SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHODS FOR 2-D HYPERBOLIC EQUATIONS*

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Abstract. This paper is concerned with superconvergence properties of discontinuous Galerkin (DG) methods for 2-D linear hyperbolic conservation laws over rectangular meshes when upwind fluxes are used. We prove, under some suitable initial and boundary discretizations, the $(2k+1)$-th order superconvergence rate of the DG approximation at the downwind points and for the cell averages, when piecewise tensor-product polynomials of degree k are used. Moreover, we prove that the gradient of the DG solution is superconvergent with a rate of $(k+1)$-th order at all interior left Radau points; and the function value approximation is superconvergent at all right Radau points with a rate of $(k+2)$-th order. Numerical experiments indicate that the aforementioned superconvergence rates are sharp.

Key words. Discontinuous Galerkin method, superconvergence, hyperbolic equations, Radau points, cell averages, initial and boundary discretizations

AMS subject classifications. 65M15, 65M60, 65N30

1. Introduction.

In this paper, we present and analyze the discontinuous Galerkin (DG) method for the two-dimensional linear hyperbolic conservation laws

$$\begin{aligned} &u_t + u_x + u_y = 0, \qquad (x, y, t) \in [0, 2\pi] \times [0, 2\pi] \times (0, T], \\ &u(x, y, 0) = u_0(x, y), \end{aligned} \tag{1.1}$$

where u_0 is sufficiently smooth. We will consider both the periodic boundary condition

$$u(0, y, t) = u(2\pi, y, t), \quad u(x, 0, t) = u(x, 2\pi, t),$$

and the Dirichlet boundary condition

$$u(0, y, t) = g_0(y, t), \quad u(x, 0, t) = g_1(x, t).$$

This paper is the fourth in a series ([10, 11, 13]) devoted to the study of superconvergence phenomena of the DG method for time-dependent partial differential equations. Superconvergence phenomena of finite element methods (or continuous Galerkin methods) were discussed as early as 1967 by Zienkiewicz and Cheung [38]. Since then the superconvergence behavior had been studied intensively. For an incomplete list of references, we refer to [7, 8, 15, 16, 21, 24, 25, 26, 27, 28, 29, 37] for finite element methods (FEM), [9, 12, 14, 20, 31] for finite volume methods (FVM), and [4, 5, 6, 18, 19, 23, 30, 32, 36] for DG methods.

In [10] and [13], we considered the discontinuous and local discontinuous Galerkin method for 1D hyperbolic conservation laws and parabolic equations when upwind fluxes (for hyperbolic conservation laws) and alternating fluxes (for parabolic equations) were used. For piecewise polynomials of degree k, these methods were shown

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to be superconvergent for the numerical traces at all nodes of the mesh, with a convergence rate of $(2k+1)$-th order (in an average sense). A pointwise $(k+2)$-th order superconvergence rate for the function value approximation and $(k+1)$-th order superconvergence rate for the derivative approximation at the (left or right) Radau points were also proved. Later, we provided in [11] a proof of $(2k+1)$-th order superconvergence rate for the cell averages and the pointwise error at nodes (downwind points for hyperbolic equations and numerical traces at nodes for parabolic equations).

In this paper, we continue our study of DG method applied to the 2-D linear hyperbolic conservation laws (1.1). To the best of our knowledge, there was not any global superconvergence result for these problems in the literature. Some previous works are based on local error estimates [1, 2, 3, 4, 5, 6], where the numerical fluxes are given as a projection of the boundary condition. Our contribution of this paper is to establish the superconvergence theory of DG methods for liner hyperbolic equations in two space dimensions. To be more precise, we shall provide a rigorous mathematical proof of the $(2k+1)$-th order superconvergence rate of the DG approximation at the downwind points and for the cell averages. We also prove the DG solution is superconvergent with a rate of $(k+2)$-th order at the right Radau points (function value approximation) and a rate of $(k+1)$-th order at the interior left Radau points (gradient approximation). As the reader may recall, these rates are the same as the counterparts in the 1D case. In other words, all superconvergence results in 1D can be extended to 2D. However, the analysis is by no means a trivial extension. Moreover, there are some new phenomena in the 2D situation, which are not shared by the 1D case.

A key step of our superconvergence analysis is the construction of a correction function. We have successfully applied the correction function idea to the DG method for 1D hyperbolic and parabolic equations (see, e.g. [10, 13]). However, when it comes to the 2D case, the procedure of constructing the correction function is more sophisticated. More special cares are needed. Especially for the Dirichlet boundary conditions, to achieve our desired superconvergence rate, e.g. $(2k+1)$-th order, both the (numerical) initial and boundary conditions need to be adjusted. This is very different from the 1D case, where only the initial condition needs to be adjusted. Our approach here is to construct a correction function w to reduce the error between the DG solution u_h and the truncated Radau expansion $P_h^- u$ of the exact solution, such that the errors $u_h - P_h^- u + w$ at the initial time and at the boundary are both convergent with order $2k+1$. By doing so, we prove that the DG solution u_h is superclose (with order $2k+1$) to $P_h^- u - w$, which yields the superconvergence properties for the cell averages and at downwind points and Radau points.

To end this introduction, we would like to point out that in principle it is straightforward to generalize the methodology we adopt in this paper to linear transport equations in higher dimensions. However, it requires very tedious and lengthy arguments to carry on the argument for 3D, 4D, etc.... in a mathematically rigorous way. On the other hand, one of our on-going project is to extend the investigation to nonlinear cases, where we require locally sufficiently smooth solution and nonlinear flux function, which will rule out possible shockwave regions.

The rest of the paper is organized as follows. In section 2, we present DG schemes for two-dimensional linear hyperbolic conservation laws. Section 3 is the most technical part, where we design a special correction function to reduce the error between the DG solution and the truncated Radau expansion of the exact solution. Section 4 is the main body of the paper, where the superconvergence results are proved with suitable initial and boundary discretizations. Some numerical examples are presented in section 5 to support our theoretical findings. Finally, we provide concluding remarks in section 6.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and the semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$. Notation $A \lesssim B$ implies that A can be bounded by B multiplied by a constant independent of the mesh size h.

2. DG schemes. Let $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{m+\frac{1}{2}} = 2\pi$ and $0 = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{n+\frac{1}{2}} = 2\pi$. For any positive integer r, we define $\mathbb{Z}_r = \{1, 2, \ldots, r\}$, and denote by $\mathcal{T}_h$ the rectangular partition of Ω. That is

$$\mathcal{T}_h = \{\tau_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] : (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n\}.$$

For any $\tau \in \mathcal{T}_h$, we denote by h_τ^x, h_τ^y the lengths of x- and y-directional edges of τ, respectively. h is the maximal length of all edges, and $h_{\min} = \min_\tau(h_\tau^x, h_\tau^y)$. We assume that the mesh $\mathcal{T}_h$ is *quasi-uniform* in the sense that there exist constants $c_1, c_2 > 0$ such that

$$h \le c_1 h_\tau^x, \quad h \le c_2 h_\tau^y \quad \forall \tau \in \mathcal{T}_h.$$

Define the finite element space

$$V_h = \{v : \ v|_\tau \in \mathbb{Q}_k(x,y) = \mathbb{P}_k(x) \times \mathbb{P}_k(y), \tau \in \mathcal{T}_h\},$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree at most k with coefficients as functions of t. The DG solution for (1.1) is to find $u_h \in V_h$ such that

$$a_\tau(u_h, v) = 0 \quad \forall \tau \in \mathcal{T}_h, v \in V_h, \tag{2.1}$$

where

$$a_\tau(u_h, v) = \int_\tau (u_{ht} v - u_h(v_x + v_y))dxdy + \int_{\partial\tau} \hat{u}_h v ds, \tag{2.2}$$

and for any $\tau = \tau_{i,j} \in \mathcal{T}_h, (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$,

$$\begin{aligned} \int_{\partial\tau_{i,j}} \hat{u}_h v ds = & \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\hat{u}_h(x_{i+\frac{1}{2}}, y) v(x_{i+\frac{1}{2}}^-, y) - \hat{u}_h(x_{i-\frac{1}{2}}, y) v(x_{i-\frac{1}{2}}^+, y)\right) dy \\ & + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\hat{u}_h(x, y_{j+\frac{1}{2}}) v(x, y_{j+\frac{1}{2}}^-) - \hat{u}_h(x, y_{j-\frac{1}{2}}) v(x, y_{j-\frac{1}{2}}^+)\right) dx. \end{aligned} \tag{2.3}$$

Here $v(x_{i-\frac{1}{2}}^-, \cdot), v(x_{i-\frac{1}{2}}^+, \cdot)$ denote the left and right limits of v across $x_{i-\frac{1}{2}}$, respectively, and $\hat{u}_h$ is the numerical flux. In this paper, we consider the upwind flux

$$\hat{u}_h = u_h^-.$$

To complete the DG scheme, we still need to define the numerical flux at the boundary $(\partial\Omega)^-$, where

$$(\partial\Omega)^- = \{(x,y) \in \partial\Omega : \mathbf{n}_0 \cdot \mathbf{n}(x,y) \le 0\}$$

with $\mathbf{n}_0 = (1,1)$ and $\mathbf{n}$ the outward normal unit vector at the boundary of a given domain. For the periodic boundary condition, the numerical flux at the boundary $(\partial\Omega)^-$ is taken as

$$\hat{u}_h(x_{\frac{1}{2}}, y) = u_h(x_{\frac{1}{2}}^-, y) = u_h(x_{m+\frac{1}{2}}^-, y) = \hat{u}_h(x_{m+\frac{1}{2}}, y),$$
$$\hat{u}_h(x, y_{\frac{1}{2}}) = u_h(x, y_{\frac{1}{2}}^-) = u_h(x, y_{n+\frac{1}{2}}^-) = \hat{u}_h(x, y_{n+\frac{1}{2}}).$$

While for the Dirichlet boundary condition, the numerical flux at $(\partial\Omega)^-$ is somewhat sophisticated. For the purpose of our superconvergence proof later, we take the numerical flux at $(\partial\Omega)^-$ as

$$u_h(x_{\frac{1}{2}}^-, y) = (P_h^- u - w)(x_{\frac{1}{2}}^-, y), \quad u_h(x, y_{\frac{1}{2}}^-) = (P_h^- u - w)(x, y_{\frac{1}{2}}^-). \tag{2.4}$$

Here $P_h^- u$ and w (defined in section 2) denote the truncated Radau expansion of u and the specially constructed correction function, respectively.

REMARK 2.1. *The special choice of the numerical flux at the boundary $(\partial\Omega)^-$ for the Dirichlet boundary condition is to guarantee that the boundary errors of DG approximation are small enough to be compatible with superconvergence error estimate, especially the $(2k+1)$-th superconvergence error at the downwind points. This choice is very different from the traditional one, which is usually taken as the L^2 projection, or the truncated Radau expansion, or the Radau interpolation function, of the exact solution u. As we shall demonstrate in the numerical experiments, the numerical fluxes at the boundary have a significant influence on the superconvergence at the downwind points.*

By denoting

$$a(u, v) = \sum_{\tau \in \mathcal{T}_h} a_\tau(u, v),$$

we obtain from a direct calculation

$$a(v, v) = (v_t, v) + \frac{1}{2}\left(\int_0^{2\pi} \sum_{i=1}^{m} [v]_{i-\frac{1}{2}}^2(y)dy + \int_0^{2\pi} \sum_{j=1}^{n} [v]_{j-\frac{1}{2}}^2(x)dx + \int_{\partial\Omega} [v^2] ds\right), \tag{2.5}$$

where

$$[v]_{i-\frac{1}{2}}(y) = v(x_{i-\frac{1}{2}}^+, y) - v(x_{i-\frac{1}{2}}^-, y), \quad [v]_{j-\frac{1}{2}}(x) = v(x, y_{j-\frac{1}{2}}^+) - v(x, y_{j-\frac{1}{2}}^-)$$

denote the jump of v across the points $(x_{i-\frac{1}{2}}, y)$ and $(x, y_{j-\frac{1}{2}})$, respectively, and

$$\int_{\partial\Omega} [v^2] ds = \int_0^{2\pi} \left(v^2(x_{m+\frac{1}{2}}^-, y) - v^2(x_{\frac{1}{2}}^-, y)\right) dy + \int_0^{2\pi} \left(v^2(x, y_{n+\frac{1}{2}}^-) - v^2(x, y_{\frac{1}{2}}^-)\right) dx.$$

If v satisfies the periodic boundary condition

$$v(x_{m+\frac{1}{2}}^-, y) = v(x_{\frac{1}{2}}^-, y), \quad v(x, y_{n+\frac{1}{2}}^-) = v(x, y_{\frac{1}{2}}^-), \tag{2.6}$$

or

$$v(x_{\frac{1}{2}}^-, y) = 0, \quad v(x, y_{\frac{1}{2}}^-) = 0, \tag{2.7}$$

then

$$\frac{1}{2}\frac{d}{dt}\|v\|_0^2 = (v_t, v) \le a(v, v). \tag{2.8}$$

3. Correction function. To study the superconvergence properties of the DG solution, we first construct a specially constructed interpolation function u_I of u such that u_I is superclose to the DG solution u_h. Then by using this super-closeness between u_I and u_h, we prove the superconvergence of the DG solution at some special points as well as for the cell averages.

In light of (2.8), by choosing $v = u_h - u_I$ and using the orthogonal property $a(u - u_h, v) = 0, v \in V_h$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_h - u_I\|_0^2 \le a(u - u_I, u_h - u_I).$$

Consequently, for all $t > 0$, the error $\|u_h - u_I\|_0(t)$ depends on two terms: $a(u - u_I, u_h - u_I)$ and the initial error $\|u_h - u_I\|_0(0)$. Since we can control the initial error by taking a special initial discretization, the superconvergence analysis of $u_h - u_I$ is reduced to the estimate of $a(u - u_I, u_h - u_I)$. Therefore, our next goal is to construct a special interpolation function u_I such that

$$a(u - u_I, v) \quad \forall v \in V_h$$

is of high order.

We begin the construction of u_I with the truncated Radau expansion $P_h^- u \in V_h$ of u. In each element $\tau_{i,j}, (i, j) \in \mathbb{Z}_m \times \mathbb{Z}_n$, suppose $u(x, y)$, for $(x, y) \in \tau_{i,j}$, has the following Radau expansion

$$u(x, y) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} u_{p,q}(L_{i,p} - L_{i,p-1})(x)(L_{j,q} - L_{j,q-1})(y), \tag{3.1}$$

where $L_{i,p}(x), L_{j,p}(y)$ denote the Legendre polynomial of degree p on the interval $\tau_i^x = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and $\tau_j^y = [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, respectively with $L_{i,-1}(x) = L_{j,-1}(y) = 0$, and

$$\begin{aligned} u_{p,q} = u(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}) &+ \sum_{l=0}^{p-1}\sum_{r=0}^{q-1} \frac{(2l+1)(2r+1)}{h_i^x h_j^y}\int_{\tau_{i,j}} u(x, y)L_{i,l}(x)L_{j,r}(y)dxdy \\ &- \sum_{l=0}^{p-1}\frac{2l+1}{h_i^x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, y^-_{j+\frac{1}{2}})L_{i,l}(x)dx - \sum_{r=0}^{q-1}\frac{2r+1}{h_j^y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x^-_{i+\frac{1}{2}}, y)L_{j,r}(y)dy \end{aligned}$$

with $h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, h_j^y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$. Then the Radau truncation $P_h^- u$ of u is defined as

$$P_h^- u(x, y) = \sum_{p=0}^{k}\sum_{q=0}^{k} u_{p,q}(L_{i,p} - L_{i,p-1})(x)(L_{j,q} - L_{j,q-1})(y).$$

A direct calculation yields

$$u - P_h^- u = E^x u + E^y u - E^x E^y u, \tag{3.2}$$

where

$$E^x u(x,y) = \sum_{p=k+1}^{\infty}\sum_{q=0}^{\infty} u_{p,q}(L_{i,p} - L_{i,p-1})(x)(L_{j,q} - L_{j,q-1})(y), \tag{3.3}$$

$$E^y u(x,y) = \sum_{p=0}^{\infty}\sum_{q=k+1}^{\infty} u_{p,q}(L_{i,p} - L_{i,p-1})(x)(L_{j,q} - L_{j,q-1})(y), \tag{3.4}$$

$$E^x E^y u(x,y) = \sum_{p=k+1}^{\infty}\sum_{q=k+1}^{\infty} u_{p,q}(L_{i,p} - L_{i,p-1})(x)(L_{j,q} - L_{j,q-1})(y). \tag{3.5}$$

The property of Legendre polynomials gives

$$E^x u(x^-_{i+\frac{1}{2}}, y) = 0, \quad \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} E^x u v_x dx = 0, \quad v \in \mathbb{P}_k(x), \tag{3.6}$$

$$E^y u(x, y^-_{j+\frac{1}{2}}) = 0, \quad \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} E^y u v_y dy = 0, \quad v \in \mathbb{P}_k(y), \tag{3.7}$$

$$E^x E^y u(x^-_{i+\frac{1}{2}}, y) = E^x E^y u(x, y^-_{j+\frac{1}{2}}) = 0, \int_{\tau_{i,j}} E^x E^y u \nabla \cdot v dxdy = 0, v \in V_h. \tag{3.8}$$

If we choose $u_I = P_h^- u$, then a straightforward analysis using the standard approximation theory yields

$$|a(u - P_h^- u, v)| \lesssim h^{k+1}, \quad v \in V_h,$$

which is far from our need for superconvergence. To achieve our superconvergence goal, we need a properly designed function w to correct the error between u and $P_h^- u$ such that

$$a(u - P_h^- u + w, v) \lesssim h^{k+1+l} \quad \forall v \in V_h$$

for some $l > 0$. Once the correction function w is designed, by letting $u_I = P_h^- u - w$, we finish the construction of the interpolation function u_I.

To construct the correction function w, we first study the term $a(u - P_h^- u, v), v \in V_h$. By the decomposition of $u - P_h^- u$ in (3.2), we have

$$a(u - P_h^- u, v) = a(E^x u, v) + a(E^y u, v) - a(E^x E^y u, v). \tag{3.9}$$

3.1. Correction function for $a(E^x u, v)$. We begin with some preliminaries. First, we define, for any $v(s) \in L^1[-1,1]$, a special Gauss-Radau projection P^- by

$$P^- v(1) = v(1), \quad \int_{-1}^{1} (P^- v - v)(s)\varphi(s) ds = 0 \quad \forall \varphi \in \mathbb{P}_{k-1},$$

and an integral operator D^{-1} by

$$D^{-1} v(s) = \int_{-1}^{s} v(s') ds'.$$

Define

$$F_1(s) = P^- D^{-1} L_k(s), \quad F_i(s) = P^- D^{-1} F_{i-1}(s), \quad 2 \le i \le k, s \in [-1,1], \tag{3.10}$$

where $L_k(s)$ denotes the Legendre polynomial of degree k on $[-1,1]$. It is proved in [13] that $F_i(s)$ has the following representation

$$F_i(s) = \sum_{p=k-i+1}^{k} b_p (L_p - L_{p-1})(s) \tag{3.11}$$

with $b_p's$ being bounded constants. By the properties of Legendre polynomials, we obtain

$$F_i(1) = 0, \quad |F_i| \lesssim 1, \quad F_i \perp \mathbb{P}_{k-i-1}, \quad i \in \mathbb{Z}_k. \tag{3.12}$$

Second, we define a special operator Q_h^x along the x-direction as follows: for any smooth function v, $Q_h^x v(x,\cdot)|_{\tau_i^x} \in \mathbb{P}_k(\tau_i^x), \tau_i^x = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and

$$Q_h^x v(x_{i-\frac{1}{2}}^+, \cdot) = v(x_{i-\frac{1}{2}}^+, \cdot), \quad Q_h^x v(x_{i+\frac{1}{2}}^-, \cdot) = v(x_{i+\frac{1}{2}}^-, \cdot), \tag{3.13}$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (v - Q_h^x v)(x,\cdot)\varphi dx = 0 \quad \forall \varphi \in \mathbb{P}_{k-2}(\tau_i^x), k \geq 2. \tag{3.14}$$

Note that in the case $k=1$, $Q_h^x v$ only satisfies the condition (3.13). It is easy to show the existence and uniqueness of $Q_h^x v$. Moreover, we have the following error estimate (c.f., [14, 17])

$$\|v - Q_h^x v\|_{p,\infty,\tau_i^x} \lesssim (h_i^x)^{l+1-p} \|v\|_{l+1,\infty,\tau_i^x}, \quad 1 \leq p \leq l. \tag{3.15}$$

Similarly, we can define the special operator Q_h^y along the y-direction.

For all $\tau = \tau_{i,j} \in \mathcal{T}_h$, recalling the definition of $a_\tau(\cdot,\cdot)$ in (2.2)-(2.3), we have, from (3.6) and the integration by parts,

$$a_\tau(E^x u, v) = \int_\tau ((E^x u)_t + (E^x u)_y)\, v dxdy + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [E^x u]_{j-\frac{1}{2}}(x) v(x, y_{j-\frac{1}{2}}^+) dx.$$

Since the function $E^x u(x,y)$ is continuous about y, we have

$$[E^x u]_{j-\frac{1}{2}}(x) = E^x u(x, y_{j-\frac{1}{2}}^+) - E^x u(x, y_{j-\frac{1}{2}}^-) = 0,$$

which yields

$$a_\tau(E^x u, v) = \int_\tau ((E^x u)_t + (E^x u)_y)\, v dxdy = \int_\tau E^x(u_t + u_y) v dxdy.$$

Consequently,

$$a(E^x u, v) = \int_\Omega E^x(u_t + u_y) v dxdy \quad \forall v \in V_h. \tag{3.16}$$

In light of (3.3), $E^x(u_t + u_y)$ has the following representation in each element $\tau_{i,j}, (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$,

$$E^x(u_t + u_y) = (\partial_t + \partial_y) E^x u = \sum_{p=k+1}^{\infty} (\partial_t + \partial_y) \bar{u}_{i,p}(y,t)(L_{i,p} - L_{i,p-1})(x),$$

where

$$\bar{u}_{i,p}(y,t) = u(x^-_{i+\frac{1}{2}}, y, t) - \sum_{l=0}^{p-1} \frac{2l+1}{h^x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,y,t)L_{i,l}(x)dx. \tag{3.17}$$

Then

$$a_{\tau_{i,j}}(E^x u, v) = \int_{\tau_{i,j}} (\partial_t + \partial_y)E^x uv dxdy = -\int_{\tau_{i,j}} (\partial_t + \partial_y)\bar{u}_{i,k+1}L_{i,k} v dxdy. \tag{3.18}$$

Now we are ready to construct the correction function corresponding to the term $a(E^x u, v)$. We define, for any positive integer l with $1 \le l \le k$,

$$w^l_1|_{\tau_{i,j}} := \sum_{p=1}^{l} w_{1,p}, \quad w_{1,p} = (\bar{h}^x_i)^p (Q^y_h G_p)(y,t)F_p(s), \tag{3.19}$$

where $\bar{h}^x_i = h^x_i/2 = (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})/2$, $s = (2x - x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})/h^x_i$ and

$$G_p = (-1)^{p-1}(\partial_t + \partial_y)^p \bar{u}_{i,k+1}, \quad p \ge 1.$$

Note that $u_{i,k+1} = 0$ when $u(x,\cdot) \in \mathbb{P}_k(\tau^x_i)$, we obtain from Bramble-Hilbert lemma and (3.17)

$$\|G_p\|_{0,\infty,\tau_{i,j}} = \|(\partial_t + \partial_y)^p \bar{u}_{i,k+1}\|_{0,\infty,\tau_{i,j}} \lesssim h^{k+1}\|u\|_{k+1+p,\infty,\tau_{i,j}}. \tag{3.20}$$

THEOREM 3.1. *Let $u \in W^{k+l+2,\infty}(\Omega), 1 \le l \le k$ be the solution of* (1.1), *and w^l_1 be the correction function defined by* (3.19). *Then*

$$w^l_1(x^-_{i+\frac{1}{2}}, y, t) = 0, \quad \|w_{1,p}\|_{0,\infty,\tau_{i,j}} \lesssim h^{k+1+p}\|u\|_{k+1+p,\infty,\tau_{i,j}}. \tag{3.21}$$

Moreover, there holds for all $v \in V_h$

$$\left|a(w^l_1, v) + a(E^x u, v)\right| \lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}\|v\|_{0,1}. \tag{3.22}$$

Proof. From the definition of (3.19), the first equation of (3.21) is a direct consequence of (3.12). On the other hand, the standard approximation theory and (3.20) give

$$\|Q^y_h G_p\|_{0,\infty,\tau_{i,j}} \lesssim \|G_p\|_{0,\infty,\tau_{i,j}} \lesssim h^{k+1}\|u\|_{k+1+p,\infty,\tau_{i,j}}.$$

Then the second inequality of (3.21) follows.

Now we consider (3.22). For all $\tau = \tau_{i,j}$, by (2.2)-(2.3), the fact that $w^l_1(x^-_{i+\frac{1}{2}}, \cdot) = 0$ and the integration by parts, we derive

$$\begin{aligned} a_\tau(w^l_1, v) &= \int_\tau \left((w^l_1)_t v - w^l_1 v_x + (w^l_1)_y v\right) dxdy + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [w^l_1]_{j-\frac{1}{2}}(x,t)v(x, y^+_{j-\frac{1}{2}})dx \\ &= \int_\tau \left((\partial_t + \partial_y)w^l_1 v - w^l_1 v_x\right)dxdy = \sum_{p=1}^{l} \int_\tau \left((\partial_t + \partial_y)w_{1,p} v - w_{1,p} v_x\right)dxdy. \end{aligned}$$

Here in the second step, we have used the fact that $G_p(y,t)$, for $p \geq 1$, is a continuous function about y, which yields

$$[w_1^l]_{j-\frac{1}{2}}(x,t) = w_1^l(x, y_{j-\frac{1}{2}}^+, t) - w_1^l(x, y_{j-\frac{1}{2}}^-, t) = 0.$$

Noticing that $\partial_t Q_h^y G_p = Q_h^y \partial_t G_p, 1 \leq p \leq l$ and

$$\begin{aligned}\partial_y Q_h^y G_p &= \partial_y G_p + O(h^{l-p}) \|\partial_y^{l+1-p} G_p\|_{0,\infty,\tau_{i,j}} \\ &= Q_h^y \partial_y G_p + O(h^{l-p}) \|\partial_y^{l+1-p} G_p\|_{0,\infty,\tau_{i,j}},\end{aligned}$$

we have

$$\begin{aligned}(\partial_t + \partial_y) Q_h^y G_p &= Q_h^y (\partial_t + \partial_y) G_p + O(h^{l-p}) \|\partial_y^{l+1-p} G_p\|_{0,\infty,\tau_{i,j}} \\ &= -Q_h^y G_{p+1} + O(h^{k+1+l-p}) \|u\|_{k+l+2,\infty,\tau_{i,j}}.\end{aligned}$$

Then for all $1 \leq p < l \leq k$,

$$\begin{aligned}\int_{\tau_{i,j}} v(\partial_t + \partial_y) w_{1,p} dxdy &= (\bar{h}_i^x)^p \int_{\tau_{i,j}} v(\partial_t + \partial_y) Q_h^y G_p F_p dxdy \\ &= -(\bar{h}_i^x)^p \int_{\tau_{i,j}} v Q_h^y G_{p+1} F_p dxdy + O(h^{k+1+l}) \|u\|_{k+l+2,\infty,\tau_{i,j}} \|v\|_{0,1,\tau_{i,j}} \\ &= (\bar{h}_i^x)^{p+1} \int_{\tau_{i,j}} v_x Q_h^y G_{p+1} D^{-1} F_p dxdy + O(h^{k+1+l}) \|u\|_{k+l+2,\infty,\tau_{i,j}} \|v\|_{0,1,\tau_{i,j}} \\ &= \int_{\tau_{i,j}} w_{1,p+1} v_x dxdy + O(h^{k+1+l}) \|u\|_{k+l+2,\infty,\tau_{i,j}} \|v\|_{0,1,\tau_{i,j}},\end{aligned}$$

where in the third step, we have used the integration by parts and the fact that

$$D^{-1} F_p(1) = D^{-1} F_p(-1) = 0, \quad 1 \leq p < l \leq k.$$

Summing over all $p, p = 1, \ldots, l$ gives

$$\begin{aligned}a_\tau(w_1^l, v) &= \sum_{p=1}^{l} \int_\tau \big((\partial_t + \partial_y) w_{1,p} v - w_{1,p} v_x\big) dxdy \\ &= \int_\tau \big((\partial_t + \partial_y) w_{1,l} v - w_{1,1} v_x\big) dxdy + O(h^{k+1+l}) \|u\|_{k+l+2,\infty,\tau} \|v\|_{0,1,\tau}.\end{aligned}$$

On the other hand, we have from (3.18)

$$\begin{aligned}a_\tau(E^x u, v) &= -\int_\tau L_{i,k} G_1 v dxdy = \bar{h}_i^x \int_\tau D^{-1} L_{i,k} G_1 v_x dxdy \\ &= \int_\tau w_{1,1} v_x dxdy + O(h^{k+1+l}) \|u\|_{k+l+2,\infty,\tau} \|v\|_{0,1,\tau}.\end{aligned}$$

Then

$$a_\tau(w_1^l + E^x u, v) = \int_\tau (\partial_t + \partial_y) w_{1,l} v dxdy + O(h^{k+1+l}) \|u\|_{k+l+2,\infty,\tau} \|v\|_{0,1,\tau}. \quad (3.23)$$

Substituting the second inequality of (3.21) into (3.23) and summing up all elements $\tau \in \mathcal{T}_h$, we obtain the desired result (3.22) directly. ▯

3.2. Correction function for $a(E^y u, v)$. Since $E^y u$ is a continuous function about x, and $E^y u(x, y^-_{j+\frac{1}{2}}) = 0$, by the same arguments as we did for $a(E^x u, v)$, we obtain

$$a(E^y u, v) = \int_\Omega E^y(u_t + u_x) v dxdy \quad \forall v \in V_h. \tag{3.24}$$

By (3.4), we have, in each element $\tau_{i,j}, (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$,

$$E^y(u_t + u_x) = (\partial_t + \partial_x) E^y u = \sum_{q=k+1}^{\infty} (\partial_t + \partial_x) \tilde{u}_{j,p}(x,t)(L_{j,q} - L_{j,q-1})(y),$$

where

$$\tilde{u}_{j,p}(x,t) = u(x, y^-_{j+\frac{1}{2}}, t) - \sum_{l=0}^{p-1} \frac{2l+1}{h^y_j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x,y,t) L_{j,l}(y) dy. \tag{3.25}$$

Then a direct calculation from (3.24) yields

$$a_{\tau_{i,j}}(E^y u, v) = \int_{\tau_{i,j}} (\partial_t + \partial_x) E^y u v dxdy = -\int_{\tau_{i,j}} (\partial_t + \partial_x) \tilde{u}_{j,k+1} L_{j,k} v dxdy.$$

The construction of the correction function $w^l_2, 1 \le l \le k$ for $a(E^y u, v)$ is similar to w^l_1, which is defined as:

$$w^l_2|_{\tau_{i,j}} := \sum_{p=1}^{l} w_{2,p}, \quad w_{2,p} = (\bar{h}^y_j)^p (Q^x_h \tilde{G}_p)(x,t) F_p(s). \tag{3.26}$$

Here $\bar{h}^y_j = h^y_j/2 = (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}})/2$, $s = (2y - y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}})/h^y_j$ and

$$\tilde{G}_p = (-1)^{p-1}(\partial_t + \partial_x)^p \tilde{u}_{i,k+1}, \quad p \ge 1.$$

Following the same line as in Theorem 3.1, we obtain

$$w^l_2(x, y^-_{j+\frac{1}{2}}, t) = 0, \quad \|w_{2,p}\|_{0,\infty,\tau_{i,j}} \lesssim h^{k+p+1} \|u\|_{k+p+1,\infty,\tau_{i,j}}, \tag{3.27}$$

and

$$\left| a(w^l_2, v) + a(E^y u, v) \right| \lesssim h^{k+l+1} \|u\|_{k+l+2,\infty} \|v\|_{0,1}. \tag{3.28}$$

3.3. Estimates. For any given l, where $1 \le l \le k$, we now define the final correction function w^l as

$$w^l = w^l_1 + w^l_2. \tag{3.29}$$

Here $w^l_i, i = \mathbb{Z}_2$ are defined in (3.19) and (3.26), respectively. We have the following estimates for the correction function w^l.

THEOREM 3.2. *Let $u \in W^{k+l+2,\infty}(\Omega), 1 \le l \le k$ be the solution of* (1.1)*, and w^l be the correction function defined by* (3.29)*,* (3.19)*, and* (3.26)*. Then*

$$w^l(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t) = 0, \quad \|w^l\|_{0,\infty} \lesssim h^{k+2} \|u\|_{k+l+1,\infty}. \tag{3.30}$$

Moreover, there holds for all $v \in V_h$

$$\left|a(w^l, v) + a(u - P_h^- u, v)\right| \lesssim h^{k+l+1} \|u\|_{k+l+2,\infty} \|v\|_{0,1}. \tag{3.31}$$

Proof. First, (3.30) is a direct consequence of (3.29), (3.21) and (3.27). By (3.9), (3.22) and (3.28),

$$\left|a(w^l, v) + a(u - P_h^- u, v)\right| \lesssim h^{k+l+1} \|u\|_{k+l+2,\infty} \|v\|_{0,1} + \left|a(E^x E^y u, v)\right|. \tag{3.32}$$

Now we estimate $a(E^x E^y u, v)$. A direct calculation from (2.3) and (3.8) yields

$$\int_{\partial \tau} E^x E^y uv ds = 0,$$

and thus

$$a(E^x E^y u, v) = \int_\Omega (E^x E^y u_t) v dxdy, \quad v \in V_h. \tag{3.33}$$

Since $u = E^x u$ when $u|_{\tau_i^x} \in \mathbb{P}_k(\tau_i^x)$, by Bramble-Hilbert lemma, we obtain

$$\|E^x u\|_{0,\infty,\tau_i^x} \lesssim (h_i^x)^r \int_{x_{i-\frac12}}^{x_{i+\frac12}} \left|\partial_x^{r+1} u\right| dx, \quad 1 \le r \le k.$$

Similarly, there holds

$$\|E^y u\|_{0,\infty,\tau_j^y} \lesssim (h_j^y)^r \int_{y_{j-\frac12}}^{y_{j+\frac12}} \left|\partial_y^{r+1} u\right| dy, \quad 1 \le r \le k.$$

Then

$$\|E^x E^y u\|_{0,\infty,\tau_{i,j}} \lesssim h^k \int_{x_{i-\frac12}}^{x_{i+\frac12}} \left|\partial_x^{k+1} E^y u\right| dx = h^k \int_{x_{i-\frac12}}^{x_{i+\frac12}} \left|E^y(\partial_x^{k+1} u)\right| dx \tag{3.34}$$

$$\lesssim h^{k+l-1} \int_{\tau_{i,j}} \left|\partial_y^l \partial_x^{k+1} u\right| dxdy \lesssim h^{k+l+1} \|u\|_{k+l+1,\infty,\tau_{i,j}}, \tag{3.35}$$

which yields

$$|a(E^x E^y u, v)| = \left|\int_\Omega (E^x E^y u_t) v dxdy\right| \lesssim h^{k+l+1} \|u\|_{k+l+2,\infty} \|v\|_{0,1}. \tag{3.36}$$

Plugging (3.36) into (3.32) yields the desired result (3.31). ∎

REMARK 3.3. *From* (3.36)*, the term* $a(E^x E^y u, v)$ *is of high order, which means that a correction function for* $a(E^x E^y u, v)$ *is not necessary.*

4. Superconvergence. In this section, we shall study superconvergence properties of the DG solution, including superconvergence for the cell averages and at some special points: downwind points and left and right Radau points.

We begin with the analysis of the super-closeness between the interpolation function $u_I^l = P_h^- u - w^l$ and the DG solution u_h.

THEOREM 4.1. *Let $u \in W^{k+l+2,\infty}(\Omega), 1 \le l \le k$ and $u_h \in V_h$ be solution of* (1.1) *and* (2.1)*, respectively. Suppose $u_I^l = P_h^- u - w^l \in V_h$ with w^l defined by* (3.29)*,* (3.19)*, and* (3.26)*. Then for both periodic and Dirichlet boundary conditions,*

$$\|u_I^l - u_h\|_0(t) \lesssim \|u_I^l - u_h\|_0(0) + th^{k+l+1}\|u\|_{k+l+2,\infty}, \quad \forall t > 0. \tag{4.1}$$

Proof. For the periodic boundary condition, we have from (3.12) and (3.19),

$$w_1^l(x^-_{m+\frac{1}{2}}, y, t) = w_1^l(x^-_{\frac{1}{2}}, y, t) = 0 \quad \forall y, t.$$

Since u satisfies the periodic boundary condition, then $P_h^- u$ satisfies the periodic boundary condition (2.6). Moreover, by (3.17),

$$\bar{u}_{i,k+1}(y^-_{n+\frac{1}{2}}, t) = \bar{u}_{i,k+1}(y^-_{\frac{1}{2}}, t) \quad \forall t \ge 0,$$

which yields

$$Q_h^y G_p(y^-_{n+\frac{1}{2}}, t) = G_p(y^-_{n+\frac{1}{2}}, t) = G_p(y^-_{\frac{1}{2}}, t) = Q_h^y G_p(y^-_{\frac{1}{2}}, t).$$

Then

$$w_1^l(x, y^-_{n+\frac{1}{2}}, t) = w_1^l(x, y^-_{\frac{1}{2}}, t) \quad \forall x, t.$$

Consequently, w_1^l satisfies the periodic boundary condition (2.6). Similar result holds true for the correction function w_2^l defined by (3.26). Since $P_h^- u, u_h$ and w^l all satisfy (2.6), then (2.6) is valid for $v = u_I^l - u_h$. For the Dirichlet boundary condition, due to the special choice of the numerical fluxes at the boundary, it is easy to see that (2.7) holds true for $v = u_I^l - u_h$. Therefore, (2.8) is valid for both periodic and Dirichlet boundary conditions with $v = u_I^l - u_h$. Then

$$\begin{aligned}
\|u_I^l - u_h\|_0 \frac{d}{dt}\|u_I^l - u_h\|_0 &\le \left|a(u_h - u_I^l, u_I^l - u_h)\right| \\
&= \left|a(u - u_I^l, u_I^l - u_h)\right| \\
&\lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}\|u_I^l - u_h\|_0,
\end{aligned}$$

which yields

$$\frac{d}{dt}\|u_I^l - u_h\|_0 \lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}.$$

Then (4.1) follows. □

REMARK 4.2. *To guarantee the superconvergence rate of $(k+l+1)$-th order for $\|u_I^l - u_h\|_0$, we know from Theorem 4.1 that the initial error should reach the same convergence rate. Namely,*

$$\|u_h(\cdot, 0) - u_I^l(\cdot, 0)\|_0 \lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}. \tag{4.2}$$

To obtain (4.2)*, a natural way of initial discretization is to choose*

$$u_h(x, y, 0) = u_I^l(x, y, 0). \tag{4.3}$$

Remark 4.3. *For the Dirichlet boundary condition, if we choose*

$$\hat{u}_h(x,y) = P_h u(x,y), \quad (x,y) \in (\partial\Omega)^-$$

instead of the choice of the numerical flux (2.4)*, where* $P_h u = R_h u, P_h^- u, I_h u$ *with* $R_h u$ *and* $I_h u$ *denoting the* L^2 *projection and the Radau interpolation function of* u *respectively, then the standard approximation theory yields*

$$\left| \int_{\partial\Omega} (u_I^l - u_h)^2 ds \right| \lesssim \|P_h^- u - P_h u\|_{0,\infty}^2 + \|w^l\|_{0,\infty}^2 \lesssim h^{2p},$$

where $p = k+1$ *for* $P_h u = R_h u$*, and* $p = k+2$ *for* $P_h u = P_h^- u, I_h u$*. This means that the boundary error of* $u_I^l - u_h$ *can not be ignored. Therefore, we have from* (2.5)*,*

$$\begin{aligned} \|u_I^l - u_h\|_0 \frac{d}{dt} \|u_I^l - u_h\|_0 &\le \left| a(u_h - u_I^l, u_I^l - u_h) \right| + \left| \int_{\partial\Omega} (u_I^l - u_h)^2 ds \right| \\ &\lesssim h^{k+l+1} \|u\|_{k+l+2,\infty} \|u_I^l - u_h\|_0 + h^{2p}. \end{aligned}$$

Thus, the choice of numerical fluxes at the boundary has an influence on the superconvergence rate for the Dirichlet boundary condition.

4.1. Superconvergence for the cell averages. We have the following superconvergence results for the cell averages.

Theorem 4.4. *Let* $u \in W^{2k+2,\infty}(\Omega)$ *be the solution of* (1.1)*, and* u_h *be the solution of* (2.1) *with the initial value* $u_h(\cdot, 0)$ *chosen such that* (4.2) *holds with* $l = k$*. Then for both the periodic and Dirichlet boundary conditions,*

$$e_{u,c} = \left(\frac{1}{nm} \sum_{\tau \in \mathcal{T}_h} \left(\frac{1}{|\tau|} \int_\tau (u - u_h) \right)^2 dxdy \right)^{\frac{1}{2}} \lesssim (1+t) h^{2k+1} \|u\|_{2k+2,\infty}. \tag{4.4}$$

Proof. Let $e_h = u - u_h$ and $u_I = u_I^k = P_h^- u - w^k$. By the special initial discretization, we have from (4.1),

$$\|u_I - u_h\|_0(t) \lesssim (1+t) h^{2k+1} \|u\|_{2k+2,\infty}. \tag{4.5}$$

On the other hand, the orthogonal property of (3.12) gives

$$\int_\tau w^k dxdy = \int_\tau \left(w_1^k + w_2^k \right) dxdy = \int_\tau (w_{1,k} + w_{2,k})\, dxdy, \ \tau \in \mathcal{T}_h.$$

Then

$$\begin{aligned} \int_\tau e_h dxdy &= \int_\tau \left(u - P_h^- u + w^k \right) dxdy + \int_\tau (u_I - u_h)\, dxdy \\ &= \int_\tau (w_{1,k} + w_{2,k})\, dxdy + \int_\tau (u_I - u_h)\, dxdy. \end{aligned}$$

In light of the estimates in (3.21) and (3.27), we derive

$$\left(\frac{1}{|\tau|} \int_\tau e_h dxdy \right)^2 \lesssim h^{2k+1} \|u\|_{2k+1,\infty,\tau} + |\tau|^{-1} \|u_I - u_h\|_{0,\tau}^2.$$

Since $\mathcal{T}_h$ is quasi-uniform, we have

$$\frac{1}{nm}|\tau|^{-1} \lesssim 1.$$

Then

$$e_{u,c} \lesssim \|u_I - u_h\|_0 + h^{2k+1}\|u\|_{2k+1,\infty} \lesssim (1+t)h^{2k+1}\|u\|_{2k+2,\infty}.$$

The proof is completed. ∎

4.2. Superconvergence at the downwind points. We are now ready to present our superconvergence result of the DG solution at the downwind points.

THEOREM 4.5. *Suppose all the conditions of Theorem 4.4 hold. Then for both the periodic and Dirichlet boundary conditions,*

$$e_{u,d} = \left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(u-u_h\right)^2(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t)\right)^{\frac{1}{2}} \lesssim (1+t)h^{2k+1}\|u\|_{2k+2,\infty}. \quad (4.6)$$

Proof. For any fixed t, $u_I - u_h \in V_h$ in each $\tau_{i,j}, (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Then the inverse inequality holds and thus,

$$\begin{aligned}\left|(u_I - u_h)(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t)\right| &\le \|u_I - u_h\|_{0,\infty,\tau_{i,j}}(t)\\ &\lesssim h^{-1}\|u_I - u_h\|_{0,\tau_{i,j}}(t).\end{aligned}$$

Then

$$\begin{aligned}\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(u_I-u_h\right)^2(x^-_{j+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t)\right)^{\frac{1}{2}} &\lesssim \left(\frac{h^{-2}}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\|u_I-u_h\|^2_{0,\tau_{i,j}}(t)\right)^{\frac{1}{2}}\\ &\lesssim \|u_I - u_h\|_0(t) \lesssim (1+t)h^{2k+1}\|u\|_{2k+2,\infty}.\end{aligned}$$

By (3.30) and the fact that $u(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t) = P^-_h u(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t)$, we have

$$u_I(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t) = u(x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t) \quad \forall (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n.$$

Then the desired result (4.6) follows. ∎

4.3. Superconvergence at the Radau points. Let $R^l_p, R^r_p, p \in \mathbb{Z}_k$ be the k interior left and right Radau points in the interval $[-1,1]$, respectively. Namely, $R^l_p, p \in \mathbb{Z}_k$ are zeros of $L_{k+1} + L_k$ except the point $s = -1$, and $R^r_p, p \in \mathbb{Z}_k$ are zeros of $L_{k+1} - L_k$ except the point $s = 1$. Then for all $\tau = \tau_{i,j} \in \mathcal{T}_h, (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$,

$$\mathcal{R}^l_\tau = \{P : P = (R^{l,x}_{\tau,p}, R^{l,y}_{\tau,q}), p, q \in \mathbb{Z}_k\}, \quad \mathcal{R}^r_\tau = \{Q : Q = (R^{r,x}_{\tau,p}, R^{r,y}_{\tau,q}), p, q \in \mathbb{Z}_k\},$$

constitute k^2 interior left and right Radau points in τ, respectively. Here

$$R^{l,x}_{\tau,p} = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}} + h^x_i R^l_p), \quad R^{l,y}_{\tau,p} = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}} + h^y_j R^l_p),$$

and $R^{r,x}_{\tau,p}, R^{r,y}_{\tau,p}$ are defined similarly. We denote by $\mathcal{R}^r = \bigcup_{\tau \in \mathcal{T}_h} \mathcal{R}^r_\tau$ the set of right Radau points on the whole domain, and

$$\mathcal{E}^l_x = \{(x,y) : x = R^{l,x}_{\tau,p}, y \in [c,d], p \in \mathbb{Z}_k, \tau \in \mathcal{T}_h\}$$

and

$$\mathcal{E}_y^l = \{(x,y) : y = R_{\tau,p}^{l,y}, x \in [a,b], p \in \mathbb{Z}_k, \tau \in \mathcal{T}_h\}$$

the set of vertical and horizontal edges of all interior left Radau points along the x-direction and the y-direction, respectively. We have the following estimates for $P_h^- u$ at the Radau points.

LEMMA 4.6. *Let $u \in W^{k+2,\infty}(\Omega)$ be the solution of (1.1). Then*

$$|(u - P_h^- u)(P)| \lesssim h^{k+2}\|u\|_{k+2,\infty} \quad \forall P \in \mathcal{R}^r, \tag{4.7}$$

and

$$|\partial_x(u - P_h^- u)(P)| + |\partial_y(u - P_h^- u)(Q)| \lesssim h^{k+1}\|u\|_{k+2,\infty} \quad \forall P \in \mathcal{E}_x^l, Q \in \mathcal{E}_y^l. \tag{4.8}$$

Proof. Recalling the decomposition of $u - P_h^- u$ in (3.2) and the estimate for $E^x E^y u$ in (3.35), we have for all $P \in \mathcal{R}_\tau^r, \tau \in \mathcal{T}_h$

$$|(u - P_h^- u)(P)| \lesssim |E^x u(P)| + |E^y u(P)| + h^{k+2}\|u\|_{k+2,\infty,\tau}.$$

In light of (3.3), we have $E^x u(P) = 0$ when $u(x,\cdot) \in \mathbb{P}_{k+1}$. By Bramble-Hilbert lemma, we obtain

$$|E^x u(P)| \lesssim h^{k+1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} |\partial_x^{k+2} u| dx \lesssim h^{k+2}\|u\|_{k+2,\infty,\tau_{i,j}} \quad \forall P \in \mathcal{R}_{\tau_{i,j}}^r.$$

Similarly,

$$|E^y u(P)| \lesssim h^{k+2}\|u\|_{k+2,\infty,\tau_{i,j}} \quad \forall P \in \mathcal{R}_{\tau_{i,j}}^r.$$

Then the desired result (4.7) follows.

Now we consider (4.8). Since it is shown in [35] that

$$(L_{k+1} - L_k)'(R_p^l) = (k+1)(L_{k+1} + L_k)(R_p^l)/(R_p^l + 1) = 0, \quad p \in \mathbb{Z}_k,$$

then for any $P = (R_{\tau,p}^{l,x}, y) \in \mathcal{E}_x^l, \tau = \tau_{i,j} \in \mathcal{T}_h$, we have $\partial_x E^x u(P) = 0$ when $u(x,\cdot) \in \mathbb{P}_{k+1}$, which yields

$$|\partial_x E^x u(P)| \lesssim h^k \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} |\partial_x^{k+2} u| dx \lesssim h^{k+1}\|u\|_{k+2,\infty,\tau_{i,j}}.$$

On the other hand,

$$|\partial_x E^y u(P)| \lesssim \|E^y u_x\|_{0,\infty,\tau_{i,j}} \lesssim h^{k+1}\|u\|_{k+2,\infty,\tau_{i,j}}.$$

Consequently,

$$\begin{aligned} |\partial_x(u - P_h^- u)(P)| &\lesssim |\partial_x E^x u(P)| + |\partial_x E^y u(P)| + \|\partial_x E^x E^y u\|_{0,\infty,\tau_{i,j}} \\ &\lesssim h^{k+1}\|u\|_{k+2,\infty,\tau_{i,j}}, \end{aligned}$$

where in the last step, we have used the fact that

$$\|\partial_x E^x E^y u\|_{0,\infty,\tau_{i,j}} \lesssim h^{-1}\|E^x E^y u\|_{0,\infty,\tau_{i,j}} \lesssim h^{k+1}\|u\|_{k+2,\infty,\tau_{i,j}}.$$

Following the same line, we obtain

$$|\partial_y(u - P_h^- u)(Q)| \lesssim h^{k+1}\|u\|_{k+2,\infty,\tau} \quad \forall Q = (x, R^{l,y}_{\tau,p}) \in \mathcal{E}^l_y.$$

Then (4.8) follows. ∎

Now we are ready to present the superconvergence properties of the DG solution at the Radau points.

THEOREM 4.7. *Let $u \in W^{k+4,\infty}(\Omega)$ be the solution of* (1.1)*, and u_h be the solution of* (2.1) *with the initial value $u_h(\cdot, 0)$ chosen such that* (4.2) *holds with $l = 2$. Then for both the periodic and Dirichlet boundary conditions,*

$$e_{u,r} = \max_{P\in\mathcal{R}^r} |(u - u_h)(P, t)| \lesssim (1 + t)h^{k+2}\|u\|_{k+4,\infty}, \tag{4.9}$$

and

$$e_{u,l} = \max_{P\in\mathcal{E}^l_x} |\partial_x(u-u_h)(P, t)| + \max_{Q\in\mathcal{E}^l_y} |\partial_y(u-u_h)(Q, t)| \lesssim (1+t)h^{k+1}\|u\|_{k+4,\infty}. \tag{4.10}$$

Proof. First, choosing $l = 2$ in (4.1) and (4.2) gives

$$\|u_h - u_I^2\|_0 \lesssim (1 + t)h^{k+3}\|u\|_{k+4,\infty}.$$

By (3.30) and the inverse inequality, we arrive at

$$\|u_h - P_h^- u\|_{0,\infty} \lesssim \|w^2\|_{0,\infty} + h^{-1}\|u_h - u_I^2\|_0 \lesssim (1 + t)h^{k+2}\|u\|_{k+4,\infty}, \tag{4.11}$$

and

$$\|u_h - P_h^- u\|_{1,\infty} \lesssim h^{-1}\|u_h - P_h^- u\|_{0,\infty} \lesssim (1 + t)h^{k+1}\|u\|_{k+4,\infty}.$$

Then (4.9)-(4.10) follow directly from (4.7)-(4.8) and the triangular inequalities. ∎

REMARK 4.8. *By* (4.10) *we know that, the x partial derivative of the numerical approximation is superconvergent at all the edges $x = R^{l,x}_{\tau,p}$, and the y partial derivative approximation is superconvergent at the edges $y = R^{l,y}_{\tau,p}$, for all $p \in \mathbb{Z}_k, \tau \in \mathcal{T}_h$. As a special case of* (4.10)*, the gradient approximation is superconvergent with an order $k + 1$ at all interior left Radau points, that is,*

$$|\nabla(u - u_h)(P, t)| \lesssim (1 + t)h^{k+1}\|u\|_{k+4,\infty} \quad \forall P = (R^{l,x}_{\tau,p}, R^{l,y}_{\tau,p}).$$

REMARK 4.9. *We observe from* (4.11)*, that the DG solution u_h is superconvergent with a rate of ($k + 2$)-th order to the truncated Radau projection $P_h^- u$.*

4.4. Initial and boundary disretizations. To end this section, we would like to demonstrate how to implement the initial and boundary discretizations. Since $u_t + u_x + u_y = 0$, we have for all integers $p \geq 1$,

$$(\partial_t + \partial_x)^p u(x, y, 0) = (-1)^p \partial_y^p u_0(x, y), \quad (\partial_t + \partial_y)^p u(x, y, 0) = (-1)^p \partial_x^p u_0(x, y).$$

Therefore, by (3.17) and (3.25), we have the derivatives

$$\begin{aligned}
\bar{G}_p(y, 0) &= (-1)^{p-1}(\partial_t + \partial_y)^p \bar{u}_{i,k+1}(y, 0) \\
&= -\partial_x^p u_0(x^-_{i+\frac{1}{2}}, y) + \sum_{l=0}^{k} \frac{2l + 1}{h_i^x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_x^p u_0(x, y)L_{i,l}(x)dx, \\
\tilde{G}_p(x, 0) &= (-1)^{p-1}(\partial_t + \partial_x)^p \tilde{u}_{i,k+1}(x, 0) \\
&= -\partial_y^p u_0(x, y^-_{j+\frac{1}{2}}) + \sum_{l=0}^{k} \frac{2l + 1}{h_j^y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \partial_y^p u_0(x, y)L_{j,l}(y)dy.
\end{aligned}$$

Now we divide the process of the initial discretization into the following steps:

1. Compute $F_p, p \in \mathbb{Z}_l$ from (3.10) .
2. In each element $\tau_{i,j}, (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$, calculate $\bar{G}_p(y,0), \tilde{G}_p(x,0)$ and choose

$$w_1^l = \sum_{p=1}^{l} (\bar{h}_i^x)^p F_p(s) Q_h^y \bar{G}_p(y,0), \quad s = (2x - x_{i-\frac{1}{2}} - x_{i+\frac{1}{2}})/h_i^x,$$

$$w_2^l = \sum_{p=1}^{l} (\bar{h}_j^y)^p F_p(s) Q_h^x \tilde{G}_p(x,0), \quad s = (2y - y_{j-\frac{1}{2}} - y_{j+\frac{1}{2}})/h_j^y.$$

3. Figure out $u_h(x,y,0) = P_h^- u_0 - w_1^l - w_2^l$.

The implementation of the boundary discretization for the Dirichlet boundary condition is similar to that of the initial discretization. Note that

$$\bar{G}_p(y_{\frac{1}{2}},t) = -\partial_x^p g_1(x,t) + \sum_{l=0}^{k} \frac{2l+1}{h_i^x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_x^p g_1(x,t) L_{i,l}(x) dx,$$

$$\tilde{G}_p(x_{\frac{1}{2}},t) = -\partial_y^p g_0(y,t) + \sum_{l=0}^{k} \frac{2l+1}{h_j^y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \partial_y^p g_0(y,t) L_{j,l}(y) dy.$$

By replacing $\bar{G}_p(y,0), \tilde{G}_p(x,0)$ with $\bar{G}_p(y_{\frac{1}{2}},t), \tilde{G}_p(x_{\frac{1}{2}},t)$ in the process of the initial discretization, we obtain $w_1^l(x,y_{\frac{1}{2}},t)$ and $w_2^l(x_{\frac{1}{2}},y,t)$. Then the numerical fluxes of u_h at the boundary $(\partial\Omega)^-$ are taken as

$$u_h(x_{\frac{1}{2}}^-,y,t) = P_h^- g_0 - w_2^l(x_{\frac{1}{2}},y,t), \quad u_h(x,y_{\frac{1}{2}}^-,t) = P_h^- g_1 - w_1^l(x,y_{\frac{1}{2}},t).$$

5. Numerical results. In this section, we use numerical examples to verify the theorems in section 4. Since all previous numerical tests in the literature (see, e.g., [1, 32]) are performed for lower order polynomials, e.g., $\mathbb{Q}_1$ and $\mathbb{Q}_2$, in order not to repeat, we only provide data for $\mathbb{Q}_3$ and $\mathbb{Q}_4$ in our numerical experiments. If not otherwise stated, the initial and boundary discretizations are given by the same way as in subsection 4.4.

Example 1. We solve the following problem

$$\begin{aligned} &u_t + u_x + u_y = 0, && (x,y,t) \in [0,2\pi] \times [0,2\pi] \times (0,0.1], \\ &u(x,y,0) = \sin(x+y), && (x,y) \in [0,2\pi] \times [0,2\pi], \end{aligned} \tag{5.1}$$

with the periodic boundary condition

$$u(0,y,t) = u(2\pi,y,t) \quad \text{and} \quad u(x,0,t) = u(x,2\pi,t).$$

Clearly, the exact solution is

$$u(x,y,t) = \sin(x+y-2t).$$

We use the ninth order SSP Runge-Kutta discretization in time [22] and take $\Delta t = 0.001 h_{\min}$ to reduce the time error. Non-uniform meshes of $m \times n$ rectangles are obtained by randomly and independently perturbing each node in the x and y axes of a uniform mesh by up to 20%. The example is tested by using $\mathbb{Q}_k$ polynomials with $k = 3, 4$. We compute the numerical solution at $t = 0.1$. In table 5.1, we compute

TABLE 5.1
Various errors with periodic boundary condition for $k = 3, 4$.

		$e_{u,c}$		$e_{u,d}$		$e_{u,r}$		$e_{u,l}$	
k	$m \times n$	error	order	error	order	error	order	error	order
3	20 ×20	2.48e-09	-	3.21e-11	-	5.45e-07	-	4.91e-05	-
	40 ×40	1.92e-11	6.93	7.89e-13	5.29	1.76e-08	4.89	3.09e-06	3.95
	80 ×80	1.49e-13	7.01	6.92e-15	6.83	6.07e-10	4.86	1.97e-07	3.95
	160×160	1.17e-15	7.07	3.77e-17	7.61	2.12e-11	4.90	1.33e-08	3.96
4	20 ×20	8.89e-13	-	1.70e-14	-	9.20e-09	-	9.32e-07	-
	40 ×40	1.68e-15	8.94	8.32e-17	7.59	1.36e-10	6.01	3.12e-08	4.93
	80 ×80	3.26e-18	9.02	1.80e-19	8.86	2.28e-12	5.90	1.01e-09	4.92
	160×160	6.45e-21	9.09	2.39e-22	9.66	3.88e-14	5.95	3.25e-11	5.05

several errors between the numerical approximation and the exact solution, which are given in Theorems 4.4-4.7.

Table 5.1 demonstrate superconvergence rates of $(2k+1)$-th order for the numerical cell averages and numerical approximation at the downwind points ($e_{u,c}$ and $e_{u,d}$), $(k+2)$-th order for the numerical solution at the right Radau points ($e_{u,r}$), and $(k+1)$-th order for the partial derivatives of the approximation at the interior left Radau points ($e_{u,l}$), which confirm our theoretical results in Theorems 4.4-4.7.

Example 2. We solve the following problem

$$\begin{array}{ll} u_t + u_x + u_y = 0, & (x,y,t) \in [0,2\pi] \times [0,2\pi] \times (0,0.1], \\ u(x,y,0) = \sin(x+y), & (x,y) \in [0,2\pi] \times [0,2\pi], \end{array} \tag{5.2}$$

with the Dirichlet boundary condition

$$u(0,y,t) = \sin(y-2t) \quad \text{and} \quad u(x,0,t) = \sin(x-2t).$$

Clearly, the exact solution is

$$u(x,y,t) = \sin(x+y-2t).$$

We use the fourth order SSP multi-step discretization in time [22] and take $\Delta t = 0.01h_{\min}^{2.5}$ to reduce the time error. The same quantities as in Example 1 on the same kind of random meshes of $m \times n$ rectangles are computed. The example is tested by using $\mathbb{Q}_k$ polynomials with $k = 3, 4$. We compute the numerical solution at $t = 0.1$. The computational results are given in Table 5.2.

From table 5.2, we can observe similar results as given in Example 1, which confirm our theoretical results in Theorems 4.4-4.7.

We also discretize the boundary condition with the L^2 projection by using Q_k polynomials with $k = 3, 4$, and the results are given in Table 5.3. We can hardly observe any of the desired superconvergence properties for the four errors given in Theorems 4.4-4.7. Actually, we can only observe the standard optimal rates of convergence.

We also consider two more discretizations of the boundary conditions: P_h^- projection and interpolation at Radau points, and the results are given in Tables 5.4-5.5. The numerical approximations at the right Radau points and the derivative approximations at the interior left Radau points are now superconvergent with $(k+2)$-th order

Table 5.2

Various errors with Dirichlet boundary condition for $k = 3, 4$.

		$e_{u,c}$		$e_{u,d}$		$e_{u,r}$		$e_{u,l}$	
k	$m \times n$	error	order	error	order	error	order	error	order
3	20 ×20	2.48e-09	-	3.11e-11	-	5.45e-07	-	4.91e-05	-
	40 ×40	1.92e-11	6.93	7.83e-13	5.26	1.76e-08	4.89	3.09e-06	3.95
	80 ×80	1.49e-13	7.01	6.91e-15	6.82	6.07e-10	4.86	1.97e-07	3.95
	160×160	1.17e-15	7.07	3.77e-17	7.61	2.12e-11	4.90	1.33e-08	3.96
4	20 ×20	9.16e-13	-	2.16e-14	-	9.20e-09	-	9.32e-07	-
	40 ×40	1.71e-15	8.97	8.24e-17	7.95	1.36e-10	6.01	3.12e-08	4.93
	80 ×80	3.28e-18	9.02	1.79e-19	8.85	2.28e-12	5.90	1.01e-09	4.92
	160×160	6.47e-21	9.09	2.40e-22	9.65	3.88e-14	5.95	3.25e-11	5.05

Table 5.3

Various errors with L^2 projection of the Dirichlet boundary condition for $k = 3,\ 4$.

		$e_{u,c}$		$e_{u,d}$		$e_{u,r}$		$e_{u,l}$	
k	$m \times n$	error	order	error	order	error	order	error	order
3	20 ×20	5.57e-08	-	4.71e-07	-	3.04e-06	-	2.72e-04	-
	40 ×40	6.73e-10	6.31	2.17e-08	4.40	1.78e-07	4.05	3.53e-05	2.91
	80 ×80	1.38e-10	2.27	3.34e-09	2.70	1.26e-08	3.81	5.03e-06	2.79
	160×160	1.09e-11	3.71	2.93e-10	3.55	7.79e-10	4.07	6.22e-07	3.07
4	20 ×20	3.70e-09	-	8.50e-07	-	8.50e-07	-	7.59e-05	-
	40 ×40	7.89e-12	8.78	2.51e-10	11.6	2.08e-09	8.58	6.32e-07	6.83
	80 ×80	1.66e-13	5.57	1.05e-11	4.58	6.05e-11	5.11	3.94e-08	3.98
	160×160	5.36e-15	5.01	5.66e-13	4.25	2.20e-12	4.84	2.68e-09	3.95

and $(k+1)$-th order, respectively. However, we cannot observe the $(2k+1)$-th order superconvergence for the numerical cell averages or for the numerical approximation at the downwind points.

Table 5.4

Various errors with P_h^- projection of the Dirichlet boundary condition for $k = 3, 4$.

		$e_{u,c}$		$e_{u,d}$		$e_{u,r}$		$e_{u,l}$	
k	$m \times n$	error	order	error	order	error	order	error	order
3	20 ×20	3.70e-09	-	1.81e-08	-	5.45e-07	-	4.91e-05	-
	40 ×40	1.43e-10	4.64	1.77e-09	3.32	1.88e-08	4.80	3.07e-06	3.95
	80 ×80	2.33e-12	5.94	7.92e-11	4.48	6.07e-10	4.95	1.97e-07	3.95
	160×160	8.93e-14	4.76	2.47e-12	5.06	2.12e-11	4.90	1.33e-08	3.96
4	20 ×20	6.64e-11	-	1.06e-08	-	1.06e-08	-	9.93e-07	-
	40 ×40	6.19e-14	9.96	5.89e-12	10.7	1.36e-10	6.22	3.12e-08	4.94
	80 ×80	4.03e-15	3.94	9.69e-14	5.92	2.28e-12	5.90	1.01e-09	4.92
	160×160	7.79e-17	5.76	2.93e-15	5.11	3.88e-14	5.95	3.25e-11	5.05

6. Concluding remarks. We have studied the superconvergence behavior of the DG solution for linear 2D hyperbolic equations using upwind fluxes and tensor

TABLE 5.5

Various errors with Radau interpolation of the Dirichlet boundary condition for $k = 3, 4$.

		$e_{u,c}$		$e_{u,d}$		$e_{u,r}$		$e_{u,l}$	
k	$m \times n$	error	order	error	order	error	order	error	order
3	20 ×20	3.29e-09	-	1.74e-08	-	5.45e-07	-	4.91e-05	-
	40 ×40	1.54e-10	4.37	1.61e-09	3.39	1.76e-08	4.89	3.09e-06	3.95
	80 ×80	3.14e-12	5.61	1.26e-10	3.68	6.07e-10	4.86	1.97e-07	3.95
	160×160	9.40e-14	5.12	3.95e-12	5.06	2.12e-11	4.90	1.33e-08	3.96
4	20 ×20	8.12e-11	-	1.76e-08	-	1.76e-08	-	1.37e-06	-
	40 ×40	6.57e-14	10.2	6.47e-12	11.3	1.36e-10	6.94	3.12e-08	5.39
	80 ×80	4.74e-15	3.80	1.64e-13	5.30	2.28e-12	5.90	1.01e-09	4.92
	160×160	6.18e-16	2.96	8.42e-15	4.33	3.88e-14	5.95	3.25e-11	5.05

product meshes and tensor product polynomials of degree k. We prove that, with suitable initial and boundary discretizations, the error between the DG solution and the exact solution converges with the rate of $(2k+1)$-th order (comparing with the standard optimal global rate of $(k+1)$-th order) for the cell averages and at the downwind points, and with rate of $(k+2)$-th order at all right Radau points. Moreover, we prove that the error for the gradient converges with the rate of $(k+1)$-th order (comparing with the standard optimal global rate of k-th order) at all interior left Radau points. Numerical experiments demonstrate that all the established error bounds above are optimal.

Finally, we would like to mention that the superconvergence analysis for $\mathbb{P}_k(x,y)$ is much more complicated than that for $\mathbb{Q}_k(x,y)$, where $\mathbb{P}_k(x,y)$ denotes the space of polynomials of degree no greater than k in each element $\tau \in \mathcal{T}_h$. Actually, this subject has been discussed under the framework of the standard C^0 finite element method for elliptic problems (see, e.g., [33, 34]), where the discussion is much more involved, and actually, most of superconvergence properties are lost for $\mathbb{P}_k(x,y)$. As for the LDG method, our numerical examples indicate that the superconvergence property will be lost without suitable initial and boundary discretizations. For $\mathbb{P}_k$ polynomials, constructing the correction function (to correct the initial and boundary errors) is difficulty and thus it deserves a separate study.

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