Analysis of sharp superconvergence of local discontinuous Galerkin method for one-dimensional linear parabolic equations [1]

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Abstract

In this paper, we study the superconvergence of the error for the local discontinuous Galerkin (LDG) finite element method for one-dimensional linear parabolic equations when the alternating flux is used. We prove that if we apply piecewise k-th degree polynomials, the error between the LDG solution and the exact solution is $(k+2)$-th order superconvergent at the Radau points with suitable initial discretization. Moreover, we also prove the LDG solution is $(k+2)$-th order superconvergent for the error to a particular projection of the exact solution. Even though we only consider periodic boundary condition, this boundary condition is not essential, since we do not use Fourier analysis. Our analysis is valid for arbitrary regular meshes and for $\mathcal{P}^k$ polynomials with arbitrary $k \geq 1$. We perform numerical experiments to demonstrate that the superconvergence rates proved in this paper are sharp.

Keywords: superconvergence, local discontinuous Galerkin method, parabolic equation, initial discretization, error estimates, Radau points

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1 Introduction

In this paper, we apply local discontinuous Galerkin (LDG) method to one-dimensional linear parabolic equation

$$\begin{array}{ll} u_t = u_{xx}, & (x,t) \in [0, 2\pi] \times [0, T], \\ u(x, 0) = u_0(x), & x \in [0, 2\pi], \end{array} \tag{1.1}$$

where the initial datum u_0 is assumed to be sufficiently smooth. For simplicity, we will consider periodic boundary condition $u(0,t) = u(2\pi, t)$. However, this assumption is not essential since the proof is not based on Fourier analysis. We use piecewise k-th degree polynomials to approximate the solution in each cell and prove that, under suitable initial discretization, the rate of convergence for the error between the LDG solution and the exact solution is of $(k+2)$-th order at the Radau points. Moreover, we also prove the $(k+2)$-th order superconvergence of the error between the LDG solution and a particular type of projection of the exact solution estimated in L^p-norm, for any $1 \le p \le \infty$.

The DG method was first introduced in 1973 by Reed and Hill [25], in the framework of neutron linear transport. Later, the method was applied by Johnson and Pitkäranta to a scalar linear hyperbolic equation and the L^p-norm error estimate was proved [23]. Subsequently, Cockburn et al. developed Runge-Kutta discontinuous Galerkin (RKDG) methods for hyperbolic conservation laws in a series of papers [18, 17, 16, 19]. In [20], Cockburn and Shu first introduced the LDG method to solve the convection-diffusion equation. Their idea was motivated by Bassi and Rebay [8], where the compressible Navier-Stokes equations were successfully solved.

The superconvergence properties have been analyzed intensively. In [2, 5], Adjerid et al. studied the ordinary differential equations and proved the $(k+2)$-th order superconvergence of the DG solutions at the downwind-biased Radau points. For hyperbolic equations, the superconvergence results have been investigated by several authors [6, 7, 12, 27, 24, 26, 10, 9]. Especially, in [26], we obtained sharp superconvergence for linear hyperbolic equations by using the dual argument, and this gives us the motivation to the prove the sharp superconvergence for linear parabolic equations. For convection-diffusion problems, in [3, 4], the authors used numerical experiments to demonstrate the superconvergence of LDG solution at the Radau points. In [11], the steady state solution was studied and the superconvergence of the numerical fluxes was proved. In [13], Cheng and Shu discussed the superconvergence property of the LDG scheme for heat equation by using piecewise linear approximations and uniform meshes. Subsequently, they proved the $(k + \frac{3}{2})$-th order superconvergence when using piecewise k-th degree polynomials with arbitrary k on arbitrary regular meshes in [14]. However, the convergence rate obtained in [14] is not sharp. Numerical tests demonstrated that the error of the DG solution towards a particular projection of the exact solution is $(k + 2)$-th order accurate, even on highly non-uniform meshes. In [14], the framework to prove the superconvergence results does not rely on Fourier analysis. Recently, in [10, 9], the authors studied the sharp superconvergence of linear hyperbolic and parabolic equations. In this paper, we give another proof for the estimate of the error between the exact and numerical solutions at the Radau points for linear parabolic equations. Motivated by [26], we adopt the dual argument to obtain the sharp rate of superconvergence and improve upon the result in [14]. The proof works for arbitrary regular meshes and schemes of any order.

The organization of this paper is as follows. In Section 2, we introduce the LDG scheme and state the main theorem. In Section 3, we present some preliminaries, including the norms we use throughout the paper, Radau polynomials, some essential properties of the finite element spaces, LDG spatial discretization as well as the error equations. Section 4 is the main body of the paper where the main theorem is proved. Numerical evidences about the sharpness of the superconvergence estimates are given in Section 5. In Section 6, we present some concluding remarks and remarks on future work. Finally, the initial discretization and properties about the test functions are given in Appendices A and B, respectively.

2 LDG scheme and the main result

In this section, we construct the LDG scheme for the linear parabolic equation (1.1). First, we divide the computational domain $\Omega = [0, 2\pi]$ into N cells

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi,$$

and define

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$$

to be the cells. Let h_j be the length of the cell I_j, and denote $h = h_{\max} = \max_j h_j$ and $h_{\min} = \min_j h_j$ to be the lengths of the largest and smallest cells, respectively.

The finite element space is defined as

$$V_h^k = \left\{ v : v|_{I_j} \in \mathcal{P}^k(I_j), j = 1, \cdots, N \right\},$$

where $\mathcal{P}^k(I_j)$ is the space of polynomials in I_j with degree no more than k. In addition, we also define

$$H_h^1 = \left\{ v : v|_{I_j} \in H^1(I_j), j = 1, \cdots, N \right\}.$$

To construct the LDG scheme, we introduce an auxiliary variable $q = u_x$, then (1.1) can be written as a first order linear system

$$\begin{aligned} u_t &= q_x, \\ q &= u_x. \end{aligned} \tag{2.1}$$

The LDG scheme we consider is the following: find $u_h,\ q_h \in V_h^k$ such that for any $v_h,\ w_h \in V_h^k$

$$\begin{aligned} ((u_h)_t, v_h)_j &= -(q_h, (v_h)_x)_j - \hat{q}_h v_h^+|_{j-\frac{1}{2}} + \hat{q}_h v_h^-|_{j+\frac{1}{2}}, \\ (q_h, w_h)_j &= -(u_h, (w_h)_x)_j - \hat{u}_h w_h^+|_{j-\frac{1}{2}} + \hat{u}_h w_h^-|_{j+\frac{1}{2}}, \end{aligned} \tag{2.2}$$

where $(u, v)_j = \int_{I_j} uv dx$, and $v_h^-|_{j+\frac{1}{2}}$ denotes the left limit of the function v_h at $x_{j+\frac{1}{2}}$. Likewise for v_h^+. $\hat{q}_h$ and $\hat{u}_h$ are the numerical fluxes. For LDG scheme, we consider the alternating fluxes

$$\hat{q}_h = q_h^+, \quad \hat{u}_h = u_h^-, \tag{2.3}$$

or

$$\hat{q}_h = q_h^-, \quad \hat{u}_h = u_h^+. \tag{2.4}$$

In this paper, we use (2.4) as the numerical flux, then the LDG scheme turns out to be

$$((u_h)_t, v_h)_j = \mathcal{H}_j^1(q_h, v_h), \qquad (q_h, w_h)_j = \mathcal{H}_j^2(u_h, w_h), \tag{2.5}$$

where

$$\mathcal{H}_j^1(q_h, v_h) = -(q_h, (v_h)_x)_j - q_h^- v_h^+|_{j-\frac{1}{2}} + q_h^- v_h^-|_{j+\frac{1}{2}}, \tag{2.6}$$

$$\mathcal{H}_j^2(u_h, w_h) = -(u_h, (w_h)_x)_j - u_h^+ w_h^+|_{j-\frac{1}{2}} + u_h^+ w_h^-|_{j+\frac{1}{2}}. \tag{2.7}$$

In this paper, we assume $k \geq 1$, and define $\mathbb{P}_- u$ and $\mathbb{P}_+ u$ as two Gauss-Radau projections of u into V_h^k such that

$$(\mathbb{P}_- u, v)_j = (u, v)_j \quad \forall v \in \mathcal{P}^{k-1}(I_j) \quad \text{and} \quad \mathbb{P}_- u(x_{j+\frac{1}{2}}^-) = u(x_{j+\frac{1}{2}}^-), \tag{2.8}$$

$$(\mathbb{P}_+ u, v)_j = (u, v)_j \quad \forall v \in \mathcal{P}^{k-1}(I_j) \quad \text{and} \quad \mathbb{P}_+ u(x_{j-\frac{1}{2}}^+) = u(x_{j-\frac{1}{2}}^+). \tag{2.9}$$

For the initial discretization, we require

$$q_h = \mathbb{P}_- q \quad \text{and} \quad \|u_h - \mathbb{P}_+ u\| = \mathcal{O}(h^{k+2}). \tag{2.10}$$

This requirement is used in our proof. However, other initial discretizations, such as $u_h = P_- u$, can still yield the same result in Theorem 2.1 in numerical experiments. The construction of the initial discretization will be given in Appendix A. Now, we can state the main result.

Theorem 2.1. *Let $u(x,t)$ be the exact solution of the linear parabolic equation (1.1) and u_h be the numerical solution of the LDG scheme (2.5). The finite element space is made up of polynomials of degree $k \geq 1$. Then at t=T, we have*

$$\max_{1 \leq j \leq N} |(u - u_h)(x_j)| \leq \begin{cases} C(1+T)h^{k+2} \ln h & k = 1 \\ C(1+T)h^{k+2} & k \geq 2 \end{cases}, \tag{2.11}$$

where x_j is any one of the left-biased Radau points in the cell I_j. The constant C does not depend on h or T, but depends on $\|u\|_{k+5,2}$ and $\|u\|_{k+3,\infty}$.

In addition, we can also prove the following corollary.

Corollary 2.1. *Suppose the conditions in the above theorem are satisfied, then we have*

$$\|\mathbb{P}_+ u - u_h\|_p \le \begin{cases} C(1+T)h^{k+2}\ln h & k=1 \\ C(1+T)h^{k+2} & k\ge 2 \end{cases}, \tag{2.12}$$

where $1 \le p \le \infty$ *is a constant, and the constant* C *does not depend on* h, T *but depends on* p, $\|u\|_{k+5,2}$ *and* $\|u\|_{k+3,\infty}$.

3 Preliminaries

3.1 Norms

In this subsection, we present some norms that will be used later.

Denote $\|u\|_{p,I_j}$ to be the standard L^p-norms of u on I_j with $1 \le p < \infty$. For any natural number $\ell \ge 1$, we consider the norm of the Sobolev space $W^{\ell,p}(I_j)$, defined by

$$\|u\|_{\ell,p,I_j} = \left\{ \sum_{0\le\alpha\le\ell} \|D^\alpha u\|_{p,I_j}^p \right\}^{1/p},$$

where $D^\alpha u = \frac{d^\alpha u}{dx^\alpha}$ is the α-th order spatial derivative. Moreover, the $W^{\ell,\infty}$-norm is defined as

$$\|u\|_{\ell,\infty,I_j} = \max_{0\le\alpha\le\ell} \|D^\alpha u\|_{\infty,I_j},$$

where $\|u\|_{\infty,I_j}$ is the standard L^∞-norm of u on I_j. Clearly, the L^∞-norm is stronger than the L^2-norm, and we have

$$\|u\|_{2,I_j} \le h_j^{1/2} \|u\|_{\infty,I_j}. \tag{3.1}$$

For convenience, if we consider the standard L^2-norm, then the corresponding index will be omitted, and we use $\|u\|_{I_j}$ to denote $\|u\|_{2,I_j}$.

Finally, we define the norms on the whole computational domain as follows:

$$\|u\|_{\ell,p,\Omega} = \left(\sum_{j=1}^N \|u\|_{\ell,p,I_j}^p \right)^{\frac{1}{p}}, \qquad \|u\|_{\ell,\infty,\Omega} = \max_{1\le j\le N} \|u\|_{\ell,\infty,I_j},$$

where $1 \le p < \infty$. For simplicity, if we consider the norm on the whole computational domain Ω, then the corresponding index will be omitted. Especially, we use $\|\cdot\|$ for the standard L^2-norm on Ω.

3.2 Radau polynomials

In this subsection, we study the properties of Radau polynomials, more details can be found in [1].

We denote

$$L^k(x) = \frac{(-1)^k}{2^k k!} \frac{d^k}{dx^k}\left[(1-x^2)^k\right]$$

to be the Legendre polynomial of degree k on $[-1,1]$. Then the left-biased Radau polynomial of degree $k+1$ is defined as

$$R_1^{k+1} = L^k + L^{k+1}.$$

Moreover, we define the scaled left-biased Radau polynomial on cell I_j as

$$R_{1,j}^{k+1}(x) = R_1^{k+1}\left(\frac{2x - x_{j-\frac{1}{2}} - x_{j+\frac{1}{2}}}{h_j}\right),$$

and the left-biased Radau points on I_j are given as the zeros of $R_{1,j}^{k+1}(x)$. In Theorem 2.1, we would like to prove the superconvergence property at these points. Similarly, we also define the right-biased Radau polynomial on $[-1,1]$ and the scaled one on I_j as

$$R_2^{k+1} = (-1)^{k+1}(L^{k+1} - L^k) \quad \text{and} \quad R_{2,j}^{k+1}(x) = R_2^{k+1}\left(\frac{2x - x_{j-\frac{1}{2}} - x_{j+\frac{1}{2}}}{h_j}\right),$$

respectively.

The following two properties are important in our analysis.

- $R_{1,j}^{k+1}(x_{j-\frac{1}{2}}) = 0$, $R_{1,j}^{k+1}(x_{j+\frac{1}{2}}) = 2$, $R_{2,j}^{k+1}(x_{j-\frac{1}{2}}) = 2$, $R_{2,j}^{k+1}(x_{j+\frac{1}{2}}) = 0$.
- For any $Q(x) \in \mathcal{P}^{k-1}(I_j)$, $\int_{I_j} R_{1,j}^{k+1}(x)Q(x)dx = 0$ and $\int_{I_j} R_{2,j}^{k+1}(x)Q(x)dx = 0$.

3.3 Properties of the finite element space

In this subsection, we state some properties of the finite element space. Let us start with the following inverse property [15].

Lemma 3.1. *Assuming $u \in V_h^k$, then there exists a constant $C > 0$ independent of h and u such that*

$$\left|u_{j+\frac{1}{2}}^-\right| + \left|u_{j-\frac{1}{2}}^+\right| \le C h_j^{-1/2} \|u\|_{I_j}. \tag{3.2}$$

In addition to the Gauss-Radau projections $\mathbb{P}_-$ and $\mathbb{P}_+$ defined in (2.8) and (2.9), we also introduce the k-th order L^2 projection of u as a function $\mathbb{P}_k u \in V_h^k$ such that:

$$(\mathbb{P}_k u, v)_j = (u, v)_j \ , \quad \forall v \in \mathcal{P}^k(I_j). \tag{3.3}$$

Suppose $\mathbb{P}_h$ is a projection, either $\mathbb{P}_k$, $\mathbb{P}_+$ or $\mathbb{P}_-$. Denote the error operator by $\mathbb{P}_h^\perp = \mathbb{I} - \mathbb{P}_h$, where $\mathbb{I}$ is the identity operator. By the scaling argument, we have the following lemma [15].

Lemma 3.2. *Suppose the function $u(x) \in C^{k+1}(I_j)$, then there exists a positive constant C independent of h and u, such that for any natural number m with $0 \le m \le k$, we have*

$$\|\mathbb{P}_h^\perp u\|_{I_j} \le C h_j^{m+1} \|D^{m+1} u\|_{I_j} \quad \text{and} \quad \|\mathbb{P}_h^\perp u\|_{\infty, I_j} \le C h_j^{m+1} \|D^{m+1} u\|_{\infty, I_j}. \tag{3.4}$$

Besides the above, we also use the following lemma for L^1-norm error estimates [15].

Lemma 3.3 (Bramble-Hilbert Lemma)**.** *Suppose the function $u(x) \in C^{k+1}(I_j)$, then there exists a positive constant C independent of h and u, such that*

$$\inf_{v \in \mathcal{P}^k} \|u - v\|_{1, I_j} \le C h_j^{k+1} \|D^{k+1} u\|_{1, I_j}. \tag{3.5}$$

Moreover, one can also prove the following superconvergence property [2].

Lemma 3.4. *Suppose $u(x) \in C^{k+2}(I_j)$, and x_j is one of the left-biased Radau points in the cell I_j, then*

$$|(u - \mathbb{P}_+ u)(x_j)| \le C h_j^{k+2} \|D^{k+2} u\|_{\infty, I_j}. \tag{3.6}$$

Now, we move on to the projection of functions depending not only on the spatial variable x but also on the time variable t. Suppose $u(x,t)$ is a function differentiable and integrable with respect to t, and t_1, t_2 are two real values such that $t_1 > t_2$, then we have

$$\mathbb{P}_h\left(u_t(x,t)\right) = \left(\mathbb{P}_h u(x,t)\right)_t, \qquad \text{and} \qquad \mathbb{P}_h\left(\int_{t_2}^{t_1} u(x,t)dt\right) = \int_{t_2}^{t_1} (\mathbb{P}_h u(x,t))dt. \tag{3.7}$$

3.4 Properties of the LDG spatial discretization

In this subsection, we present some basic properties about the bilinear forms $\mathcal{H}_j^1$ and $\mathcal{H}_j^2$. The definitions of the two Gauss-Radau projections (2.8) and (2.9) lead to the following lemma.

Lemma 3.5. *Suppose $v_h \in V_h^k$ and $p(x) \in H_h^1$. The two Gauss-Radau projections satisfy the following properties*

$$\mathcal{H}_j^1(\mathbb{P}_-^\perp p(x), v_h) = 0, \qquad \mathcal{H}_j^1(v_h, \mathbb{P}_+^\perp p(x)) = 0, \tag{3.8}$$

and

$$\mathcal{H}_j^2(\mathbb{P}_+^\perp p(x), v_h) = 0, \qquad \mathcal{H}_j^2(v_h, \mathbb{P}_-^\perp p(x)) = 0. \tag{3.9}$$

Moreover, we define the bilinear forms and inner product on the whole computational domain Ω as

$$\mathcal{H}^1(p,q) = \sum_j \mathcal{H}_j^1(p,q), \qquad \mathcal{H}^2(p,q) = \sum_j \mathcal{H}_j^2(p,q), \quad \text{and} \quad (p,q) = \sum_j (p,q)_j$$

to obtain the following corollary directly.

Corollary 3.1. *Suppose $p(x) \in H_h^1$ and $v_h \in V_h^k$, there holds*

$$\mathcal{H}^1(\mathbb{P}_-^\perp p(x), v_h) = 0, \qquad \mathcal{H}^1(v_h, \mathbb{P}_+^\perp p(x)) = 0, \tag{3.10}$$

and

$$\mathcal{H}^2(\mathbb{P}_+^\perp p(x), v_h) = 0, \qquad \mathcal{H}^2(v_h, \mathbb{P}_-^\perp p(x)) = 0. \tag{3.11}$$

3.5 The error equations

In this subsection, we proceed to construct the error equations. Denote the error between the exact solution and the LDG numerical solution to be $e_u = u - u_h$. As the usual treatment in finite element analysis, we divide the considered error into the form $e_u = \eta_u - \xi_u$, where

$$\eta_u = u - \mathbb{P}_+ u, \quad \text{and} \quad \xi_u = u_h - \mathbb{P}_+ u.$$

Similarly, we also define $e_q = q - q_h$, and rewrite the error as $e_q = \eta_q - \xi_q$, where

$$\eta_q = q - \mathbb{P}_- q, \quad \text{and} \quad \xi_q = q_h - \mathbb{P}_- q.$$

Lemma 3.5 yields the following error equations of the LDG scheme. For any $v_h, w_h \in V_h^k$,

$$\begin{aligned}
((e_u)_t, v_h)_j &= \mathcal{H}_j^1(e_q, v_h) \\
&= -\mathcal{H}_j^1((\xi_q, v_h) \\
&= (\xi_q, (v_h)_x)_j - \xi_q^- v_h^-|_{j+\frac{1}{2}} + \xi_q^- v_h^+|_{j-\frac{1}{2}} && (3.12) \\
&= -((\xi_q)_x, v_h)_j - [\xi_q] v_h^+|_{j-\frac{1}{2}}, && (3.13)
\end{aligned}$$

and similarly

$$\begin{aligned}
(e_q, w_h)_j &= (\xi_u, (w_h)_x)_j - \xi_u^+ w_h^-|_{j+\frac{1}{2}} + \xi_u^+ w_h^+|_{j-\frac{1}{2}} && (3.14) \\
&= -((\xi_u)_x, w_h)_j - [\xi_u] w_h^-|_{j+\frac{1}{2}}, && (3.15)
\end{aligned}$$

where $[v]_{j+\frac{1}{2}} = v(x_{j+\frac{1}{2}}^+) - v(x_{j+\frac{1}{2}}^-)$ is the jump of v across $x_{j+\frac{1}{2}}$. Equations (3.12)-(3.15) are fundamental in our analysis. For example, we can obtain the estimates of e_q and $(e_u)_t$. The proof follows from [14] with some minor changes, so we skip it and state the results in the following lemma.

Lemma 3.6. *Let $u(x,t)$, $q(x,t) = u_x(x,t)$ be the exact solution of the linear parabolic equation* (1.1) *and u_h, q_h be the numerical solution of the LDG scheme* (2.5). *The finite element space is made up of polynomials of degree $k \geq 1$ and the initial discretization satisfies* (2.10). *Then by using flux* (2.4), *we have*

$$\|e_q(t)\| \leq Ch^{k+1}(1+t), \tag{3.16}$$

and

$$\|(e_u)_t(t)\| \leq Ch^{k+1}(1+t), \tag{3.17}$$

where $C = C(\|u\|_{k+5,2})$ is independent of h and t.

Let us finish this section by providing the following lemma whose proof follows from Lemma 3.7 in [26] with some minor changes.

Lemma 3.7. *Suppose $\bar{\xi}_u$ and $\bar{\xi}_q$ are the cell averages of ξ_u and ξ_q, respectively. Then we have*

$$\|\xi_u - \bar{\xi}_u\|_{I_j} \le Ch_j\|(\xi_u)_x\|_{I_j} \le Ch_j\|\mathbb{P}_k e_q\|_{I_j} \le Ch_j\|e_q\|_{I_j}, \tag{3.18}$$

$$\|\xi_q - \bar{\xi}_q\|_{I_j} \le Ch_j\|(\xi_q)_x\|_{I_j} \le Ch_j\|\mathbb{P}_k (e_u)_t\|_{I_j} \le Ch_j\|(e_u)_t\|_{I_j}. \tag{3.19}$$

4 Proof of the main result

In this section, we proceed to prove Theorem 2.1 and give the estimate of $e_u(x_j)$. The following is the basic idea. Because of Lemma 3.4, only $\xi_u(x_j)$ is considered. Noticing that ξ_u is a polynomial in each cell, by the Gauss-Radau quadrature, $\xi_u(x_j)$ can be written as an inner product of ξ_u and a suitable polynomial Φ in cell I_j. By extending Φ to the whole computational domain Ω, we are able to give the estimate of (ξ_u, Φ) by using the dual problem of (1.1). Notice that Φ may not be smooth on Ω at the final time $t = T$. Finally, we will prove Corollary 2.1 at the end of this section. Now, we give the details of the estimate of $\xi_u(x_j)$.

Denote the left-biased Radau points of the cell I_j as x_j^i, $0 \le i \le k$. Also denote Φ_j^i to be a piecewise polynomial function on the real line, such that

- Φ_j^i is continuous on Ω.
- Φ_j^i is supported on the union of I_{j-1}, I_j and I_{j+1}. For convenience, we denote I_0 and I_{N+1} to be I_N and I_1, respectively.
- on I_j, $\Phi_j^i \in \mathcal{P}^k(I_j)$ and
$$\Phi_j^i(x_\ell) = \begin{cases} 1 & x_\ell = x_j^i \\ 0 & x_\ell \ne x_j^i \end{cases}.$$
- on I_{j-1}, $\Phi_j^i = \frac{C_1 R_{1,j-1}^{k+2}}{2}$, where $C_1 = \Phi_j^i(x_{j-\frac{1}{2}}^+)$ and $R_{1,j-1}^{k+2}$ is the left-biased Radau polynomial of degree $k+2$.

- on I_{j+1}, $\Phi_j^i = \frac{C_2 R_{2,j+1}^{k+2}}{2}$, where $C_2 = \Phi_j^i(x_{j+\frac{1}{2}}^-)$ and $R_{2,j+1}^{k+2}$ is the right-biased Radau polynomial of degree $k+2$.

Clearly, we have

$$\|\Phi_j^i\| \le Ch^{\frac{1}{2}}, \quad \|(\Phi_j^i)_x\| \le Ch^{-\frac{1}{2}},$$

where the constant C does not depend on h, i or j. By the Gauss-Radau quadrature

$$(\xi_u, \Phi_j^i) = \sum_{\ell=0}^{k} \omega_\ell \xi_u(x_j^\ell)\Phi_j^i(x_j^\ell)h_j = \omega_i h_j \xi_u(x_j^i),$$

where the constant ω_ℓ is the weight of the quadrature at the ℓth left-biased Radau point on the reference interval $[-\frac{1}{2}, \frac{1}{2}]$. Therefore, $\xi_u(x_j^i) = \frac{1}{\omega_i h_j}(\xi_u, \Phi_j^i)$. Motivated by [22], we consider the dual problem of (1.1): Find a function $\phi_j^i(x,t)$ which satisfies

$$\begin{array}{ll} (\phi_j^i)_t + (\phi_j^i)_{xx} = 0, & (x,t) \in \Omega \times [0,T], \\ \phi_j^i(x,T) = \Phi_j^i(x), & x \in \Omega, \end{array} \tag{4.1}$$

with periodic boundary condition $\phi(0,t) = \phi(2\pi, t)$. For convenience, we drop the superscript i as well as the subscript j, and denote ϕ to be ϕ_j^i and Φ to be Φ_j^i. Some properties of the test function ϕ are given in the following lemma.

Lemma 4.1. *Suppose t_1, t_2 are two real numbers with $t_1 > t_2$ and $\ell \ge 0$ is a natural number, then we have*

$$\|D^\ell \phi(t_1)\|^2 = \|D^\ell \phi(t_2)\|^2 + 2\int_{t_2}^{t_1} \|D^{\ell+1}\phi\|^2 dt, \tag{4.2}$$

$$\int_{T-h}^{T} \|\phi_x\| dt \le Ch, \qquad \int_{T-h}^{T} \|\phi_{xx}\| dt \le C, \tag{4.3}$$

$$\|\phi(0)\| \le Ch, \tag{4.4}$$

$$\|D^\ell \phi(t)\|_1 \le C(T-t)^{-\frac{\ell}{2}} h, \qquad \|D^\ell \phi(t)\| \le C(T-t)^{-\frac{\ell}{2}} h^{\frac{1}{2}}. \tag{4.5}$$

The proof of this lemma will be given in Appendix B.

In this section, for simplicity, if $p(x,t)$ and $q(x,t)$ are two functions depend on x and t, then we denote $(p,q)(t) = (p(\cdot,t), q(\cdot,t)) = \sum_{j=1}^N (p(\cdot,t), q(\cdot,t))_j$. With all the above preparation, we can proceed to the proof of Theorem 2.1.

Proof. Following [22]

$$(e_u, \phi)(T) = (e_u, \phi)(0) + \int_0^T ((e_u), \phi)_t dt. \tag{4.6}$$

We apply the two Gauss-Radau projections (2.8) and (2.9) to deal with the integrand.

$$\begin{aligned}
(e_u, \phi)_t &= ((e_u)_t, \phi) + (e_u, \phi_t) \\
&= ((e_u)_t, \mathbb{P}_+^\perp \phi) - \mathcal{H}^1(\xi_q, \mathbb{P}_+ \phi) + (e_u, \phi_t) \\
&= ((e_u)_t, \mathbb{P}_+^\perp \phi) - \mathcal{H}^1(\xi_q, \phi) + (e_u, \phi_t) \\
&= ((e_u)_t, \mathbb{P}_+^\perp \phi) + (\xi_q, \phi_x) + (e_u, \phi_t)
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
(\xi_q, \phi_x) &= (\xi_q, \mathbb{P}_-^\perp \phi_x) + (\xi_q, \mathbb{P}_- \phi_x) \\
&= (\xi_q, \mathbb{P}_-^\perp \phi_x) + (\eta_q, \mathbb{P}_- \phi_x) + \mathcal{H}^2(\xi_u, \mathbb{P}_- \phi_x) \\
&= (\xi_q, \mathbb{P}_-^\perp \phi_x) + (\eta_q, \mathbb{P}_- \phi_x) + \mathcal{H}^2(\xi_u, \phi_x) \\
&= (\xi_q, \mathbb{P}_-^\perp \phi_x) + (\eta_q, \mathbb{P}_- \phi_x) - (\xi_u, \phi_{xx}).
\end{aligned} \tag{4.8}$$

Plug (4.8) into (4.7), then plug (4.7) into (4.6). We have

$$(e_u, \phi)(T) = (e_u, \phi)(0) + \int_0^T ((e_u)_t, \mathbb{P}_+^\perp \phi) + (\xi_q, \mathbb{P}_-^\perp \phi_x) + (\eta_q, \mathbb{P}_- \phi_x) + (\eta_u, \phi_t) dt. \tag{4.9}$$

We use integration by parts on the last term of the right-hand side of (4.9),

$$\int_0^T (\eta_u, \phi_t) dt = (\eta_u, \phi)(T) - (\eta_u, \phi)(0) - \int_0^T ((\eta_u)_t, \phi) dt. \tag{4.10}$$

Plugging (4.10) into (4.9) and noticing the fact that $e_u = \eta_u - \xi_u$, we obtain

$$\begin{aligned}
(\xi_u\, \phi)(T) &= (\xi_u, \phi)(0) - \int_0^T ((e_u)_t, \mathbb{P}_+^\perp \phi) - ((\eta_u)_t, \phi) + (\xi_q, \mathbb{P}_-^\perp \phi_x) + (\eta_q, \mathbb{P}_- \phi_x) dt \\
&= T_1 - T_2 - \cdots - T_6,
\end{aligned} \tag{4.11}$$

where

$$\begin{array}{ll}
T_1 = (\xi_u, \phi)(0), & T_2 = \int_{T-h}^T ((e_u)_t, \mathbb{P}_+^\perp \phi) - ((\eta_u)_t, \phi) dt, \\
T_3 = \int_{T-h}^T (\xi_q, \mathbb{P}_-^\perp \phi_x) dt, & T_4 = \int_0^{T-h} ((e_u)_t, \mathbb{P}_+^\perp \phi) - (e_q, \mathbb{P}_-^\perp \phi_x) dt, \\
T_5 = \int_{T-h}^T (\eta_q, \mathbb{P}_- \phi_x) dt, & T_6 = \int_0^{T-h} (\eta_q, \phi_x) - ((\eta_u)_t, \phi) dt.
\end{array} \tag{4.12}$$

In (4.11), we separate the time interval into two parts $[0, T-h]$ and $[T-h, T]$. This is because ϕ is not smooth when $t \in [T-h, T]$. Therefore, we can hardly use the regularity of ϕ. Fortunately, the length of the interval is h, hence it does not affect the superconvergence result. In what follows, we will give the estimate of each term in (4.12). For any natural numbers m and n, we denote C_m^n as a constant that does not depend on h or T, but may depend on $\|u\|_{k+m,2}$ and $\|u\|_{k+n,\infty}$. For convenience, if $m = 0$ or $n = 0$, then the corresponding index will be omitted. Especially, we use C for a constant that is independent of u.

Using the Cauchy-Schwarz inequality, (2.10) and (4.4), we have

$$T_1 \le \|\xi(0)\| \|\phi(0)\| \le C_2 h^{k+3}. \tag{4.13}$$

For T_2

$$\begin{aligned} T_2 &= \int_{T-h}^{T} ((e_u)_t, \mathbb{P}_+^{\perp}\phi) - ((\eta_u)_t, \phi - \overline{\phi}) dt \\ &\le Ch \int_{T-h}^{T} \|(e_u)_t\| \|\phi_x\| + \|(\eta_u)_t\| \|\phi_x\| dt \\ &\le C_5(1+T)h^{k+2} \int_{T-h}^{T} \|\phi_x\| dt \\ &\le C_5(1+T)h^{k+3}, \end{aligned} \tag{4.14}$$

where the first inequality follows from Lemma 3.2 and Cauchy-Schwarz inequality, the second inequality is based on (3.17) and the last inequality is given in (4.3). Similarly, we have the estimates for T_3 and T_4. Actually, using Lemma 3.7, (3.17) and (4.3) we have

$$\begin{aligned} T_3 &= \int_{T-h}^{T} (\xi_q - \overline{\xi}_q, \mathbb{P}_-^{\perp}\phi_x) dt \\ &\le Ch^2 \int_{T-h}^{T} \|(e_u)_t\| \|\phi_{xx}\| dt \\ &\le C_5(1+T)h^{k+3}. \end{aligned} \tag{4.15}$$

For T_4 we have

$$T_4 \le C_5(1+T)h^{2k+2} \int_0^{T-h} \|D^{k+1}\phi\| + \|D^{k+2}\phi\| dt$$

$$
\begin{aligned}
&\le C_5(1+T)h^{2k+5/2}\int_0^{T-h}(T-t)^{-\frac{k+2}{2}}+(T-t)^{-\frac{k+1}{2}}dt \\
&\le C_5(1+T)h^{k+3}. \qquad (4.16)
\end{aligned}
$$

Here the first inequality follows from (3.16), (3.17) and Lemma 3.2, the second inequality is based on (4.5) and the last one is direct computation. The estimates of T_5 and T_6 are more complicated. Let us consider T_6 first. Suppose $p_1, p_2 \in V_h^{k-1}$, by using Hölder inequality, we have

$$
\begin{aligned}
T_6 &= \int_0^{T-h}(\eta_q, \phi_x - p_1) - ((\eta_u)_t, \phi - p_2)dt \\
&\le \int_0^{T-h}\|\eta_q\|_\infty\|\phi_x - p_1\|_1 + \|(\eta_u)_t\|_\infty\|\phi - p_2\|_1 dt.
\end{aligned}
$$

Notice the fact that p_1 and p_2 are arbitrarily chosen in V_h^{k-1}, by Lemma 3.3,

$$
\begin{aligned}
T_6 &\le \int_0^{T-h}\|\eta_q\|_\infty \inf_{p_1}\|\phi_x - p_1\|_1 + \|(\eta_u)_t\|_\infty \inf_{p_2}\|\phi - p_2\|_1 dt \\
&\le C^3 h^{2k+1}\int_0^{T-h}\|D^{k+1}\phi\|_1 + \|D^k\phi\|_1 dt \\
&\le C^3 h^{2k+2}\int_0^{T-h}(T-t)^{-\frac{k+1}{2}} + (T-t)^{-\frac{k}{2}}dt \\
&\le \begin{cases} C^3(1+\sqrt{T})h^{k+3}\ln h, & k=1, \\ C^3(1+\sqrt{T})h^{k+3}, & k\ge 2, \end{cases} \qquad (4.17)
\end{aligned}
$$

where the second inequality follows from Lemmas 3.2 and 3.3, the third one is based on (4.5), and the last one follows from direct computation. Finally, we proceed to the estimate of T_5, which is the most complicated. We first write T_5 into two parts $T_5 = T_{51} + T_{52}$, where

$$
T_{51} = \int_{T-h}^{T}(\eta_q - \eta_q(T), \mathbb{P}_-\phi_x)dt, \qquad T_{52} = \int_{T-h}^{T}(\eta_q(T), \mathbb{P}_-\phi_x)dt.
$$

T_{51} is easy to deal with,

$$
\begin{aligned}
T_{51} &\le \int_{T-h}^{T}\|\eta_q - \eta_q(T)\|\|\mathbb{P}_-\phi_x - \phi_x + \phi_x - \overline{\phi}_x\|dt \\
&\le Ch^{k+2}\int_{T-h}^{T}\|D^{k+1}\left[q(t) - q(T)\right]\|\|\phi_{xx}\|dt \\
&\le C_4 h^{k+3}\int_{T-h}^{T}\|\phi_{xx}\|dt
\end{aligned}
$$

$$\leq C_4 h^{k+3}. \tag{4.18}$$

In the first two inequalities, we use Cauchy-Schwarz inequality and Lemma 3.2, while the last two follow from Taylor expansion and (4.3). Before proceeding to the estimate of T_{52}, we define

$$\psi(x,t) = \int_0^x \phi(s,t)ds. \tag{4.19}$$

Then ψ satisfies

$$\psi_t + \psi_{xx} = \phi_x(0,t).$$

Therefore,

$$\begin{aligned} T_{52} &= (\eta_q(T), \mathbb{P}_- \int_{T-h}^{T} \psi_{xx} dt) \\ &= -(\eta_q(T), \mathbb{P}_- \int_{T-h}^{T} \psi_t dt) \\ &= (\eta_q(T), \mathbb{P}_- \psi(T-h)) - (\eta_q(T), \mathbb{P}_- \psi(T)). \end{aligned} \tag{4.20}$$

Let us estimate the first term on the right-hand side of (4.20). Actually,

$$\begin{aligned} (\eta_q(T), \mathbb{P}_- \psi(T-h)) &= (\eta_q(T), \mathbb{P}_- \psi(T-h) - \psi(T-h) + \psi(T-h) - \bar{\psi}(T-h)) \\ &\leq Ch^2 \|\eta_q(T)\| \|D^2 \psi(T-h)\| + Ch \|\eta_q(T)\|_\infty \|D\psi(T-h)\|_1 \\ &\leq C_2 h^{k+3} \|\phi_x(T-h)\| + C^2 h^{k+2} \|\phi(T-h)\|_1 \\ &\leq C_2^2 h^{k+3}, \end{aligned}$$

where the first inequality follows from Hölder inequality, Lemma 3.2 and Lemma 3.3, the second one is based on Lemma 3.2, the third one follows from (4.19) and the fact that $\eta_u = u - \mathbb{P}_- u$, finally in the last one we use (4.5). The estimate of the second term on the right-hand side of (4.20) is trivial. Using Cauchy-Schwarz inequality and Lemma 3.2, we have

$$\begin{aligned} (\eta_q(T), \mathbb{P}_- \psi(T)) &= (\eta_q(T), \mathbb{P}_- \psi(T) - \psi(T) + \psi(T) - \bar{\psi}(T)) \\ &\leq Ch \sum_{i=0}^{N} \|\eta_q(T)\|_{I_i} \|\phi(T)\|_{I_i} \end{aligned}$$

$$= \sum_{i=j-1}^{j+1} Ch^{\frac{3}{2}} \|\eta_q(T)\|_{\infty,I_i} \|\phi(T)\|_{I_i}$$
$$\le C^2 h^{k+3}.$$

Now we finish the estimate of T_{52}, i.e.

$$T_{52} = (\eta_q(T), \mathbb{P}_-(\psi(T-h)) - (\eta_q(T), \mathbb{P}_-\psi(T)) \le C_2^2 h^{k+3}. \tag{4.21}$$

Combining (4.18) and (4.21), we obtain

$$T_5 = T_{51} + T_{52} \le C_4^2 h^{k+3}. \tag{4.22}$$

We have now completed the estimates of T_i, and finished the proof of Theorem 2.1. □

Now we proceed to the proof of Corollary 2.1. For simplicity, we only prove for $p = \infty$, as the cases for other p follow from the same lines. We consider the assertion in cell I_j and define a special norm in $\mathcal{P}^k(I_j)$ as

$$|||v||| = \max_{0 \le i \le k} \left\{ v(x_j^i), x_j^i \text{ are the left-biased Radau points in cell } I_j \right\}.$$

It is not difficult to show this is indeed a norm and the analysis in this section implies

$$|||\xi_u||| \le \begin{cases} C_5^3(1+T)h^{k+2} \ln h & k = 1 \\ C_5^3(1+T)h^{k+2} & k \ge 2 \end{cases}.$$

Since all norms in $\mathcal{P}^k$ are equivalent, we have

$$\|\mathbb{P}_+ u - u_h\|_{\infty,I_j} = \|\xi_u\|_{\infty,I_j} \le C|||\xi_u|||,$$

which further implies Corollary 2.1.

5 Numerical examples

In this section, we use numerical experiments to verify our main result, Theorem 2.1 and Corollary 2.1. In this section, we use λ to denote the ratio of the length of the largest cell to that of the smallest one.

Example 1. We solve the following problem

$$\begin{array}{ll} u_t = u_{xx}, & (x,t) \in [0,2\pi] \times (0,1], \\ u(x,0) = \sin(x), & x \in [0,2\pi], \end{array} \tag{5.1}$$

with periodic boundary condition $u(0,t) = u(2\pi,t)$. Clearly, the exact solution is

$$u(x,t) = e^{-t}\sin(x).$$

We use ninth order strong-stability-preserving (SSP) Runge-Kutta discretization in time [21] and take $\Delta t = 0.01h_{\min}^2$ to reduce the time error. Non-uniform meshes which are obtained by randomly and independently perturbing each node in a uniform mesh by up to 20% are used, and the example is tested with both $\mathcal{P}^1$ and $\mathcal{P}^2$ polynomials. The special error in Theorem 2.1 at different left-biased Radau points at $t = 1$ on random meshes of N cells are computed. In Table 5.1, we can observe $(2k+1)$-th order superconvergence at the left end point and $(k+2)$-th order superconvergence at other Radau points. The initial solution is obtained by exactly the same way as mentioned in Appendix A. The left-biased Radau points on the interval [-1,1] are -1 and $\frac{1}{3}$ for $\mathcal{P}^1$ polynomials, and are -1, $\frac{1-\sqrt{6}}{5}$ and $\frac{1+\sqrt{6}}{5}$ for $\mathcal{P}^2$ ones.

Table 5.1: The error e_u at the Radau points for equation (5.1) when using $\mathcal{P}^1$ and $\mathcal{P}^2$ polynomials.

				left end point		2^{nd} Radau point		3^{rd} Radau point	
Polynomial	N	h_{max}	λ	error	order	error	order	error	order
$\mathcal{P}^1$	20	0.406	1.813	5.74E-04	-	4.70E-04	-		
	40	0.217	2.146	7.89E-05	3.15	6.48E-05	3.15		
	80	0.107	2.089	9.36E-06	3.02	7.57E-06	3.04		
	160	5.300e-02	2.106	1.20E-06	2.92	9.81E-07	2.91		
$\mathcal{P}^2$	20	0.406	1.813	3.58E-07	-	3.43E-06	-	5.87E-06	-
	40	0.217	2.146	1.60E-08	4.94	3.76E-07	3.52	4.81E-07	3.98
	80	0.107	2.089	3.94E-10	5.24	2.19E-08	4.02	2.95E-08	3.95
	160	5.300e-02	2.106	1.11E-11	5.10	1.19E-09	4.16	1.59E-09	4.17

Table 5.2 shows the rate of convergence of the error ξ_u in L^∞-norm. We observe that the order is $k+2$, indicating that the estimate in equation (2.12) is sharp.

Table 5.2: The error ξ_u for equation (5.1) when using $\mathcal{P}^1$ and $\mathcal{P}^2$ polynomials.

L^∞ norm of ξ_u			$\mathcal{P}^1$ Polynomial		$\mathcal{P}^2$ Polynomial	
N	h_{max}	λ	L^∞ error	order	L^∞ error	order
20	0.406	1.813	5.74E-04	-	2.44E-05	-
40	0.217	2.146	7.89E-05	3.15	2.17E-06	3.85
80	0.107	2.089	9.36E-06	3.02	1.31E-07	3.97
160	5.300e-02	2.106	1.20E-06	2.92	7.06E-09	4.16

6 Concluding remarks

We have studied the behavior of the error between the LDG solution and the exact solution for sufficiently smooth solutions of linear parabolic equations when the alternating flux is used. We prove that under suitable initial discretization, the error between the LDG solution and the exact solution is $(k+2)$-th order superconvergent at the Radau points. We also prove that the LDG solution is superconvergent with the rate $k+2$ towards a particular projection of the exact solution estimated in L^p-norm. Moreover, numerical experiments demonstrate that the rates of convergence are sharp.

In future work, we will attempt to prove the superconvergent property for general initial conditions, and apply the superconvergence at the Radau points for adaptive methods.

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A Initial discretization

In this appendix we consider the suitable discretization of the initial datum. As mentioned in Section 2, we would like to have the initial solution satisfy $\xi_q = 0$ and $\|\xi_u\|_\Omega \le Ch^{k+2}$, see (2.10). We assume $\xi_q = 0$ and construct a special numerical initial solution which also satisfies the second requirement $\|\xi_u\| \le Ch^{k+2}$. For simplicity, in this appendix we use ξ and e for ξ_u and e_u, respectively. Let us start from the following lemma. Taking $w_h = 1$ in equation (3.14), we have

Lemma A.1. *$\int_{I_j} e_q dx = 0$, $\forall\ 1 \le j \le N$ if and only if $\xi^+_{j-\frac{1}{2}}$ is a constant which does not depend on j.*

Denote S to be the constant given in the previous lemma. To control $\|\xi\|_{I_j}$, we have to take a small S, and this is shown in the following lemma.

Lemma A.2. *Suppose $\|e_q\| \le Ch^{k+1}$, then $\|\xi\| \le Ch^{k+2}$ if $S \le Ch^{k+2}$.*

Proof: Suppose $S \le Ch^{k+2}$, then by Lemma 3.7

$$\begin{aligned}
\bar{\xi}_j &= \xi^-_{j+\frac{1}{2}} - (\xi - \bar{\xi}_j)^-_{j+\frac{1}{2}} \\
&\le S + Ch_j^{-1/2}\|\xi - \bar{\xi}\|_{I_j} \\
&\le S + Ch_j^{1/2}\|e_q\|_{I_j},
\end{aligned}$$

where $\bar{\xi}_j$ is the cell average of ξ in cell I_j. Then

$$\begin{aligned}
\|\bar{\xi}\|^2 &= \sum_j \|\bar{\xi}\|^2_{I_j} \\
&\le \sum_j S^2 h_j + \sum_j Ch_j^2\|e_q\|^2_{I_j} \\
&\le S^2 + Ch^2\|e_q\|^2 \\
&\le Ch^{2k+4}.
\end{aligned}$$

Therefore,

$$\|\xi\| \le \|\bar{\xi}\| + \|\xi - \bar{\xi}\| \le Ch^{k+2} + h\|e_q\| \le Ch^{k+2}.$$

Remark: Because we require $\xi_q = 0$, we have $\|e_q\| \le Ch^{k+1}\|D^{k+2}u\|$. Therefore, the assumption $\|e_q\| \le Ch^{k+1}$ in the lemma is true. We will also use this estimate of e_q later in this appendix.

Now let us proceed to construct the initial solution u_h from $\xi_q = 0$.

Lemma A.3. *Suppose $\int_{I_j} e_q = 0$, then ξ_x is uniquely determined by $\mathbb{P}_k e_q$ in the cell I_j.*

Proof: Let $w_h^-|_{j+\frac{1}{2}} = 0$ in equation (3.15), then we have

$$(\mathbb{P}_k e_q, w_h)_j = -(\xi_x, w_h)_j. \tag{A.1}$$

By the linearity of the equation above, we need only prove the uniqueness. That is, suppose $(\mathbb{P}_k e_q, w_h)_j = 0,\ \forall\ w_h \in V_h^k$ and $w_h^-|_{j+\frac{1}{2}} = 0$, then we have $\xi_x = 0$. To show this, we take $w_h = p - p^-_{j+\frac{1}{2}}$, where $p(x)$ is an arbitrary polynomial of degree no more than k. Then

$$(\mathbb{P}_k e_q, p)_j = (\mathbb{P}_k e_q, p - p^-_{j+\frac{1}{2}})_j = 0.$$

Therefore, $\mathbb{P}_k e_q = 0$, which further implies $\xi_x = 0$ by Lemma 3.7.

Now, we determine the value of the constant $S = \xi^+_{j-\frac{1}{2}}$. By Lemma A.2 we can simply take $S = 0$. However, such S violates the conservation of mass. We can construct a special S such that $\int_\Omega \xi = 0$ and such S satisfies the property $S \leq Ch^{k+2}$. Actually,

$$0 = \int_\Omega \xi dx = \sum_{j=1}^N \bar{\xi}_j h_j = \sum_{j=1}^N \left(S - (\xi - \bar{\xi})^-_{j+\frac{1}{2}} \right) h_j,$$

which yields

$$S|\Omega| = \sum_{j=1}^N (\xi - \bar{\xi})^-_{j+\frac{1}{2}} h_j. \tag{A.2}$$

Then we obtain

$$S \leq \frac{C}{|\Omega|} \sum_{j=1}^N \|e_q\|_{I_j} h_j^{3/2} \leq \frac{Ch}{\sqrt{|\Omega|}} \|e_q\| \leq \frac{C}{\sqrt{|\Omega|}} h^{k+2} \|D^{k+2} u\|. \tag{A.3}$$

In the first inequality in (A.3) we use Lemma 3.1 and Lemma 3.7. For the second inequality we use Cauchy-Schwarz inequality and the fact that $\sum h_j = |\Omega|$. The last inequality follows from the estimate $\|e_t\| \leq Ch_j^{k+1} \|D^{k+2} u\|$ which is obtained in the remark after Lemma A.2.

Now we summarize the procedure to implement the initial discretization. We divide the process into the following steps:

(1) Let $\xi_q = 0$, then compute the value of e_q;

(2) Find out ξ_x by using Lemma A.3;

(3) Compute $\xi - \bar{\xi}$ in each cell from the expression of ξ_x and the fact that $\int_{I_j}(\xi - \bar{\xi})dx = 0$;

(4) Work out S by using (A.2) or simply by taking $S = 0$;

(5) Calculate ξ from the expressions of S and ξ_x;

(6) Figure out $u_h = \xi + \mathbb{P}_+ u$.

From the process mentioned above, we can observe that the initial solution is uniquely determined by the requirements $\xi_q = 0$ and $\int_\Omega \xi dx = 0$ or $\xi^+_{j-\frac{1}{2}} = 0$.

B Proof of Lemma 4.1

In this appendix, we proceed to the proof of Lemma 4.1. The following lemma is useful in this appendix.

Lemma B.1. *Suppose $f(x) \in L^1(R)$ and $g(x) \in L^p(\Omega)$. If we extend $g(x)$ periodically on R, and also use $g(x)$ to denote the extended function, then $\|f * g\|_p \leq \|f\|_{1,R}\|g\|_p$*

Proof: The proof directly follows from that of Young's inequality with some minor changes, so we omit it here.

Now we start the proof of Lemma 4.1. In (4.2), we take $\ell = 0$ only, and the cases for $\ell \geq 1$ follows from the same lines.

$$\begin{aligned}
\|\phi\|^2(t_1) &= \|\phi\|^2(t_2) + \int_{t_2}^{t_1} \frac{d}{dt}(\phi, \phi)dt \\
&= \|\phi\|^2(t_2) + 2\int_{t_2}^{t_1} (\phi_t, \phi)dt \\
&= \|\phi\|^2(t_2) - 2\int_{t_2}^{t_1} (\phi_{xx}, \phi)dt \\
&= \|\phi\|^2(t_2) + 2\int_{t_2}^{t_1} \|\phi_x\|^2 dt. \qquad \text{(B.1)}
\end{aligned}$$

If we take $t_1 = T$ and $t_2 = T - h$ in (B.1), then we have

$$\int_{T-h}^{T} \|\phi_x\|^2 dt \leq \frac{1}{2}\|\phi\|^2(T) \leq Ch.$$

Then by Hölder inequality,

$$\int_{T-h}^{T} \|\phi_x\| dt \leq h^{\frac{1}{2}} \left(\int_{T-h}^{T} \|\phi_x\|^2 dt\right)^{\frac{1}{2}} \leq Ch.$$

Similarly, we can also prove

$$\int_{T-h}^{T} \|\phi_{xx}\| dt \le h^{\frac{1}{2}} \left(\int_{T-h}^{T} \|\phi_{xx}\|^2 dt \right)^{\frac{1}{2}} \le C h^{\frac{1}{2}} \|\phi_x\|(T) \le C.$$

Now, we finish the proof of (4.2) and (4.3). To prove (4.4) and (4.5), we have to periodically extend ϕ to the entire real line R. Then

$$\phi(x,t) = \Gamma(x, T-t) * \phi(x,T),$$

where $\Gamma(x,t)$ is the fundamental solution of the heat equation given as

$$\Gamma(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

Clearly, there exits some polynomial P with degree ℓ such that

$$D^\ell \Gamma(x,t) = \frac{1}{\sqrt{\pi}} (4t)^{-\frac{\ell+1}{2}} e^{-\frac{x^2}{4t}} P(x/\sqrt{4t}).$$

Then for $t \in [0, T-h]$ we have

$$\|D^\ell \Gamma(x,t)\|_{1,R} \le C t^{-\frac{\ell}{2}}.$$

We consider the L^p-norm of $D^\ell \phi$, and

$$\begin{aligned} \|D^\ell \phi(x,t)\|_p &= \|(D^\ell \Gamma(x, T-t)) * \phi(x,T)\|_p \\ &\le \|D^\ell \Gamma(\cdot, T-t)\|_{1,R} \|\phi(\cdot, T)\|_p \\ &\le C(T-t)^{-\frac{\ell}{2}} \|\phi\|_p. \end{aligned} \tag{B.2}$$

If we take $p = 1$ and 2, we obtain (4.5).

Finally, we prove (4.4). Actually

$$|\phi(x,0)| = \left| \int_R \frac{1}{\sqrt{4\pi T}} e^{-\frac{(x-y)^2}{4T}} \phi(y,T) dy \right| \le \frac{1}{\sqrt{4\pi T}} \|\phi(y,T)\|_1 \le Ch.$$

Now we finish the proof of Lemma 4.1.