-Kutta (RK) discretization in time [19, 20, 21]. For the detailed description of this method, we refer readers to the lecture notes [2] and the review paper [9].

We now make a brief review on related error estimate results of the DG methods in the literature. For smooth solutions to linear conservation laws, error estimates have been given in [15, 13, 18, 16] for full DG discretization of steady problems or by using space-time finite element spaces for time dependent problems, and for the semi-discrete version of the DG method for time dependent problems in [7]. The first result for error estimates to smooth solutions of fully discrete RKDG methods for nonlinear scalar conservation laws was obtained by Zhang and Shu in [25] where the second-order Runge-Kutta time discretization is used. The main techniques are Taylor expansion and energy estimate. With careful treatment to boundary terms, a priori error estimates of $O(h^{k+1/2} + \tau^2)$ are obtained for general monotone fluxes, and optimal error estimates of $O(h^{k+1} + \tau^2)$ are obtained for upwind fluxes for the P^k finite element spaces, under standard CFL condition for $k = 1$ and more restrictive

CFL conditions for $k > 1$. Later, the same authors presented a priori error estimates for symmetrizable systems and the third-order Runge-Kutta time discretization under standard CFL conditions in [26] and [27]. For more details, please refer to [27].

In practical computations, one has to replace the integrals over elements and (in multi-dimensional cases) edges by quadrature rules in the DG scheme. However, there exists very limited work discussing the effects of quadrature rules. In [3], the truncation error of numerical integrations is analyzed for the semi-discrete DG method for nonlinear conservation laws with sufficiently smooth u and $\mathbf{f}$, under the assumption that the quadrature rules over the elements and edges are exact for polynomials of degree $(2k)$ and $(2k+1)$, respectively. Recently, in [23, 22], the authors discussed the effects of quadratures in DG method for non-linear convection-diffusion equations in 2D and 3D cases. We remark that the estimates in [23, 22] could not be applied to the pure convection case (i.e., nonlinear conservation laws), since the constant in their estimates would blow up as the diffusion coefficient tends to zero.

In this paper, we perform a priori error estimate to smooth solutions of semi-discrete DG methods with the P^k finite element space of piecewise kth degree polynomials and quadrature rules for scalar conservation laws (1.1). The main techniques we use in this paper are Taylor expansion and energy estimate first introduced by Zhang and Shu in [25]. We first establish an energy equality as is customary in error analysis in finite element methods, and then split the equality into several parts. The first part only involves error with exact integrals and has been treated in [25]. The second term is somewhat like "truncation error" and has been estimated in Lemma 2.2 in [3]. Our main contribution is the careful treatment to terms on quadrature errors in elements and edges. For the terms on quadrature errors in elements, we perform the Taylor expansion and split into several parts and then estimate them one by one using a key lemma which characterizes error of numerical integrations. The same technique could also be applied to the terms on quadrature errors in edges, provided that the numerical flux is sufficiently smooth (e.g. Lax-Friedrichs flux). However, most numerical fluxes are not smooth enough but only locally Lipschitz continuous (e.g. Godunov flux) or only have up to first-order derivative (e.g. Engquist-Osher flux). Thus, the same technique could not be applied here. To fix this problem, we borrow some terms from the first part on the error with exact integrals to eliminate some "trouble" terms here. Finally, under the assumptions that u and $\mathbf{f}$ are sufficiently smooth, and the quadrature over elements is exact for polynomials of degree $(2k)$ and the one over edges is exact for polynomials of degree $(2k+1)$ (in multidimensional case), the L^2-norm error estimate is established.

The paper is organized as follows. In Section 2, we introduce some basic notations, the semi-discrete DG method with quadrature rules, and some useful lemmas. In Section 3, we derive the error equations and some key lemmas for the error estimate, while the detailed proofs of some lemmas are left in the appendix. Numerical results are reported in Section 4. Some concluding remarks are given in Section 5.

2 Preliminaries

2.1 Basic notations

Let us assume that the domain Ω is polygonal, and let $\mathscr{T}_h$ for $h > 0$ be a family of quasi-uniform triangulations of Ω with shape-regular elements K. Let $\mathscr{E}_h$ denote the union of edges (called points in 1D and faces in 3D) of elements $K \in \mathscr{T}_h$, i.e., $\mathscr{E}_h = \cup_{K\in\mathscr{E}_h} \partial K$.

Noticing the periodic or zero (compactly supported) boundary conditions, we ignore the boundaries and assume that every $\Gamma \in \mathscr{E}_h$ is shared by two elements from $\mathscr{T}_h$ for convenience. Following the notations in [24], we could choose a fixed vector β, which is not parallel with any edge $\Gamma \in \mathscr{E}_h$, and then for each Γ, define a fixed unit normal vector $\mathbf{n}_\Gamma$ which satisfies $\mathbf{n}_\Gamma \cdot \beta > 0$, and designate the "plus" side to be the side that $\mathbf{n}_\Gamma$ is the inner normal vector and the "minus" one to be the opposite side. For piecewise smooth function $v \in L^2(\Omega)$, we denote trace of $v|_{K_\Gamma^-}$ and $v|_{K_\Gamma^+}$ on Γ by $v|_\Gamma^-$ and $v|_\Gamma^+$, respectively. We introduce the jump on the edge Γ

$$[v]_\Gamma = v|_\Gamma^+ - v|_\Gamma^-,$$

and the average

$$\bar{v}_\Gamma = \frac{1}{2}(v|_\Gamma^+ + v|_\Gamma^-).$$

In what follows the standard notations in Sobolev spaces $W^{k,p}(\Omega)$ and $H^k(\Omega)$ are used. The notation $\|\cdot\|_X$ and $|\cdot|_X$ denote a norm and a semi-norm in the space X, respectively. For simplicity, we use $\|\cdot\|_p$ denote the norm in $L^p(\Omega)$ for $1 \le p \le \infty$. Specifically for $p = 2$, we let $\|\cdot\|$ and $(\cdot, \cdot)$ denote the norm and inner product in the space $L^2(\Omega)$. We shall often use the following broken Sobolev spaces with respect to the triangulation $\mathscr{T}_h$,

$$W^{k,p}(\mathscr{T}_h) = \{v \in L^1(\Omega) : v|_K \in W^{k,p}(K), \quad \forall K \in \mathscr{T}_h\},$$

equipped with the norm

$$\|v\|_{W^{k,p}(\mathscr{T}_h)} = \left(\sum_{K \in \mathscr{T}_h} \|v\|^p_{W^{k,p}(K)} \right)^{1/p},$$

and the semi-norm

$$|v|_{W^{k,p}(\mathscr{T}_h)} = \left(\sum_{K \in \mathscr{T}_h} |v|^p_{W^{k,p}(K)} \right)^{1/p},$$

for $1 \le p < \infty$, and

$$\|v\|_{W^{k,\infty}(\mathscr{T}_h)} = \max_{K \in \mathscr{T}_h} \|v\|_{W^{k,\infty}(K)},$$

$$|v|_{W^{k,\infty}(\mathscr{T}_h)} = \max_{K \in \mathscr{T}_h} |v|_{W^{k,\infty}(K)}.$$

For the piecewise smooth function $v \in L^2(\Omega)$, we define the L^p-norm on the edge $\Gamma \in \mathscr{E}_h$ for $1 \le p < \infty$:

$$\|v\|_{L^p(\Gamma)} = \left(\|v^-\|^p_{L^p(\Gamma)} + \|v^+\|^p_{L^p(\Gamma)} \right)^{1/p},$$

and

$$\|v\|_{L^p(\mathscr{E}_h)} = \left(\sum_{\Gamma \in \mathscr{E}_h} \|v\|^p_{L^p(\Gamma)} \right)^{1/p}.$$

The finite element space is defined by

$$V_h^k = \{v \in L^2(\Omega) : v|_K \in P^k(K), \quad \forall K \in \mathscr{T}_h\},$$

where $P^k(K)$ denotes the space of polynomials in K of degree at most $k \ge 0$.

For $v \in L^2(\Omega)$, we denote by $\mathbb{P}v \in V_h^k$ the L^2-projection of v on V_h^k which satisfies

$$(\mathbb{P}v - v, \varphi_h) = 0, \quad \forall \varphi_h \in V_h^k. \tag{2.2}$$

2.2 Semi-discrete discontinuous Galerkin methods

In this part, we follow [3] and define the semi-discrete DG methods for (1.1). The approximation solution of (1.1) for $t > 0$ is determined by,

$$\int_K (u_h)_t v_h \mathrm{d}x - \int_K \mathbf{f}(u_h) \cdot \nabla v_h \mathrm{d}x + \int_{\partial K} \widehat{\mathbf{f} \cdot \mathbf{n}_K}(u_h) v_h \mathrm{d}s = 0, \quad \forall v_h \in V_h^k, \quad \forall K \in \mathscr{T}_h, \tag{2.3}$$

where $\mathbf{n}_K$ is the outward unit normal vector to ∂K, the boundary of K. The numerical flux $\widehat{\mathbf{f} \cdot \mathbf{n}_K}(u_h)$ depends on the normal vector $\mathbf{n}_K$ and traces u_h^{int} and u_h^{ext} which are evaluated from the inside and outside (inside the neighboring element) of the element K, respectively, i.e.,

$$\widehat{\mathbf{f} \cdot \mathbf{n}_K}(u_h) \equiv \widehat{\mathbf{f} \cdot \mathbf{n}_K}(u_h^{int}, u_h^{ext}).$$

The function $\widehat{\mathbf{f} \cdot \mathbf{n}}(u, v)$ satisfies the following conditions:

1. It is locally Lipschitz continuous with respect to u and v.

2. It is consistent with the flux $\mathbf{f}(u)$, i.e.

$$\widehat{\mathbf{f} \cdot \mathbf{n}}(v, v) = \mathbf{f}(v) \cdot \mathbf{n}.$$

3. It is nondecreasing with respect to u and nonincreasing with respect to v.

4. It is conservative, i.e.,

$$\widehat{\mathbf{f} \cdot \mathbf{n}}(u, v) + \widehat{\mathbf{f} \cdot (-\mathbf{n})}(v, u) = 0.$$

In practical computation for multidimensional problems, the integrals in (2.3) often have to be approximated by quadrature rules

$$\int_K F \mathrm{d}x \approx \sum_{j=1}^{M} \omega_j F(x_{Kj}) \left| K \right|, \tag{2.4a}$$

$$\int_\Gamma G \mathrm{d}s \approx \sum_{j=1}^{L} \underline{\omega}_j G(x_{\Gamma j}) \left|\Gamma\right|, \tag{2.4b}$$

for $F \in C(K)$ and $G \in C(\Gamma)$. Here ω_j, $\underline{\omega}_j$ are integration weights and $x_{Kj} \in K$, $x_{\Gamma j} \in \Gamma$ are integration points. Finally, the semi-discrete DG scheme with numerical integrations is obtained

$$\begin{aligned}\int_K (u_h)_t v_h \mathrm{d}x - \sum_{j=1}^{M} \omega_j \mathbf{f}(u_h(t, x_{Kj})) \cdot \nabla v_h(x_{Kj}) \left|K\right| \\ + \sum_{\Gamma \in \partial K} \sum_{j=1}^{L} \underline{\omega}_j \widehat{\mathbf{f} \cdot \mathbf{n}_K}(u_h(t, x_{\Gamma j})) v_h(x_{\Gamma j}) \left|\Gamma\right| = 0, \quad \forall v_h \in V_h^k, \quad \forall K \in \mathscr{T}_h.\end{aligned} \tag{2.5}$$

We remark that the quadrature in edges is not needed in the 1D case. As usual, the initial value of u_h is taken as the L^2-projection of u_0, i.e., $u_h = \mathbb{P} u_0$.

2.3 Some auxiliary results

Some useful lemmas are listed in this part. The L^2 norm in the boundary ∂K of a function could be bounded by some norm in K with the following multiplicative trace inequality:

Lemma 2.1 (Multiplicative trace inequality). *There exists a constant $C > 0$, independent of v, h and K, such that for all $K \in \mathscr{T}_h$, $v \in H^1(K)$ and $h \in (0, h_0)$,*

$$\|v\|_{L^2(\partial K)}^2 \le C(\|v\|_{L^2(K)} |v|_{H^1(K)} + h^{-1} \|v\|_{L^2(K)}^2).$$

Cf. Ref. [10] Lemma 3.1 for a detailed proof.

For the L^2-projection $\mathbb{P}$ defined in (2.2), it is easy to show (cf. Theorem 3.1.4 in [1])

Lemma 2.2 (Interpolation inequalities). *Given an integer $0 \le m \le k+1$, there exists a constant $C > 0$, independent of h, such that for any $v \in W^{k+1,\infty}(\Omega)$,*

$$\|v - \mathbb{P}v\| \le C h^{k+1} |v|_{H^{k+1}(\Omega)}, \tag{2.6a}$$

$$|v - \mathbb{P}v|_{W^{m,\infty}(\Omega)} \le C h^{k-m+1} |v|_{W^{k+1,\infty}(\Omega)}. \tag{2.6b}$$

We also present some inverse properties of the finite element space V_h^k that will be used in our analysis. For more details, we refer reader to Theorem 3.2.6 in [1].

Lemma 2.3 (Inverse inequalities)**.** *There exists a constant $C > 0$, independent of h, such that for any $v_h \in V_h^k$,*

$$|v_h|_{H^1(\mathscr{T}_h)} \leq Ch^{-1} \|v_h\|, \tag{2.7a}$$

$$\|v_h\|_{L^2(\mathscr{E}_h)} \leq Ch^{-1/2} \|v_h\|, \tag{2.7b}$$

$$\|v_h\|_\infty \leq Ch^{-d/2} \|v_h\|. \tag{2.7c}$$

Here d is the dimension of the space $\Omega \subset \mathbb{R}^d$.

For convenience, some notations denoting the error of quadrature (2.4) are introduced

$$E_K(F) = \int_K F\mathrm{d}x - \sum_{j=1}^{M} \omega_j F(x_{Kj}) |K|, \tag{2.8a}$$

$$E_\Gamma(G) = \int_\Gamma G\mathrm{d}s - \sum_{j=1}^{L} \underline{\omega}_j G(x_{\Gamma j}) |\Gamma|, \tag{2.8b}$$

for $F \in C(K)$ and $G \in C(\Gamma)$. With the aid of the Bramble-Hilbert lemma (cf. Theorem 4.1.3 in [1]), it is easy to obtain the following lemma which is useful for estimating the error of numerical integration (cf. Lemma 4.7 in [22]).

Lemma 2.4 (Error of numerical integration)**.** *Let $s \geq 1$, $p \geq 0$ be integers, and $1 \leq q \leq \infty$.*

(i) Assume that the quadrature over element (2.4a) *is exact for $P^{s+p-1}(K)$. Then there exists a constant $C > 0$, such that for any $Q \in W^{s,\infty}(K)$ and $v \in P^p(K)$,*

$$|E_K(Qv)| \leq Ch^{s+d(1-\frac{1}{q})} |Q|_{W^{s,\infty}(K)} \|v\|_{L^q(K)}. \tag{2.9}$$

Here h denotes the diameter of K.

(ii) Assume that the quadrature over edge (2.4b) *is exact for $P^{s+p-1}(\Gamma)$. Then there exists a constant $C > 0$, such that for any $G \in W^{s,\infty}(\Gamma)$ and $w \in P^p(\Gamma)$,*

$$|E_\Gamma(Gw)| \leq Ch^{s+(d-1)(1-\frac{1}{q})} |G|_{W^{s,\infty}(\Gamma)} \|w\|_{L^q(\Gamma)}. \tag{2.10}$$

Here h denotes the diameter of Γ.

3 Error estimates of the semi-discrete DG methods with quadrature rules

In this section, we present the main theorem and give the detailed proof.

Theorem 3.1 (the main results). *Let u be the exact solution of problem* (1.1) *and u_h be the numerical solution of the semi-discrete DG scheme* (2.5) *with the piecewise polynomial finite element space of degree $k \geq 1$ and quadrature rules. Assume that u and the physical flux $\mathbf{f}$ are both sufficiently smooth that $u \in C^{k+2}(\Omega)$ and $\mathbf{f} \in C^{k+3}(\mathbb{R})$. Denote the corresponding numerical error by $e(x,t) = u(x,t) - u_h(x,t)$. For regular triangulations of polygonal domain $\Omega \subset \mathbb{R}^d$, if the quadrature over the elements is exact for polynomials of degree* $(2k)$*, and that over the edges is exact for* $(2k+1)$ *(in multidimensional case), and all the quadrature weights are non-negative, then for small enough h, there holds the following error estimates:*

(1) For the 1D problem,

a) For general monotone numerical flux and $k \geq 1$,

$$\max_{0 \leq t \leq T} \|e(t, \cdot)\|_{L^2(\Omega)} \leq C h^{k+\frac{1}{2}}. \tag{3.11}$$

b) For upwind numerical flux and $k \geq 1$,

$$\max_{0 \leq t \leq T} \|e(t, \cdot)\|_{L^2(\Omega)} \leq C h^{k+1}. \tag{3.12}$$

(2) For 2D and 3D problems,

a) For general monotone numerical flux and $k \geq 3$,

$$\max_{0 \leq t \leq T} \|e(t, \cdot)\|_{L^2(\Omega)} \leq C h^{k}. \tag{3.13}$$

b) For monotone and sufficiently smooth numerical flux and $k \geq 2$,

$$\max_{0 \leq t \leq T} \|e(t, \cdot)\|_{L^2(\Omega)} \leq C h^{k+\frac{1}{2}}. \tag{3.14}$$

Here the positive constant C is independent of h and the approximation solution u_h.

Remark 3.1. *For 2D and 3D problems, the estimate of $O(k+\frac{1}{2})$ order holds for "sufficiently smooth" numerical flux, in the sense that $\widehat{\mathbf{f}\cdot\mathbf{n}}'(u,u) \in C^{k+2}(\mathscr{E}_h)$, $\widehat{\mathbf{f}\cdot\mathbf{n}}''(u,u) \in C^2(\mathscr{E}_h)$ and $\widehat{\mathbf{f}\cdot\mathbf{n}}''' \in C(\mathbb{R}^2)$. Note that $\widehat{\mathbf{f}\cdot\mathbf{n}} = \widehat{\mathbf{f}\cdot\mathbf{n}}(v,w)$ is a function of two variables and thus the derivatives should be interpreted as the usual multiindex notation (see Lemma 3.4 for details). Under our assumption on the smoothness of u and $\mathbf{f}$ in Theorem 3.1, the local Lax-Friedrichs flux with the constant α uniform in each edge and the global Lax-Friedrichs flux meet this condition.*

Remark 3.2. *Our proof does not work for the finite element spaces of low order degrees for multidimensional problems (we need $k \geq 3$ for general monotone flux with $d = 2,3$ and $k \geq 2$ for monotone and sufficiently smooth numerical flux with $d = 2,3$). Such restrictive assumptions are purely needed for the a priori assumption. In practice, it does not seem necessary as the numerical results in Section 4 show.*

Now we give the detailed proof of Theorem 3.1. At the beginning of the proof, we make the following customary modification on the flux $\mathbf{f}(u)$. Suppose the initial solution $u_0(x)$ lies in $[m_0, M_0]$. Then, the exact solution $u(x,t)$ is also in this range by the maximum principle. Thus, with no harm, we could choose the modified flux function $\tilde{\mathbf{f}}$, which is equal to $\mathbf{f}$ on $[m_0, M_0]$, vanishes outside $[m_0 - 1, M_0 + 1]$ and belongs to $C^3(\mathbb{R})$. For notation convenience, throughout this paper, we will still denote this modified flux by $\mathbf{f}$ and assume that $\mathbf{f} \in C_b^3(\mathbb{R}) \equiv C^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$.

Following [25], the following notations for piecewise smooth functions $p, q \in L^2(\Omega)$ are introduced,

$$H_K(p,q) = \int_K \mathbf{f}(p(x)) \cdot \nabla q(x) \mathrm{d}x - \sum_{\Gamma\in\partial K} \int_\Gamma \widehat{\mathbf{f}\cdot\mathbf{n}_K}(p(x))q(x)\mathrm{d}s,$$

and

$$\tilde{H}_K(p,q) = \sum_{j=1}^{M} \omega_j \mathbf{f}(p(x_{Kj})) \cdot \nabla q(x_{Kj}) \left|K\right| - \sum_{\Gamma\in\partial K} \sum_{j=1}^{L} \underline{\omega}_j \widehat{\mathbf{f}\cdot\mathbf{n}_K}(p(x_{\Gamma j}))q(x_{\Gamma j}) \left|\Gamma\right|.$$

Then the semi-discrete DG scheme with numerical integration (2.5) can be rewritten as

$$\int_K (u_h)_t v_h \mathrm{d}x = \tilde{H}_K(u_h, v_h), \quad \forall v_h \in V_h^k, \tag{3.15}$$

and the sufficiently smooth exact solution $u = u(t, x)$ satisfies

$$\int_K u_t v_h \mathrm{d}x = H_K(u, v_h), \quad \forall v_h \in V_h^k. \tag{3.16}$$

We would like to estimate the error $e = u - u_h$. As is customary in error analysis of finite element methods, we denote $\eta := \mathbb{P}u - u$ and $\xi := \mathbb{P}u - u_h$ where $\mathbb{P}$ is the L^2-projection defined in (2.2). By taking the difference of (3.15) and (3.16), taking the test function $v_h = \xi$ and making summation over all triangulations of $\mathscr{T}_h$, we obtain the energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t,\cdot)\|^2 = \sum_{K\in\mathscr{T}_h}(H_K(u,\xi) - \tilde{H}_K(u_h,\xi)). \tag{3.17}$$

With periodic or zero boundary conditions and the conservation of numerical flux, it is easy to obtain

$$\sum_{K\in\mathscr{T}_h}(H_K(p,q) - \tilde{H}_K(p,q)) = \sum_{K\in\mathscr{T}_h} E_K(\mathbf{f}(p)\cdot\nabla q) + \sum_{\Gamma\in\mathscr{E}_h} E_\Gamma(\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(p^-,p^+)[q]).$$

Thus the terms in the RHS of (3.17) could be split into

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t,\cdot)\|^2 &= \sum_{K\in\mathscr{T}_h}(H_K(u,\xi) - H_K(u_h,\xi)) + \sum_{K\in\mathscr{T}_h}(H_K(u_h,\xi) - \tilde{H}_K(u_h,\xi)),\\
&= \sum_{K\in\mathscr{T}_h}(H_K(u,\xi) - H_K(u_h,\xi)) + \sum_{K\in\mathscr{T}_h} E_K(\mathbf{f}(u_h)\cdot\nabla\xi) + \sum_{\Gamma\in\mathscr{E}_h} E_\Gamma(\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-,u_h^+)[\xi]),\\
&= \sum_{K\in\mathscr{T}_h}(H_K(u,\xi) - H_K(u_h,\xi)) + \left(\sum_{K\in\mathscr{T}_h} E_K(\mathbf{f}(u)\cdot\nabla\xi) + \sum_{\Gamma\in\mathscr{E}_h} E_\Gamma(\mathbf{f}(u)\cdot\mathbf{n}_\Gamma[\xi])\right)\\
&\quad + \sum_{K\in\mathscr{T}_h} E_K((\mathbf{f}(u_h) - \mathbf{f}(u))\cdot\nabla\xi) + \sum_{\Gamma\in\mathscr{E}_h} E_\Gamma((\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-,u_h^+) - \mathbf{f}(u)\cdot\mathbf{n}_\Gamma)[\xi]),\\
&\triangleq \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4.
\end{aligned}$$

The first term $\mathcal{S}_1$ only involves the error with exact integral and has been well treated in [25]. The main technique is the Taylor expansion with careful treatment to the boundary terms. We summarize the results in the following lemma (cf. Lemma 5.4 in [25] and Lemma 5.6 in [14] for the multidimensional case).

Lemma 3.1. *Assume* $\mathbf{f}$ *and* u *are sufficiently smooth functions that* $\mathbf{f} \in C_b^3(\mathbb{R})$ *and* $u \in H^{k+1}(\Omega)$*. Then there exists a constant* $C > 0$*, independent of* h*,* t*, and* u_h*, such that*

$$\sum_{K \in \mathscr{T}_h} (H_K(u, \xi) - H_K(u_h, \xi)) \leq C(1 + h^{-1} \|e\|_\infty^2)(h^{2k+1} + \|\xi\|^2). \tag{3.18}$$

The second term $\mathcal{S}_2$ is somewhat like the "truncation error" and has been analyzed in Proposition 2.1 in [3]. We list the results in the following lemma.

Lemma 3.2. *Assume* $\mathbf{f}$ *and* u *are sufficiently smooth functions that* $\mathbf{f}(u) \in W^{k+2,\infty}(\Omega)$*. If the quadrature rule over the element* (2.4a) *is exact for polynomials of degree* $(2k)$*, and the one over the edge* (2.4b) *is exact for polynomials of degree* $(2k+1)$*, then there exists a constant* $C > 0$*, independent of* h*,* t *and* ξ*, such that,*

$$\left| \sum_{K \in \mathscr{T}_h} E_K(\mathbf{f}(u) \cdot \nabla \xi) \right| + \left| \sum_{\Gamma \in \mathscr{E}_h} E_\Gamma(\mathbf{f}(u) \cdot \mathbf{n}_\Gamma [\xi]) \right| \leq C(h^{2k+2} + \|\xi\|^2). \tag{3.19}$$

The main technique used in the estimate of the third term $\mathcal{S}_3$ is Taylor expansion. By using Taylor expansion on $\mathbf{f}(u_h)$ up to second order and (2.9) in Lemma 2.4, we can prove the following lemma. The details of the technical proof are left for the appendix.

Lemma 3.3. *Assume* $\mathbf{f}$ *and* u *are sufficiently smooth functions that* $\mathbf{f}'(u) \in W^{k+2,\infty}(\Omega)$*,* $\mathbf{f}''(u) \in W^{2,\infty}(\Omega)$*,* $\mathbf{f}''' \in C_b(\mathbb{R})$ *and* $u \in W^{k+1,\infty}(\Omega)$*. If the quadrature rule over the element* (2.4a) *is exact for polynomials of degree* $(2k)$ *and the integration weights in* (2.4a) *are nonnegative, then there exists a constant* $C > 0$*, independent of* h*,* t *and* u_h*, such that, for* $k = 1$*,*

$$\left| \sum_{K \in \mathscr{T}_h} E_K((\mathbf{f}(u_h) - \mathbf{f}(u)) \cdot \nabla \xi) \right| \leq C(1 + h^{-1} \|e\|_\infty^2)(h^{2k+2} + \|\xi\|^2), \tag{3.20}$$

and for $k \geq 2$*,*

$$\left| \sum_{K \in \mathscr{T}_h} E_K((\mathbf{f}(u_h) - \mathbf{f}(u)) \cdot \nabla \xi) \right| \leq C(1 + h^{-1} \|e\|_\infty)(h^{2k+2} + \|\xi\|^2). \tag{3.21}$$

As for the estimate of the fourth term $\mathcal{S}_4$, if the numerical flux is sufficiently smooth, the same technique in the proof of Lemma 3.3 could be applied. With the aid of the multiplicative trace inequality in Lemma 2.1 and (2.10) in Lemma 2.4, we obtain the following result and the proof is presented in the appendix.

Lemma 3.4. *Assume the numerical flux* $\widehat{\mathbf{f}\cdot\mathbf{n}} = \widehat{\mathbf{f}\cdot\mathbf{n}}(v,w)$ *and* u *are sufficiently smooth functions that* $\widehat{\mathbf{f}\cdot\mathbf{n}}'(u,u) \in W^{k+2,\infty}(\mathscr{E}_h)$, $\widehat{\mathbf{f}\cdot\mathbf{n}}''(u,u) \in W^{2,\infty}(\mathscr{E}_h)$, $\widehat{\mathbf{f}\cdot\mathbf{n}}''' \in C_b(\mathbb{R}^2)$ *and* $u \in W^{k+1,\infty}(\Omega)$*. If the quadrature rule over the edge* (2.4b) *is exact for polynomials of degree* $(2k+1)$ *and the integration weights in* (2.4b) *are nonnegative, then there exists a constant* $C > 0$*, independent of* h*,* t *and* u_h*, such that, for* $k = 1$*,*

$$\left|\sum_{\Gamma\in\mathscr{E}_h} E_\Gamma((\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-,u_h^+) - \mathbf{f}(u)\cdot\mathbf{n}_\Gamma)[\xi])\right| \le C(1+h^{-1}\|e\|_\infty^2)(h^{2k+2}+\|\xi\|^2), \tag{3.22}$$

and for $k \ge 2$*,*

$$\left|\sum_{\Gamma\in\mathscr{E}_h} E_\Gamma((\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-,u_h^+) - \mathbf{f}(u)\cdot\mathbf{n}_\Gamma)[\xi])\right| \le C(1+h^{-1}\|e\|_\infty)(h^{2k+2}+\|\xi\|^2). \tag{3.23}$$

However, most numerical fluxes are not smooth enough. They are only locally Lipschitz continuous (e.g. Godunov flux) or only have up to first-order derivative (e.g. Engquist-Osher flux). Thus, the Taylor expansion technique could not be used in the estimate of the fourth term $\mathcal{S}_4$. In this case, we would put the terms $\mathcal{S}_1$ and $\mathcal{S}_4$ together and split them as follows:

$$\begin{aligned}
\mathcal{S}_1 + \mathcal{S}_4 = &\sum_{K\in\mathscr{T}_h}\int_K((\mathbf{f}(u)-\mathbf{f}(u_h))\cdot\nabla\xi) + \sum_{\Gamma\in\mathscr{E}_h}\int_\Gamma((\mathbf{f}(u)-\mathbf{f}(\bar u_h))\cdot\mathbf{n}_\Gamma[\xi]) \\
&+ \sum_{\Gamma\in\mathscr{E}_h}\int_\Gamma((\mathbf{f}(\bar u_h) - \widehat{\mathbf{f}\cdot\mathbf{n}}_\Gamma(u_h^-,u_h^+))\cdot\mathbf{n}_\Gamma[\xi]) \\
&+ \sum_{\Gamma\in\mathscr{E}_h} E_\Gamma((\mathbf{f}(\bar u_h)-\mathbf{f}(u))\cdot\mathbf{n}_\Gamma)[\xi]) + \sum_{\Gamma\in\mathscr{E}_h}\int_\Gamma((\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-,u_h^+) - \mathbf{f}(\bar u_h)\cdot\mathbf{n}_\Gamma)[\xi]) \\
&- \sum_{\Gamma\in\mathscr{E}_h}\sum_j \underline{\omega}_j((\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-,u_h^+) - \mathbf{f}(\bar u_h)\cdot\mathbf{n}_\Gamma)[\xi])(x_{\Gamma j})\,|\Gamma| \\
= &\sum_{K\in\mathscr{T}_h}\int_K((\mathbf{f}(u)-\mathbf{f}(u_h))\cdot\nabla\xi) + \sum_{\Gamma\in\mathscr{E}_h}\int_\Gamma((\mathbf{f}(u)-\mathbf{f}(\bar u_h))\cdot\mathbf{n}_\Gamma[\xi]) \\
&+ \sum_{\Gamma\in\mathscr{E}_h} E_\Gamma((\mathbf{f}(\bar u_h)-\mathbf{f}(u))\cdot\mathbf{n}_\Gamma)[\xi]) - \sum_{\Gamma\in\mathscr{E}_h}\sum_j \underline{\omega}_j((\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-,u_h^+) - \mathbf{f}(\bar u_h)\cdot\mathbf{n}_\Gamma)[\xi])(x_{\Gamma j})\,|\Gamma|, \\
\triangleq &\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4.
\end{aligned}$$

Note that here $\mathcal{S}_1$ is split into three terms, and the third one balances the second "trouble" term of $\mathcal{S}_4$. Under the assumption that $\mathbf{f} \in C_b^3(\mathbb{R})$ and $u \in H^{k+1}(\Omega)$, we have the estimate of $\mathcal{W}_1$ and $\mathcal{W}_2$ with similar approach in the proof of Lemma 5.4 and Lemma 5.7 in [25]:

$$\mathcal{W}_1 + \mathcal{W}_2 \le C(1+h^{-1}\|e\|_\infty)(h^{2k}+\|\xi\|^2). \tag{3.24}$$

For the third term $\mathcal{W}_3$, with Taylor expansion technique in the proof of Lemma 3.4, the smoothness assumption $\mathbf{f}'(u) \in W^{k+2,\infty}(\Omega)$, $\mathbf{f}''(u) \in W^{2,\infty}(\Omega)$, $\mathbf{f}''' \in C_b(\mathbb{R})$ and $u \in W^{k+1,\infty}(\Omega)$, and assuming that the quadrature rule over the edge (2.4b) is exact for polynomials of degree $(2k+1)$ and quadrature weights in (2.4b) are non-negative, the similar result is obtained:

$$|\mathcal{W}_3| \leq C(1 + h^{-1} \|e\|_\infty)(h^{2k+2} + \|\xi\|^2). \tag{3.25}$$

For the estimate of $\mathcal{W}_4$, on each edge Γ, we define (cf. [12])

$$\alpha(u_h)_\Gamma = \begin{cases} [u_h]^{-1}(\mathbf{f}(\bar{u}_h) \cdot \mathbf{n}_\Gamma - \widehat{\mathbf{f} \cdot \mathbf{n}_\Gamma}(u_h^-, u_h^+)), & \text{if} \quad [u_h] \neq 0, \\ |\mathbf{f}'(\bar{u}_h) \cdot \mathbf{n}_\Gamma|, & \text{if} \quad [u_h] = 0. \end{cases}$$

The monotonicity and the Lipschitz continuity of the numerical flux imply the nonnegative and the bounded property of $\alpha(u_h)_\Gamma$ (cf. Lemma 3.1 in [25]). With this notation and remembering that $[u_h] = -[e] = [\eta] - [\xi]$, we have that

$$\begin{aligned} \mathcal{W}_4 =& \sum_{\Gamma \in \mathscr{E}_h} \sum_j \underline{\omega}_j \alpha(u_h)_\Gamma ([\eta][\xi] - [\xi]^2)(x_{\Gamma j}) |\Gamma|, \\ \leq& \sum_{\Gamma \in \mathscr{E}_h} \sum_j \underline{\omega}_j \alpha(u_h)_\Gamma \frac{1}{4} [\eta]^2 (x_{\Gamma j}) |\Gamma|, \\ \leq& C \|\eta\|_\infty^2 \sum_{\Gamma \in \mathscr{E}_h} |\Gamma| \leq C h^{2k+1}. \end{aligned}$$

Here in the first inequality, the assumption is made that the integration weights in (2.4b) are all nonnegative. We summarize the above results in the following lemma.

Lemma 3.5. *Assume* $\mathbf{f}$ *and* u *are sufficiently smooth functions that* $\mathbf{f}'(u) \in W^{k+2,\infty}(\Omega)$, $\mathbf{f}''(u) \in W^{2,\infty}(\Omega)$, $\mathbf{f} \in C_b^3(\mathbb{R})$ *and* $u \in W^{k+1,\infty}(\Omega)$. *If the quadrature rule over the edge* (2.4b) *is exact for polynomials of degree* $(2k+1)$ *and the integration weights in* (2.4b) *are nonnegative, then there exists a constant* $C > 0$, *independent of* h, t *and* u_h, *such that,*

$$\sum_{K \in \mathscr{T}_h} (H_K(u, \xi) - H_K(u_h, \xi)) + \sum_{\Gamma \in \mathscr{E}_h} E_\Gamma((\widehat{\mathbf{f} \cdot \mathbf{n}_\Gamma}(u_h^-, u_h^+) - \mathbf{f}(u) \cdot \mathbf{n}_\Gamma)[\xi]) \leq C(1 + h^{-1} \|e\|_\infty)(h^{2k} + \|\xi\|^2).$$

In the last lemma, we list the estimate of $\mathcal{S}_1$ for the upwind flux in the 1D case in [25]. If we replace the L^2-projection by the Gauss-Radau projection, then we have the following lemma (cf. Lemma 5.7 in [25]). Moreover, we would like to mention that, the interpolation inequalities in Lemma 2.2 also hold for the Gauss-Radau projection. Hence, we could also prove Lemma 3.3 with the Gauss-Radau projection with the same token.

Lemma 3.6. *Assume* $\mathbf{f}$ *and* u *are sufficiently smooth functions that* $\mathbf{f} \in C_b^3(\mathbb{R})$ *and* $u \in H^{k+1}(\Omega)$ *and the numerical flux is upwind in the 1D case. If the Gauss-Radau projection is used, there exists a constant* $C > 0$*, independent of* h*,* t*, and* u_h*, such that*

$$\sum_{K \in \mathscr{T}_h} (H_K(u, \xi) - H_K(u_h, \xi)) \leq C(1 + h^{-1} \|e\|_\infty)(h^{2k+2} + \|\xi\|^2). \tag{3.26}$$

Now we are going to prove our main theorem 3.1. For simplicity, we will only give the detailed proof for one case in Theorem 3.1, namely $d = 3$ and $k \geq 3$ with general monotone flux, as other cases follow along the same lines. Following [25], we first make an *a priori* assumption that, for small enough h, there holds the inequality:

$$\|e(t, \cdot)\| \leq h^{5/2}, \tag{3.27}$$

for $0 \leq t \leq T$. Then by the triangle inequality, the interpolation property (2.6a)-(2.6b) and the inverse property (2.7c), we have that

$$\|e(t, \cdot)\|_\infty \leq Ch, \tag{3.28}$$

for $0 \leq t \leq T$. From Lemma 3.2, Lemma 3.3 and Lemma 3.5, an estimate on the RHS of (3.17) is obtained

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\xi(t, \cdot)\|^2 \leq C \|\xi(t, \cdot)\|^2 + Ch^{2k}.$$

Thus it follows that

$$\|e\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch^k. \tag{3.29}$$

To complete the proof, let us verify the a priori assumption (3.27). For fixed $k \geq 3$, we consider h small enough such that $Ch^k < \frac{1}{2}h^{5/2}$ with the constant C in (3.29). Then, define

$t^* := \sup\{t : \|e(t,\cdot)\| \le h^{5/2}\}$, and immediately we get $\|e(t^*,\cdot)\| = h^{5/2}$ by continuity if t^* is finite. On the other hand, our proof shows that $\|e(t,\cdot)\| \le Ch^k$ for $t \le t^*$, in particular $\|e(t^*,\cdot)\| \le Ch^k < \frac{1}{2}h^{5/2}$. This reaches a contradiction if $t^* < T$. Hence $t^* \ge T$ and our a priori assumption (3.27) is justified.

4 Numerical results

In this section, we display some numerical results to validate our error estimate in section 3. The TVD Runge-Kutta time discretization [20] is used here: second-order RK method for the piecewise linear finite element space ($k = 1$) and third-order RK method for the piecewise quadratic finite element space ($k = 2$). For two-dimensional problems, the triangular meshes are used and the triangulation is constructed by adding diagonals linking the left-bottom and right-top vertexes in a uniform square mesh.

Consider the two-dimensional Burgers' equation with periodic boundary conditions:

$$\begin{cases} \partial_t u + \partial_x \left(\dfrac{u^2}{2}\right) + \partial_y \left(\dfrac{u^2}{2}\right) = 0, \quad (x,y,t) \in \Omega \times (0,T), \\ u(t=0,x,y) = \dfrac{1}{2}\sin(x+y), \quad (x,y) \in \Omega, \end{cases}$$

where the domain Ω is the square $[0,2\pi] \times [0,2\pi]$. The mesh size is chosen as $h = \pi/N$ with N the number of elements. We compute up to time $T = 0.8$ with time step $\tau = 0.2h$.

To validate our analysis, we use quadrature rules to approximate integrals over elements and edges which are exact for polynomials of degree $(2k)$ and $(2k+1)$, respectively. To be more precise, for $k = 1$, we use the Gaussian quadrature of degree 2 for the standard triangle $T_{st} := \{(\xi,\eta) : 0 \le \xi, \eta, \xi + \eta \le 1\}$:

$$\iint_{T_{st}} g(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta = \frac{1}{6}\left(g(\frac{1}{6},\frac{1}{6}) + g(\frac{2}{3},\frac{1}{6}) + g(\frac{1}{6},\frac{2}{3})\right),$$

which is accurate for $g(\xi,\eta) = \xi^i\eta^j$ with $0 \le i, j, i+j \le 2$. For $k = 2$, we use the Gaussian quadrature of degree 4 which is exact for $g(\xi,\eta) = \xi^i\eta^j$ with $0 \le i, j, i+j \le 4$; see e.g. [11]. For integrals over edges, we use the Gauss-Legendre quadrature rules with 2 points and 3 points for $k = 1$ and $k = 2$, respectively. In Table 4.1 and Table 4.2, the L^1, L^2 and L^∞ errors are displayed for $k = 1$ and $k = 2$ with quadrature rules over elements and edges of degree $(2k)$ and $(2k+1)$. From Table 4.1, we observe that, the schemes with $k = 1$ behaves a clear second-order accuracy in L^1- and L^2-error with different types of numerical fluxes, but not very clean in L^∞-norm. For $k = 2$, the convergence orders are almost three, except that for the global Lax-Friedrichs flux, the order is around 2.7 in average. To figure out the reason for this order reduction phenomenon, we perform tests with the same computational parameters but with the quadrature rules over elements and edges replaced by exact integrals (namely we use the original DG method). The convergence order with the global Lax-Friedrichs flux and exact integrals is still around 2.7 as the mesh is refined. This indicates that the order reduction phenomenon is not caused by the quadrature rules. A detailed study to pin down the root of this phenomenon is left for the future.

We have also performed numerical experiments with lower order quadrature rules. The errors with the Godunov flux and $k = 2$ are reported in Table 4.3. As before, we use Gaussian quadrature rules over both elements and edges. In the table, "Elements 4 and edges 5" means that we have used the quadrature rules which are exact for polynomials of degree up to 4 and 5 over elements and edges, respectively. In the first three cases, the accuracy over edges is 5, which is high enough to keep accuracy according to our main theorem. In the last case, we use quadrature rules which is accurate enough over elements but not accurate enough over edges. From these cases, we observe loss of accuracy if the integration is not accurate enough. Moreover, we have performed numerical tests with quadrature rules over elements and edges which are exact for polynomials of degree up to 4 and 1, respectively. In that case, the numerical solution actually blows up, indicating an instability of the algorithm. It seems that the DG methods is more sensitive to the quality of the quadrature rules used over edges than to that used in the elements.

Table 4.1: 2D Burgers' equation at $T = 0.8$, $k = 1$.

numerical flux	N	L^1-error	order	L^2-error	order	L^∞-error	order
Godunov flux	10×10	7.34e-01	-	2.10e-01	-	2.61e-01	-
	20×20	2.31e-01	1.67	8.15e-02	1.37	1.32e-01	0.98
	40×40	6.79e-02	1.77	2.86e-02	1.51	5.01e-02	1.40
	80×80	1.90e-02	1.84	8.85e-03	1.69	1.98e-02	1.34
	160×160	5.02e-03	1.92	2.48e-03	1.83	6.29e-03	1.65
	320×320	1.29e-03	1.96	6.69e-04	1.89	1.83e-03	1.79
Global Lax-Friedrichs flux	10×10	8.23e-01	-	2.30e-01	-	2.66e-01	-
	20×20	2.52e-01	1.71	9.03e-02	1.35	1.29e-01	1.05
	40×40	7.15e-02	1.82	3.13e-02	1.53	4.96e-02	1.37
	80×80	1.94e-02	1.88	9.25e-03	1.76	1.98e-02	1.33
	160×160	5.11e-03	1.93	2.46e-03	1.91	4.96e-03	1.99
	320×320	1.30e-03	1.98	6.31e-04	1.96	1.25e-03	1.99
Local Lax-Friedrichs flux	10×10	8.07e-01	-	2.29e-01	-	2.57e-01	-
	20×20	2.43e-01	1.73	8.71e-02	1.39	1.33e-01	0.95
	40×40	6.90e-02	1.82	2.96e-02	1.56	5.08e-02	1.38
	80×80	1.89e-02	1.87	8.92e-03	1.73	2.06e-02	1.30
	160×160	5.00e-03	1.92	2.48e-03	1.85	6.30e-03	1.71
	320×320	1.29e-03	1.96	6.65e-04	1.90	1.81e-03	1.80
Enquist-Osher flux	10×10	7.34e-01	-	2.10e-01	-	2.61e-01	-
	20×20	2.31e-01	1.67	8.15e-02	1.37	1.32e-01	0.98
	40×40	6.79e-02	1.77	2.86e-02	1.51	5.01e-02	1.40
	80×80	1.90e-02	1.84	8.85e-03	1.69	1.98e-02	1.34
	160×160	5.02e-03	1.92	2.48e-03	1.83	6.29e-03	1.65
	320×320	1.29e-03	1.96	6.69e-04	1.89	1.83e-03	1.79

Table 4.2: 2D Burgers' equation at $T = 0.8$, $k = 2$.

numerical flux	N	L^1-error	order	L^2-error	order	L^∞-error	order
Godunov flux	10×10	2.93e-01	-	1.28e-01	-	1.21e-01	-
	20×20	6.43e-02	2.19	3.97e-02	1.69	5.87e-02	1.05
	40×40	1.03e-02	2.64	8.36e-03	2.25	2.25e-02	1.38
	80×80	1.33e-03	2.96	1.17e-03	2.83	5.33e-03	2.08
	160×160	1.87e-04	2.82	1.54e-04	2.93	8.14e-04	2.71
	320×320	2.51e-05	2.90	2.09e-05	2.89	1.02e-04	3.00
Global Lax-Friedrichs flux	10×10	3.11e-01	-	1.30e-01	-	1.20e-01	-
	20×20	6.87e-02	2.18	4.02e-02	1.69	5.63e-02	1.10
	40×40	1.17e-02	2.55	8.58e-03	2.23	1.95e-02	1.53
	80×80	1.66e-03	2.82	1.29e-03	2.73	4.11e-03	2.25
	160×160	2.62e-04	2.66	1.98e-04	2.71	5.99e-04	2.78
	320×320	3.84e-05	2.77	3.09e-05	2.68	9.17e-05	2.71
Local Lax-Friedrichs flux	10×10	3.11e-01	-	1.30e-01	-	1.22e-01	-
	20×20	6.66e-02	2.22	4.00e-02	1.70	5.80e-02	1.07
	40×40	1.04e-02	2.68	8.35e-03	2.26	2.25e-02	1.37
	80×80	1.37e-03	2.92	1.19e-03	2.81	5.41e-03	2.05
	160×160	1.95e-04	2.82	1.60e-04	2.89	8.39e-04	2.69
	320×320	2.57e-05	2.92	2.15e-05	2.89	1.06e-04	2.99
Enquist-Osher flux	10×10	2.93e-01	-	1.28e-01	-	1.21e-01	-
	20×20	6.43e-02	2.19	3.97e-02	1.69	5.87e-02	1.05
	40×40	1.03e-02	2.64	8.36e-03	2.25	2.25e-02	1.38
	80×80	1.33e-03	2.96	1.17e-03	2.83	5.33e-03	2.08
	160×160	1.87e-04	2.82	1.54e-04	2.93	8.14e-04	2.71
	320×320	2.51e-05	2.90	2.09e-05	2.89	1.02e-04	3.00

Table 4.3: Comparison for lower quadrature rules. 2D Burgers' equation with Godunov flux and $k = 2$.

Accuracy of quadrature rules	N	L^1-error	order	L^2-error	order	L$^\infty$-error	order
Elements 4 and edges 5	10×10	2.93e-01	-	1.28e-01	-	1.21e-01	-
	20×20	6.43e-02	2.19	3.97e-02	1.69	5.87e-02	1.05
	40×40	1.03e-02	2.64	8.36e-03	2.25	2.25e-02	1.38
	80×80	1.33e-03	2.96	1.17e-03	2.83	5.33e-03	2.08
	160×160	1.87e-04	2.82	1.54e-04	2.93	8.14e-04	2.71
	320×320	2.51e-05	2.90	2.09e-05	2.89	1.02e-04	3.00
Elements 3 and edges 5	10×10	4.63e-01	-	1.65e-01	-	1.29e-01	-
	20×20	1.33e-01	1.80	6.72e-02	1.29	9.49e-02	0.45
	40×40	3.08e-02	2.11	2.09e-02	1.69	4.69e-02	1.02
	80×80	5.35e-03	2.52	4.51e-03	2.21	1.38e-02	1.76
	160×160	7.83e-04	2.77	7.25e-04	2.64	3.09e-03	2.16
	320×320	1.03e-04	2.92	1.02e-04	2.83	4.93e-04	2.65
Elements 2 and edges 5	10×10	4.94e-01	-	1.82e-01	-	1.83e-01	-
	20×20	1.30e-01	1.93	6.13e-02	1.57	8.10e-02	1.18
	40×40	2.68e-02	2.28	1.53e-02	2.00	2.83e-02	1.51
	80×80	5.95e-03	2.17	3.86e-03	1.99	9.76e-03	1.54
	160×160	1.73e-03	1.78	1.44e-03	1.42	4.99e-03	0.97
	320×320	5.12e-04	1.76	5.56e-04	1.38	3.37e-03	0.57
Elements 4 and edges 3	10×10	4.81e-01	-	1.41e-01	-	1.26e-01	-
	20×20	1.55e-01	1.64	5.26e-02	1.43	7.86e-02	0.69
	40×40	5.08e-02	1.61	1.94e-02	1.44	3.65e-02	1.11
	80×80	1.59e-02	1.68	7.52e-03	1.36	1.53e-02	1.26
	160×160	4.95e-03	1.68	2.94e-03	1.36	9.36e-03	0.70
	320×320	1.53e-03	1.69	1.12e-03	1.39	5.33e-03	0.81

5 Concluding remarks

In this work, we perform error estimates to smooth solutions of semi-discrete discontinuous Galerkin (DG) methods with the P^k finite element space of piecewise kth degree polynomials and quadrature rules for scalar conservation laws (1.1). Assuming that the exact solution u and the physical flux $\mathbf{f}$ are sufficiently smooth, we show that, in 1D problems, if the quadrature over elements is exact for polynomials of degree $(2k)$, error estimates of $O(h^{k+1/2})$ are obtained for general monotone fluxes, while optimal estimates of $O(h^{k+1})$ are obtained for upwind fluxes. For multidimensional problems, if we further assume that quadrature over edges is exact for polynomials of degree $(2k+1)$, error estimates of $O(h^{k+1/2})$ are obtained for sufficiently smooth numerical fluxes. For the general monotone fluxes in the multidimensional case, error estimate of $O(h^k)$ are proved if we further assume that the quadrature weights are non-negative.

Even though we have considered only semi-discrete schemes for scalar conservation laws in this paper, the analysis can be generalized to Runge-Kutta DG schemes and to symmetrizable systems of conservation laws following the ideas in [26, 27]. We hope to report our progress in the near future.

A Proof of Lemma 3.3

In this appendix, we give the proof of Lemma 3.3. We would like to use the following Taylor expansion:

$$\begin{aligned}\mathbf{f}(u_h) - \mathbf{f}(u) =& \eta\mathbf{f}'(u) - \xi\mathbf{f}'(u) + \frac{1}{2}\eta^2\mathbf{f}''(u) - \eta\xi\mathbf{f}''(u) + \frac{1}{2}\xi^2\mathbf{f}''(u) - \frac{1}{6}e^3\mathbf{f}'''_u,\\ \triangleq& \phi_1 + \phi_2 + \cdots + \phi_6,\end{aligned}$$

where $\mathbf{f}'''_u$ is the mean value. The linearity of integrals and quadrature rules leads to the representation:

$$\sum_{K\in\mathscr{T}_h} E_K((\mathbf{f}(u_h) - \mathbf{f}(u))\cdot\nabla\xi) = X_1 + X_2 + \cdots + X_6,$$

where X_i given by

$$X_i = \sum_{K\in\mathscr{T}_h} E_K(\phi_i \cdot \nabla\xi), \quad i = 1, 2, \cdots, 6,$$

will be estimated one by one later.

By taking $q = 2$, $p = k-1$, $s = k+2$, $Q = \eta\mathbf{f}'(u)$ and $v = \nabla\xi$ in (2.9) in Lemma 2.4 and remembering that the quadrature over element K is exact for $P^{2k}(K)$, together with interpolation property (2.6b), the inverse property (2.7a) and Cauchy's inequality, we have the following estimate:

$$\begin{aligned}
|X_1| \leq &\sum_{K\in\mathscr{T}_h} |E_K(\phi_1 \cdot \nabla\xi)|, \\
\leq &\sum_{K\in\mathscr{T}_h} Ch^{k+2+d/2} \left|\eta\mathbf{f}'(u)\right|_{W^{k+2,\infty}(K)} \left\|\nabla\xi\right\|_{L^2(K)}, \\
\leq &\sum_{K\in\mathscr{T}_h} Ch^{k+2+d/2} \left\|\eta\right\|_{W^{k+2,\infty}(K)} \left\|\mathbf{f}'(u)\right\|_{W^{k+2,\infty}(K)} \left\|\nabla\xi\right\|_{L^2(K)}, \\
\leq &C \left|u\right|_{W^{k+1,\infty}(\Omega)} \left\|\mathbf{f}'(u)\right\|_{W^{k+2,\infty}(\Omega)} h^{k+2+d/2} \sum_{K\in\mathscr{T}_h} \left\|\nabla\xi\right\|_{L^2(K)}, \\
\leq &C \left|u\right|_{W^{k+1,\infty}(\Omega)} \left\|\mathbf{f}'(u)\right\|_{W^{k+2,\infty}(\Omega)} (h^{2k+2} + \|\xi\|^2).
\end{aligned}$$

Here in the fourth inequality we use the relation

$$\begin{aligned}
\|\eta\|_{W^{k+2,\infty}(K)} \leq &\left|\eta\right|_{W^{0,\infty}(K)} + \left|\eta\right|_{W^{1,\infty}(K)} + \cdots + \left|\eta\right|_{W^{k+1,\infty}(K)} + \left|\eta\right|_{W^{k+2,\infty}(K)}, \\
\leq &C \left|u\right|_{W^{k+1,\infty}(\Omega)} (h^{k+1} + h^k + \cdots + 1) + \left|u\right|_{W^{k+1,\infty}(\Omega)} \leq C \left|u\right|_{W^{k+1,\infty}(\Omega)}.
\end{aligned}$$

In the similar approach, by taking $q = 1$, $p = 2k-1$, $s = 1$, $Q = \mathbf{f}'(u)$ and $v = \xi\nabla\xi$ in (2.9) in Lemma 2.4, Hölder's inequality and the inverse property (2.7a), it can be easily shown that

$$|X_2| \leq \sum_{K\in\mathscr{T}_h} |E_K(\phi_2 \cdot \nabla\xi)| \leq Ch \sum_{K\in\mathscr{T}_h} \left|\mathbf{f}'(u)\right|_{W^{1,\infty}(K)} \left\|\xi\nabla\xi\right\|_{L^1(K)} \leq C \left|\mathbf{f}'(u)\right|_{W^{1,\infty}(\Omega)} \|\xi\|^2. \tag{A.30}$$

Similarly as the estimate of X_1 and X_2, for X_3 and X_4, we have

$$|X_3| \leq \sum_{K\in\mathscr{T}_h} |E_K(\phi_3 \cdot \nabla\xi)|,$$

$$
\begin{aligned}
&\le \sum_{K\in\mathscr{T}_h} Ch^{1+d/2}\left|\eta^2\mathbf{f}''(u)\right|_{W^{1,\infty}(K)}\|\nabla\xi\|_{L^2(K)}\,,\\
&\le Ch^{1+d/2}\left\|\eta^2\right\|_{W^{1,\infty}(\Omega)}\|\mathbf{f}''(u)\|_{W^{1,\infty}(\Omega)}\sum_{K\in\mathscr{T}_h}\|\nabla\xi\|_{L^2(K)}\,,\\
&\le Ch^{1+d/2}(\|\eta\|^2_{L^\infty(\Omega)}+|\eta|_{W^{1,\infty}(\Omega)}\|\eta\|_{L^\infty(\Omega)})\|\mathbf{f}''(u)\|_{W^{1,\infty}(\Omega)}\sum_{K\in\mathscr{T}_h}\|\nabla\xi\|_{L^2(K)}\,,\\
&\le Ch^{2k+2+d/2}|u|^2_{W^{k+1,\infty}(\Omega)}\|\mathbf{f}''(u)\|_{W^{1,\infty}(\Omega)}\sum_{K\in\mathscr{T}_h}\|\nabla\xi\|_{L^2(K)}\,,\\
&\le C|u|^2_{W^{k+1,\infty}(\Omega)}\|\mathbf{f}''(u)\|_{W^{1,\infty}(\Omega)}(h^{4k+2}+\|\xi\|^2),
\end{aligned}
$$

and

$$
\begin{aligned}
|X_4| &\le \sum_{K\in\mathscr{T}_h}|E_K(\phi_4\cdot\nabla\xi)|,\\
&\le \sum_{K\in\mathscr{T}_h} Ch\,|\eta\mathbf{f}''(u)|_{W^{1,\infty}(K)}\|\xi\nabla\xi\|_{L^1(K)}\,,\\
&\le Ch\,\|\eta\|_{W^{1,\infty}(\Omega)}\|\mathbf{f}''(u)\|_{W^{1,\infty}(\Omega)}\sum_{K\in\mathscr{T}_h}\|\xi\|_{L^2(K)}\|\nabla\xi\|_{L^2(K)}\,,\\
&\le Ch^k|u|_{W^{k+1,\infty}(\Omega)}\|\mathbf{f}''(u)\|_{W^{1,\infty}(\Omega)}\|\xi\|^2\,.
\end{aligned}
$$

As for the estimate of X_5, setting $q=1$, $p=2k-1$, $s=2$, $Q=\xi\mathbf{f}''(u)$ and $v=\xi\nabla\xi$ in (2.9) in Lemma 2.4, we find that

$$
|X_5| \le \sum_{K\in\mathscr{T}_h}|E_K(\phi_5\cdot\nabla\xi)| \le Ch^2\|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)}\sum_{K\in\mathscr{T}_h}\|\xi\|_{W^{2,\infty}(K)}\|\xi\|_{L^2(K)}\|\nabla\xi\|_{L^2(K)}\,.
$$

For a more careful estimate, we make the discussion in two cases:

(a) For $k=1$, i.e., the P^1 finite element space, we have $|\xi|_{W^{2,\infty}(K)}=0$. By virtue of Cauchy's inequality and inverse property (2.7a), we deduce that

$$
\begin{aligned}
|X_5| &\le Ch^2\|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)}\sum_{K\in\mathscr{T}_h}\|\xi\|_{W^{1,\infty}(K)}\|\xi\|_{L^2(K)}\|\nabla\xi\|_{L^2(K)}\,,\\
&\le Ch^2\|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)}\sum_{K\in\mathscr{T}_h}h^{-1}\|\xi\|_{L^\infty(K)}\|\xi\|_{L^2(K)}\|\nabla\xi\|_{L^2(K)}\,,\\
&\le Ch\,\|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)}\|\xi\|_\infty\|\xi\|\,\|\nabla\xi\|\,,\\
&\le C\|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)}(\|e\|_\infty+h^{k+1}|u|_{W^{k+1,\infty}(\Omega)})\|\xi\|^2\,.
\end{aligned}
$$

Here in the first inequality we use the generalized inverse property (cf. Theorem 3.2.6 in [1]):

$$|\xi|_{W^{1,\infty}(K)} \leq Ch^{-1} \|\xi\|_{L^\infty(K)}.$$

(b) For $k \geq 2$, in the same approach, we get the estimate

$$\begin{aligned}
|X_5| \leq & Ch^2 \|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)} \sum_{K\in\mathscr{T}_h} \|\xi\|_{W^{2,\infty}(K)} \|\xi\|_{L^2(K)} \|\nabla\xi\|_{L^2(K)}, \\
\leq & Ch^2 \|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)} \sum_{K\in\mathscr{T}_h} h^{-2} \|\xi\|_{L^\infty(K)} \|\xi\|_{L^2(K)} \|\nabla\xi\|_{L^2(K)}, \\
\leq & C \|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)} \|\xi\|_\infty \|\xi\| \|\nabla\xi\|, \\
\leq & Ch^{-1} \|\mathbf{f}''(u)\|_{W^{2,\infty}(\Omega)} (\|e\|_\infty + h^{k+1} |u|_{W^{k+1,\infty}(\Omega)}) \|\xi\|^2.
\end{aligned}$$

The last term X_6 is divided into two parts and will be estimated separately:

$$|X_6| \leq \sum_{K\in\mathscr{T}_h} |E_K(\phi_6 \cdot \nabla\xi)| \leq \sum_{K\in\mathscr{T}_h} \left| \int_K \frac{1}{6} e^3 \mathbf{f}_u''' \cdot \nabla\xi \mathrm{d}x \right| + \sum_{K\in\mathscr{T}_h} \left| \sum_{j=1}^M \omega_j (\frac{1}{6} e^3 \mathbf{f}_u''' \cdot \nabla\xi)(x_{Kj}) \right| |K|.$$

For the first integral term, by Cauchy's inequality, inverse property (2.7a) and interpolation property (2.6a), we can easily show that

$$\sum_{K\in\mathscr{T}_h} \left| \int_K \frac{1}{6} e^3 \mathbf{f}_u''' \cdot \nabla\xi \mathrm{d}x \right| \leq Ch^{-1} \|\mathbf{f}_u'''\|_\infty \|e\|_\infty^2 (h^{2k+2} |u|_{H^{k+1}(\Omega)}^2 + \|\xi\|^2).$$

Due to the facts that the quadrature over the element K is exact for polynomials of degree $(2k)$ and the quadrature weights are non-negative, the second term can be estimated:

$$\begin{aligned}
& \sum_{K\in\mathscr{T}_h} \left| \sum_{j=1}^M \omega_j (\frac{1}{6} e^3 \mathbf{f}_u''' \cdot \nabla\xi)(x_{Kj}) \right| |K| \\
\leq & \sum_{K\in\mathscr{T}_h} \left((\sum_{j=1}^M |\omega_j| \left| (\frac{1}{6} e^3 \mathbf{f}_u''')(x_{Kj}) \right|^2 |K|)^{1/2} (\sum_{j=1}^M |\omega_j| |\nabla\xi(x_{Kj})|^2 |K|)^{1/2} \right), \\
= & \sum_{K\in\mathscr{T}_h} \left((\sum_{j=1}^M |\omega_j| \left| (\frac{1}{6} e^3 \mathbf{f}_u''')(x_{Kj}) \right|^2 |K|)^{1/2} \|\nabla\xi\|_{L^2(K)} \right), \\
\leq & C \|\mathbf{f}_u'''\|_\infty \|e\|_\infty^2 \sum_{K\in\mathscr{T}_h} \|e\|_{L^\infty(K)} |K|^{1/2} \|\nabla\xi\|_{L^2(K)}, \\
\leq & C \|\mathbf{f}_u'''\|_\infty \|e\|_\infty^2 \sum_{K\in\mathscr{T}_h} (\|\xi\|_{L^\infty(K)} + \|\eta\|_{L^\infty(K)}) h^{d/2} h^{-1} \|\xi\|_{L^2(K)},
\end{aligned}$$

$$
\begin{aligned}
&\le C h^{d/2-1} \|\mathbf{f}_u'''\|_\infty \|e\|_\infty^2 \big(\sum_{K\in\mathscr{T}_h} h^{-d/2} \|\xi\|_{L^2(K)} \|\xi\|_{L^2(K)} + h^{k+1} |u|_{W^{k+1,\infty}(\Omega)} \sum_{K\in\mathscr{T}_h} \|\xi\|_{L^2(K)}\big),\\
&\le C h^{-1} \|\mathbf{f}_u'''\|_\infty \|e\|_\infty^2 (1 + h^{k+1} |u|_{W^{k+1,\infty}(\Omega)}) \|\xi\|^2 .
\end{aligned}
$$

Hence, we have that

$$
\begin{aligned}
|X_6| \le & C h^{-1} \|\mathbf{f}_u'''\|_\infty \|e\|_\infty^2 \left(h^{2k+2} |u|_{H^{k+1}(\Omega)}^2 + (1 + h^{k+1} |u|_{W^{k+1,\infty}(\Omega)}) \|\xi\|^2\right),\\
\le & C h^{-1} \|\mathbf{f}_u'''\|_\infty \|e\|_\infty^2 (1 + h^{2k+2} |u|_{W^{k+1,\infty}(\Omega)}^2) \|\xi\|^2 .
\end{aligned}
$$

Finally, we collect the above estimates about $X_1, X_2, \cdots, X_6$ to complete the proof of Lemma 3.3. □

B Proof of Lemma 3.4

By Taylor expansion at (u,u) for the numerical flux $\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-, u_h^+)$, we have

$$
\begin{aligned}
\widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}(u_h^-, u_h^+) - \mathbf{f}(u)\cdot\mathbf{n}_\Gamma = & \eta^-\hat{\mathbf{f}}_1' - \xi^-\hat{\mathbf{f}}_1' + \frac{1}{2}(\eta^-)^2\hat{\mathbf{f}}_{11}'' - \eta^-\xi^-\hat{\mathbf{f}}_{11}'' + \frac{1}{2}(\xi^-)^2\hat{\mathbf{f}}_{11}'' - \frac{1}{6}(e^-)^3\hat{\mathbf{f}}_{111}'''\\
& + \eta^+\hat{\mathbf{f}}_2' - \xi^+\hat{\mathbf{f}}_2' + \frac{1}{2}(\eta^+)^2\hat{\mathbf{f}}_{22}'' - \eta^+\xi^+\hat{\mathbf{f}}_{22}'' + \frac{1}{2}(\xi^+)^2\hat{\mathbf{f}}_{22}'' - \frac{1}{6}(e^+)^3\hat{\mathbf{f}}_{222}'''\\
& + (\eta^- - \xi^-)(\eta^+ - \xi^+)\hat{\mathbf{f}}_{12}'' - \frac{1}{2}(e^-)^2 e^+\hat{\mathbf{f}}_{112}''' - \frac{1}{2}e^-(e^+)^2\hat{\mathbf{f}}_{122}''',
\end{aligned}
$$

where $\hat{\mathbf{f}} \equiv \widehat{\mathbf{f}\cdot\mathbf{n}_\Gamma}$ for short. The subscripts 1 and 2 denote the partial derivative with respect to the first and the second argument of $\hat{\mathbf{f}}$. The omitted argument in the first-order and second-order derivatives is (u,u). For instance, $\hat{\mathbf{f}}_{12}'' \equiv \frac{\partial^2\hat{\mathbf{f}}}{\partial u\partial v}(u,u)$. The third-order derivatives are the mean values.

Notice that the L^2 norm of v^- or v^+ $(v = \eta, \xi)$ on the edge will be controlled by the norm of v in $L^2(\mathscr{E}_h)$. Thus, for notation convenience, in the following estimate, we will not distinguish v^- or v^+ and write it as v in a uniformly way. That is, we will perform the estimate for

$$
Y_i := \sum_{\Gamma\in\mathscr{E}_h} |E_\Gamma(\psi_i[\xi])|, \quad i = 1, 2, \cdots, 6,
$$

with

$$
\eta\hat{\mathbf{f}}' + \xi\hat{\mathbf{f}}' + \eta^2\hat{\mathbf{f}}'' + \eta\xi\hat{\mathbf{f}}'' + \xi^2\hat{\mathbf{f}}'' + e^3\hat{\mathbf{f}}''' \triangleq \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6. \tag{B.31}
$$

By taking $q=2$, $p=k$, $s=k+2$, $G=\eta\hat{\mathbf{f}}'$ and $w=[\xi]$ in (2.10) in Lemma 2.4, and remembering that the quadrature over edges is exact for polynomials of degree $(2k+1)$, we obtain the estimate of Y_1:

$$\begin{aligned}
Y_1 \leq & \sum_{\Gamma\in\mathscr{E}_h} Ch^{k+2+(d-1)/2} \left|\eta\hat{\mathbf{f}}'\right|_{W^{k+2,\infty}(\Gamma)} \|[\xi]\|_{L^2(\Gamma)}, \\
\leq & Ch^{k+d/2+3/2} \|\eta\|_{W^{k+2,\infty}(\mathscr{E}_h)} \left\|\hat{\mathbf{f}}'\right\|_{W^{k+2,\infty}(\mathscr{E}_h)} \sum_{\Gamma\in\mathscr{E}_h} \|\xi\|_{L^2(\Gamma)}, \\
\leq & Ch^{k+3/2} |u|_{W^{k+1,\infty}(\Omega)} \left\|\hat{\mathbf{f}}'\right\|_{W^{k+2,\infty}(\mathscr{E}_h)} \|\xi\|_{L^2(\mathscr{E}_h)}, \\
\leq & C |u|_{W^{k+1,\infty}(\Omega)} \left\|\hat{\mathbf{f}}'\right\|_{W^{k+2,\infty}(\mathscr{E}_h)} (h^{2k+2}+\|\xi\|^2),
\end{aligned}$$

where we use Lemma 2.1 and the inverse property (2.7a) in the last inequality.

Now we proceed to the estimate of term Y_2. Taking $q=1$, $p=2k$, $s=1$, $G=\hat{\mathbf{f}}'$ and $w=\xi[\xi]$ in (2.10) in Lemma 2.4 obtains

$$\begin{aligned}
Y_2 \leq & \sum_{\Gamma\in\mathscr{E}_h} Ch \left|\hat{\mathbf{f}}'\right|_{W^{1,\infty}(K)} \left\|\xi^2\right\|_{L^1(\Gamma)}, \\
\leq & Ch \left|\hat{\mathbf{f}}'\right|_{W^{1,\infty}(\mathscr{E}_h)} \sum_{\Gamma\in\mathscr{E}_h} \|\xi\|^2_{L^2(\Gamma)} \leq C \left|\hat{\mathbf{f}}'\right|_{W^{1,\infty}(\mathscr{E}_h)} \|\xi\|^2.
\end{aligned}$$

Similar to the estimate of Y_1 and Y_2, it can be easily shown that

$$Y_3 \leq C |u|^2_{W^{k+1,\infty}(\Omega)} \left\|\hat{\mathbf{f}}''\right\|_{W^{1,\infty}(\mathscr{E}_h)} (h^{4k+2}+\|\xi\|^2), \tag{B.32}$$

$$Y_4 \leq Ch^k |u|_{W^{k+1,\infty}(\Omega)} \left\|\hat{\mathbf{f}}''\right\|_{W^{1,\infty}(\mathscr{E}_h)} \|\xi\|^2, \tag{B.33}$$

$$Y_5 \leq \begin{cases} C \left\|\hat{\mathbf{f}}''\right\|_{W^{2,\infty}(\mathscr{E}_h)} (\|e\|_\infty + h^{k+1} |u|_{W^{k+1,\infty}(\Omega)}) \|\xi\|^2, & k=1, \\ Ch^{-1} \left\|\hat{\mathbf{f}}''\right\|_{W^{2,\infty}(\mathscr{E}_h)} (\|e\|_\infty + h^{k+1} |u|_{W^{k+1,\infty}(\Omega)}) \|\xi\|^2. & k\geq 2. \end{cases} \tag{B.34}$$

Note that some careful treatment should be used in the estimate of Y_5, which is similar to that of X_5 in the proof of Lemma 3.3.

The last term Y_6 is divided into two part and the first integral term is estimated as follows:

$$\sum_{\Gamma\in\mathscr{E}_h} \left|\int_\Gamma e^3\hat{\mathbf{f}}'''[\xi]\mathrm{d}x\right| \leq C \|e\|^2_\infty \left\|\hat{\mathbf{f}}'''\right\|_\infty \sum_{\Gamma\in\mathscr{E}_h} \left|\int_\Gamma |e|\,|\xi|\,\mathrm{d}x\right|,$$

$$\leq C \|e\|_\infty^2 \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_{L^2(\mathscr{E}_h)} \|\xi\|_{L^2(\mathscr{E}_h)},$$
$$\leq C h^{-1} \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_\infty^2 (h^{2k+2} |u|_{H^{k+1}(\Omega)}^2 + \|\xi\|^2),$$

where we use Lemma 2.1, the interpolation inequality (2.6a) and the inverse property (2.7a) in the last step. The second quadrature term could be bounded by

$$\sum_{\Gamma \in \mathscr{E}_h} \left| \sum_{j=1}^{L} \underline{\omega}_j (e^3 \hat{\mathbf{f}}'''[\xi])(x_{\Gamma j}) \right| |\Gamma|$$
$$\leq \sum_{\Gamma \in \mathscr{E}_h} \left((\sum_{j=1}^{L} |\underline{\omega}_j| \left|(e^3 \hat{\mathbf{f}}''')(x_{\Gamma j})\right|^2 |\Gamma|)^{1/2} (\sum_{j=1}^{L} |\underline{\omega}_j| \, |[\xi](x_{\Gamma j})|^2 |\Gamma|)^{1/2} \right),$$
$$= \sum_{\Gamma \in \mathscr{E}_h} \left((\sum_{j=1}^{L} |\underline{\omega}_j| \left|(e^3 \hat{\mathbf{f}}''')(x_{\Gamma j})\right|^2 |\Gamma|)^{1/2} \|[\xi]\|_{L^2(\Gamma)} \right),$$
$$\leq C \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_\infty^2 \sum_{\Gamma \in \mathscr{E}_h} \|e\|_{L^\infty(\Gamma)} |\Gamma|^{1/2} \|[\xi]\|_{L^2(\Gamma)},$$
$$\leq C \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_\infty^2 \sum_{K \in \mathscr{T}_h} \|e\|_{L^\infty(K)} |\partial K|^{1/2} \|\xi\|_{L^2(\partial K)},$$
$$\leq C \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_\infty^2 \sum_{K \in \mathscr{T}_h} \|e\|_{L^\infty(K)} (h^{d-1})^{1/2} h^{-1/2} \|\xi\|_{L^2(K)},$$
$$\leq C \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_\infty^2 \sum_{K \in \mathscr{T}_h} (\|\xi\|_{L^\infty(K)} + \|\eta\|_{L^\infty(K)}) h^{d/2-1} \|\xi\|_{L^2(K)},$$
$$\leq C h^{d/2-1} \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_\infty^2 (\sum_{K \in \mathscr{T}_h} h^{-d/2} \|\xi\|_{L^2(K)} \|\xi\|_{L^2(K)} + h^{k+1} |u|_{W^{k+1,\infty}(\Omega)} \sum_{K \in \mathscr{T}_h} \|\xi\|_{L^2(K)}),$$
$$\leq C h^{-1} \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_\infty^2 (1 + h^{k+1} |u|_{W^{k+1,\infty}(\Omega)}) \|\xi\|^2.$$

In the equality, we use the fact that the quadrature weights $\underline{\omega}_j \geq 0$ and the quadrature rule over edges is exact for polynomials of degree $(2k+1)$. Hence, we have

$$Y_6 \leq C h^{-1} \left\|\hat{\mathbf{f}}'''\right\|_\infty \|e\|_\infty^2 (1 + h^{2k+2} |u|_{W^{k+1,\infty}(\Omega)}^2) \|\xi\|^2. \tag{B.35}$$

Combining the estimate of Y_1, Y_2, $\cdots$, Y_6 completes the proof of Lemma 3.4.

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