Stability analysis and error estimates of Lax-Wendroff discontinuous Galerkin methods for linear conservation laws[1]

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Abstract

In this paper, we analyze the Lax-Wendroff discontinuous Galerkin (LWDG) method for solving linear conservation laws. The method was originally proposed by Guo et al. in [11], where they applied local discontinuous Galerkin (LDG) techniques to approximate high order spatial derivatives in the Lax-Wendroff time discretization. We show that, under the standard CFL condition $\tau \leq \lambda h$ (where τ and h are the time step and the maximum element length respectively and $\lambda > 0$ is a constant) and uniform or non-increasing time steps, the second order schemes with piecewise linear elements and the third order schemes with arbitrary piecewise polynomial elements are stable in the L^2 norm. The specific type of stability may differ with different choices of numerical fluxes. Our stability analysis includes multidimensional problems with divergence-free coefficients. Besides solving the equation itself, the LWDG method also gives approximations to its time derivative simultaneously. We obtain optimal error estimates for both the solution u and its first order time derivative u_t in one dimension, and numerical examples are given to validate our analysis.

Keywords: Discontinuous Galerkin method; Lax-Wendroff time discretization; linear conservation laws; L^2-stability; error estimates

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1 Introduction

In this paper, we present stability analysis and error estimates for the Lax-Wendroff discontinuous Galerkin (LWDG) method solving linear conservation laws. We concentrate our attention in the scalar case, although the analysis can be easily generalized to one-dimensional linear hyperbolic systems and multidimensional symmetric linear systems. To be more specific, we are interested in the following initial value problem

$$\begin{cases} u_t = \nabla\cdot(\boldsymbol{\beta} u), & (\boldsymbol{x},t)\in\Omega\times(0,T), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega. \end{cases} \tag{1.1}$$

For simplicity, periodic boundary conditions are assumed, but our analysis does not depend on periodicity and can be extended to other types of boundary conditions. Here, $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{x})$ is a vector-valued function with the divergence-free condition $\nabla\cdot\boldsymbol{\beta}(\boldsymbol{x}) = 0$. In this setting, the L^2 energy of the solution to (1.1) is conserved, which facilitates the study of LWDG schemes using energy methods. Note that the divergence-free condition forces the coefficient in (1.1) to be constant in the one-dimensional case, but it allows variable coefficients in multidimensions.

The discontinuous Galerkin (DG) methods are a class of finite element methods using discontinuous piecewise polynomial spaces, which were originally designed and most suitable for solving hyperbolic conservation laws. The discontinuity at cell interfaces brings extra degrees of freedom to the choice of numerical fluxes and to enforce conservation locally. On the other hand, the finite-element nature allows the methods to suit complicated geometries and boundary conditions. A major development of DG methods for solving time-dependent nonlinear conservation laws was carried out by Cockburn et al. in a series of papers [6, 5, 4, 3, 7]. Later on, to adapt the methods to equations with diffusion terms, Cockburn and Shu developed local discontinuous Galerkin (LDG) methods [8], based on the successful numerical experiments of Bassi and Rebay in [1], in which they introduced auxiliary variables to transform the original high order equation into a first order system. The LDG technique can be used to handle even higher order spatial derivatives, more related work can be found in, e.g. [15, 14].

The DG spatial discretization can be combined with different time integrators. The most widely used ones are the strong-stability-preserving Runge-Kutta methods [10]. As an alternative, one can also choose the Lax-Wendroff type time discretization, which relies on converting time derivatives in a truncated Taylor expansion into spatial derivatives by using the differential equation repeatedly, resulting in LWDG methods. The first LWDG method for solving hyperbolic conservation laws was proposed by Qiu et al. in [13]. In their schemes, the high order derivatives are approximated by directly differentiating the numerical polynomials. In [11], Guo et al. pointed out that this method does not exhibit the superconvergence property, so they developed a new LWDG method by applying the LDG techniques. More specifically, they introduced new auxiliary variables and used DG spatial discretization to approximate the spatial derivatives converted from time derivatives in the Lax-Wendroff procedure. In this paper, we focus on the LWDG method in [11] and study its stability and accuracy properties for solving linear scalar conservation laws.

Firstly, let us remark that there exist close relationships between the LWDG method and the Runge-Kutta discontinuous Galerkin (RKDG) method. Both of these methods introduce auxiliary or intermediate variables. In the RKDG method, the stage variables correspond to the solution u at different internal time stages between two time levels; while in the LWDG method, the intermediate variables approximate time derivatives of u. In general, these two sets of variables do not contain the same information. But for linear conservation laws not explicitly depending on time (including multidimensional systems), if we use the same fluxes throughout the LWDG scheme, then it will be equivalent to the RKDG method after one full time step. More specifically, in this case, the approximations of time derivatives in a LWDG scheme and the stage variables in a RKDG scheme can be expressed as a linear combination of each other, and they lead to the same numerical solution at the next time level. More details are given in the appendix. Therefore, there are strong connections between our analysis and the work by Zhang and Shu in [17, 20, 19], where they have provided the stability analysis and a prior error estimates for the second and third order RKDG schemes. We will comment on these connections later in the paper.

Next, we move on to the stability analysis. In the LWDG schemes in [11], Guo et al. applied alternating fluxes as that in [8, 15]. However, it can be shown that the LWDG schemes based on these alternating fluxes cannot preserve strong stability under the standard CFL condition $\tau \leq \lambda h$, where τ and h are the time step and the maximum element length respectively and $\lambda > 0$ is a constant. This reminds us of the well-known fact that the choice of numerical fluxes has an essential influence on the types of stability. We assume the numerical fluxes can be either upwind or downwind for each variable. After a detailed analysis, we find that, if uniform or non-increasing time steps are used, for the second order schemes (LW2DG) with piecewise linear elements, and the third order schemes (LW3DG) with arbitrary $\mathcal{P}_k$ elements, as long as we apply the upwind flux for u, then the schemes will be stable in the L^2 norm. Furthermore, if we also use the upwind flux for the second variable p (which approximates u_t), then the schemes will be strongly stable, that is $\|u_h^{n+1}\| \leq \|u_h^n\|$. Notice that, for the linear cases, the RKDG method belongs to this class. However, if we use the downwind flux for p (e.g. according to the choice of alternating fluxes), then the scheme is only stable in a weaker sense, namely $\|u_h^n\| \leq C\|u_h^0\|$, where C is a constant which depends on the CFL number and the inverse estimate constants, but is independent of the total time T. Therefore, with both choices of the fluxes, the energy of the solution is well-controlled after long time integration.

Finally, we perform error estimates under the same framework of the stability analysis in one dimension. We highlight that optimal error estimates of both u_h and p_h can be obtained, where u_h and p_h are numerical approximations to u and u_t respectively. To be more precise, assuming u is a smooth solution to the one-dimensional problem, under a proper CFL condition $\tau \leq \lambda h$, we have both $\|u - u_h\|_{L^2} \leq C(\tau^2 + h^2)$ and $\|u_t - p_h\| \leq C(\tau^2 + h^2)$ for the LW2DG schemes with piecewise linear elements; and $\|u - u_h\|_{L^2} \leq \mathcal{O}(\tau^3 + h^{k+1})$ and $\|u_t - p_h\| \leq \mathcal{O}(\tau^3 + h^{k+1})$ for the LW3DG schemes with $\mathcal{P}_k$ elements. The error estimates hold for both choices of the numerical fluxes.

The organization of this paper is as follows. In section 2, the notations and preliminaries are introduced for the one-dimensional analysis. In section 3, we present the stability analysis of the LW2DG schemes and the LW3DG schemes for one-dimensional advection equations, as well as the extension to multidimensions with divergence-free coefficients. In section 4, we state the error estimates for one-dimensional problems and give a detailed proof only for the second order schemes. In section 5, numerical examples are given to validate our error estimates.

2 Notations and preliminaries

2.1 Model problem and notations

We study the following model problem in our one-dimensional analysis

$$\begin{cases} u_t = u_x, & (x,t) \in (0,2\pi) \times (0,T), \\ u(x,0) = u_0(x), & x \in (0,2\pi). \end{cases} \tag{2.1}$$

with periodic boundary conditions.

Let $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ be a regular partition of the domain. We denote the cell length by $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, and define $h = \max_j h_j$. We use the notation $(w,v)_j = \int_{I_j} wv\, dx$ for the L^2 inner product on I_j, and define $(w,v) = \sum_j (w,v)_j$. Without specification, the notation $\|\cdot\|$ refers to the L^2 norm.

The associated discontinuous finite element space is defined as

$$V_h = \{v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_k(I_j), \forall j\},$$

where $\mathcal{P}_k(I_j)$ is the space of polynomials on I_j of degree at most k. Note that functions in V_h can be double-valued at cell interfaces, so we use v^+ and v^- to represent the right and left limits of v respectively. The jump is denoted by $[v] = v^+ - v^-$. We define the jump semi-norm on V_h as $[\![v]\!] = \sqrt{\sum_j [v]^2_{j+\frac{1}{2}}}$, and the associated bilinear form as $[w,v] = \sum_j [w]_{j+\frac{1}{2}}[v]_{j+\frac{1}{2}}$.

One has the inverse estimate for polynomials, which states $\forall v \in \mathcal{P}_k(I_j)$,

$$h\|v_x\|_{I_j} + h^{\frac{1}{2}}\|v\|_{\partial I_j} \leq \mu\|v\|_{I_j},$$

where $\|v\|_{\partial I_j} = \sqrt{(v^+_{j-\frac{1}{2}})^2 + (v^-_{j+\frac{1}{2}})^2}$ and μ is the inverse estimate constant, which is independent of h. We denote a constant independent of h but depending on μ by C_μ. Similarly, we may use $C_{\mu,\lambda}$, $C_{\mu,\lambda,\boldsymbol{\beta}}$ and so on. The subscripts indicate the dependency of the constant C on these parameters.

2.2 The LWDG method

For simplicity, we assume the time step τ is a constant for different time levels throughout this paper. The strong stability results however do not need this restriction, while the weaker stability results (the energy is bounded but not necessarily non-increasing) do need the restriction that the time steps are non-increasing, for which the constant time step is a special case.

For the second order LWDG method, we first approximate the original equation (2.1) with the following first order system

$$\begin{aligned} u^{n+1} &= u^n + \tau u^n_x + \frac{\tau^2}{2}p^n_x, \\ p^n &= u^n_x. \end{aligned} \tag{2.2}$$

Then the fully discretized scheme can be obtained by replacing the spatial derivatives by their DG discretizations. For convenience, we introduce the linear operators $\mathcal{H}^+$ and $\mathcal{H}^-$,

$$\mathcal{H}^{\pm}(w,v) = \sum_j \mathcal{H}^{\pm}_j(w,v) = \sum_j \left((w,v_x)_j - w^{\pm}_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}} + w^{\pm}_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}}\right).$$

As a convention, $\mathcal{H}^{\pm 1} = \mathcal{H}^{\pm}$. Then, the schemes will take the form

$$\begin{aligned} (u^{n+1}_h, \varphi_h) &= (u^n_h, \varphi_h) - \tau\mathcal{H}^{\alpha_u}(u^n_h, \varphi_h) - \frac{\tau^2}{2}\mathcal{H}^{\alpha_p}(p^n_h, \varphi_h), \\ (p^n_h, \psi_h) &= -\mathcal{H}^{\widehat{\alpha_u}}(p^n_h, \psi_h), \end{aligned} \tag{2.3}$$

where $\varphi_h, \psi_h \in V_h$, and α_u, α_p and $\widehat{\alpha_u}$ are equal to $+1$ or -1, corresponding to different numerical fluxes, to be specified later.

In our analysis, we consider different choices of the numerical fluxes, and require $\alpha_u = \widehat{\alpha_u}$. When $\alpha_u = \widehat{\alpha_u} = +1$ and $\alpha_p = -1$, we recover the alternating fluxes used in [11]; when $\alpha_u = \widehat{\alpha_u} = \alpha_p = +1$, we can easily verify that this is actually the classical second order RKDG scheme. The only superficial difference is that, for the RKDG scheme, one introduces the stage variable $u_h^{(1)} = u_h^n + \tau p_h^n$ instead of p_h^n in our context.

Similarly, one can write down the third order schemes,

$$\begin{aligned} (u_h^{n+1}, \varphi_h) &= (u_h^n, \varphi_h) - \tau \mathcal{H}^{\alpha_u}(u_h^n, \varphi_h) - \frac{\tau^2}{2}\mathcal{H}^{\alpha_p}(p_h^n, \varphi_h) - \frac{\tau^3}{6}\mathcal{H}^{\alpha_q}(q_h^n, \varphi_h), \\ (p_h^n, \psi_h) &= -\mathcal{H}^{\widehat{\alpha_u}}(u_h^n, \psi_h), \\ (q_h^n, \eta_h) &= -\mathcal{H}^{\widehat{\alpha_p}}(p_h^n, \eta_h), \end{aligned} \tag{2.4}$$

where φ_h, ψ_h and η_h are test functions from V_h, and α_u, α_p, $\widehat{\alpha_u}$ and $\widehat{\alpha_p}$ are equal to $+1$ or -1, corresponding to different numerical fluxes. We would again require $\alpha_u = \widehat{\alpha_u}$ and $\alpha_p = \widehat{\alpha_p}$. When $\alpha_u = +1$ and $\alpha_p = \alpha_q = -1$, we restore the alternating fluxes in [11]; while if $\alpha_u = \alpha_p = \alpha_q = +1$, we can easily verify that this is actually the classical third order RKDG scheme, where the stage variables are $u_h^{(1)} = u_h^n + \tau p_h^n$ and $u_h^{(2)} = u_h^n + \frac{\tau}{2} p_h^n + \frac{\tau^2}{4} q_h^n$.

We are interested in the stability of these schemes. We call a scheme to be stable, if there is a constant C independent of τ and h, but may depend on the total time $T = n\tau$, such that $\|u_h^n\| \leq C\|u_h^0\|$. We say a scheme is strongly stable, if $\|u_h^{n+1}\| \leq \|u_h^n\|$ for any n. As a corollary, we have $C = 1$ for a strongly stable scheme in $\|u_h^n\| \leq C\|u_h^0\|$.

2.3 Properties of $\mathcal{H}$

In this subsection, we familiarize the readers with the properties of the operator pair $\mathcal{H}^+$ and $\mathcal{H}^-$, which are fundamental for the analyses later.

The first two lemmas focus on the operators themselves. Lemma 2.1 describes the relationships between $\mathcal{H}^+$ and $\mathcal{H}^-$. It states the anti-symmetry, the semi-definiteness and the difference of the operator pair. Lemma 2.2 estimates the L^2 operator norms of $\mathcal{H}^\pm(w, \cdot)$ and $\mathcal{H}^\pm(\cdot, v)$ on V_h. These lemmas are classical and can be proved straightforwardly, we refer interested readers to, e.g. [16].

Lemma 2.1. *Suppose $w, v \in V_h$, then*

$$\mathcal{H}^-(w,w) = -\frac{1}{2}[\![w]\!]^2, \tag{2.5}$$

$$\mathcal{H}^-(w,v) = -\mathcal{H}^+(v,w), \tag{2.6}$$

$$\mathcal{H}^+(w,v) = \mathcal{H}^-(w,v) + [w,v]. \tag{2.7}$$

Lemma 2.2. *Suppose $w, v \in V_h$, then*

$$|\mathcal{H}^\pm(w,v)| \le (\|w_x\| + C_\mu h^{-\frac{1}{2}}[\![w]\!])\|v\|, \tag{2.8a}$$

$$|\mathcal{H}^\pm(w,v)| \le (\|v_x\| + C_\mu h^{-\frac{1}{2}}[\![v]\!])\|w\|. \tag{2.8b}$$

The next two lemmas describe the numerical spatial derivatives from *DG* and *LDG* discretizations. Lemma 2.3 gives a crude bound for the first order derivative, and it follows directly from Lemma 2.2 and the inverse estimate. Lemma 2.4 establishes the connections among different orders of derivatives, which is essentially the discretized version of integration by parts. To cover different choices of fluxes, we introduce undetermined parameters α's in this lemma, which can be either $+1$ or -1. The detailed proof is omitted, as it just amounts to repeatedly using Lemma 2.1.

Lemma 2.3. *Let $u, p \in V_h$. For any test functions $\psi \in V_h$, $(p,\psi) = -\mathcal{H}^{\alpha_u}(u,\psi)$. Then*

$$\|p\| \le C_\mu h^{-1}\|u\|. \tag{2.9}$$

Lemma 2.4. *Let $u, p, q, r \in V_h$, such that for any test functions $\psi, \eta, \zeta \in V_h$,*

$$(p,\psi) = -\mathcal{H}^{\alpha_u}(u,\psi), \tag{2.10a}$$

$$(q,\eta) = -\mathcal{H}^{\alpha_p}(p,\eta), \tag{2.10b}$$

$$(r,\zeta) = -\mathcal{H}^{\alpha_q}(q,\zeta), \tag{2.10c}$$

where $\alpha_u, \alpha_p, \alpha_q = -1, +1$. Then we have,

(i)

$$(p,u) = -\frac{\alpha_u}{2}[\![u]\!]^2, \tag{2.11a}$$

$$(q,u) = -\|p\|^2 - \frac{\alpha_{up}}{2}[u,p], \tag{2.11b}$$

$$(r,u) = \frac{\alpha_p}{2}[\![p]\!]^2 - \frac{\alpha_{uq}}{2}[u,q], \tag{2.11c}$$

(ii)

$$(q,p) = -\frac{\alpha_p}{2}[\![p]\!]^2, \tag{2.12a}$$

$$(r,p) = -\|q\|^2 - \frac{\alpha_{pq}}{2}[p,q], \tag{2.12b}$$

where $\alpha_{wv} = \alpha_w + \alpha_v,\ w,v = u,p,q.$

3 Stability analysis

3.1 Second order schemes

The LW2DG schemes can be rewritten as follows. Given u_h^n, find $u_h^{n+1} \in V_h$, such that

$$(u_h^{n+1}, \varphi_h) = (u_h^n, \varphi_h) + \tau(p_h^n, \varphi_h) + \frac{\tau^2}{2}(q_h^n, \varphi_h), \tag{3.1a}$$

$$(p_h^n, \psi_h) = -\mathcal{H}^{\alpha_u}(u_h^n, \psi_h), \tag{3.1b}$$

$$(q_h^n, \eta_h) = -\mathcal{H}^{\alpha_p}(p_h^n, \eta_h), \tag{3.1c}$$

for any $\varphi_h, \psi_h, \eta_h \in V_h$.

Note that (3.1a) implies

$$u_h^{n+1} = u_h^n + \tau p_h^n + \frac{\tau^2}{2} q_h^n. \tag{3.2}$$

In our analysis, we restrict the finite element space to be piecewise linear. This is because when one uses upwind fluxes for both u_h^n and p_h^n, the scheme is equivalent to the second order RKDG scheme, which, according to the von Neumann analysis in [9], is unstable with high order ($k > 1$) piecewise polynomial spaces when τ/h is a constant.

The key ingredient for the stability analysis is to use the specialty of piecewise linear elements. In this case, the L^2 norm of $(p_h^n)_x$ can be bounded by the jump of u_h^n, which is not necessarily true in general. Zhang and Shu first used this technique in [19] to analyze the second order RKDG scheme, but for a different auxiliary variable.

Here is the outline for the stability analysis of (3.1). We first prove Lemma 3.1 to connect $\|(p_h^n)_x\|$ with $[\![u_h^n]\!]$. Then, with this lemma, precise estimates for $\|u_h^{n+1} - u_h^n - \tau p_h^n\|$ and $\|p_h^{n+1} - p_h^n\|$ can be obtained, which are stated in Lemma 3.2 and Lemma 3.3 respectively. Finally, we carry out the proof of Theorem 3.1 to establish the stability of the LW2DG schemes. The first part of this theorem, which essentially rebuilds the result by Zhang and Shu in [19], only uses Lemma 3.2. For the second part with alternating fluxes, Lemma 3.3 is used as well.

Lemma 3.1. *With $\mathcal{P}_1$ elements, $\|(p_h^n)_x\|_j = 6h_j^{-\frac{3}{2}}|[u_h^n]_{j+\frac{\alpha_u}{2}}|$.*

Proof. We only prove the lemma for $\alpha_u = +1$, the case $\alpha_u = -1$ follows along the same line.

For simplicity, we drop all the subscripts h and superscripts n. Let $\{\phi^0, \phi^1\}$ be the normalized Legendre polynomials basis on I_j, and ϕ^i is of degree i. Suppose $p = p_0\phi^0 + p_1\phi^1$ on I_j. Then we have $p_x = 2\sqrt{3}h^{-\frac{3}{2}}p_1$, where p_1 can be obtained through (3.1b),

$$p_1 = (p, \phi^1)_j = -(u, \phi_x^1)_j + u^+_{j+\frac{1}{2}}(\phi^1)^-_{j+\frac{1}{2}} - u^+_{j-\frac{1}{2}}(\phi^1)^+_{j-\frac{1}{2}}.$$

With integration by parts, we get

$$p_1 = (u_x, \phi^1)_j - u^-_{j+\frac{1}{2}}(\phi^1)^-_{j+\frac{1}{2}} + u^+_{j+\frac{1}{2}}(\phi^1)^-_{j+\frac{1}{2}} = [u]_{j+\frac{1}{2}}(\phi^1)^-_{j+\frac{1}{2}}, \tag{3.3}$$

where the second equality holds due to the fact that u_x is a constant. Note that $(\phi^1_{j+\frac{1}{2}})^- = \sqrt{3}h_j^{-\frac{1}{2}}$. Now (3.3) implies $p_1 = \sqrt{3}h_j^{-\frac{1}{2}}[u]_{j+\frac{1}{2}}$. Therefore $p_x = 6h_j^{-2}[u]_{j+\frac{1}{2}}$ and $\|p_x\|_j = 6h_j^{-\frac{3}{2}}|[u]_{j+\frac{1}{2}}|$. □

The following statements hold based on Lemma 3.1.

Lemma 3.2.

$$\|u_h^{n+1} - u_h^n - \tau p_h^n\|^2 \le C_\mu(\tau\lambda^3[\![u_h^n]\!]^2 + \tau^3\lambda[\![p_h^n]\!]^2).$$

Proof. Rearranging (3.1) gives

$$(u_h^{n+1} - u_h^n - \tau p_h^n, \varphi_h) = -\frac{\tau^2}{2}\mathcal{H}^{\alpha_p}(p_h^n, \varphi_h).$$

Exploiting Lemma 2.2, we obtain

$$(u_h^{n+1} - u_h^n - \tau p_h^n, \varphi_h) \le \frac{\tau^2}{2}\big(\|(p_h^n)_x\| + C_\mu h^{-\frac{1}{2}}[\![p_h^n]\!]\big)\|\varphi_h\|.$$

Then the proof is completed by applying Lemma 3.1 and setting $\varphi_h = u_h^{n+1} - u_h^n - \tau p_h^n$. □

Lemma 3.3.

$$\|p_h^{n+1} - p_h^n\|^2 \le C_\mu(h^{-1}\lambda^2[\![u_h^n]\!]^2 + \tau\lambda[\![p_h^n]\!]^2 + \tau^2\lambda^2\|q_h^n\|^2).$$

Proof. According to (3.2), the following relationship holds

$$(p_h^{n+1} - p_h^n, \psi_h) = -\mathcal{H}^{\alpha_u}(u_h^{n+1} - u_h^n, \psi_h) = -\mathcal{H}^{\alpha_u}(\tau p_h^n + \frac{\tau^2}{2}q_h^n, \psi_h). \tag{3.4}$$

Then we apply Lemma 2.2 and Lemma 2.3 to obtain

$$(p_h^{n+1} - p_h^n, \psi_h) \le C_\mu(\tau\|(p_h^n)_x\| + \tau h^{-\frac{1}{2}}[\![p_h^n]\!]^2 + \tau\lambda\|q_h^n\|)\|\psi_h\|.$$

The proof is completed by applying Lemma 3.1 and plugging in $\psi_h = p_h^{n+1} - p_h^n$. □

We are now ready to state our main theorem for the stability of the LW2DG schemes.

Theorem 3.1 (Stability of the LW2DG schemes)**.** *There exists a constant λ, such that when $\tau \le \lambda h$, the numerical solution of schemes* (3.1) *satisfies,*

(i) $\|u_h^{n+1}\| \le \|u_h^n\|$, *if* $\alpha_u = \alpha_p = +1$,

(ii) $\|u_h^n\| \le C_{\mu,\lambda}\|u_h^0\|$, *if* $\alpha_u = +1$ *and* $\alpha_p = -1$,

where $C_{\mu,\lambda}$ is a constant depending on λ and the inverse estimate constant μ, but is independent of the total time $T = n\tau$.

Proof. Take $\varphi_h = u_h^n$ in (3.1a). Using (2.4) we get

$$\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 + \frac{\tau\alpha_u}{2}[\![u_h^n]\!]^2 + \frac{\tau^2\alpha_{up}}{2}[u_h^n, p_h^n] = \frac{1}{2}\|u_h^{n+1} - u_h^n\|^2 - \frac{\tau^2}{2}\|p_h^n\|^2.$$

Note that

$$\|u_h^{n+1} - u_h^n\|^2 - \tau^2\|p_h^n\|^2 \;=\; \|u_h^{n+1} - u_h^n - \tau p_h^n\|^2 + 2\tau(u_h^{n+1} - u_h^n - \tau p_h^n, p_h^n),$$

where $(u_h^{n+1}-u_h^n-\tau p_h^n, p_h^n)=\frac{\tau^2}{2}(q_h^n,p_h^n)=-\frac{\tau^2\alpha_p}{4}[\![p_h^n]\!]^2$. Hence

$$\|u_h^{n+1}\|^2-\|u_h^n\|^2+\tau\alpha_u[\![u_h^n]\!]^2+\tau^2\alpha_{up}[u_h^n,p_h^n]+\frac{\tau^3\alpha_p}{2}[\![p_h^n]\!]^2=\|u_h^{n+1}-u_h^n-\tau p_h^n\|^2. \tag{3.5}$$

Exploiting Lemma 3.2, we have

$$\|u_h^{n+1}\|^2-\|u_h^n\|^2+\tau(\alpha_u-C_\mu\lambda^3)[\![u_h^n]\!]^2+\tau^2\alpha_{up}[u_h^n,p_h^n]+\tau^3(\frac{\alpha_p}{2}-C_\mu\lambda)[\![p_h^n]\!]^2\le 0. \tag{3.6}$$

(i) When $\alpha_u=\alpha_p=+1$, (3.6) can be rewritten as

$$\|u_h^{n+1}\|^2-\|u_h^n\|^2+\tau(\frac{1}{4}-C_\mu\lambda^3)[\![u_h^n]\!]^2+\tau^3(\frac{1}{6}-C_\mu\lambda)[\![p_h^n]\!]^2+\tau[\![\frac{\sqrt{3}}{2}u_h^n+\tau\frac{\sqrt{3}}{3}p_h^n]\!]^2\le 0,$$

which implies $\|u_h^{n+1}\|\le\|u_h^n\|$ when λ is sufficiently small.

(ii) When $\alpha_u=+1$ and $\alpha_p=-1$, (3.6) gives

$$\|u_h^{n+1}\|^2-\|u_h^n\|^2+\tau(1-C_\mu\lambda^3)[\![u_h^n]\!]^2-\tau^3(\frac{1}{2}+C_\mu\lambda)[\![p_h^n]\!]^2\le 0. \tag{3.7}$$

The coefficient of $[\![p_h^n]\!]^2$ is negative, and one needs to bring in new terms to balance it. To this end, we plug in $\psi_h=p_h^n$ in (3.4) to get

$$\frac{1}{2}\|p_h^{n+1}\|^2-\frac{1}{2}\|p_h^n\|^2-\frac{1}{2}\|p_h^{n+1}-p_h^n\|^2=-\frac{\tau}{2}[\![p_h^n]\!]^2-\frac{\tau^2}{2}\|q_h^n\|^2.$$

By Lemma 3.3, this gives

$$\|p_h^{n+1}\|^2-\|p_h^n\|^2-C_\mu\lambda^2h^{-1}[\![u_h^n]\!]^2\le-\tau(1-C_\mu\lambda)[\![p_h^n]\!]^2-\tau^2(1-C_\mu\lambda)\|q_h^n\|^2. \tag{3.8}$$

Multiplying (3.8) with τ^2 and adding it to (3.7), we have

$$\begin{aligned}(\|u_h^{n+1}\|^2+\tau^2\|p_h^{n+1}\|^2)-(\|u_h^n\|^2+\tau^2\|p_h^n\|^2)+\tau(1-C_\mu\lambda^3)[\![u_h^n]\!]^2\\ +\tau^3(\frac{1}{2}-C_\mu\lambda)[\![p_h^n]\!]^2+\tau^4(1-C_\mu\lambda)\|q_h^n\|^2\le 0.\end{aligned}$$

When λ is small enough, this gives

$$\|u_h^{n+1}\|^2+\tau^2\|p_h^{n+1}\|^2\le\|u_h^n\|^2+\tau^2\|p_h^n\|^2.$$

Specially, when uniform time steps are used,

$$\|u_h^n\|^2+\tau^2\|p_h^n\|^2\le\|u_h^0\|^2+\tau^2\|p_h^0\|^2\le(1+C_\mu\lambda^2)\|u_h^0\|^2.$$

The last inequality holds due to Lemma 2.3. This implies the stability result $\|u_h^n\| \leq \sqrt{1 + C_\mu \lambda^2}\|u_h^0\|$. Clearly, the result also holds if the time steps are non-increasing, namely $\tau_{n+1} \leq \tau_n$. □

Remark 3.1. *We require that the time steps are non-increasing to ensure the stability in (ii). One should note this is only a sufficient but not necessary condition. Let $\{\tau_n\}$ be the sequence of time steps and assume $0 < \lambda_0 < \frac{\tau_n}{h} \leq \lambda$. The scheme is still stable in the following cases.*

Case 1: The time steps increase occasionally. Suppose $\tau_{n+1} \leq \tau_n$ except for $n = n_0$, then

$$\|u_h^{n_0+1}\|^2 + \tau_{n_0+1}^2\|p_h^{n_0+1}\|^2 \leq \frac{\tau_{n_0+1}^2}{\tau_{n_0}^2}(\|u_h^{n_0+1}\|^2 + \tau_{n_0}^2\|p_h^{n_0+1}\|^2) \leq \frac{\tau_{n_0+1}^2}{\tau_{n_0}^2}(\|u_h^{n_0}\|^2 + \tau_{n_0}^2\|p_h^{n_0}\|^2).$$

One can continue the inequality by including the factor $\frac{\tau_{n_0+1}^2}{\tau_{n_0}^2}$, and finally get

$$\|u_h^n\|^2 \leq \frac{\tau_{n_0+1}^2}{\tau_{n_0}^2}(1 + C_\mu\lambda^2)\|u_h^0\|^2 \leq \frac{\lambda^2}{\lambda_0^2}(1 + C_\mu\lambda^2)\|u_h^0\|^2.$$

Similarly, if there is a fixed number m, such that the time steps increase no more than m times, one can show that $\|u_h^n\|^2 \leq \frac{\lambda^{2m}}{\lambda_0^{2m}}(1 + C_\mu\lambda^2)\|u_h^0\|^2$.

Case 2: The time steps increase monotonically. Using the argument in case 1, one has

$$\|u_h^n\|^2 + \tau_n^2\|p_h^n\|^2 \leq (\prod_{i=1}^{n}\frac{\tau_i^2}{\tau_{i-1}^2})(\|u_h^0\|^2 + \tau_0^2\|p_h^0\|^2) = \frac{\tau_n^2}{\tau_0^2}(\|u_h^0\|^2 + \tau_0^2\|p_h^0\|^2) \leq \frac{\lambda^2}{\lambda_0^2}(1 + C\lambda^2)\|u_h^0\|^2,$$

which is still stable.

However, these arguments can not be used for general time steps $\{\tau_n\}$. For example, $\tau_{2n+1} = \lambda h$ and $\tau_{2n+2} = \frac{\lambda}{2}h$. The factor will blow up when h goes to 0. One would need to make more detailed discussions to analyze the stability.

In the following sections, we will also assume the non-increasing time steps when analyzing the stability of the LWDG schemes with alternating fluxes. One should note the argument above will still work, and we will not repeat it.

Remark 3.2. *We note that the alternating fluxes in (ii) actually do not preserve the strong stability. To see this, one can plug in a continuous initial condition u_h^0 into (3.1). Then, by (3.5), $\|u_h^1\|^2 - \|u_h^0\|^2 = \frac{\tau^3}{2}[\![p_h^n]\!]^2 + \|u_h^{n+1} - u_h^n - \tau p_h^n\|^2 > 0$.*

3.2 Third order schemes

We can rewrite LW3DG schemes as follows. In each step, find $u_h^{n+1} \in V_h$, such that

$$(u_h^{n+1}, \varphi_h) = (u_h^n, \varphi_h) + \tau(p_h^n, \varphi_h) + \frac{\tau^2}{2}(q_h^n, \varphi_h) + \frac{\tau^3}{6}(r_h^n, \varphi_h), \tag{3.9a}$$

$$(p_h^n, \psi_h) = -\mathcal{H}^{\alpha_u}(u_h^n, \psi_h), \tag{3.9b}$$

$$(q_h^n, \eta_h) = -\mathcal{H}^{\alpha_p}(p_h^n, \eta_h), \tag{3.9c}$$

$$(r_h^n, \zeta_h) = -\mathcal{H}^{\alpha_q}(q_h^n, \zeta_h), \tag{3.9d}$$

for any $\varphi_h, \psi_h, \eta_h, \zeta_h \in V_h$. Note that (3.9a) implies

$$u_h^{n+1} = u_h^n + \tau p_h^n + \frac{\tau^2}{2} q_h^n + \frac{\tau^3}{6} r_h^n. \tag{3.10}$$

The proof of the third order schemes are actually easier, since there is inherent numerical viscosity from the time discretization. The stability will hold for piecewise polynomial elements of arbitrary degrees. In this section, $\lambda \le 1$ is assumed for simplicity.

Before going into the main theorem, we first prove the following lemma.

Lemma 3.4.

$$\|p_h^{n+1} - p_h^n\|^2 - \tau^2 \|q_h^n\|^2 \le (\frac{\tau}{4} + C_\mu \lambda)[\![p_h^n]\!]^2 + \tau^2 C_\mu \lambda \|q_h^n\|^2.$$

Proof. According to (3.9b) and (3.10), the following relationship holds

$$(p_h^{n+1} - p_h^n, \psi_h) = -\mathcal{H}^{\alpha_u}(u_h^{n+1} - u_h^n, \psi_h) = -\mathcal{H}^{\alpha_u}(\tau p_h^n + \frac{\tau^2}{2} q_h^n + \frac{\tau^3}{6} r_h^n, \psi_h). \tag{3.11}$$

Hence

$$\begin{aligned}(p_h^{n+1} - p_h^n - \tau q_h^n, \psi_h) &= \tau \frac{\alpha_p - \alpha_u}{2}[p_h^n, \psi_h^n] - \frac{\tau^2}{2}\mathcal{H}^{\alpha_u}(q_h^n, \psi_h^n) - \frac{\tau^3}{6}\mathcal{H}^{\alpha_u}(r_h^n, \psi_h^n) \\ &\le C_\mu(\tau h^{-\frac{1}{2}}[\![p_h^n]\!] + \tau\lambda\|q_h^n\| + \tau^2\lambda\|r_h^n\|)\|\psi_h^n\|.\end{aligned}$$

Using Lemma 2.3, we know $\tau^2 \lambda \|r_h^n\| \le \tau C_\mu \lambda^2 \|q_h^n\|$. Hence

$$(p_h^{n+1} - p_h^n - \tau q_h^n, \psi_h) \le C_\mu(\tau h^{-\frac{1}{2}}[\![p_h^n]\!] + \tau\lambda\|q_h^n\|)\|\psi_h\|. \tag{3.12}$$

Note that

$$\|p_h^{n+1}-p_h^n\|^2-\tau^2\|q_h^n\|^2=\|p_h^{n+1}-p_h^n-\tau q_h^n\|^2+2\tau(p_h^{n+1}-p_h^n-\tau q_h^n,q_h^n). \tag{3.13}$$

By (3.12),

$$\|p_h^{n+1}-p_h^n-\tau q_h^n\|^2\leq C_\mu(\tau\lambda[\![p_h^n]\!]^2+\tau^2\lambda\|q_h^n\|^2),$$

$$(p_h^{n+1}-p_h^n-\tau q_h^n,q_h^n)\leq C_\mu(\tau h^{-\frac{1}{2}}[\![p_h^n]\!]+\tau\lambda\|q_h^n\|)\|q_h^n\|\leq\frac{1}{8}[\![p_h^n]\!]^2+\tau C_\mu\lambda\|q_h^n\|^2.$$

The proof is completed by substituting the estimates above into (3.13). □

Theorem 3.2 (Stability of the LW3DG schemes)**.** *There exists a constant λ, such that when $\tau\leq\lambda h$, the solution of schemes* (3.9) *satisfies,*

(i) $\|u_h^{n+1}\|\leq\|u_h^n\|$, if $\alpha_u=\alpha_p=+1$, $\alpha_q=\pm1$,

(ii) $\|u_h^n\|\leq C_{\mu,\lambda}\|u_h^0\|$, if $\alpha_u=+1$, $\alpha_p=-1$ and $\alpha_q=\pm1$,

where $C_{\mu,\lambda}$ is a constant dependent upon λ and the inverse estimate constant μ, but is independent of the total time $T=n\tau$.

Proof. Substituting $\varphi_h=u_h^n$ into (3.9) and summing over j, we have

$$\begin{aligned}\frac{1}{2}\|u_h^{n+1}\|^2-\frac{1}{2}\|u_h^n\|^2+\frac{\tau\alpha_u}{2}[\![u_h^n]\!]^2-\frac{\tau^3\alpha_p}{12}[\![p_h^n]\!]^2+\frac{\tau^2\alpha_{up}}{4}[u_h^n,p_h^n]+\frac{\tau^3\alpha_{uq}}{12}[u_h^n,q_h^n]\\=\frac{1}{2}\|u_h^{n+1}-u_h^n\|^2-\frac{\tau^2}{2}\|p_h^n\|^2.\end{aligned} \tag{3.14}$$

Note that

$$\|u_h^{n+1}-u_h^n\|^2-\tau^2\|p_h^n\|^2=\|u_h^{n+1}-u_h^n-\tau p_h^n\|^2+2\tau(u_h^{n+1}-u_h^n-\tau p_h^n,p_h^n),$$

where

$$(u_h^{n+1}-u_h^n-\tau p_h^n,p_h^n)=\frac{\tau^2}{2}(q_h^n,p_h^n)+\frac{\tau^3}{6}(r_h^n,p_h^n)=-\frac{\tau^2\alpha_p}{4}[\![p_h^n]\!]^2-\frac{\tau^3}{6}\|q_h^n\|^2-\frac{\tau^3\alpha_{pq}}{12}[p_h^n,q_h^n].$$

Hence

$$\|u_h^{n+1}-u_h^n\|^2-\tau^2\|p_h^n\|^2=\|u_h^{n+1}-u_h^n-\tau p_h^n\|^2-\frac{\tau^3\alpha_p}{2}[\![p_h^n]\!]^2-\frac{\tau^4}{3}\|q_h^n\|^2-\frac{\tau^4\alpha_{pq}}{6}[p_h^n,q_h^n]. \tag{3.15}$$

Using (3.15), the energy estimate (3.14) becomes

$$\|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \tau\alpha_u [\![u_h^n]\!]^2 + \frac{\tau^3\alpha_p}{3}[\![p_h^n]\!]^2 + \frac{\tau^2\alpha_{up}}{2}[u_h^n, p_h^n] + \frac{\tau^3\alpha_{uq}}{6}[u_h^n, q_h^n] + \frac{\tau^4\alpha_{pq}}{6}[p_h^n, q_h^n]$$
$$= \|u_h^{n+1} - u_h^n - \tau p_h^n\|^2 - \frac{\tau^4}{3}\|q_h^n\|^2.$$

Since

$$(u_h^{n+1} - u_h^n - \tau p_h^n, \varphi_h) = \frac{\tau^2}{2}(q_h^n, \varphi_h) + \frac{\tau^3}{6}(r_h^n, \varphi_h),$$

by Lemma 2.3, $\|u_h^{n+1} - u_h^n - \tau p_h^n\| \leq \tau^2(\frac{1}{2} + C_\mu\lambda)\|q_h^n\|$. Suppose $\lambda \leq 1$, then the estimate becomes

$$\begin{aligned}\|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \tau\alpha_u [\![u_h^n]\!]^2 + \frac{\tau^3\alpha_p}{3}[\![p_h^n]\!]^2 + \frac{\tau^2\alpha_{up}}{2}[u_h^n, p_h^n] + \frac{\tau^3\alpha_{uq}}{6}[u_h^n, q_h^n]\\ + \frac{\tau^4\alpha_{pq}}{6}[p_h^n, q_h^n] \leq \tau^4(-\frac{1}{12} + C_\mu\lambda)\|q_h^n\|^2.\end{aligned} \tag{3.16}$$

Different from (3.1), the right hand side of (3.16) is automatically negative when λ is small. This is due to the numerical dissipation from the third order time discretization. One can avoid detailed arguments as in Lemma 3.1 to absorb terms with jumps.

(i) When $\alpha_u = \alpha_p = +1$, by using the inequality $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$, we obtain

$$\|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \tau C[\![u_h^n]\!]^2 + \tau^3 C[\![p_h^n]\!]^2 \leq \tau^4(-\frac{1}{12} + C_\mu\lambda)\|q_h^n\|^2 + \tau^5 C[\![q_h^n]\!]^2,$$

where the C's are positive constants. Using the inverse estimate, $\tau^5 C[\![q_h^n]\!]^2 \leq \tau^4 C_\mu\lambda\|q_h^n\|^2$. Hence $\|u_h^{n+1}\|^2 - \|u_h^n\|^2 \leq \tau^4(-\frac{1}{12} + C'_\mu\lambda)\|q_h^n\|^2$. The strong stability holds as long as $\lambda \leq \frac{1}{12C'_\mu}$.

(ii) When $\alpha_u = +1$ and $\alpha_p = -1$, the estimate becomes

$$\|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \tau[\![u_h^n]\!]^2 - \frac{\tau^3}{3}[\![p_h^n]\!]^2 + \frac{\tau^3\alpha_{uq}}{6}[u_h^n, q_h^n] + \frac{\tau^4\alpha_{pq}}{6}[p_h^n, q_h^n] \leq \tau^4(-\frac{1}{12} + C_\mu\lambda)\|q_h^n\|^2. \tag{3.17}$$

Use Lemma 2.3 and the inverse estimate,

$$\frac{\tau^3\alpha_{uq}}{6}[u_h^n, q_h^n] \leq \frac{\tau}{2}[\![u_h^n]\!]^2 + \tau^4 C_\mu\lambda\|q_h^n\|^2,$$

$$\frac{\tau^3\alpha_{uq}}{6}[p_h^n, q_h^n] \leq \frac{\tau}{6}[\![p_h^n]\!]^2 + \tau^4 C_\mu\lambda\|q_h^n\|^2.$$

Hence

$$\|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \frac{\tau}{2}[\![u_h^n]\!]^2 - \frac{\tau^3}{2}[\![p_h^n]\!]^2 \le \tau^4(-\frac{1}{12} + C_\mu\lambda)\|q_h^n\|^2. \tag{3.18}$$

Now the jump term of p_h^n on the left hand side is negative. As we have done in Theorem 3.1, we would introduce $\|p_h^n\|$ to balance this term. Substituting $\psi_h = p_h^n$ into (3.11), we have

$$\frac{1}{2}\|p_h^{n+1}\|^2 - \frac{1}{2}\|p_h^n\|^2 - \frac{1}{2}\|p_h^{n+1} - p_h^n\|^2 = -\frac{\tau}{2}[\![p_h^n]\!]^2 - \frac{\tau^2}{2}\|q_h^n\|^2 - \frac{\tau^3}{6}\mathcal{H}^+(r_h^n, p_h^n). \tag{3.19}$$

Note that

$$\frac{\tau^3}{6}\mathcal{H}^+(r_h^n, p_h^n) = \frac{\tau^3}{6}(q_h^n, r_h^n) \le \frac{\tau^3}{6}\|q_h^n\|\|r_h^n\| \le \tau^2 C_\mu\lambda\|q_h^n\|^2, \tag{3.20}$$

where Lemma 2.3 is used in the last inequality. Therefore, by using Lemma 3.4 and (3.20) in (3.19), one can get

$$\|p_h^{n+1}\|^2 - \|p_h^n\|^2 \le (C_\mu\lambda - \frac{3}{4})\tau[\![p_h^n]\!]^2 + \tau^2 C_\mu\lambda\|q_h^n\|^2. \tag{3.21}$$

Multiplying (3.21) by τ^2 and adding it to (3.18), we get

$$(\|u_h^{n+1}\|^2+\tau^2\|p_h^{n+1}\|^2)-(\|u_h^n\|^2+\tau^2\|p_h^n\|^2)+\frac{\tau}{2}[\![u_h^n]\!]^2+\tau^3(\frac{1}{4}-C_\mu\lambda)[\![p_h^n]\!]^2 \le \tau^4(-\frac{1}{12}+C_\mu\lambda)\|q_h^n\|^2.$$

Therefore, when λ is sufficiently small, all jump terms are nonnegative, and the right hand side is less than zero. So when uniform time steps are used, we get

$$\|u_h^n\|^2 + \tau^2\|p_h^n\|^2 \le \|u_h^n\|^2 + \tau^2\|p_h^n\|^2 \le \|u_h^n\|^2 + \tau^2\|p_h^0\|^2.$$

Using Lemma 2.3, one has $\tau^2\|p_h^n\|^2 \le C_\mu\lambda^2\|u_h^n\|^2$. Hence $\|u_h^n\| \le \sqrt{1 + C_\mu\lambda^2}\|u_h^0\|$. Clearly, the result also holds with non-increasing time steps, namely $\tau_{n+1} \le \tau_n$. □

3.3 Multidimensional cases

3.3.1 Notations and preliminaries

Let us now consider the general linear scalar conservation law (1.1). Recall that $\nabla\cdot\boldsymbol{\beta}(\boldsymbol{x}) = 0$, hence $\nabla\cdot(\boldsymbol{\beta}u) = (\boldsymbol{\beta}\cdot\nabla)u$.

Suppose $\mathcal{K} = \{K\}$ is a quasi-uniform partition of the domain Ω with triangular (or rectangular) element K, where $h = \max_K h_K$, with h_K being the diameter of element K. The collection of cell interfaces is denoted by $\mathcal{E}$. We use the notation $(u, v)_K = \int_K uv\ dxdy$ for the L^2 inner product on each element, and $(u, v) = \sum_K (u, v)_K$ for the whole domain. Besides, we denote the integration on element interfaces by $\langle u, v\rangle_e = \int_e uv\ dl$ and jumps by $[u]_e = (u^+ - u^-)_e$. Here, $u^+ = \lim_{\varepsilon\to 0^+} u(\boldsymbol{x} + \varepsilon\boldsymbol{\beta})$, while $u^- = \lim_{\varepsilon\to 0^+} u(\boldsymbol{x} - \varepsilon\boldsymbol{\beta})$. Furthermore, let $[\![u]\!]_{\boldsymbol{\beta},e} = \sqrt{\int_e [u]_e^2 |\boldsymbol{\beta}\cdot\boldsymbol{n}| dl}$ and $[\![u]\!]_{\boldsymbol{\beta}} = \sqrt{\sum_{e\in\mathcal{E}} [\![u]\!]_{\boldsymbol{\beta},e}^2}$. The corresponding cross term is denoted by $[u, v]_{\boldsymbol{\beta}} = \sum_{e\in\mathcal{E}} \langle [u], [v]|\boldsymbol{\beta}\cdot\boldsymbol{n}|\rangle$.

Similar to our one-dimensional cases, we introduce the operators $\mathcal{H}^{\pm}_{\boldsymbol{\beta}}$ for convenience,

$$\mathcal{H}^{\pm}_{\boldsymbol{\beta}}(w, v) = \sum_K \mathcal{H}^{\pm}_{\boldsymbol{\beta},K}(w, v) = \sum_K \big((w, \nabla\cdot(\boldsymbol{\beta}v))_K - \langle w^{\pm}, v\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial K}\big).$$

When $\boldsymbol{\beta}(\boldsymbol{x})$ is parallel to the cell interfaces, $u^{\pm}$ are not well-defined. But in this case, $\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n} = 0$, hence the value of $u^{\pm}$ will not make any difference. When $\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}$ changes sign on the edge, $\mathcal{H}^{\pm}_{\boldsymbol{\beta}}(w, v)$ can still be defined as above. But one should note $w^{\pm}$ are no longer polynomials and one would need to be careful when applying the inverse estimates. See appendix B.2 for details.

As before, we introduce the auxiliary variables $p = u_t = \nabla\cdot(\boldsymbol{\beta}u)$, $q = u_{tt} = \nabla\cdot(\boldsymbol{\beta}p)$ and $r = u_{ttt} = \nabla\cdot(\boldsymbol{\beta}q)$ (for the third order schemes). Then the numerical schemes can be defined as in the one-dimensional cases in (3.1) and (3.9), except for replacing $\mathcal{H}^{\pm}$ by $\mathcal{H}^{\pm}_{\boldsymbol{\beta}}$. Moreover, we use the discontinuous finite element space $V_h = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{K}\}$, where $\mathcal{P}_k(K)$ denotes the space of polynomials on K of degree no more than k.

For $\nabla\cdot\boldsymbol{\beta}(\boldsymbol{x}) = 0$ and $w, v \in V_h$, the following relationships hold, which extend Lemma 2.1 and Lemma 2.2 to multidimensions. The proof of Lemma 3.5 and 3.6 is given in the appendix.

Lemma 3.5.

$$\mathcal{H}^{-}_{\boldsymbol{\beta}}(w, w) = -\frac{1}{2}[\![w]\!]^2_{\boldsymbol{\beta}}, \tag{3.22}$$

$$\mathcal{H}^{-}_{\boldsymbol{\beta}}(w, v) = -\mathcal{H}^{+}_{\boldsymbol{\beta}}(v, w), \tag{3.23}$$

$$\mathcal{H}_{\boldsymbol{\beta}}^{+}(w,v) \;\; = \;\; \mathcal{H}_{\boldsymbol{\beta}}^{-}(w,v) + [w,v]_{\boldsymbol{\beta}}. \tag{3.24}$$

Lemma 3.6.

$$|\mathcal{H}_{\boldsymbol{\beta}}^{-}(w,v)| \;\; \leq \;\; (\|(\boldsymbol{\beta}\cdot\nabla)w\| + C_{\mu,\boldsymbol{\beta}}h^{-\frac{1}{2}}[\![w]\!]_{\boldsymbol{\beta}})\|v\|, \tag{3.25}$$

$$|\mathcal{H}_{\boldsymbol{\beta}}^{-}(w,v)| \;\; \leq \;\; (\|(\boldsymbol{\beta}\cdot\nabla)v\| + C_{\mu,\boldsymbol{\beta}}h^{-\frac{1}{2}}[\![v]\!]_{\boldsymbol{\beta}})\|w\|. \tag{3.26}$$

With the relationships above, we can state a similar lemma to Lemma 2.4.

Lemma 3.7. *Lemma 2.4 holds after replacing $\mathcal{H}$ with $\mathcal{H}_{\boldsymbol{\beta}}$, $[\cdot,\cdot]$ with $[\cdot,\cdot]_{\boldsymbol{\beta}}$ and $[\![\cdot]\!]$ with $[\![\cdot]\!]_{\boldsymbol{\beta}}$.*

Lemma 3.1 plays a key role in the stability analysis of the LW2DG schemes. Similarly, we need the following lemma to bound $\|(\boldsymbol{\beta}\cdot\nabla)p_h^n\|_K$ with $[\![u_h^n]\!]_{\boldsymbol{\beta},\partial K}$. The proof of this lemma is given in the appendix.

Lemma 3.8. *For the piecewise linear polynomial space in $\mathbb{R}^d$,*

(i) If $\boldsymbol{\beta}$ is a constant vector, then $\|(\boldsymbol{\beta}\cdot\nabla)p_h^n\|_K \leq C_{\mu,\boldsymbol{\beta}}h^{-\frac{3}{2}}[\![u_h^n]\!]_{\boldsymbol{\beta},\partial K}$,

(ii) If $\boldsymbol{\beta}$ is a vector-valued function with $\nabla\cdot\boldsymbol{\beta}(\boldsymbol{x}) = 0$, then

$$\|(\boldsymbol{\beta}\cdot\nabla)p_h^n\|_K \leq C_{\mu,\boldsymbol{\beta}}h^{-\frac{3}{2}}[\![u_h^n]\!]_{\boldsymbol{\beta},\partial K} + C_{\mu,\boldsymbol{\beta}}h^{-1}\|u_h^n\|_K.$$

Remark 3.3. *Note that our proof only holds for $\mathcal{P}_1$ elements. It is difficult to extend the proof to $\mathcal{Q}_1$ elements, which are defined by tensor products on rectangular meshes.*

3.3.2 Stability results

Theorem 3.3 (Stability of the LW2DG schemes in multidimensions)**.** *For multidimensional cases, let u_h^n be the numerical solution of the LW2DG schemes with $\mathcal{P}_1$ elements. Then under the standard CFL condition $\tau \leq \lambda h$,*

(i) When $\boldsymbol{\beta}$ is constant, $\|u_h^{n+1}\| \leq \|u_h^n\|$ if $\alpha_u = \alpha_p = +1$; $\|u_h^n\| \leq C_{\mu,\lambda,\boldsymbol{\beta}}\|u_h^0\|$ if $\alpha_u = +1$ and $\alpha_p = -1$, where $C_{\mu,\lambda,\boldsymbol{\beta}}$ depends on λ, μ and $\boldsymbol{\beta}$ but is independent of T.

(ii) When $\boldsymbol{\beta}(\boldsymbol{x})$ is a function with $\nabla\cdot\boldsymbol{\beta}(\boldsymbol{x}) = 0$, if $\alpha_u = +1$ and $\alpha_p = \pm 1$, then $\|u_h^n\| \leq C_{\mu,\lambda,\boldsymbol{\beta},T}\|u_h^0\|$, where $C_{\mu,\lambda,\boldsymbol{\beta},T}$ depends on λ, μ, $\boldsymbol{\beta}$ and T.

Proof. For constant $\boldsymbol{\beta}$, we proceed in the same way as we have done for Theorem 3.1, except for using Lemma 3.7 instead of Lemma 2.4, and Lemma 3.8 instead of Lemma 3.1. For non-constant function $\boldsymbol{\beta}$, we will get an extra $\|u\|_K$ term in Lemma 3.8. A similar argument will lead to $\|u_h^{n+1}\| \leq (1+\tau C_{\tau,\lambda,\boldsymbol{\beta}})\|u_h^n\|$ when $\alpha_p = +1$, which will end up with $\|u_h^n\| \leq e^{C_{\mu,\lambda,\boldsymbol{\beta}}T}\|u_h^0\|$. For the same reason, the constant will also depend on T when $\alpha_p = -1$. □

Theorem 3.4 (Stability of the LW3DG schemes in multidimensions)**.** *For multidimensional cases, assume $\nabla \cdot \boldsymbol{\beta}(\boldsymbol{x}) = 0$. Let u_h^n be the numerical solution of the LW3DG schemes with $\mathcal{P}_k$ elements. Then under the standard CFL condition $\tau \leq \lambda h$,*

(i) $\|u_h^{n+1}\| \leq \|u_h^n\|$, if $\alpha_u = \alpha_p = +1$ and $\alpha_q = \pm 1$,

(ii) $\|u_h^n\| \leq C_{\mu,\lambda,\boldsymbol{\beta}}\|u_h^0\|$, if $\alpha_u = +1$, $\alpha_p = -1$ and $\alpha_q = \pm 1$,

where $C_{\mu,\lambda,\boldsymbol{\beta}}$ depends on λ, μ and $\boldsymbol{\beta}$, but is independent of T.

Proof. The proof follows the same lines as in the one-dimensional case and is thus omitted.

□

Remark 3.4. *By now, we have spent our main effort on some special coefficients $\boldsymbol{\beta}$. Actually, the LWDG schemes can be modified to suit general $\boldsymbol{\beta}(\boldsymbol{x}, t)$, where $\boldsymbol{\beta}$ may depend on time and can have a non-zero divergence. Furthermore, as long as $\boldsymbol{\beta}$ is sufficiently smooth, we can obtain the weak stability result $\|u_h^n\| \leq C\|u_h^0\|$, where C depends on $\boldsymbol{\beta}$ and T, as well as μ and λ if alternating fluxes are used.*

We use the LW2DG scheme with upwinding fluxes for a brief illustration. The revised scheme then becomes

$$\begin{aligned}
(u_h^{n+1}, \varphi_h) &= (u_h^n + \tau p_h^n + \frac{\tau^2}{2} q_h^n, \varphi_h), \\
(p_h^n, \psi_h) &= -\mathcal{H}_{\boldsymbol{\beta}}^+(u_h^n, \psi_h) + (\nabla \cdot \boldsymbol{\beta} u_h^n, \psi_h), \\
(q_h^n, \eta_h) &= -\mathcal{H}_{\boldsymbol{\beta}}^+(p_h^n, \eta_h) + (\nabla \cdot \boldsymbol{\beta} p_h^n, \eta_h) - \mathcal{H}_{\boldsymbol{\beta}_t}^+(u_h^n, \eta_h) + (\nabla \cdot \boldsymbol{\beta}_t u_h^n, \eta_h).
\end{aligned} \tag{3.27}$$

Here we have the extra terms $(\nabla \cdot \boldsymbol{\beta} u_h^n, \psi_h)$, $(\nabla \cdot \boldsymbol{\beta} p_h^n, \eta_h)$, $\mathcal{H}_{\boldsymbol{\beta}_t}^+(u_h^n, \eta_h)$ and $(\nabla \cdot \boldsymbol{\beta}_t u_h^n, \eta_h)$. Furthermore, the properties of $\mathcal{H}$ has been changed and there will be more terms coming out when one follows the previous approach. But compared with the exponents of τ before them, *the order of derivatives for all these new terms are relatively low. One can apply the Cauchy-Schwartz inequality, Lemma 3.6 and the inverse estimates to show they can be bounded by* $C\tau\|u_h^n\|^2$ *with some proper constant* C*. Hence we will finally obtain* $\|u_h^{n+1}\|^2 \leq (1+C\tau)\|u_h^n\|^2$*, which indicates the weak stability result.*

4 Error estimates

For the one-dimensional problem (2.1), denote $u^n(x) = u(x, t_n)$ and $u_t^n(x) = u_t(x, t_n)$. Assuming the exact solution $u(x,t)$ is smooth, we have the following error estimates.

Theorem 4.1 (Error estimates of the LW2DG schemes)**.** *Under the CFL condition* $\tau \leq \lambda h$ *for a proper constant* λ*, using* $\mathcal{P}_1$ *elements, the numerical approximations* u_h^n *and* p_h^n *from the LW2DG schemes* (3.1) *satisfy the following error estimates,*
(i) $\|u^n - u_h^n\| \leq C(\tau^2 + h^2)$*,*
(ii) $\|u_t^n - p_h^n\| \leq C(\tau^2 + h^2)$*,*
where C *depends on* μ*,* λ*,* T*,* u*,* u_t *and their derivatives but is independent of* τ *and* h*.*

Theorem 4.2 (Error estimates of the LW3DG schemes)**.** *Under the CFL condition* $\tau \leq \lambda h$ *for a proper constant* λ*, using* $\mathcal{P}_k$ *elements (*k *is arbitrary), the numerical approximations* u_h^n *and* p_h^n *from the LW3DG schemes* (3.9) *satisfy the following error estimates,*
(i) $\|u^n - u_h^n\| \leq C(\tau^3 + h^{k+1})$*,*
(ii) $\|u_t^n - p_h^n\| \leq C(\tau^3 + h^{k+1})$*,*
where C *depends on* μ*,* λ*,* T*,* u*,* u_t *and their derivatives but is independent of* τ *and* h*.*

The error estimate of u follows from the stability results through a standard argument, while the error estimate of u_t relies on the relationship between u_h^n and p_h^n. Here we only give the details of the proof of Theorem 4.1 and omit the details of the proof of Theorem 4.2, as they can be obtained following similar lines.

We will need the following lemma for the error estimates. The proof of (4.1) is straightforward, and (4.2) follows from the standard approximation theorem [2].

Lemma 4.1. *Suppose w is a smooth function, then*

$$\mathcal{H}^{\pm}(\pi^{\pm}w - w, v) = 0, \qquad \forall v \in V_h, \tag{4.1}$$

$$\|\pi^{\pm}w - w\| \leq C_w h^{k+1}, \tag{4.2}$$

where C_w is a constant depending on w; $\pi^{\pm}$ are the Gauss-Radau projections to V_h. More precisely, $\pi^{\pm}w$ is the unique element in V_h such that

$$(\pi^{\pm}w - w, v)_j = 0 \quad \forall v \in \mathcal{P}_{k-1}(I_j), \quad (\pi^{\pm}w)^{\pm}_{j\mp\frac{1}{2}} = w^{\pm}_{j\mp\frac{1}{2}}.$$

Proof of Theorem 4.1: Let u be the exact solution of (1.1), $p = u_t$ and $q = u_{tt}$. We denote the truncation error by

$$\omega^n = u^{n+1} - u^n - \tau p^n - \frac{\tau^2}{2} q^n. \tag{4.3}$$

By the consistency of the schemes, for any $\varphi_h, \psi_h, \eta_h \in V_h$, the following equalities hold

$$(u^{n+1}, \varphi_h) = (u^n + \tau p^n + \frac{\tau^2}{2} q^n, \varphi_h) + (\omega^n, \varphi_h), \tag{4.4a}$$

$$(p^n, \psi_h) = -\mathcal{H}^{\alpha_u}(u^n, \psi_h), \tag{4.4b}$$

$$(q^n, \eta_h) = -\mathcal{H}^{\alpha_p}(p^n, \eta_h). \tag{4.4c}$$

Subtract the system (4.4) from the numerical schemes (3.1), then we get the error equation

$$(u^{n+1} - u_h^{n+1}, \varphi_h) = (u^n - u_h^n + \tau(p^n - p_h^n) + \frac{\tau^2}{2}(q^n - q_h^n, \varphi_h) + (\omega^n, \varphi_h), \tag{4.5a}$$

$$(p^n - p_h^n, \psi_h) = -\mathcal{H}^{\alpha_u}(u^n - u_h^n, \psi_h), \tag{4.5b}$$

$$(q^n - q_h^n, \eta_h) = -\mathcal{H}^{\alpha_p}(p^n - p_h^n, \eta_h). \tag{4.5c}$$

Denote $e_v^n = \pi^{\alpha_v} v - v_h^n$ and $\varepsilon_v^n = v - \pi^{\alpha_v} v$, with $v = u, p, q$. Applying Lemma 4.1, the error equation can be simplified as follows

$$(e_u^{n+1} - e_u^n - \tau e_p^n - \frac{\tau^2}{2} e_q^n, \varphi_h) = -(\varepsilon_u^{n+1} - \varepsilon_u^n - \tau \varepsilon_p^n - \frac{\tau^2}{2} \varepsilon_q^n, \varphi_h) + (\omega^n, \varphi_h), \tag{4.6a}$$

$$(e_p^n, \psi_h) = -\mathcal{H}^{\alpha_u}(e_u^n, \psi_h) - (\varepsilon_p^n, \psi_h), \tag{4.6b}$$

$$(e_q^n, \eta_h) = -\mathcal{H}^{\alpha_p}(e_p^n, \eta_h) - (\varepsilon_q^n, \eta_h). \tag{4.6c}$$

Furthermore, by using (4.3) and (4.6b), one can also obtain the error equation for u_t

$$(e_p^{n+1} - e_p^n, \psi_h) = -\mathcal{H}^{\alpha_u}(\tau e_p^n + \frac{\tau^2}{2} e_q^n + w, \psi_h) - (\varepsilon_p^{n+1} - \varepsilon_p^n, \psi_h). \tag{4.7}$$

Our error estimates are based on the framework of stability analysis. We need lemmas comparable to Lemma 2.3, Lemma 2.4, Lemma 3.1, Lemma 3.2 and Lemma 3.3. Among them, Lemma 4.2, Lemma 4.3 and Lemma 4.4 can be proved by using (4.6) and similar techniques as before, and the details are omitted.

Lemma 4.2.

$$\|e_p^n\| \leq C_\mu h^{-1}\|e_u^n\| + \|\varepsilon_p^n\|, \tag{4.8a}$$

$$\|e_q^n\| \leq C_\mu h^{-1}\|e_p^n\| + \|\varepsilon_q^n\|. \tag{4.8b}$$

Lemma 4.3.

$$(e_p^n, e_u^n) = -\frac{\alpha_u}{2}[\![e_u^n]\!]^2 - (\varepsilon_p^n, e_u^n), \tag{4.9a}$$

$$(e_q^n, e_u^n) = -\|e_p^n\|^2 - \frac{\alpha_{up}}{2}[e_u^n, e_p^n] - (\varepsilon_q^n, e_u^n) - (\varepsilon_p^n, e_p^n), \tag{4.9b}$$

$$(e_q^n, e_p^n) = -\frac{\alpha_p}{2}[\![e_p^n]\!]^2 - (\varepsilon_q^n, e_p^n), \tag{4.9c}$$

where $\alpha_{wv} = \alpha_w + \alpha_v,\ w, v = \pm 1$.

Lemma 4.4. *For piecewise linear elements,* $\|(e_p^n)_x\| \leq C_\mu h^{-\frac{3}{2}}[\![e_u^n]\!] + C_\mu h^{-1}\|\varepsilon_p^n\|$.

Lemma 4.5.

$$\|e_u^{n+1} - e_u^n - \tau e_p^n\| \leq C_\mu \tau^2 (h^{-\frac{3}{2}}[\![e_u^n]\!] + h^{-\frac{1}{2}}[\![e_p^n]\!]) + C(\tau^3 + \tau h^2). \tag{4.10}$$

Proof. Substituting (4.6b) into (4.6a), we get

$$\begin{aligned}
&(e_u^{n+1} - e_u^n - \tau e_p^n, \varphi_h) = -(\varepsilon_u^{n+1} - \varepsilon_u^n - \tau\varepsilon_p^n, \varphi_h) - \frac{\tau^2}{2}\mathcal{H}^{\alpha_p}(e_p^n, \varphi_h) + (\omega^n, \varphi_h) \\
&\leq \left(\|\varepsilon_u^{n+1} - \varepsilon_u^n\| + \tau\|\varepsilon_p^n\| + \|\omega^n\| + \tau^2 C_\mu(\|(e_p^n)_x\| + h^{-\frac{1}{2}}[\![e_p^n]\!])\right)\|\varphi_h\| \\
&\leq \left(\|\varepsilon_u^{n+1} - \varepsilon_u^n\| + \tau(1 + C_\mu\lambda)\|\varepsilon_p^n\| + \|\omega^n\| + \tau^2 C_\mu(h^{-\frac{3}{2}}[\![e_u^n]\!] + h^{-\frac{1}{2}}[\![e_p^n]\!])\right)\|\varphi_h\|,
\end{aligned} \tag{4.11}$$

where we have used Lemma 4.4 in the last inequality. One can see that

$$\|\varepsilon_u^{n+1} - \varepsilon_u^n\| + \tau(1 + C_\mu\lambda)\|\varepsilon_p^n\| + \|\omega^n\| \le C(\tau^3 + \tau h^2). \tag{4.12}$$

The proof can be completed by using the estimate (4.12) and substituting $\varphi_h = e_u^{n+1} - e_u^n - \tau e_p^n$ into (4.11). □

Lemma 4.6.

$$\|e_p^{n+1} - e_p^n\| \le C_\mu(\tau h^{-\frac{3}{2}}[\![e_u^n]\!] + \tau h^{-\frac{1}{2}}[\![e_p^n]\!] + \tau\lambda\|e_q^n\|) + C(\tau^2 + h^2). \tag{4.13}$$

Proof. By (4.7), one has

$$\begin{aligned}
&(e_p^{n+1} - e_p^n, \psi_h) = -\mathcal{H}^{\alpha_u}(\tau e_p^n + \frac{\tau^2}{2} e_q^n + w, \psi_h) - (\varepsilon_p^{n+1} - \varepsilon_p^n, \psi_h) \\
&\le \left(C_\mu(\tau\|(e_p^n)_x\| + \tau h^{-\frac{1}{2}}[\![e_p^n]\!] + \tau\lambda\|e_q^n\| + h^{-1}\|\omega^n\|) + \|\varepsilon_p^{n+1} - \varepsilon_p^n\| \right)\|\psi_h\| \\
&\le \left(C_\mu(\tau h^{-\frac{3}{2}}[\![e_u^n]\!] + \tau h^{-\frac{1}{2}}[\![e_p^n]\!] + \tau\lambda\|e_q^n\| + \lambda\|\varepsilon_p^n\| + h^{-1}\|\omega^n\|) + \|\varepsilon_p^{n+1} - \varepsilon_p^n\| \right)\|\psi_h\|
\end{aligned} \tag{4.14}$$

where we have used Lemma 4.4 in the last inequality. We see that

$$\lambda\|\varepsilon_p^n\| + h^{-1}\|\omega^n\| + \|\varepsilon_p^{n+1} - \varepsilon_p^n\| \le C(\tau^2 + h^2). \tag{4.15}$$

The proof is completed by using the estimate (4.15) and substituting $\varphi_h = e_p^{n+1} - e_p^n$ into (4.14). □

Step 1: To prove $\|e_u^n\| \le C(\tau^2 + h^2)$.

Substitute $\varphi_h = e_u^n$ into (4.6) and use (4.3). It follows that

$$\|e_u^{n+1}\|^2 - \|e_u^n\|^2 + \tau\alpha_u[\![e_u^n]\!]^2 + \tau^2\alpha_{up}[e_u^n, e_p^n] + \frac{\tau^3\alpha_p}{2}[\![e_p^n]\!]^2 = \|e_u^{n+1} - e_u^n - \tau e_p^n\|^2 + \gamma_u,$$

where $\gamma_u = -2\tau(\varepsilon_p^n, e_u^n) - 2\tau^2(\varepsilon_q^n, e_u^n) + 2\tau^2(\varepsilon_p^n, e_p^n) - \tau^3(\varepsilon_q^n, e_p^n) + (\varepsilon_u^{n+1} - \varepsilon_u^n, e_u^n) + (\omega^n, e_u^n)$. Using Lemma 4.2, one can get an estimate for γ_u

$$\gamma_u \le C_\mu\tau\|e_u^n\|^2 + \tau C(\|\varepsilon_p^n\|^2 + \|\varepsilon_q^n\|^2) + \frac{1}{\tau}\|\varepsilon_u^{n+1} - \varepsilon_u^n\|^2 + \frac{1}{\tau}\|\omega\|^2 \le C_\mu\tau\|e_u^n\|^2 + C(\tau^5 + \tau h^4).$$

Hence, with the estimate above, by using Lemma 4.5, we get

$$\|e_u^{n+1}\|^2+\tau(\alpha_u-C_\mu\lambda)[\![e_u^n]\!]^2+\tau^2\alpha_{up}[e_u^n,e_p^n]+\tau^3(\frac{\alpha_p}{2}-C_\mu\lambda)[\![e_p^n]\!]^2 \le (1+\tau C_\mu)\|e_u^n\|^2+C(\tau^5+\tau h^4). \tag{4.16}$$

Case 1: When λ is small and $\alpha_u=\alpha_p=+1$, all the jump terms can be removed to yield a much simpler inequality

$$\|e_u^{n+1}\|^2 \le (1+\tau C_\mu)\|e_u^n\|^2+C(\tau^5+\tau h^4). \tag{4.17}$$

Note that $e_u^0=0$. (4.16) implies

$$\|e_u^n\| \le C(\tau^2+h^2). \tag{4.18}$$

Case 2: When λ is small, $\alpha_u=+1$ and $\alpha_p=-1$, we have

$$\|e_u^{n+1}\|^2+\tau(1-C_\mu\lambda)[\![e_u^n]\!]^2+\tau^3(-\frac{1}{2}-C_\mu\lambda)[\![e_p^n]\!]^2 \le (1+\tau C_\mu)\|e_u^n\|^2+C(\tau^5+\tau h^4). \tag{4.19}$$

Plug in $\psi_h=e_p^n$ in (4.7). It follows that

$$\|e_p^{n+1}\|^2-\|e_p^n\|^2=\|e_p^{n+1}-e_p^n\|^2-\tau^2\|e_q^n\|^2-\tau\alpha_u[\![e_p^n]\!]^2+\gamma_p, \tag{4.20}$$

where $\gamma_p=-\tau^2(\varepsilon_q^n,e_q^n)-\mathcal{H}^+(\omega,e_p^n)-(\varepsilon_p^{n+1}-\varepsilon_p^n,e_p^n)$. Note that

$$\gamma_p \le \tau C_\mu\|e_p^n\|^2+\tau\|\varepsilon_q^n\|^2+\frac{1}{\tau}\|\varepsilon_u^{n+1}-\varepsilon_u^n\|^2+\frac{1}{\tau h^2}\|\omega\|^2 \le C_\mu\tau\|e_p^n\|^2+C(\tau^5h^{-2}+\tau h^4).$$

Hence, using Lemma 4.6, we get

$$\|e_p^{n+1}\|^2 \le (1+\tau C_\mu)\|e_p^n\|^2+C_\mu h^{-1}\lambda^2[\![e_u^n]\!]^2-\tau^2(1-C_\mu\lambda^2)\|e_q^n\|^2-\tau(1-C_\mu\lambda)[\![e_p^n]\!]^2+C(\tau^5h^{-2}+\tau^4+h^4). \tag{4.21}$$

(4.19) and (4.21) lead to the following inequality

$$\|e_u^{n+1}\|^2+\tau^2\|e_p^{n+1}\|^2 \le (1+\tau C_\mu)(\|e_u^n\|^2+\tau^2\|e_p^n\|^2)+C(\tau^5+\tau h^4). \tag{4.22}$$

Note that $\|e_u^0\|=0$, $\|e_p^0\|\le\|\varepsilon_h^0\|\le Ch^2$. Hence we have,

$$\|e_u^n\|^2+\tau^2\|e_p^n\|^2 \le C(\tau^4+h^4), \tag{4.23}$$

which implies $\|e_u^n\| \leq C(\tau^2 + h^2)$.

Step 2: We claim that $\|e_u^{n+1} - e_u^n\| \leq C(\tau^3 + \tau h^2)$.

Note that $e_u^{n+1} - e_u^n$ satisfies the same error equation as that of e_u^n, except for $\|\omega^n\| \leq C\tau^5$. Noting also that $\|e_u^1 - e_u^0\| = \|e_u^1\| \leq C\tau h^2$ by estimating (4.6a) directly, we can prove the statement in the same way as that in Step 1.

Step 3: (4.6) gives that

$$(e_u^{n+1} - e_u^n - \tau e_p^n - \frac{\tau^2}{2} e_q^n, e_p^n) = -(\nu^n, e_p^n), \tag{4.24}$$

where $\nu^n = \varepsilon_u^{n+1} - \varepsilon_u^n - \tau\varepsilon_p^n - \frac{\tau^2}{2}\varepsilon_q^n + \omega^n$. Hence

$$\|\nu^n\| \leq \|\varepsilon_u^{n+1} - \varepsilon_u^n\| + \tau\|\varepsilon_p^n + \frac{\tau}{2}\varepsilon_q^n\| + \|\omega^n\| \leq C(\tau^3 + \tau h^2).$$

Rearranging the equality, we have

$$\begin{aligned} \tau\|e_p^n\|^2 &= (e_u^{n+1} - e_u^n - \frac{\tau^2}{2} e_q^n, e_p^n) - (\nu^n, e_p^n) \\ &\Rightarrow \tau\|e_p^n\| \leq \|e_u^{n+1} - e_u^n\| + \frac{\tau^2}{2}\|e_q^n\| + \|\nu^n\| \\ &\Rightarrow \tau(1 - C_\mu\lambda)\|e_p^n\| \leq \|e_u^{n+1} - e_u^n\| + \frac{\tau^2}{2}\|\varepsilon_q^n\| + \|\nu^n\|, \end{aligned} \tag{4.25}$$

where Lemma 4.2 is used in the last step. Using the results in Step 1 and Step 2, one can get

$$\|e_p^n\| \leq C(\tau^2 + h^2).$$

Hence by Lemma 4.1,

$$\|u^n - u_h^n\| \leq \|\varepsilon_u^n\| + \|e_u^n\| \leq C(\tau^2 + h^2),$$

$$\|u_t^n - p_h^n\| \leq \|\varepsilon_p^n\| + \|e_p^n\| \leq C(\tau^2 + h^2).$$

□

Sketch of the proof for Theorem 4.2: We adopt similar notations as before. By using the same lines as those in the stability analysis, one can show $\|e_u^n\| \leq C(\tau^3 + h^{k+1})$ and $\|e_u^{n+1} - e_u^n\| \leq C(\tau^4 + \tau h^{k+1})$. This already implies the error estimate of u. Like what we have done in Step 3 above, $\tau\|e_p^n\| \leq \|e_u^{n+1} - e_u^n\| + \frac{\tau^2}{2}\|e_q^n\| + \frac{\tau^3}{6}\|e_r^n\| + \|\nu^n\|$, where $\nu^n = \varepsilon_u^{n+1} - \varepsilon_u^n - \tau\varepsilon_p^n - \frac{\tau^2}{2}\varepsilon_q^n - \frac{\tau^3}{6}\varepsilon_r^n + \omega^n$. We can bound $\|e_p^n\|$ by estimating each term here.

□

Remark 4.1. *Note that our optimal error estimates are only one-dimensional results. Since there are no proper Gauss-Radau projections for $\mathcal{P}_k$ elements in multidimensions to eliminate the cell boundary terms, one may lose the optimal order of accuracy when directly applying trace inverse inequalities. As for rectangular meshes with $\mathcal{Q}_k$ elements, we may obtain optimal error estimates for the third order schemes using similar arguments as in the one-dimensional case.*

5 Numerical tests

The main purpose of this section is to numerically validate the error estimates in section 4. We list the error tables of the second order schemes with fluxes $(+,+)$ and $(+,-)$, and the third order schemes with numerical fluxes $(+,+,+)$ and $(+,-,-)$. Here $(+,+)$ means $\alpha_u = +1$ and $\alpha_p = +1$. Others are defined analogously. The piecewise linear polynomial space is used for the second order schemes and the piecewise quadratic polynomial space is used for the third order schemes. Furthermore, we use the uniform time steps and uniform spatial meshes for the numerical tests. For two-dimensional problems, the triangular meshes are used and the triangulation is constructed by adding diagonals linking the left-bottom and right-top vertexes in a uniform square mesh.

5.1 One-dimensional example with constant coefficients

Consider

$$\begin{cases} u_t = u_x, & (x,t) \in (0,2\pi) \times (0,T), \\ u(x,0) = e^{\sin(x)}, & x \in (0,2\pi), \end{cases} \tag{5.1}$$

with periodic boundary conditions. The mesh size is chosen as $h = 2\pi/N$, where N is the number of cells and $N = 40, 80, 160, 320, 640$ are used. We compute up to the time $T = \pi/2$, with $\tau = 0.05h$. The numerical results are listed in Table 5.1. The convergence is as we have predicted in section 4.

Table 5.1: 1D, constant coefficients, $T = \frac{\pi}{2}$, $\lambda = 0.05$

scheme	N	u L^2 error	order	u_t L^2 error	order
$(+,+)$	40	4.3721E-03	-	7.7284E-03	-
	80	1.0993E-03	1.99	1.8998E-03	2.02
	160	2.7654E-04	1.99	4.7316E-04	2.01
	320	6.9407E-05	1.99	1.1824E-04	2.00
	640	1.7389E-05	2.00	2.9565E-05	2.00
$(+,-)$	40	4.3813E-03	-	6.6884E-03	-
	80	1.0999E-03	1.99	1.6147E-03	2.05
	160	2.7657E-04	1.99	4.0014E-04	2.01
	320	6.9409E-05	1.99	9.9854E-05	2.00
	640	1.7390E-05	2.00	2.4958E-05	2.00
$(+,+,+)$	40	9.0552E-05	-	1.8916E-04	-
	80	1.1336E-05	3.00	2.4017E-05	2.98
	160	1.4173E-06	3.00	3.0286E-06	2.99
	320	1.7717E-07	3.00	3.8029E-07	2.99
	640	2.2146E-08	3.00	4.7644E-08	3.00
$(+,-,-)$	40	9.0252E-05	-	1.9211E-04	-
	80	1.1327E-05	2.99	2.4465E-05	2.97
	160	1.4170E-06	3.00	3.0816E-06	2.99
	320	1.7716E-07	3.00	3.8645E-07	3.00
	640	2.2146E-08	3.00	4.8377E-08	3.00

5.2 One-dimensional example with variable coefficients

Using the same parameters, we solve the following problem with a variable coefficient

$$\begin{cases} u_t = \sin^2(x)\ u_x, & (x,t) \in (0,2\pi) \times (0,T), \\ u(x,0) = \sin(x), & x \in (0,2\pi), \end{cases} \tag{5.2}$$

where periodic boundary conditions are assumed. The exact solution to (5.2) is $u(x,t) = \sin\left(\cot^{-1}(\cot(x) - t)\right)$. Note that our analysis does not cover the problems with variable coefficients in one dimension. This numerical test is given only for showing the generality of our results. The numerical results are listed in Table 5.2. One can observe the designed order of accuracy for all the schemes and for both u and u_t.

Table 5.2: 1D, variable coefficients, $T = \frac{\pi}{2}$, $\lambda = 0.05$

scheme	N	u L^2 error	order	u_t L^2 error	order
$(+,+)$	40	4.3310E-02	-	1.2344E-01	-
	80	1.2661E-02	1.77	3.2003E-02	1.95
	160	3.4131E-03	1.89	8.6104E-03	1.89
	320	8.8808E-04	1.94	2.2432E-03	1.94
	640	2.2659E-04	1.97	5.7261E-04	1.97
$(+,-)$	40	4.3517E-02	-	1.2376E-01	-
	80	1.2705E-02	1.78	3.1334E-02	1.98
	160	3.4193E-03	1.89	8.3235E-03	1.91
	320	8.8883E-04	1.94	2.1586E-03	1.95
	640	2.2668E-04	1.97	5.5026E-04	1.97
$(+,+,+)$	40	6.1574E-03	-	1.1386E-02	-
	80	8.8437E-04	2.80	3.0574E-03	1.90
	160	1.1798E-04	2.91	4.1416E-04	2.88
	320	1.5238E-05	2.95	5.3551E-05	2.95
	640	1.9373E-06	2.98	6.8084E-06	2.98
$(+,-,-)$	40	6.1434E-03	-	1.1103E-02	-
	80	8.8038E-04	2.80	2.9849E-03	1.90
	160	1.1764E-04	2.90	4.1036E-04	2.86
	320	1.5217E-05	2.95	5.3397E-05	2.94
	640	1.9360E-06	2.97	6.8012E-06	2.97

5.3 Two-dimensional example with constant coefficients

Consider

$$\begin{cases} u_t = \frac{\sqrt{2}}{2}\, u_x + \frac{\sqrt{2}}{2} u_y, & (x,y,t) \in (0,2\pi)\times(0,2\pi)\times(0,T), \\ u(x,0) = \sin(x+y), & (x,y) \in (0,2\pi)\times(0,2\pi), \end{cases} \tag{5.3}$$

with periodic boundary conditions. The mesh is constructed by adding diagonals to the uniform square mesh with $N \times N$ elements, where $N = 20, 40, 80, 160, 320$. We compute up to time $T = 1$, with $\tau = 0.05h$. The numerical results are listed in Table 5.3. Once again, the designed order of accuracy can be observed.

5.4 Two-dimensional example with variable coefficients

We then compute a two-dimensional problem with variable coefficients. Consider

$$\begin{cases} u_t = (-\frac{y}{\pi}+1)u_x + (\frac{x}{\pi}-1)u_y, & (x,y,t) \in (0,2\pi)\times(0,2\pi)\times(0,T), \\ u(x,0) = e^{-2(x-\pi)^2-4(y-\pi)^2}, & (x,y) \in (0,2\pi)\times(0,2\pi), \end{cases} \tag{5.4}$$

Table 5.3: 2D, constant coefficients, $T = 1$, $\lambda = 0.05$

scheme	$N \times N$	u L^2 error	order	u_t L^2 error	order
$(+,+)$	20×20	1.2879E-01	-	1.8160E-01	-
	40×40	3.1837E-02	2.02	4.4990E-02	2.01
	80×80	7.8797E-03	2.01	1.1142E-02	2.01
	160×160	1.9579E-03	2.01	2.7689E-03	2.01
	320×320	4.8786e-04	2.00	6.8994e-04	2.00
$(+,-)$	20×20	1.2891E-01	-	1.7886E-01	-
	40×40	3.1844E-02	2.02	4.4179E-02	2.02
	80×80	7.8802E-03	2.01	1.0930E-02	2.02
	160×160	1.9580E-03	2.01	2.7148E-03	2.01
	320×320	4.8786E-04	2.00	6.7647E-03	2.00
$(+,+,+)$	20×20	2.7288E-03	-	3.8230E-03	-
	40×40	3.3286E-04	3.04	4.7406E-04	3.01
	80×80	4.1561E-05	3.00	5.8895E-05	3.01
	160×160	5.1948E-06	3.00	7.3615E-06	3.00
	320×320	6.4942E-07	3.00	9.2031E-07	3.00
$(+,-,-)$	20×20	2.7068E-03	-	3.6877E-03	-
	40×40	3.3274E-04	3.02	4.5206E-04	3.03
	80×80	4.1550E-05	3.00	5.6536E-05	3.00
	160×160	5.1941E-06	3.00	7.0576E-06	3.00
	320×320	6.4938E-07	3.00	8.8436E-07	3.00

with periodic boundary conditions. The same parameters are used as that of (5.3), except for the time steps $\tau = 0.02h$. When calculating the exact solution, we remove the periodic boundary conditions and compute by tracing the characteristic lines. For general initial inputs, this does cause inconsistency, and the error can be from the mismatch of both u and $\boldsymbol{\beta}$ outside of the domain. However, since our initial condition is intentionally chosen as a Gaussian centered at (π, π), which decays quickly near the boundary, the difference between the exact solution we use and the real exact solution is negligibly small. For simplicity of implementation, we apply upwind and downwind fluxes in terms of the wind direction at the midpoint of each edge, instead of strictly using the definition in our analysis. Since the coefficients are smooth, this should cause negligible difference. The L^2 error of u and u_t is listed in Table 5.4. We can observe the designed convergence of u. As for u_t, the LW2DG schemes exhibit a clear second order convergence, and the order of accuracy for the LW3DG schemes are also close to the third order.

Table 5.4: 2D, variable coefficients, $T = 1$, $\lambda = 0.02$

scheme	$N \times N$	u L^2 error	order	u_t L^2 error	order
$(+,+)$	20×20	6.0750E-02	-	2.5668E-02	-
	40×40	1.6066E-02	1.92	6.0414E-03	2.09
	80×80	4.0954E-03	1.97	1.4221E-03	2.09
	160×160	1.0300E-03	1.99	3.4356E-04	2.05
	320×320	2.5821e-04	2.00	8.4534E-05	2.02
$(+,-)$	20×20	6.0747e-02	-	2.5584e-02	-
	40×40	1.6067e-02	1.92	6.0305e-03	2.08
	80×80	4.0956e-03	1.97	1.4201e-03	2.09
	160×160	1.0301e-03	1.99	3.4311e-04	2.05
	320×320	2.5821e-04	2.00	8.4422e-05	2.02
$(+,+,+)$	20×20	5.6089e-03	-	6.5544e-03	-
	40×40	6.4540e-04	3.12	9.3527e-04	2.81
	80×80	7.4533e-05	3.11	9.4578e-05	3.31
	160×160	9.0299e-06	3.05	8.2682e-06	3.52
	320×320	1.1214e-06	3.01	7.0247e-07	3.56
$(+,-,-)$	20×20	5.6014E-03	-	6.5005E-03	-
	40×40	6.4485E-04	3.12	9.2575E-04	2.81
	80×80	7.4506E-05	3.11	9.3600E-05	3.31
	160×160	9.0289E-06	3.04	8.2128E-06	3.51
	320×320	1.1213E-06	3.01	6.9989E-07	3.55

Finally, we remark that, even though the choices of the numerical fluxes may have influence on the specific types of stability, they only cause minor differences on the L^2 error according to our numerical tests.

6 Concluding remarks

We analyze the stability and estimate the error of the Lax-Wendroff discontinuous Galerkin (LWDG) method for linear scalar conservation laws. Assume uniform or non-increasing time steps, one can choose between upwind and downwind fluxes for each variable. As we have shown, for both second order LW2DG schemes with $\mathcal{P}_1$ elements and third order LW3DG schemes with arbitrary $\mathcal{P}_k$, as long as we use the upwind flux for u, the scheme will be stable. Furthermore, if we also use the upwind flux for p (which approximates u_t), then the scheme will be strongly stable. On the other hand, if the downwind flux is used for

p, then the energy of the numerical solution will be bounded by the initial energy times a constant, which is independent of the total time T. In both cases we have a good control of the L^2 energy after long time integration. The stability results can be extended to high dimensions easily for problems with constant coefficients. For variable coefficient problems with a divergence-free condition $\nabla \cdot \boldsymbol{\beta}(x) = 0$, the previous results still hold for the third order schemes. But for the second order schemes, we can only prove the stability in a weaker sense, namely $\|u_h^n\| \leq C\|u_h^0\|$ where C may depend on T.

For the error estimates, we analyze one-dimensional problems with a smooth solution. We obtain optimal error estimates for both u and u_t. Once we choose the upwind flux for u, whatever fluxes we use for the remaining variables, our error estimates will hold. The LW2DG scheme with $\mathcal{P}_1$ elements is of the second order accuracy, and the LW3DG scheme with P_2 elements is of the third order accuracy.

Even though we have considered only scalar problems in this paper, the analysis can be easily generalized to one-dimensional linear hyperbolic systems and multidimensional symmetric linear systems. The method also generalizes to nonlinear scalar equations or systems, however stability analysis is expected to be difficult. On the other hand, error estimates for smooth solutions should carry through, along the lines in [18, 12] for RKDG methods.

A Relationships between RKDG and LWDG methods

We only consider the third order temporal discretization. The relationships of the second order schemes follow from the same lines.

Consider the time dependent problem $\boldsymbol{u}_t = L\boldsymbol{u}$. One can apply Lax-Wendroff procedure to write down a semi-discretized scheme with respect to time.

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \tau \boldsymbol{p}^n + \frac{\tau^2}{2}\boldsymbol{q}^n + \frac{\tau^3}{6}L\boldsymbol{q}^n,$$

$$\boldsymbol{p}^n = L\boldsymbol{u}^n, \qquad \boldsymbol{q}^n = L\boldsymbol{p}^n,$$

where τ is the time step.

In general, to obtain a fully discretized scheme, one can replace L with different consistent spatial discretization operators. For example, for the DG method, one can apply different numerical fluxes, and the final scheme will still approximate the original equation.

Now we assume the same spatial discretization L_N is used for each L. Then the scheme becomes

$$\begin{aligned} \boldsymbol{u}^{n+1} &= \boldsymbol{u}^n + \tau \boldsymbol{p}^n + \frac{\tau^2}{2}\boldsymbol{q}^n + \frac{\tau^3}{6} L_N \boldsymbol{q}^n, \\ \boldsymbol{p}^n &= L_N \boldsymbol{u}^n, \qquad \boldsymbol{q}^n = L_N \boldsymbol{p}^n, \end{aligned} \tag{A.1}$$

which corresponds to the truncated Taylor expansion for the semi-discretized scheme $\boldsymbol{u}_t = L_N \boldsymbol{u}$.

Assume L_N to be linear and independent of t. If the third order SSP Runge-Kutta method is applied for time stepping, one will get

$$\begin{aligned} \boldsymbol{u}^{(1)} &= \boldsymbol{u}^n + \tau L_N \boldsymbol{u}^n = \boldsymbol{u}^n + \tau \boldsymbol{p}^n, \\ \boldsymbol{u}^{(2)} &= \frac{3}{4}\boldsymbol{u}^n + \frac{1}{4}(\boldsymbol{u}^{(1)} + \tau L_N \boldsymbol{u}^{(1)}) \\ &= \frac{3}{4}\boldsymbol{u}^n + \frac{1}{4}(\boldsymbol{u}^n + \tau \boldsymbol{p}^n + \tau L_N \boldsymbol{u}^n + \tau^2 L_N \boldsymbol{p}^n) \\ &= \boldsymbol{u}^n + \frac{\tau}{2}\boldsymbol{p}^n + \frac{\tau^2}{4}\boldsymbol{q}^n, \\ \boldsymbol{u}^{n+1} &= \frac{1}{3}\boldsymbol{u}^n + \frac{2}{3}(\boldsymbol{u}^{(2)} + \tau L \boldsymbol{u}^{(2)}) \\ &= \frac{1}{3}\boldsymbol{u}^n + \frac{2}{3}(\boldsymbol{u}^n + \frac{\tau}{2}\boldsymbol{p}^n + \frac{\tau^2}{4}\boldsymbol{q}^n + \tau L_N \boldsymbol{u}^n + \frac{\tau^2}{2} L_N \boldsymbol{p}^n + \frac{\tau^3}{4} L_N \boldsymbol{q}^n) \\ &= \boldsymbol{u}^n + \tau \boldsymbol{p}^n + \frac{\tau^2}{2}\boldsymbol{q}^n + \frac{\tau^3}{6} L_N \boldsymbol{q}^n. \end{aligned} \tag{A.2}$$

After comparing (A.1) and (A.2), we will see, if the same spatial discretization L_N is used, with L_N being linear and independent of time t, then the fully discretized schemes obtain from the Lax-Wendroff procedure and the Runge-Kutta method are equivalent after one full time step.

In particular, let L_N corresponds to the DG discretization for multidimensional system $\boldsymbol{u}_t = \sum_{i=1}^{d} (A_i(\boldsymbol{x})\boldsymbol{u})_{x_i}$. One can check this specific L_N also satisfies the assumptions above. If this L_N is used through out the Lax-Wendroff procedure, then the scheme will be equivalent to the corresponding RKDG scheme.

B Proofs of several lemmas

B.1 Proof of Lemma 3.5

Proof. We restore the integral notation in this proof.

(1) In $\mathcal{H}_{\boldsymbol{\beta}}^-$, $w = w^+$ if $\boldsymbol{\beta}\cdot\boldsymbol{n} < 0$, $w = w^-$ if $\boldsymbol{\beta}\cdot\boldsymbol{n} > 0$. Hence

$$\begin{aligned}
\mathcal{H}_{\boldsymbol{\beta}}^-(w,w) &= \sum_{K\in\mathcal{K}} \frac{1}{2}\int_K (\boldsymbol{\beta}\cdot\nabla)w^2 dx - \sum_{K\in\mathcal{K}}\int_{\partial K} w^- w\boldsymbol{\beta}\cdot\boldsymbol{n} dl \\
&= \sum_{K\in\mathcal{K}}\int_{\partial K} (\frac{1}{2}w - w^-) w\boldsymbol{\beta}\cdot\boldsymbol{n} dl \\
&= \sum_{e\in\mathcal{E}}\int_e -\frac{1}{2}(w^-)^2|\boldsymbol{\beta}\cdot\boldsymbol{n}| - (\frac{1}{2}w^+ - w^-)w^+|\boldsymbol{\beta}\cdot\boldsymbol{n}| dl \\
&= \sum_{e\in\mathcal{E}}\int_e -\frac{1}{2}(w^- - w^+)^2|\boldsymbol{\beta}\cdot\boldsymbol{n}| dl \\
&= -\frac{1}{2}[\![w]\!]_{\boldsymbol{\beta}}^2.
\end{aligned} \tag{B.1}$$

(2) Since $\nabla\cdot\boldsymbol{\beta} = 0$, $(\boldsymbol{\beta}\cdot\nabla)v = \nabla\cdot(\boldsymbol{\beta}v)$

$$\begin{aligned}
\mathcal{H}_{\boldsymbol{\beta}}^-(w,v) &= \sum_{K\in\mathcal{K}}\int_K w\nabla\cdot(\boldsymbol{\beta}v)dx - \sum_{K\in\mathcal{K}}\int_{\partial K} w^- v\boldsymbol{\beta}\cdot\boldsymbol{n} dl \\
&= -\sum_{K\in\mathcal{K}}\int_K v(\boldsymbol{\beta}\cdot\nabla)w dx - \sum_{K\in\mathcal{K}}\int_{\partial K} (w^- - w)v\boldsymbol{\beta}\cdot\boldsymbol{n} dl \\
&= -\sum_{K\in\mathcal{K}}\int_K v(\boldsymbol{\beta}\cdot\nabla)w dx + \sum_{e\in\mathcal{E}}\int_e (w^- - w^+)v^+|\boldsymbol{\beta}\cdot\boldsymbol{n}| dl \\
&= -\sum_{K\in\mathcal{K}}\int_K v(\boldsymbol{\beta}\cdot\nabla)w dx + \sum_{K\in\mathcal{K}}\int_{\partial K} wv^+\boldsymbol{\beta}\cdot\boldsymbol{n} dl \\
&= -\mathcal{H}_{\boldsymbol{\beta}}^+(v,w).
\end{aligned} \tag{B.2}$$

(3)

$$\begin{aligned}\mathcal{H}^-_{\boldsymbol{\beta}}(w,v) &= \sum_{K\in\mathcal{K}}\int_K w(\boldsymbol{\beta}\cdot\nabla)v dx - \sum_{K\in\mathcal{K}}\int_{\partial K} w^- v\boldsymbol{\beta}\cdot\boldsymbol{n}dl \\ &= \sum_{K\in\mathcal{K}}\int_K w(\boldsymbol{\beta}\cdot\nabla)v dx - \sum_{K\in\mathcal{K}}\int_{\partial K} w^+ v\boldsymbol{\beta}\cdot\boldsymbol{n}dl + \sum_{K\in\mathcal{K}}\int_{\partial K}[w]v\boldsymbol{\beta}\cdot\boldsymbol{n}dl \\ &= \mathcal{H}^+_{\boldsymbol{\beta}}(w,v) + \sum_{e\in\mathcal{E}}\int_e [w](v^- - v^+)|\boldsymbol{\beta}\cdot\boldsymbol{n}|dl \\ &= \mathcal{H}^+_{\boldsymbol{\beta}}(w,v) - [w,v]_{\boldsymbol{\beta}}.\end{aligned} \tag{B.3}$$

□

B.2 Proof of Lemma 3.6

Proof. (1) According to (B.2),

$$\mathcal{H}^-_{\boldsymbol{\beta}}(w,v) = -\sum_{K\in\mathcal{K}}\int_K v(\boldsymbol{\beta}\cdot\nabla)w dx - \sum_{e\in\mathcal{E}}\int_e [w]v^+|\boldsymbol{\beta}\cdot\boldsymbol{n}|dl. \tag{B.4}$$

Therefore, by the Cauchy-Schwartz inequality,

$$\begin{aligned}|\mathcal{H}^-_{\boldsymbol{\beta}}(w,v)| &\le \sum_{K\in\mathcal{K}} \|v\|_K\|(\boldsymbol{\beta}\cdot\nabla)w\|_K + \sqrt{\sum_{e\in\mathcal{E}}\int_e [w]^2|\boldsymbol{\beta}\cdot\boldsymbol{n}|dl}\sqrt{\sum_{e\in\mathcal{E}}\int_e (v^+)^2|\boldsymbol{\beta}\cdot\boldsymbol{n}|dl} \\ &\le \|(\boldsymbol{\beta}\cdot\nabla)w\|\|v\| + C_{\boldsymbol{\beta}}[\![w]\!]_{\boldsymbol{\beta}}\sqrt{\sum_{e\in\mathcal{E}}\int_e (v^+)^2 dl}.\end{aligned} \tag{B.5}$$

We would like to bound the term $\sqrt{\sum_{e\in\mathcal{E}}\int_e (v^+)^2 dl}$ with $\|v\|$. However, v^+ will no longer be a polynomial if $\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}$ changes sign along the edge, and in this case one can not directly apply the trace inverse inequality. Therefore, we introduce v^l and v^r to denote the left and right limit of v along the edge. Then it gives

$$\begin{aligned}|\mathcal{H}^-_{\boldsymbol{\beta}}(w,v)| &\le \|(\boldsymbol{\beta}\cdot\nabla)w\|\|v\| + C_{\boldsymbol{\beta}}[\![w]\!]_{\boldsymbol{\beta}}\sqrt{\sum_{e\in\mathcal{E}}\int_e 2(v^l)^2 + 2(v^r)^2 dl}, \\ &\le (\|(\boldsymbol{\beta}\cdot\nabla)w\| + C_{\mu,\boldsymbol{\beta}}h^{-\frac{1}{2}}[\![w]\!]_{\boldsymbol{\beta}})\|v\|.\end{aligned} \tag{B.6}$$

(2) Use $\mathcal{H}^-_{\boldsymbol{\beta}}(w,v) = -\mathcal{H}^+_{\boldsymbol{\beta}}(v,w)$. The remaining steps are similar. □

B.3 Proof of Lemma 3.8

Proof. For simplicity, we drop all the subscripts h and superscripts n. Let $\boldsymbol{x}_0$ be the centroid of the element K in d-dimensions. Denote $\boldsymbol{\beta}_0 = \boldsymbol{\beta}(\boldsymbol{x}_0)$. Let $\mathcal{P}_1(K) = V_0 + V_1 + V_2$, where

$$V_0 = \{y \in \mathcal{P}_1(K) | y \text{ is a constant}\},$$

$$V_1 = \{y \in \mathcal{P}_1(K) | (y, z)_K = 0, \forall z \in V_0 \text{ and } (\boldsymbol{\beta}_0 \cdot \nabla) y = 0\},$$

$$V_2 = \{y \in \mathcal{P}_1(K) | (y, z)_K = 0, \forall z \in V_0 \cup V_1\}.$$

Then we can assume $p = p_0\phi^0 + p_1\phi^1 + p_2\phi^2$, where $\phi^i \in V_i$ and $\|\phi^i\|_K = 1$. Note that $\nabla\phi^0 = \vec{0}$ and $(\boldsymbol{\beta}_0 \cdot \nabla)\phi^1 = 0$. Using the inverse inequality, we get

$$\begin{aligned} \|(\boldsymbol{\beta} \cdot \nabla)p\|_K &\le \|p_2(\boldsymbol{\beta} \cdot \nabla)\phi^2\|_K + \|p_1((\boldsymbol{\beta} - \boldsymbol{\beta}_0) \cdot \nabla)\phi^1\|_K \\ &\le C_{\mu,\boldsymbol{\beta}} h^{-1} |p_2| \|\phi^2\|_K + C_\mu h^{-1} \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{L^\infty} |p_1| \|\phi^1\|_K \\ &\le C_{\mu,\boldsymbol{\beta}} h^{-1} |p_2| + C_\mu h^{-1} \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{L^\infty} |p_1|. \end{aligned} \tag{B.7}$$

Denote $e^+ = \{x \in \partial K | \boldsymbol{\beta}(\boldsymbol{x}) \cdot \boldsymbol{n} > 0\}$, and $e^- = \partial K \backslash e^+$. We integrate by parts to get p_2.

$$\begin{aligned} p_2 = (p, \phi^2)_K &= -(u, (\boldsymbol{\beta} \cdot \nabla)\phi^2)_K + \langle u^+, \phi^2 \boldsymbol{\beta} \cdot \boldsymbol{n} \rangle_{\partial K} \\ &= ((\boldsymbol{\beta} \cdot \nabla)u, \phi^2)_K - \langle u, \phi^2 \boldsymbol{\beta} \cdot \boldsymbol{n} \rangle_{\partial K} + \langle u^+, \phi^2 \boldsymbol{\beta} \cdot \boldsymbol{n} \rangle_{\partial K} \\ &= ((\boldsymbol{\beta} \cdot \nabla)u, \phi^2)_K - \langle u^-, \phi^2 |\boldsymbol{\beta} \cdot \boldsymbol{n}| \rangle_{e^+} + \langle u^+, \phi^2 |\boldsymbol{\beta} \cdot \boldsymbol{n}| \rangle_{e^-} + \langle u^+, \phi^2 \boldsymbol{\beta} \cdot \boldsymbol{n} \rangle_{\partial K} \\ &= (((\boldsymbol{\beta} - \boldsymbol{\beta}_0) \cdot \nabla)u, \phi^2)_K + \langle [u], \phi^2 |\boldsymbol{\beta} \cdot \boldsymbol{n}| \rangle_{e^+}, \end{aligned}$$

where we have used the fact $(\boldsymbol{\beta}_0 \cdot \nabla)u \in V_0$. By the trace inverse inequality,

$$\begin{aligned} |p_2| &\le C_{\mu,\boldsymbol{\beta}} [\![u]\!]_{\boldsymbol{\beta},\partial K} \|\phi^2\|_{\partial K} + C_\mu h^{-1} \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{L^\infty} \|u\|_K \\ &\le C_{\mu,\boldsymbol{\beta}} h^{-\frac{1}{2}} [\![u]\!]_{\boldsymbol{\beta},\partial K} + C_\mu h^{-1} \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{L^\infty} \|u\|_K. \end{aligned} \tag{B.8}$$

Similarly, one can show

$$|p_1| \le C_{\mu,\boldsymbol{\beta}} h^{-\frac{1}{2}} [\![u]\!]_{\boldsymbol{\beta},\partial K} + C_\mu h^{-1} \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{L^\infty} \|u\|_K. \tag{B.9}$$

(i) When $\boldsymbol{\beta}$ is a constant, $\boldsymbol{\beta} - \boldsymbol{\beta}_0 = 0$. Therefore, $\|(\boldsymbol{\beta} \cdot \nabla)p\|_K \le C_{\mu,\boldsymbol{\beta}} h^{-1} |p_2| \le C_{\mu,\boldsymbol{\beta}} h^{-\frac{3}{2}} [\![u]\!]_{\boldsymbol{\beta},\partial K}$.

(ii) For non-constant $\boldsymbol{\beta}$, by the regularity of $\boldsymbol{\beta}$, $\|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{L^\infty} \le C_{\boldsymbol{\beta}} h$. Therefore

$$\|(\boldsymbol{\beta} \cdot \nabla)p\|_K \le C_{\mu,\boldsymbol{\beta}} h^{-1} |p_2| + C_{\mu,\boldsymbol{\beta}} |p_1|$$

$$|p_2| \le C_{\mu,\boldsymbol{\beta}} h^{-\frac{1}{2}} [\![u]\!]_{\boldsymbol{\beta},\partial K} + C_{\mu,\boldsymbol{\beta}} \|u\|_K$$

$$|p_1| \le C_{\mu,\boldsymbol{\beta}} h^{-\frac{1}{2}} [\![u]\!]_{\boldsymbol{\beta},\partial K} + C_{\mu,\boldsymbol{\beta}} \|u\|_K,$$

which implies $\|(\boldsymbol{\beta} \cdot \nabla)p\|_K \le C_{\mu,\boldsymbol{\beta}} h^{-\frac{3}{2}} [\![u]\!]_{\boldsymbol{\beta},\partial K} + C_{\mu,\boldsymbol{\beta}} h^{-1} \|u\|_K$. □

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