STABILITY AND ERROR ESTIMATES OF LOCAL DISCONTINUOUS GALERKIN METHODS WITH IMPLICIT-EXPLICIT TIME-MARCHING FOR ADVECTION-DIFFUSION PROBLEMS

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Abstract. The main purpose of this paper is to analyze the stability and error estimates of the local discontinuous Galerkin (LDG) methods coupled with carefully chosen implicit-explicit (IMEX) Runge-Kutta time discretization up to third order accuracy, for solving one-dimensional linear advection-diffusion equations. In the time discretization the advection term is treated explicitly and the diffusion term implicitly. There are three highlights of this work. The first is that we establish an important relationship between the gradient and interface jump of the numerical solution with the independent numerical solution of the gradient in the LDG methods. The second is that, by aid of the aforementioned relationship and the energy method, we show that the IMEX LDG schemes are unconditionally stable for the linear problems, in the sense that the time-step τ is only required to be upper-bounded by a constant which depends on the ratio of the diffusion and the square of the advection coefficients and is independent of the mesh-size h, even though the advection term is treated explicitly. The last is that under this time step condition, we obtain optimal error estimates in both space and time for the third order IMEX Runge-Kutta time-marching coupled with LDG spatial discretization. Numerical experiments are also given to verify the main results.

Key words. local discontinuous Galerkin method, implicit-explicit Runge-Kutta time-marching scheme, advection-diffusion equation, stability, error estimate, energy method.

AMS subject classifications. 65M12, 65M15, 65M60

1. Introduction.

In this paper we perform a fully-discrete analysis on advection-diffusion problems. For simplicity, we concentrate on a linear advection-diffusion problem with periodic boundary condition in one dimension. In order to alleviate the stringent time step restriction of explicit time discretization for diffusion terms, we consider a class of implicit-explicit time discretization which treats the advection terms explicitly and the diffusion terms implicitly. The spatial discretization is the standard local discontinuous Galerkin (LDG) method.

The LDG method was introduced by Cockburn and Shu for solving convection-diffusion problems in [9], motivated by the work of Bassi and Rebay [3] for solving the compressible Navier-Stokes equations. This scheme shares some of the advantages of the Runge-Kutta discontinuous Galerkin (RKDG) schemes for solving hyperbolic conservation laws [10], such as high order accuracy, flexibility of h-p adaptivity, flexibility on complex geometry, and so on. Besides, it is locally solvable, that is, the auxiliary variables approximating the gradient of the solution can be locally eliminated [19, 5].

Over the past years, there has been extensive study of the LDG methods, such as for elliptic problems [5], convection-diffusion problems [6], the Stokes system [8], the KdV type equations [21], Hamilton-Jacobi equations [20], time-dependent fourth order problems [12], etc.. More recently, there has been analysis on the fully discretized LDG schemes, e.g. in [22, 17]. The time discretization used in [22, 17] is the third order explicit total variation diminishing Runge-Kutta (TVDRK) time march-

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ing. Such explicit methods are stable, efficient and accurate for solving hyperbolic conservation laws and convection-dominated problems, but for convection-diffusion equations which are not convection-dominated, explicit time discretization will suffer from a stringent time step restriction for stability [18]. When it comes to such problems, a natural consideration to overcome the small time step restriction is to use implicit time marching. However, in many applications the convection terms are often nonlinear, hence it would be desirable to treat them explicitly while using implicit time discretization only for the diffusion terms. Such time discretizations are called implicit-explicit (IMEX) time discretizations [1]. Even for nonlinear diffusion terms, IMEX time discretizations would show their advantages in obtaining an elliptic algebraic system, which is easy to solve by many iterative methods. If both convection and diffusion are treated implicitly, the resulting algebraic system will be far from elliptic and convergence of many iterative solvers will suffer.

There are many IMEX schemes designed for different purposes in the literature, such as [1, 2, 4, 15, 11, 14, 16, 25]. Among them, the schemes in [2, 14] are multistep schemes, the remaining are Runge-Kutta (RK) type IMEX methods. In [11] the authors constructed pairs of additive Runge-Kutta methods up to order four which are combinations of explicit RK methods and implicit A-stable RK methods, and they use the maximum number of zero diagonal elements for the implicit part to ensure A-stability. However, these methods do not satisfy the necessary stability condition presented in [4], where a Fourier analysis was given to study the stability property of IMEX RK methods for solving linear advection-diffusion problems. The property given in [4] is an extension of the concept of L-stability, which ensures the stability of the scheme when refining the spatial mesh provided that the time step $\tau \leq \tau_0$, where τ_0 only depends on the coefficients of the advection and diffusion terms. The schemes of Pareschi and Russo [16] treat the explicit part by a strong-stability-preserving scheme [13] and the implicit part by an L-stable diagonally implicit RK method. Their schemes have good asymptotic preserving property, however, several of this type of schemes use more stages for the implicit part than the minimum required for the designed accuracy, thus affecting their efficiency.

In this paper, we will consider three specific Runge-Kutta type IMEX schemes given in [1] and [4], from first to third order accuracy. Coupling with the LDG spatial discretization, we give the stability analysis by the energy method. Our analysis indicates that the corresponding IMEX LDG methods are all stable, provided the time step τ is bounded by a positive constant which is proportional to d/c^2, where c and d are the advection and diffusion coefficients respectively, regardless of the mesh size h. We also perform error estimates for the LDG scheme with the third order IMEX RK time marching, showing optimal convergence rates in both space and time, when the time step τ and the mesh size h go to zero independently. Our analysis relies heavily on a crucial relationship that we establish in Lemma 2.4.

The paper is organized as follows. In Section 2 we present the semi-discrete LDG scheme for the model problem and give some preliminary results. Section 3 is devoted to the presentation of several IMEX Runge-Kutta schemes, and to the proof of stability of the corresponding IMEX LDG schemes. In Section 4 we give optimal error estimates for the third order scheme. Several numerical results are presented in Section 5 to verify the main results. Finally, we give concluding remarks in Section 6.

2. The LDG method and some preliminaries.

2.1. The semi-discrete LDG scheme. In this subsection we present the definition of semi-discrete LDG schemes for the linear advection-diffusion problem

$$U_t + cU_x - dU_{xx} = 0, \qquad (x,t) \in Q_T = (a,b) \times (0,T], \tag{2.1a}$$
$$U(x,0) = U_0(x), \qquad x \in \Omega = (a,b), \tag{2.1b}$$

with periodic boundary condition, where $d > 0$ is the diffusion coefficient and c is the velocity of the flow field. Without loss of generality, we assume $c > 0$ in this paper. The initial solution $U_0(x)$ is assumed to be in $L^2(\Omega)$.

Let $Q = \sqrt{d}U_x$ and define $(h_U, h_Q) := (cU - \sqrt{d}Q, -\sqrt{d}U)$. The LDG scheme starts from the following equivalent first-order differential system

$$U_t + (h_U)_x = 0, \quad Q + (h_Q)_x = 0, \qquad (x,t) \in Q_T, \tag{2.2}$$

with the same initial condition (2.1b) and boundary condition.

Let $\mathcal{T}_h = \{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}_{j=1}^N$ be the partition of Ω, where $x_{\frac{1}{2}} = a$ and $x_{N+\frac{1}{2}} = b$ are the two boundary endpoints. Denote the cell length as $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $j = 1, \ldots, N$, and define $h = \max_j h_j$. We assume $\mathcal{T}_h$ is quasi-uniform in this paper, that is, there exists a positive constant ρ such that for all j there holds $h_j/h \geq \rho$, as h goes to zero.

Associated with this mesh, we define the discontinuous finite element space

$$V_h = \left\{ v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_k(I_j), \, \forall j = 1, \ldots, N \right\}, \tag{2.3}$$

where $\mathcal{P}_k(I_j)$ denotes the space of polynomials in I_j of degree at most $k \geq 1$. Note that the functions in this space are allowed to have discontinuities across element interfaces. At each element interface point, for any piecewise function p, there are two traces along the right-hand and left-hand, denoted by p^+ and p^-, respectively. As usual, the jump is denoted by $[\![p]\!] = p^+ - p^-$.

The semi-discrete LDG scheme is defined as follows: for any $t > 0$, find the numerical solution $\boldsymbol{w}(t) := (u(t), q(t)) \in V_h \times V_h$ (where the argument x is omitted), such that the variational forms

$$(u_t, v)_j = c\mathcal{H}_j^-(u,v) - \sqrt{d}\mathcal{H}_j^+(q,v); \tag{2.4a}$$
$$(q, r)_j = -\sqrt{d}\mathcal{H}_j^-(u,r), \tag{2.4b}$$

hold in each cell I_j, $j = 1, 2, \ldots, N$, for any test functions $\boldsymbol{z} = (v, r) \in V_h \times V_h$. Here $(\cdot,\cdot)_j$ is the usual inner product in $L^2(I_j)$ and

$$\mathcal{H}_j^\pm(v,r) = (v, r_x)_j - v^\pm_{j+\frac{1}{2}} r^-_{j+\frac{1}{2}} + v^\pm_{j-\frac{1}{2}} r^+_{j-\frac{1}{2}}. \tag{2.5}$$

The alternating numerical flux [9] and the upwind numerical flux for the advection part are used here. In (2.4) and below, we drop the argument t if there is no confusion. In the later analysis, we will also use the following equivalent form of $\mathcal{H}_j^\pm$:

$$\mathcal{H}_j^-(v,r) = -[(v_x, r)_j + [\![v]\!]_{j-\frac{1}{2}} r^+_{j-\frac{1}{2}}], \quad \mathcal{H}_j^+(v,r) = -[(v_x, r)_j + [\![v]\!]_{j+\frac{1}{2}} r^-_{j+\frac{1}{2}}]. \tag{2.6}$$

The initial condition $u(x,0)$ can be taken as any approximation of the given initial solution $U_0(x)$, for example, the local Gauss-Radau projections of $U_0(x)$. These projections are defined in Subsection 2.2, equation (2.12). We have now defined the semi-discrete LDG scheme.

For the convenience of analysis, we denote by $(q,r)=\sum_{j=1}^{N}(q,r)_j$ the inner product in $L^2(\Omega)$. Summing up the variational formulations (2.4) over $j=1,2,\ldots,N$, we can write the above semi-discrete LDG scheme in the global form: for any $t>0$, find the numerical solution $\boldsymbol{w}=(u,q)\in V_h\times V_h$ such that the variation equations

$$(u_t,v)=c\mathcal{H}^-(u,v)-\sqrt{d}\mathcal{H}^+(q,v); \tag{2.7a}$$

$$(q,r)=-\sqrt{d}\mathcal{H}^-(u,r), \tag{2.7b}$$

hold for any $\boldsymbol{z}=(v,r)\in V_h\times V_h$. Here $\mathcal{H}^\pm=\sum_{j=1}^{N}\mathcal{H}_j^\pm$. Furthermore, for the simplicity of notations, we would like to denote

$$\mathcal{H}=c\mathcal{H}^-,\quad \mathcal{L}=-\sqrt{d}\mathcal{H}^+,\quad \text{and}\quad \mathcal{K}=-\sqrt{d}\mathcal{H}^-. \tag{2.8}$$

Then the variational formulation becomes

$$(u_t,v)=\mathcal{H}(u,v)+\mathcal{L}(q,v); \tag{2.9a}$$

$$(q,r)=\mathcal{K}(u,r). \tag{2.9b}$$

2.2. Preliminaries. In this section, we first present some notations and norms which will be used throughout this paper, and then we will present some properties of the finite element space and the LDG spatial discretizations.

2.2.1. Notations and norms. We use the standard norms and semi-norms in Sobolev spaces. For example, for any integer $s\geq 0$, we use $H^s(D)$ to represent the space equipped with the norm $\|\cdot\|_{H^s(D)}$, in which the function itself and the derivatives up to the s-th order are all in $L^2(D)$. In particular, $H^0(D)=L^2(D)$ and the associated L^2-norm is denoted as $\|\cdot\|_D$ for simplicity of notation. If $D=\Omega$, we omit the subscript Ω for convenience. We also use the notation $L^\infty(0,T;H^s(D))$ to represent the set of functions v such that $\max_{0\leq t\leq T}\|v(\cdot,t)\|_{H^s(D)}<\infty$. Furthermore, we would like to consider the (mesh-dependent) broken Sobolev space

$$H^1(\mathcal{T}_h)=\left\{\phi\in L^2(\Omega):\phi|_{I_j}\in H^1(I_j),\ \forall j=1,\ldots,N\right\}, \tag{2.10}$$

which contains the discontinuous finite element space V_h. Associated with the space $H^1(\mathcal{T}_h)$, we would like to define a so called "jump semi-norm" $[\![v]\!]^2=\sum_{j=1}^{N}[\![v]\!]^2_{j-\frac12}$, for arbitrary $v\in H^1(\mathcal{T}_h)$.

2.2.2. The inverse and projection properties. Now we present the following inverse property with respect to the finite element space V_h. For any function $v\in V_h$, there exists a positive constant $\mu>0$ independent of v,h and j such that

$$\|v\|_{\partial I_j}\leq\sqrt{\mu h^{-1}}\|v\|_{I_j}, \tag{2.11}$$

where $\|v\|_{\partial I_j}=\sqrt{(v^+_{j-\frac12})^2+(v^-_{j+\frac12})^2}$ is the L^2-norm on the boundary of I_j. We call μ the inverse constant.

In this paper we will use two Gauss-Radau projections, from $H^1(\mathcal{T}_h)$ to V_h, denoted by π_h^- and π_h^+ respectively. For any function $p\in H^1(\mathcal{T}_h)$, the projection $\pi_h^\pm p$ is defined as the unique element in V_h such that, in each element $I_j=(x_{j-\frac12},x_{j+\frac12})$

$$(\pi_h^-p-p,v)_{I_j}=0,\quad\forall v\in\mathcal{P}_{k-1}(I_j),\quad(\pi_h^-p)^-_{j+\frac12}=p^-_{j+\frac12}; \tag{2.12a}$$

$$(\pi_h^+p-p,v)_{I_j}=0,\quad\forall v\in\mathcal{P}_{k-1}(I_j),\quad(\pi_h^+p)^+_{j-\frac12}=p^+_{j-\frac12}. \tag{2.12b}$$

In view of the exact collocation on one endpoint of each element, the Gauss-Radau projections provide a great help to obtain the optimal error estimates.

Denote by $\eta = p - \pi_h^{\pm} p$ the projection error. By a standard scaling argument [7], it is easy to obtain the following approximation property

$$\|\eta\|_{I_j} + h^{1/2}\|\eta\|_{\partial I_j} \le Ch^{\min(k+1,s)}\|p\|_{H^s(I_j)}, \quad \forall j, \tag{2.13}$$

where the bounding constant $C > 0$ is independent of h and j.

In what follows we will mainly use the inverse inequalities (2.11) and the approximation property (2.13) in the global form by summing up the above local inequalities over every $j = 1, 2, \ldots, N$. The conclusions are almost the same as their local counterparts, so they are omitted here.

2.2.3. The properties of the LDG spatial discretization. We will first present several properties of the operators $\mathcal{H}^{\pm}$ defined in Subsection 2.1. The proofs are routine so we omit them to save space; for readers who are interested in the details, we refer to [22].

Lemma 2.1. *For any $w, v \in H^1(\mathcal{T}_h)$, there hold the equalities*

$$\mathcal{H}^-(v, v) = -\frac{1}{2}[\![v]\!]^2, \tag{2.14}$$

$$\mathcal{H}^-(w, v) = -\mathcal{H}^+(v, w). \tag{2.15}$$

Lemma 2.2. *For any $w, v \in V_h$, there hold the following inequalities*

$$|\mathcal{H}^-(w, v)| \le \left(\|w_x\| + \sqrt{\mu h^{-1}}[\![w]\!]\right)\|v\|, \tag{2.16a}$$

$$|\mathcal{H}^-(w, v)| \le \left(\|v_x\| + \sqrt{\mu h^{-1}}[\![v]\!]\right)\|w\|. \tag{2.16b}$$

Lemma 2.3. *For any $w \in H^1(\mathcal{T}_h)$ and $v \in V_h$, there hold*

$$\mathcal{H}^{\pm}(\pi_h^{\pm} w - w, v) = 0. \tag{2.17}$$

The next lemma establishes the important relationship between $\|u_x\|$, $[\![u]\!]$ and $\|q\|$, which plays a key role in obtaining the good stability of the IMEX LDG scheme in the next section.

Lemma 2.4. *Suppose $\boldsymbol{w} = (u, q) \in V_h \times V_h$ is the solution of the scheme (2.9), then there exists a positive constant C_μ, which is independent of h and d but may depend on the inverse constant μ, such that*

$$\|u_x\| + \sqrt{\mu h^{-1}}[\![u]\!] \le \frac{C_\mu}{\sqrt{d}}\|q\|. \tag{2.18}$$

Proof. Let L_k be the standard Legendre polynomial of degree k in $[-1, 1]$, we have $L_k(-1) = (-1)^k$ and L_k is orthogonal to any polynomials with degree at most $k - 1$. First we take

$$r(x)|_{I_j} = u_x(x) - (-1)^k u_x^+(x_{j-\frac{1}{2}})L_k(\xi),$$

in (2.4b), with $\xi = \frac{2(x-x_j)}{h_j}$. Clearly, there hold $r^+_{j-\frac{1}{2}} = 0$, and $(u_x, r)_j = (u_x, u_x)_j$ since $(u_x, L_k)_j = 0$. Hence by (2.6) we have

$$(q, r)_j = \sqrt{d}\left[(u_x, r)_j + [\![u]\!]_{j-\frac{1}{2}} r^+_{j-\frac{1}{2}}\right] = \sqrt{d}\|u_x\|^2_{I_j}.$$

Thus

$$\|u_x\|_{I_j}^2 \leq \frac{1}{\sqrt{d}}\|q\|_{I_j}\left(\|u_x\|_{I_j} + |u_x^+(x_{j-\frac{1}{2}})|\|L_k(\xi)\|_{I_j}\right) \leq \frac{C_\mu}{\sqrt{d}}\|q\|_{I_j}\|u_x\|_{I_j},$$

where the first inequality is obtained by using the Cauchy-Schwarz inequality and the second is derived by using the inverse inequality (2.11) and the fact that

$$\|L_k(\xi)\|_{I_j}^2 = \int_{I_j} L_k(\xi)^2 \mathrm{d}x \leq h_j \max_{\xi\in[-1,1]} |L_k(\xi)|^2 \leq Ch_j \leq Ch.$$

Therefore

$$\|u_x\|_{I_j} \leq \frac{C_\mu}{\sqrt{d}}\|q\|_{I_j}. \tag{2.19}$$

Next we take $r = 1$ in (2.4b) and again by (2.6) we can obtain

$$[\![u]\!]_{j-\frac{1}{2}} = \frac{1}{\sqrt{d}}(q, 1)_j - (u_x, 1)_j.$$

As a consequence, by the Cauchy-Schwarz inequality and (2.19) we have

$$|[\![u]\!]_{j-\frac{1}{2}}| \leq h^{1/2}\left(\frac{1}{\sqrt{d}}\|q\|_{I_j} + \|u_x\|_{I_j}\right) \leq \frac{C_\mu h^{1/2}}{\sqrt{d}}\|q\|_{I_j}. \tag{2.20}$$

Finally, by summing over all elements we get the desired result (2.18). □

3. The IMEX RK fully discrete schemes and their stability analysis. Instead of adopting explicit Runge-Kutta time marching schemes to solve the semi-discrete LDG scheme introduced in the above section, we prefer a type of implicit-explicit Runge-Kutta methods, which can not only relax the severe time step restriction due to the implicit integrator for the linear diffusion part, but also be easy to implement for potentially nonlinear convection part since we use explicit discretization for this term.

Typically, implicit schemes coupled with proper spatial discretizations (such as finite difference discretizations) are unconditionally stable for solving the pure diffusion equation. For IMEX schemes coupled with an LDG spatial discretization for solving convection-diffusion problems, in which we treat the diffusion term implicitly and the convection term explicitly, we can reasonably expect that the schemes are stable under the standard CFL condition for explicit RKDG schemes for convection equations, $\tau \leq Ch$ for a constant C, where τ is the time step. However, we would like to have stability under a much weaker condition $\tau \leq C$, where the constant C depends only on the diffusion and advection coefficients and not on the mesh size h. In this section, we would like to explore a kind of IMEX Runge-Kutta schemes which would allow us to achieve such stability.

For a detailed introduction to IMEX RK schemes, we refer the readers to [1] and [4]. In this paper, we would like to adopt the forms given in [4]. To give a brief introduction of the scheme, let us consider the system of ordinary differential equations

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = L(t, \mathbf{y}) + N(t, \mathbf{y}), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{3.1}$$

where $\mathbf{y} = [y_1, y_2, \cdots, y_d]^\top$, $L(t, \mathbf{y})$ is derived from the spatial discretization of the diffusion term, and $N(t, \mathbf{y})$ arises from the discretization of the convection term. By applying the general s-stage IMEX RK time marching scheme, the solution of (3.1) advanced from time t^n to $t^{n+1} = t_n + \Delta t$ is given by:

$$\begin{aligned}
&\mathbf{Y}_1 = \mathbf{y}_n, \\
&\mathbf{Y}_i = \mathbf{y}_n + \Delta t \sum_{j=2}^{i} a_{ij} L(t_n^j, \mathbf{Y}_j) + \Delta t \sum_{j=1}^{i-1} \hat{a}_{ij} N(t_n^j, \mathbf{Y}_j), \quad 2 \le i \le s+1, \\
&\mathbf{y}_{n+1} = \mathbf{y}_n + \Delta t \sum_{i=2}^{s+1} b_i L(t_n^i, \mathbf{Y}_i) + \Delta t \sum_{i=1}^{s+1} \hat{b}_i N(t_n^i, \mathbf{Y}_i),
\end{aligned}$$

where $\mathbf{Y}_i$ denotes the intermediate stages, $c_i = \sum_{j=2}^{i} a_{ij} = \sum_{j=1}^{i-1} \hat{a}_{ij}$, and $t_n^j = t_n + c_j \Delta t$. Denote $A = (a_{ij}), \hat{A} = (\hat{a}_{ij}) \in \mathbb{R}^{(s+1)\times(s+1)}$, $\boldsymbol{b}^\top = [0, b_2, \cdots, b_{s+1}], \hat{\boldsymbol{b}}^\top = [\hat{b}_1, \cdots, \hat{b}_{s+1}]$ and $\boldsymbol{c}^\top = [0, c_2, \cdots, c_{s+1}]$, then we can express the general s-stage IMEX RK scheme as the following Butcher tableau

$$\begin{array}{c|c|c} \boldsymbol{c} & A & \hat{A} \\ \hline & \boldsymbol{b}^\top & \hat{\boldsymbol{b}}^\top \end{array} \tag{3.2}$$

In the above tableau, the pair $(A \,|\, \boldsymbol{b})$ determines an s-stage diagonally implicit Runge-Kutta method and $(\hat{A} \,|\, \hat{\boldsymbol{b}})$ defines an $(s+1)$-stage (s-stage if $\hat{b}_{s+1} = 0$) explicit Runge-Kutta method. In this paper, we call the scheme an s-stage scheme if it has s stages for the implicit part. We will not distinguish whether it has one more explicit stage or not. Generally, an s-stage scheme does not necessarily have order s, especially for $s \ge 4$. In this work, we only consider three specific s-stage s-th order schemes, for $s = 1, 2, 3$. Among them, the scheme has s explicit stages for $s = 1, 2$. For $s = 3$, we have not been able to find a third order scheme with 3 explicit stages to fit our purpose, therefore we use one with 4 stages for the explicit part.

The first order IMEX method is taking the forward Euler discretization for the explicit part and the backward Euler discretization for the implicit part. The second order scheme that we consider is the L-stable, two-stage, second-order DIRK(2,2,2) scheme given in Ascher et al. [1]. The third order scheme we adopt is from Calvo et al. [4]. In the following we present them in the form (3.2).

First order:

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ \hline & 0 & 1 & 1 & 0 \end{array} \tag{3.3}$$

Second order:

$$\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \gamma & 0 & \gamma & 0 & 0 \\ 1 & 0 & 1-\gamma & \gamma & \delta & 1-\delta & 0 \\ \hline & 0 & 1-\gamma & \gamma & \delta & 1-\delta & 0 \end{array} \tag{3.4}$$

where $\gamma = 1 - \frac{\sqrt{2}}{2}$ and $\delta = 1 - \frac{1}{2\gamma}$.

Third order:

$$
\begin{array}{c|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma & 0 & \gamma & 0 & 0 & \gamma & 0 & 0 & 0 \\
\frac{1+\gamma}{2} & 0 & \frac{1-\gamma}{2} & \gamma & 0 & \frac{1+\gamma}{2}-\alpha_1 & \alpha_1 & 0 & 0 \\
1 & 0 & \beta_1 & \beta_2 & \gamma & 0 & 1-\alpha_2 & \alpha_2 & 0 \\
\hline
 & 0 & \beta_1 & \beta_2 & \gamma & 0 & \beta_1 & \beta_2 & \gamma
\end{array}
\tag{3.5}
$$

In this scheme, γ is the middle root of $6x^3-18x^2+9x-1=0$, $\gamma \approx 0.435866521508459$. $\beta_1 = -\frac{3}{2}\gamma^2 + 4\gamma - \frac{1}{4}$, $\beta_2 = \frac{3}{2}\gamma^2 - 5\gamma + \frac{5}{4}$. The parameter α_1 is chosen as -0.35 in [4] and $\alpha_2 = \frac{\frac{1}{3}-2\gamma^2-2\beta_2\alpha_1\gamma}{\gamma(1-\gamma)}$.

In what follows, we would like to analyze the stability of the above three schemes with the LDG spatial discretization. Let $\{t^n = n\tau\}_{n=0}^M$ be the uniform partition of the time interval $[0,T]$, with time step τ. The time step could actually change from step to step, but in this paper we take the time step as a constant for simplicity. Given u^n, hence (u^n, q^n), we would like to find the numerical solution at the next time level t^{n+1}, (maybe through several intermediate stages $t^{n,\ell}$), by the above IMEX RK methods.

3.1. First order scheme. The LDG scheme with the first order IMEX time-marching scheme (3.3) is given in the following form:

$$(u^{n+1}, v) = (u^n, v) + \tau\mathcal{H}(u^n, v) + \tau\mathcal{L}(q^{n+1}, v); \tag{3.6a}$$

$$(q^{n+1}, r) = \mathcal{K}(u^{n+1}, r), \tag{3.6b}$$

for any function $(v, r) \in V_h \times V_h$.

Proposition 3.1. *There exists a positive constant τ_0 independent of h, such that if $\tau \le \tau_0$, then the solution of scheme (3.6) satisfies*

$$\|u^n\| \le \|u^0\|, \qquad \forall n. \tag{3.7}$$

Proof. Taking $v = u^{n+1}$ in (3.6a), we get

$$(u^{n+1} - u^n, u^{n+1}) = \tau\mathcal{H}(u^n, u^{n+1}) - \tau\|q^{n+1}\|^2, \tag{3.8}$$

where we have used the property

$$\mathcal{L}(q, u) = -\mathcal{K}(u, q) = -\|q\|^2, \tag{3.9}$$

due to (2.15) and (2.9b). Noting that

$$(u^{n+1} - u^n, u^{n+1}) = \frac{1}{2}\|u^{n+1}\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 - \frac{1}{2}\|u^n\|^2.$$

Then (3.8) is equivalent to

$$\underbrace{\tfrac{1}{2}\|u^{n+1}\|^2 - \tfrac{1}{2}\|u^n\|^2 + \tfrac{1}{2}\|u^{n+1} - u^n\|^2 + \tau\|q^{n+1}\|^2}_{LHS} = \underbrace{\tau\mathcal{H}(u^n, u^{n+1})}_{RHS}. \tag{3.10}$$

This provides two stability terms $\frac{1}{2}\|u^{n+1} - u^n\|^2$ and $\tau\|q^{n+1}\|^2$, which can be used to estimate the only remaining term RHS. We add and subtract a term $\tau\mathcal{H}(u^{n+1}, u^{n+1})$ to obtain

$$
\begin{aligned}
RHS &= \tau\mathcal{H}(u^{n+1}, u^{n+1}) - \tau\mathcal{H}(u^{n+1} - u^n, u^{n+1}) \\
&= -\frac{c}{2}\tau[\![u^{n+1}]\!]^2 - \tau\mathcal{H}(u^{n+1} - u^n, u^{n+1}),
\end{aligned}
$$

where the last equality holds by the property (2.14). Thus by (2.16b), we have

$$RHS \le \tau|\mathcal{H}(u^{n+1}-u^n, u^{n+1})| \le c\tau \left(\|u_x^{n+1}\| + \sqrt{\mu h^{-1}}[\![u^{n+1}]\!]\right)\|u^{n+1}-u^n\|. \tag{3.11}$$

Then exploiting Lemma 2.4 and the Young's inequality directly, we obtain

$$RHS \le \frac{C_\mu c}{\sqrt{d}}\tau\|q^{n+1}\|\|u^{n+1}-u^n\| \le \tau\|q^{n+1}\|^2 + \frac{C_\mu^2 c^2}{4d}\tau\|u^{n+1}-u^n\|^2. \tag{3.12}$$

Consequently, if $\frac{C_\mu^2 c^2}{4d}\tau \le \frac{1}{2}$, i.e, $\tau \le \tau_0 = \frac{2d}{C_\mu^2 c^2}$, then we have

$$RHS \le \tau\|q^{n+1}\|^2 + \frac{1}{2}\|u^{n+1}-u^n\|^2.$$

Hence from (3.10) we have $\|u^{n+1}\| \le \|u^n\| \le \cdots \le \|u^0\|$. ∎

Remark 3.1. *From the proof we can see that τ_0 is proportional to d/c^2, where c, d are the advection and diffusion coefficients, respectively. If we introduce the Reynolds number Re which is proportional to c/d, then τ_0 is proportional to $1/(c\,Re)$. We use the notation τ_0 as a generic time step bound for the stability analysis and error estimates in this paper, it may have different values in each occurrence.*

3.2. Second order scheme. The LDG scheme with the second order IMEX time marching scheme (3.4) is given as:

$$(u^{n,1}, v) = (u^n, v) + \gamma\tau\mathcal{H}(u^n, v) + \gamma\tau\mathcal{L}(q^{n,1}, v), \tag{3.13a}$$

$$\begin{aligned}(u^{n+1}, v) = &(u^n, v) + \delta\tau\mathcal{H}(u^n, v) + (1-\delta)\tau\mathcal{H}(u^{n,1}, v)\\ &+ (1-\gamma)\tau\mathcal{L}(q^{n,1}, v) + \gamma\tau\mathcal{L}(q^{n+1}, v);\end{aligned} \tag{3.13b}$$

$$(q^{n,\ell}, r) = \mathcal{K}(u^{n,\ell}, r), \quad \ell = 1, 2, \tag{3.13c}$$

for any function $(v, r) \in V_h \times V_h$, where $\gamma = 1 - \frac{\sqrt{2}}{2}$, $\delta = 1 - \frac{1}{2\gamma}$. Here $w^{n,2} = w^{n+1}$.

PROPOSITION 3.2. *Under the condition of Proposition 3.1, the solution of the scheme (3.13) satisfies (3.7).*

Proof. From (3.13a) and (3.13b), we get

$$(u^{n,1}-u^n, v) = \gamma\tau\mathcal{H}(u^n, v) + \gamma\tau\mathcal{L}(q^{n,1}, v), \tag{3.14a}$$

$$\begin{aligned}(u^{n+1}-u^{n,1}, v) = &(\delta-\gamma)\tau\mathcal{H}(u^n, v) + (1-\delta)\tau\mathcal{H}(u^{n,1}, v)\\ &+ (1-2\gamma)\tau\mathcal{L}(q^{n,1}, v) + \gamma\tau\mathcal{L}(q^{n+1}, v).\end{aligned} \tag{3.14b}$$

By taking $v = u^{n,1}, u^{n+1}$ in (3.14a) and (3.14b), respectively, and adding them together, we obtain

$$\underbrace{\tfrac{1}{2}\|u^{n+1}\|^2 - \tfrac{1}{2}\|u^n\|^2 + \tfrac{1}{2}\|u^{n+1}-u^{n,1}\|^2 + \tfrac{1}{2}\|u^{n,1}-u^n\|^2}_{LHS} = R_1 + R_2,$$

where

$$\begin{aligned}R_1 &= \gamma\tau\mathcal{H}(u^n, u^{n,1}) + (\delta-\gamma)\,\tau\mathcal{H}(u^n, u^{n+1}) + (1-\delta)\tau\mathcal{H}(u^{n,1}, u^{n+1}),\\ R_2 &= \gamma\tau\mathcal{L}(q^{n,1}, u^{n,1}) + (1-2\gamma)\tau\mathcal{L}(q^{n,1}, u^{n+1}) + \gamma\tau\mathcal{L}(q^{n+1}, u^{n+1})\\ &= -\gamma\tau\|q^{n,1}\|^2 - (1-2\gamma)\tau(q^{n,1}, q^{n+1}) - \gamma\tau\|q^{n+1}\|^2.\end{aligned}$$

To obtain R_2, we have used the property (3.9) and the similar property

$$\mathcal{L}(q_1,u_2) = -\mathcal{K}(u_2,q_1) = -(q_2,q_1) = -(q_1,q_2), \tag{3.15}$$

for any pairs (u_1,q_1) and (u_2,q_2), also owing to (2.15) and (2.9b).

In order to use the stability terms provided by LHS and R_2 to estimate R_1, we rewrite R_1 in the following equivalent form:

$$\begin{aligned} R_1 = &\gamma\tau\mathcal{H}(u^{n,1},u^{n,1}) + (1-\gamma)\tau\mathcal{H}(u^{n+1},u^{n+1}) - \gamma\tau\mathcal{H}(u^{n,1}-u^n,u^{n,1}) \\ &- (1-\gamma)\tau\mathcal{H}(u^{n+1}-u^{n,1},u^{n+1}) - (\delta-\gamma)\,\tau\mathcal{H}(u^{n,1}-u^n,u^{n+1}). \end{aligned}$$

Noting that $\delta-\gamma=-1$, and by the property (2.14) we have

$$\begin{aligned} R_1 = &-\frac{c}{2}\gamma\tau[\![u^{n,1}]\!]^2 - \frac{c}{2}(1-\gamma)\tau[\![u^{n+1}]\!]^2 - \gamma\tau\mathcal{H}(u^{n,1}-u^n,u^{n,1}) \\ &- (1-\gamma)\tau\mathcal{H}(u^{n+1}-u^{n,1},u^{n+1}) + \tau\mathcal{H}(u^{n,1}-u^n,u^{n+1}). \end{aligned}$$

We will proceed in a similar argument as (3.11)-(3.12) to estimate R_1. Exploiting (2.16b), Lemma 2.4 and the Young's inequality successively, we can derive

$$R_1 \le \frac{\gamma}{4}\tau(\|q^{n,1}\|^2 + \|q^{n+1}\|^2) + \frac{C_\gamma C_\mu^2 c^2}{d}\tau\left(\|u^{n,1}-u^n\|^2 + \|u^{n+1}-u^{n,1}\|^2\right),$$

where C_γ is a positive constant depending on γ. As a consequence, we obtain

$$LHS + S \le \frac{C_\gamma C_\mu^2 c^2}{d}\tau\left(\|u^{n,1}-u^n\|^2 + \|u^{n+1}-u^{n,1}\|^2\right),$$

where

$$S = \frac{3}{4}\gamma\tau\|q^{n,1}\|^2 + (1-2\gamma)\tau(q^{n,1},q^{n+1}) + \frac{3}{4}\gamma\tau\|q^{n+1}\|^2.$$

We denote by $\mathbf{x}^\top = (q^{n,1},q^{n+1})$, then $S = \tau\int_\Omega \mathbf{x}^\top\mathbb{M}\mathbf{x}\,\mathrm{d}x$, with

$$\mathbb{M} = \begin{pmatrix} \frac{3}{4}\gamma & \frac{1}{2}-\gamma \\ \frac{1}{2}-\gamma & \frac{3}{4}\gamma \end{pmatrix}.$$

It is easy to check that $\mathbb{M}$ is positive definite, so $S \ge 0$, which leads to

$$LHS \le \frac{C_\gamma C_\mu^2 c^2}{d}\tau\left(\|u^{n,1}-u^n\|^2 + \|u^{n+1}-u^{n,1}\|^2\right).$$

Consequently, if $\frac{C_\gamma C_\mu^2 c^2}{d}\tau \le \frac{1}{2}$, i.e, $\tau \le \tau_0 = \frac{d}{2C_\gamma C_\mu^2 c^2}$, then we have

$$\|u^{n+1}\| \le \|u^n\| \le \cdots \le \|u^0\|. \qquad \square$$

3.3. Third order scheme. The LDG scheme with the third order IMEX time marching scheme (3.5) reads: for any function $(v,r)\in V_h\times V_h$,

$$(u^{n,1},v)=(u^n,v)+\gamma\tau\mathcal{H}(u^n,v)+\gamma\tau\mathcal{L}(q^{n,1},v), \tag{3.16a}$$

$$\begin{aligned}(u^{n,2},v)=&(u^n,v)+\left(\frac{1+\gamma}{2}-\alpha_1\right)\tau\mathcal{H}(u^n,v)+\alpha_1\tau\mathcal{H}(u^{n,1},v)\\&+\frac{1-\gamma}{2}\tau\mathcal{L}(q^{n,1},v)+\gamma\tau\mathcal{L}(q^{n,2},v),\end{aligned} \tag{3.16b}$$

$$\begin{aligned}(u^{n,3},v)=&(u^n,v)+(1-\alpha_2)\tau\mathcal{H}(u^{n,1},v)+\alpha_2\tau\mathcal{H}(u^{n,2},v)\\&+\beta_1\tau\mathcal{L}(q^{n,1},v)+\beta_2\tau\mathcal{L}(q^{n,2},v)+\gamma\tau\mathcal{L}(q^{n,3},v),\end{aligned} \tag{3.16c}$$

$$\begin{aligned}(u^{n+1},v)=&(u^n,v)+\beta_1\tau\mathcal{H}(u^{n,1},v)+\beta_2\tau\mathcal{H}(u^{n,2},v)+\gamma\tau\mathcal{H}(u^{n,3},v)\\&+\beta_1\tau\mathcal{L}(q^{n,1},v)+\beta_2\tau\mathcal{L}(q^{n,2},v)+\gamma\tau\mathcal{L}(q^{n,3},v);\end{aligned} \tag{3.16d}$$

$$(q^{n,\ell},r)=\mathcal{K}(u^{n,\ell},r),\quad \ell=1,2,3 \tag{3.16e}$$

where the coefficients are given below the tableau (3.5).

For the convenience of analysis, we would like to introduce a series of notations

$$\begin{aligned}&\mathbb{E}_1w^n=w^{n,1}-w^n, &&\mathbb{E}_2w^n=w^{n,2}-2w^{n,1}+w^n,\\&\mathbb{E}_3w^n=2w^{n,3}+w^{n,2}-3w^{n,1}, &&\mathbb{E}_4w^n=w^{n+1}-w^{n,3},\end{aligned} \tag{3.17}$$

for arbitrary w, and rewrite the above scheme into the following compact form. In the following we denote $\boldsymbol{u}^n=(u^n,u^{n,1},u^{n,2},u^{n,3})$ and $\boldsymbol{q}^n=(q^{n,1},q^{n,2},q^{n,3})$.

$$(\mathbb{E}_\ell u^n,v)=\Phi_\ell(\boldsymbol{u}^n,v)+\Psi_\ell(\boldsymbol{q}^n,v),\quad\text{for}\quad \ell=1,2,3,4; \tag{3.18a}$$

$$(q^{n,\ell},r)=\mathcal{K}(u^{n,\ell},r),\quad\text{for}\quad \ell=1,2,3 \tag{3.18b}$$

where (noting that $\beta_1+\beta_2+\gamma=1$)

$$\Phi_1(\boldsymbol{u}^n,v)=\gamma\tau\mathcal{H}(u^n,v), \tag{3.19a}$$

$$\Phi_2(\boldsymbol{u}^n,v)=\left(\frac{1-3\gamma}{2}-\alpha_1\right)\tau\mathcal{H}(u^n,v)+\alpha_1\tau\mathcal{H}(u^{n,1},v), \tag{3.19b}$$

$$\begin{aligned}\Phi_3(\boldsymbol{u}^n,v)=&\left(\frac{1-5\gamma}{2}-\alpha_1\right)\tau\mathcal{H}(u^n,v)+(2(1-\alpha_2)+\alpha_1)\tau\mathcal{H}(u^{n,1},v)\\&+2\alpha_2\tau\mathcal{H}(u^{n,2},v),\end{aligned} \tag{3.19c}$$

$$\begin{aligned}\Phi_4(\boldsymbol{u}^n,v)=&(\alpha_2-\beta_2-\gamma)\tau\mathcal{H}(u^{n,1},v)+(\beta_2-\alpha_2)\tau\mathcal{H}(u^{n,2},v)\\&+\gamma\tau\mathcal{H}(u^{n,3},v);\end{aligned} \tag{3.19d}$$

and

$$\Psi_1(\boldsymbol{q}^n,v)=\gamma\tau\mathcal{L}(q^{n,1},v), \tag{3.20a}$$

$$\Psi_2(\boldsymbol{q}^n,v)=\gamma\tau\mathcal{L}(q^{n,2}-2q^{n,1},v)+\frac{1-\gamma}{2}\tau\mathcal{L}(q^{n,1},v), \tag{3.20b}$$

$$\begin{aligned}\Psi_3(\boldsymbol{q}^n,v)=&2\gamma\tau\mathcal{L}(q^{n,3},v)+2\left(1-\beta_1-\frac{\gamma}{2}\right)\tau\mathcal{L}(q^{n,2}-2q^{n,1},v)\\&+2\left(\frac{9}{4}-\frac{11}{4}\gamma-\beta_1\right)\tau\mathcal{L}(q^{n,1},v),\end{aligned} \tag{3.20c}$$

$$\Psi_4(\boldsymbol{q}^n,v)=0. \tag{3.20d}$$

PROPOSITION 3.3. *Under the condition of Proposition 3.1, the solution of the scheme (3.16) satisfies (3.7).*

Proof. The proof is a bit more tricky than that for the first and second order cases. By taking $v = u^{n,1}, u^{n,2} - 2u^{n,1}, u^{n,3}$ and $2u^{n+1}$ in (3.18), for $\ell = 1,2,3,4$, respectively, we can derive:

$$\frac{1}{2}\|u^{n,1}\|^2 + \frac{1}{2}\|u^{n,1} - u^n\|^2 - \frac{1}{2}\|u^n\|^2 = \Phi_1(\boldsymbol{u}^n, u^{n,1}) + \Psi_1(\boldsymbol{q}^n, u^{n,1}), \tag{3.21a}$$

$$\begin{aligned}&\frac{1}{2}\|u^{n,2} - 2u^{n,1}\|^2 + \frac{1}{2}\|u^{n,2} - 2u^{n,1} + u^n\|^2 - \frac{1}{2}\|u^n\|^2\\ &= \Phi_2(\boldsymbol{u}^n, u^{n,2} - 2u^{n,1}) + \Psi_2(\boldsymbol{q}^n, u^{n,2} - 2u^{n,1}),\end{aligned} \tag{3.21b}$$

$$\begin{aligned}&\|u^{n,3}\|^2 + \frac{1}{2}\|u^{n,3} + u^{n,2} - 2u^{n,1}\|^2 + \frac{1}{2}\|u^{n,3} - u^{n,1}\|^2\\ &- \frac{1}{2}\|u^{n,2} - 2u^{n,1}\|^2 - \frac{1}{2}\|u^{n,1}\|^2 = \Phi_3(\boldsymbol{u}^n, u^{n,3}) + \Psi_3(\boldsymbol{q}^n, u^{n,3}),\end{aligned} \tag{3.21c}$$

$$\|u^{n+1}\|^2 + \|u^{n+1} - u^{n,3}\|^2 - \|u^{n,3}\|^2 = 2\Phi_4(\boldsymbol{u}^n, u^{n+1}) + 2\Psi_4(\boldsymbol{q}^n, u^{n+1}). \tag{3.21d}$$

To obtain (3.21c), we have divided $\mathbb{E}_3 u^n$ into two parts: $u^{n,3} + u^{n,2} - 2u^{n,1}$ and $u^{n,3} - u^{n,1}$, with the purpose of canceling the terms $\frac{1}{2}\|u^{n,1}\|^2$ and $\frac{1}{2}\|u^{n,2} - 2u^{n,1}\|^2$ with (3.21a) and (3.21b).

Adding (3.21) together leads to

$$\|u^{n+1}\|^2 - \|u^n\|^2 + \mathcal{S} = \mathcal{T}_{\mathrm{c}} + \mathcal{T}_{\mathrm{d}}, \tag{3.22}$$

where

$$\begin{aligned}\mathcal{S} = &\|u^{n+1} - u^{n,3}\|^2 + \frac{1}{2}\|u^{n,3} + u^{n,2} - 2u^{n,1}\|^2 + \frac{1}{2}\|u^{n,3} - u^{n,1}\|^2\\ &+ \frac{1}{2}\|u^{n,2} - 2u^{n,1} + u^n\|^2 + \frac{1}{2}\|u^{n,1} - u^n\|^2\end{aligned} \tag{3.23a}$$

provides one part of the stability for the scheme. The terms $\mathcal{T}_{\mathrm{c}}$ and $\mathcal{T}_{\mathrm{d}}$ are related to the advection and diffusion discretizations, respectively, which have the following forms:

$$\mathcal{T}_{\mathrm{c}} = \Phi_1(\boldsymbol{u}^n, u^{n,1}) + \Phi_2(\boldsymbol{u}^n, u^{n,2} - 2u^{n,1}) + \Phi_3(\boldsymbol{u}^n, u^{n,3}) + 2\Phi_4(\boldsymbol{u}^n, u^{n+1}), \tag{3.23b}$$

$$\mathcal{T}_{\mathrm{d}} = \Psi_1(\boldsymbol{q}^n, u^{n,1}) + \Psi_2(\boldsymbol{q}^n, u^{n,2} - 2u^{n,1}) + \Psi_3(\boldsymbol{q}^n, u^{n,3}) + 2\Psi_4(\boldsymbol{q}^n, u^{n+1}). \tag{3.23c}$$

We will first consider the term $\mathcal{T}_{\mathrm{d}}$. By the properties (3.9) and (3.15) we have

$$\begin{aligned}\mathcal{T}_{\mathrm{d}} = &-\gamma\tau\|q^{n,1}\|^2 - \gamma\tau\|q^{n,2} - 2q^{n,1}\|^2 - 2\gamma\tau\|q^{n,3}\|^2 - \frac{1-\gamma}{2}\tau(q^{n,1}, q^{n,2} - 2q^{n,1})\\ &- 2\left(\frac{9}{4} - \frac{11}{4}\gamma - \beta_1\right)\tau(q^{n,1}, q^{n,3}) - 2\left(1 - \beta_1 - \frac{\gamma}{2}\right)\tau(q^{n,2} - 2q^{n,1}, q^{n,3}).\end{aligned} \tag{3.24}$$

We denote $\mathbf{w}^\top = (q^{n,1}, q^{n,2} - 2q^{n,1}, q^{n,3})$, then

$$\mathcal{T}_{\mathrm{d}} = -\tau\int_\Omega \mathbf{w}^\top \mathbb{A}\mathbf{w}\,\mathrm{d}x, \tag{3.25}$$

where

$$\mathbb{A} = \begin{pmatrix} \gamma & \frac{1-\gamma}{4} & \frac{9}{4} - \frac{11}{4}\gamma - \beta_1\\ \frac{1-\gamma}{4} & \gamma & 1 - \beta_1 - \frac{\gamma}{2}\\ \frac{9}{4} - \frac{11}{4}\gamma - \beta_1 & 1 - \beta_1 - \frac{\gamma}{2} & 2\gamma \end{pmatrix}. \tag{3.26}$$

It can be verified that $\mathbb{A}$ is positive definite by verifying the principal minor determinants of $\mathbb{A}$ are all positive, so $\mathcal{T}_{\rm d} \le 0$, which implies that if $c = 0$ then the scheme is unconditionally stable, since in this case $\mathcal{T}_{\rm c} = 0$. Note that $\mathcal{T}_{\rm d}$ provides another part of stability for the scheme.

For the general case $c \neq 0$, towards the goal of estimating the term $\mathcal{T}_{\rm c}$, we will follow the approach in the estimate for the RHS term in the proof of Proposition 3.1, adding and subtracting some terms to rewrite the operators Φ_i into the following equivalent forms, for the purpose of using the stability provided by $\mathcal{S}$ and $\mathcal{T}_{\rm d}$.

$$\Phi_1(\boldsymbol{u}^n, v) = \gamma\tau\mathcal{H}(u^{n,1}, v) - \gamma\tau\mathcal{H}(u^{n,1} - u^n, v), \tag{3.27a}$$

$$\begin{aligned}\Phi_2(\boldsymbol{u}^n, v) = {}& \frac{3\gamma-1}{2}\tau\mathcal{H}(u^{n,2} - 2u^{n,1}, v) + \alpha_1\tau\mathcal{H}(u^{n,1} - u^n, v)\\ &+ \frac{1-3\gamma}{2}\tau\mathcal{H}(u^{n,2} - 2u^{n,1} + u^n, v),\end{aligned} \tag{3.27b}$$

$$\begin{aligned}\Phi_3(\boldsymbol{u}^n, v) = {}& \frac{5(1-\gamma)}{2}\tau\mathcal{H}(u^{n,3}, v) + \left(\alpha_1 + 2\alpha_2 - \frac{1-5\gamma}{2}\right)\tau\mathcal{H}(u^{n,1} - u^n, v)\\ &+ 2\alpha_2\tau\mathcal{H}(u^{n,2} - 2u^{n,1} + u^n, v) - \frac{5(1-\gamma)}{2}\tau\mathcal{H}(u^{n,3} - u^{n,1}, v),\end{aligned} \tag{3.27c}$$

$$\begin{aligned}\Phi_4(\boldsymbol{u}^n, v) = {}& (\beta_2 - \alpha_2)\tau\mathcal{H}(u^{n,1} - u^n, v) + (\beta_2 - \alpha_2)\tau\mathcal{H}(u^{n,2} - 2u^{n,1} + u^n, v)\\ &+ \gamma\tau\mathcal{H}(u^{n,3} - u^{n,1}, v).\end{aligned} \tag{3.27d}$$

In addition, to deal with the last term $\Phi_4(\boldsymbol{u}^n, u^{n+1})$ in $\mathcal{T}_{\rm c}$, we add and subtract a term $\Phi_4(\boldsymbol{u}^n, u^{n,3})$ to obtain

$$\Phi_4(\boldsymbol{u}^n, u^{n+1}) = \Phi_4(\boldsymbol{u}^n, u^{n,3}) + \Phi_4(\boldsymbol{u}^n, u^{n+1} - u^{n,3}),$$

for the same purpose. Then after some tedious manipulation, we can rewrite $\mathcal{T}_{\rm c}$ as:

$$\begin{aligned}\mathcal{T}_{\rm c} = {}& \gamma\tau\mathcal{H}(u^{n,1}, u^{n,1}) + \frac{3\gamma-1}{2}\tau\mathcal{H}(u^{n,2} - 2u^{n,1}, u^{n,2} - 2u^{n,1})\\ &+ \frac{5(1-\gamma)}{2}\tau\mathcal{H}(u^{n,3}, u^{n,3}) + \sum_{i=1}^{3} T_i\\ = {}& -\frac{c}{2}\tau\left([\![u^{n,1}]\!]^2 + \frac{3\gamma-1}{2}[\![u^{n,2} - 2u^{n,1}]\!]^2 + \frac{5(1-\gamma)}{2}[\![u^{n,3}]\!]^2\right) + \sum_{i=1}^{3} T_i\,,\end{aligned} \tag{3.28}$$

where we have used the property (2.14), and T_i are given as

$$\begin{aligned}T_1 = {}& 2(\beta_2 - \alpha_2 - \gamma)\tau\mathcal{H}(u^{n,1}, u^{n+1} - u^{n,3}) - \gamma\tau\mathcal{H}(u^{n,1} - u^n, u^{n,1}),\\ T_2 = {}& 2(\beta_2 - \alpha_2)\tau\mathcal{H}(u^{n,2} - 2u^{n,1}, u^{n+1} - u^{n,3}) + \alpha_1\tau\mathcal{H}(u^{n,1} - u^n, u^{n,2} - 2u^{n,1}),\\ &+ \frac{1-3\gamma}{2}\tau\mathcal{H}(u^{n,2} - 2u^{n,1} + u^n, u^{n,2} - 2u^{n,1})\\ T_3 = {}& 2\gamma\tau\mathcal{H}(u^{n,3}, u^{n+1} - u^{n,3}) + 2\beta_2\mathcal{H}(u^{n,2} - 2u^{n,1} + u^n, u^{n,3})\\ &+ \left(\alpha_1 + 2\beta_2 - \frac{1-5\gamma}{2}\right)\tau\mathcal{H}(u^{n,1} - u^n, u^{n,3}) - \frac{5-9\gamma}{2}\tau\mathcal{H}(u^{n,3} - u^{n,1}, u^{n,3}).\end{aligned}$$

Denote $C_\star$ as the maximum of the absolute value of all the coefficients in the expression of T_i for $i = 1, 2, 3$, and denote

$$T_0 = \|u^{n+1} - u^{n,3}\| + \|u^{n,1} - u^n\| + \|u^{n,2} - 2u^{n,1} + u^n\| + \|u^{n,3} - u^{n,1}\|,$$

then by the aid of Lemmas 2.2 and 2.4, we can derive

$$|T_1| \le C_\star c\tau \left(\|(u^{n,1})_x\| + \sqrt{\mu h^{-1}} [\![u^{n,1}]\!] \right) T_0 \le \frac{C_\star C_\mu c}{\sqrt{d}} \tau \|q^{n,1}\| T_0.$$

Similarly,

$$|T_2| \le \frac{C_\star C_\mu c}{\sqrt{d}} \tau \|q^{n,2} - 2q^{n,1}\| T_0, \qquad |T_3| \le \frac{C_\star C_\mu c}{\sqrt{d}} \tau \|q^{n,3}\| T_0.$$

Then using the Young's inequality, we obtain

$$\begin{aligned} |\sum_{i=1}^{3} T_i| &\le \frac{\gamma}{4}\tau \left(\|q^{n,1}\|^2 + \|q^{n,2} - 2q^{n,1}\|^2 + \|q^{n,3}\|^2 \right) + 3\frac{C_\star^2 C_\mu^2 c^2}{d\gamma} \tau T_0^2 \\ &\le \frac{\gamma}{4}\tau \left(\|q^{n,1}\|^2 + \|q^{n,2} - 2q^{n,1}\|^2 + \|q^{n,3}\|^2 \right) + 24\frac{C_\star^2 C_\mu^2 c^2}{d\gamma} \tau \mathcal{S}, \end{aligned} \tag{3.29}$$

where $\mathcal{S}$ is defined in (3.23a). Owing to (3.22), (3.25), (3.28) and (3.29) we have

$$\|u^{n+1}\|^2 - \|u^n\|^2 + \mathcal{S} \le -\tau \int_\Omega \mathbf{w}^\top (\mathbb{A} - \mathbb{B}) \mathbf{w} \, \mathrm{d}x + 24\frac{C_\star^2 C_\mu^2 c^2}{d\gamma} \tau \mathcal{S}, \tag{3.30}$$

where $\mathbb{B} = \frac{\gamma}{4}\mathbb{I}$, with $\mathbb{I}$ being the identity matrix. It can also be verified that $\mathbb{A} - \mathbb{B}$ is positive definite by the same way as for $\mathbb{A}$. Thus $\|u^{n+1}\| \le \|u^n\| \le \cdots \le \|u^0\|$, if $24\frac{C_\star^2 C_\mu^2 c^2}{d\gamma}\tau \le 1$, that is, $\tau \le \tau_0 = \frac{d\gamma}{24 C_\star^2 C_\mu^2 c^2}$. ∎

Remark 3.2. *We remark that $C_\star$ depends on the choice of the parameter α_1. After a simple manipulation we know that, if $\alpha_1 \in (-0.481, -0.108)$, then $C_\star$ attains its minimum value $C_\star = |2\beta_2| \approx 1.2887$.*

4. Error estimates. With the stability result in the previous section, it is conceptually straightforward, although still technical, to obtain error estimates for smooth solutions. We will only give the error estimates for the third order IMEX LDG scheme (3.16) as an example. Following [24], we introduce four reference functions, denoted by $\boldsymbol{W}^{(\ell)} = (U^{(\ell)}, Q^{(\ell)})$, $\ell = 0, 1, 2, 3$, associated with the third order IMEX RK time discretization (3.5). In detail, $U^{(0)} = U$ is the exact solution of the problem (2.1) and then we define

$$U^{(1)} = U^{(0)} - \gamma\tau c U_x^{(0)} + \gamma\tau\sqrt{d} Q_x^{(1)}, \tag{4.1a}$$

$$\begin{aligned} U^{(2)} = & \; U^{(0)} - \left(\frac{1+\gamma}{2} - \alpha_1 \right) \tau c U_x^{(0)} - \alpha_1 \tau c U_x^{(1)} \\ & + \frac{1-\gamma}{2}\tau\sqrt{d} Q_x^{(1)} + \gamma\tau\sqrt{d} Q_x^{(2)}, \end{aligned} \tag{4.1b}$$

$$\begin{aligned} U^{(3)} = & \; U^{(0)} - (1-\alpha_2)\tau c U_x^{(1)} - \alpha_2 \tau c U_x^{(2)} \\ & + \beta_1 \tau\sqrt{d} Q_x^{(1)} + \beta_2 \tau\sqrt{d} Q_x^{(2)} + \gamma\tau\sqrt{d} Q_x^{(3)}; \end{aligned} \tag{4.1c}$$

where

$$Q^{(\ell)} = \sqrt{d} U_x^{(\ell)}, \quad \text{for} \quad \ell = 1, 2, 3. \tag{4.2}$$

For any indexes n and ℓ under consideration, the reference function at each stage time level is defined as $\boldsymbol{W}^{n,\ell} = (U^{n,\ell}, Q^{n,\ell}) = \boldsymbol{W}^{(\ell)}(x, t^n)$.

At each stage time, we denote the error between the exact (reference) solution and the numerical solution by $\boldsymbol{e}^{n,\ell} = (e_u^{n,\ell}, e_q^{n,\ell}) = (U^{n,\ell} - u^{n,\ell}, Q^{n,\ell} - q^{n,\ell})$. As the standard treatment in finite element analysis, we would like to divide the error in the form $\boldsymbol{e} = \boldsymbol{\xi} - \boldsymbol{\eta}$, where

$$\boldsymbol{\eta} = (\eta_u, \eta_q) = (\pi_h^- U - U, \pi_h^+ Q - Q), \quad \boldsymbol{\xi} = (\xi_u, \xi_q) = (\pi_h^- U - u, \pi_h^+ Q - q), \tag{4.3}$$

here we have dropped the superscripts n and ℓ for simplicity.

We would like to assume that the exact solution U has the following smoothness,

$$U \in L^\infty(0,T;H^{k+2}), \quad D_t U \in L^\infty(0,T;H^{k+1}), \quad \text{and} \quad D_t^4 U \in L^\infty(0,T;L^2), \tag{4.4}$$

where $D_t^\ell U$ is the ℓ-th order time derivative of U.

By the smoothness assumption (4.4), it follows from (2.13) and the linearity of the projections $\pi_h^\pm$ that the stage projection errors and their evolutions satisfy

$$\|\eta_u^{n,\ell}\| + \|\eta_q^{n,\ell}\| \le Ch^{k+1}, \quad \|\mathbb{E}_{\ell+1}\eta_u^n\| \le Ch^{k+1}\tau, \tag{4.5}$$

for any n and $\ell = 0, 1, 2, 3$ under consideration. Here the bounding constant $C > 0$ depends solely on the smoothness of the exact solution and is independent of n, h, τ. In the remaining of this section, we also use C to represent a generic positive constant which is independent of c, d and n, h, τ, if there is no special explanation. It may have a different value in each occurrence.

In what follows we will focus our attention on the estimate of the error in the finite element space, say, $\boldsymbol{\xi} \in V_h \times V_h$. To this end, we need to set up the error equations about $\boldsymbol{\xi}^{n,\ell}$. This process is based on the following lemma.

LEMMA 4.1. *Let $\boldsymbol{W} = (U, Q)$ be the sufficiently smooth solution of problem (2.1). Assume U satisfies the smoothness assumption (4.4). Denote $\boldsymbol{Q}^n = (Q^{n,1}, Q^{n,2}, Q^{n,3})$ and $\boldsymbol{U}^n = (U^n, U^{n,1}, U^{n,2}, U^{n,3})$. Then for any function $(v, r) \in V_h \times V_h$, there hold the following variational forms*

$$(\mathbb{E}_\ell U^n, v) = \Phi_\ell(\boldsymbol{U}^n, v) + \Psi_\ell(\boldsymbol{Q}^n, v) + \delta_{4\ell}(\zeta^n, v), \quad \textit{for} \quad \ell = 1, 2, 3, 4; \tag{4.6a}$$

$$(Q^{n,\ell}, r) = \mathcal{K}(U^{n,\ell}, r) \quad \textit{for} \quad \ell = 1, 2, 3. \tag{4.6b}$$

Here $\delta_{4\ell}$ is the Kronecker symbol and ζ^n is the local truncation error in each step of the third order IMEX RK time-marching (4.1). Besides, there exists a bounding constant $C > 0$ depending on the regularity of U, independent of n, h and τ, such that

$$\|\zeta^n\| \le C\tau^4. \tag{4.7}$$

Proof. The proof is trivial by the considered PDE and the definitions of the reference functions (4.1), so we omit it. Similar analysis can be found in [23]. ◻

Subtracting those variational forms in Lemma 4.1 from those in the scheme (3.18), in the same order, we will obtain the following error equations

$$(\mathbb{E}_\ell \xi_u^n, v) = \Phi_\ell(\boldsymbol{\xi}_{\boldsymbol{u}}^n, v) + \Psi_\ell(\boldsymbol{\xi}_{\boldsymbol{q}}^n, v) + (\mathbb{E}_\ell \eta_u^n + \delta_{4\ell}\zeta^n, v), \quad \text{for} \quad \ell = 1, 2, 3, 4; \tag{4.8a}$$

$$(\xi_q^{n,\ell}, r) = \mathcal{K}(\xi_u^{n,\ell}, r) + (\eta_q^{n,\ell}, r), \quad \text{for} \quad \ell = 1, 2, 3, \tag{4.8b}$$

since the projection error related terms in Φ_ℓ, Ψ_ℓ and $\mathcal{K}$ vanish by Lemma 2.3.

The above error equations are critical to obtain the final error estimate. The process is the similar to but more complicated than the stability analysis. Towards the goal of the final error estimate for the scheme (3.16), we would like to give the following two lemmas.

LEMMA 4.2. *There exist positive constant C independent of n, h, τ, and τ_0 independent of h, such that, if $\tau \le \tau_0$, then*

$$\|\xi_u^{n+1}\|^2 - \|\xi_u^n\|^2 \le \tau \sum_{\ell=1}^{4} \|\xi_u^{n,\ell}\|^2 + \frac{C}{\tau} \sum_{\ell=1}^{4} \|\mathbb{E}_\ell \eta_u^n + \delta_{4\ell}\zeta^n\|^2 + C\tau \sum_{\ell=1}^{3} \|\eta_q^{n,\ell}\|^2, \tag{4.9}$$

where $\xi_u^{n,4} = \xi_u^{n+1}$.

Proof. Similar to the proof of Proposition 3.3, we take $v = \xi_u^{n,1}, \xi_u^{n,2} - 2\xi_u^{n,1}, \xi_u^{n,3}$ and $2\xi_u^{n+1}$ in (4.8), for $\ell = 1, 2, 3, 4$ respectively, and add them together, to obtain the energy equation

$$\|\xi_u^{n+1}\|^2 - \|\xi_u^n\|^2 + \mathcal{S}' = \mathcal{T}_{\rm c}' + \mathcal{T}_{\rm d}' + \mathcal{T}_{\rm p}, \tag{4.10}$$

where $\mathcal{S}'$, $\mathcal{T}_{\rm c}'$ and $\mathcal{T}_{\rm d}'$ are replacing $(u, \boldsymbol{u}^n, \boldsymbol{q}^n)$ with $(\xi_u, \boldsymbol{\xi}_{\boldsymbol{u}}^n, \boldsymbol{\xi}_{\boldsymbol{q}}^n)$ in $\mathcal{S}$, $\mathcal{T}_{\rm c}$ and $\mathcal{T}_{\rm d}$, which are defined in (3.23). $\mathcal{T}_{\rm p}$ is related to the projection errors which is given as

$$\mathcal{T}_{\rm p} = (\mathbb{E}_1\eta_u^n, \xi_u^{n,1}) + (\mathbb{E}_2\eta_u^n, \xi_u^{n,2} - 2\xi_u^{n,1}) + (\mathbb{E}_3\eta_u^n, \xi_u^{n,3}) + (\mathbb{E}_4\eta_u^n + \zeta^n, 2\xi_u^{n+1}). \tag{4.11}$$

A simple use of the Cauchy-Schwarz inequality and the Young's inequality leads to

$$\mathcal{T}_{\rm p} \le \tau \sum_{\ell=1}^{4} \|\xi_u^{n,\ell}\|^2 + \frac{C}{\tau} \sum_{\ell=1}^{4} \|\mathbb{E}_\ell \eta_u^n + \delta_{4\ell}\zeta^n\|^2. \tag{4.12}$$

The estimate for $\mathcal{T}_{\rm d}'$ is a little different from the estimate of $\mathcal{T}_{\rm d}$ in (3.24). Similar as the properties (3.9) and (3.15), we have

$$\mathcal{L}(\xi_q, \xi_u) = -\,\mathcal{K}(\xi_u, \xi_q) = -\|\xi_q\|^2 + (\eta_q, \xi_q); \tag{4.13}$$

$$\mathcal{L}(\xi_q^1, \xi_u^2) = -\,\mathcal{K}(\xi_u^2, \xi_q^1) = -(\xi_q^2, \xi_q^1) + (\eta_q^2, \xi_q^1), \tag{4.14}$$

for any pairs of (ξ_u, ξ_q), (ξ_u^1, ξ_q^1) and (ξ_u^2, ξ_q^2), due to (2.15) and (4.8b). Hence, by using (4.13) and (4.14) to estimate $\mathcal{T}_{\rm d}'$, we get

$$\mathcal{T}_{\rm d}' = -\tau \int_\Omega \mathbf{v}^\top \mathbb{A} \mathbf{v} \,\mathrm{d}x + V, \tag{4.15}$$

where $\mathbf{v}^\top = (\xi_q^{n,1}, \xi_q^{n,2} - 2\xi_q^{n,1}, \xi_q^{n,3})$, $\mathbb{A}$ is defined in (3.26) and V is related to the projection errors in the form:

$$\begin{aligned} V =\,& \gamma\tau(\eta_q^{n,1}, \xi_q^{n,1}) + \gamma\tau(\eta_q^{n,2} - 2\eta_q^{n,1}, \xi_q^{n,2} - 2\xi_q^{n,1}) + 2\gamma\tau(\eta_q^{n,3}, \xi_q^{n,3}) \\ &+ \frac{1-\gamma}{2}\tau(\eta_q^{n,2} - 2\eta_q^{n,1}, \xi_q^{n,1}) + 2\left(\frac{9}{4} - \frac{11}{4}\gamma - \beta_1\right)\tau(\eta_q^{n,3}, \xi_q^{n,1}) \\ &+ 2\left(1 - \beta_1 - \frac{\gamma}{2}\right)\tau(\eta_q^{n,3}, \xi_q^{n,2} - 2\xi_q^{n,1}). \end{aligned}$$

A simple use of the Cauchy-Schwarz and the Young's inequalities leads to

$$V \le \varepsilon\tau \int_\Omega \mathbf{v}^\top \mathbf{v} \,\mathrm{d}x + C_\varepsilon \tau \sum_{\ell=1}^{3} \|\eta_q^{n,\ell}\|^2,$$

for arbitrary $\varepsilon > 0$, where C_ε is a positive constant only depending on ε. As a consequence,

$$\mathcal{T}_{\rm d}' \le -\tau \int_\Omega \mathbf{v}^\top(\mathbb{A} - \varepsilon\mathbb{I})\mathbf{v}\,\mathrm{d}x + C_\varepsilon\tau\sum_{\ell=1}^{3}\|\eta_q^{n,\ell}\|^2. \tag{4.16}$$

Note that $\mathcal{T}_{\rm c}'$ has the similar expression as $\mathcal{T}_{\rm c}$ defined in (3.28), replacing u with ξ_u. We denote the corresponding terms to T_i as T_i'. We use Lemma 2.2 again and the relationship

$$\|(\xi_u)_x\| + \sqrt{\mu h^{-1}}[\![\xi_u]\!] \le \frac{C_\mu}{\sqrt{d}}(\|\xi_q\| + \|\eta_q\|), \tag{4.17}$$

which is similar to (2.18), to estimate T_i'. This process gives rise to

$$\mathcal{T}_{\rm c}' \le |\sum_{i=1}^{3} T_i'| \le \tau\int_\Omega \mathbf{v}^\top\mathbb{B}\mathbf{v}\,\mathrm{d}x + C\tau\sum_{\ell=1}^{3}\|\eta_q^{n,\ell}\|^2 + \frac{CC_\mu^2c^2}{d}\tau\mathcal{S}', \tag{4.18}$$

where $\mathbb{B} = \frac{\gamma}{4}\mathbb{I}$ has been defined in Subsection 3.3.

As a result, from (4.10), (4.12), (4.16) and (4.18) we have

$$\|\xi_u^{n+1}\|^2 - \|\xi_u^n\|^2 + \mathcal{S}' \le -\tau\int_\Omega \mathbf{v}^\top(\mathbb{A} - \mathbb{B} - \varepsilon\mathbb{I})\mathbf{v}\,\mathrm{d}x + \frac{CC_\mu^2c^2}{d}\tau\mathcal{S}' + \mathcal{P}, \tag{4.19}$$

where

$$\mathcal{P} = \tau\sum_{\ell=1}^{4}\|\xi_u^{n,\ell}\|^2 + \frac{C}{\tau}\sum_{\ell=1}^{4}\|\mathbb{E}_\ell\eta_u^n + \delta_{4\ell}\zeta^n\|^2 + C\tau\sum_{\ell=1}^{3}\|\eta_q^{n,\ell}\|^2.$$

Choosing ε small enough, for example $\varepsilon = \frac{\gamma}{12}$, we can verify $\mathbb{A} - \mathbb{B} - \varepsilon\mathbb{I}$ is also positive definite. Noting that $\mathcal{S}' \ge 0$, hence if $\tau \le \tau_0 = \frac{d}{CC_\mu^2c^2}$, we have

$$\|\xi_u^{n+1}\|^2 - \|\xi_u^n\|^2 \le \mathcal{P}. \qquad \square$$

The estimate for the stage values $\|\xi_u^{n,\ell}\|$ emerging in the first term of $\mathcal{P}$ can be obtained along the similar argument as the proof for Lemma 4.2. So we omit the details and only state it in the following lemma.

Lemma 4.3. *Under the condition of Lemma 4.2, we have, for* $\ell = 1, 2, 3, 4$

$$\|\xi_u^{n,\ell}\|^2 \le C\left(\|\xi_u^n\|^2 + \sum_{\ell=1}^{4}\|\mathbb{E}_\ell\eta_u^n + \delta_{4\ell}\zeta^n\|^2 + \tau\sum_{\ell=1}^{3}\|\eta_q^{n,\ell}\|^2\right). \tag{4.20}$$

Combining Lemmas 4.2 and 4.3 and by the aid of the discrete Gronwall's inequality, we can derive

$$\|\xi_u^n\|^2 \le e^{Cn\tau}\left(\|\xi_u^0\|^2 + \sum_{m=0}^{n-1}\left\{\frac{1}{\tau}\sum_{\ell=1}^{4}\|\mathbb{E}_\ell\eta_u^m + \delta_{4\ell}\zeta^m\|^2 + \tau\sum_{\ell=1}^{3}\|\eta_q^{m,\ell}\|^2\right\}\right), \tag{4.21}$$

provided that τ is small enough such that $\tau \le \tau_0$.

Noting that $\xi_u^0 = 0$, and owing to (4.5), (4.7) and the triangle inequality, we get the main error estimate, which is presented in the following Theorem.

THEOREM 4.4. *Let u be the numerical solution of scheme (3.16). The finite element space V_h is the space of piecewise polynomials with degree $k \geq 1$ on the quasi-uniform triangulations of $\Omega = (a, b)$. Let U be the exact solution of problem* (2.1) *which satisfies the smoothness assumption* (4.4)*, then there exists a positive constant τ_0 depending only on the advection and diffusion coefficients and not on h, such that if $\tau \leq \tau_0$, there holds the following error estimate*

$$\max_{n\tau \leq T} \|U(t^n) - u^n\| \leq C(h^{k+1} + \tau^3), \tag{4.22}$$

where T is the final computing time and the bounding constant $C > 0$ is independent of h and τ.

5. Numerical experiments. The purpose of this section is to numerically validate the stability for the three IMEX LDG schemes given in Section 3, and error estimates for the second and third order IMEX LDG schemes (3.13) and (3.16). For the third order scheme, we take the parameter $\alpha_1 = -0.35$ as the choice in [4].

First we consider the exact solution

$$u(x, t) = e^{-dt} \sin(x - ct), \tag{5.1}$$

for the problem (2.1), in the interval $(a, b) = (-\pi, \pi)$. The periodic boundary condition is given by the exact solution. The finite element space is piecewise constant, piecewise linear and piecewise quadratic polynomials for the first, second and third order schemes, respectively. Table 1 lists the maximum time step τ_0 which can be

TABLE 1
The maximum time step τ_0 to ensure that the L^2-norm decreases with time for the schemes.

	$d = 0.01$			$c = 0.5$			ν
scheme	$c = 0.05$	$c = 0.1$	$c = 0.2$	$d = 0.01$	$d = 0.02$	$d = 0.04$	
first (3.6)	8.537	2.119	0.550	0.099	0.179	0.341	2.119
second (3.13)	5.540	1.385	0.346	0.055	0.110	0.221	1.385
third (3.16)	19.45	4.862	1.215	0.194	0.388	0.777	4.862

chosen to ensure the stability of the schemes (in the sense that the L^2-norm decreases with time) for solving this problem on uniform meshes, with mesh size $h = (b-a)/N$, where N is the number of cells. In this test, we take $N = 640$. The final computing time is $T = 5000$ in the cases of the $d = 0.01$ column and $T = 2000$ in the cases of the $c = 0.5$ column. The result shows that $\tau_0 \approx \nu d/c^2$ for some constant ν, which validate our stability properties stated in Propositions 3.1, 3.2 and 3.3.

Tables 2 and 3 are the L^2 errors and orders of accuracy for the schemes (3.13) and (3.16) for solving (2.1) on nonuniform meshes, respectively. The nonuniform meshes are obtained by randomly perturbing each node in the uniform mesh by up 20%. We take $\tau = h$ in all the tests. We can clearly observe the designed orders of accuracy from both tables.

Next we consider the viscous Burgers' equation with a source term

$$U_t + UU_x = dU_{xx} + g(x, t), \qquad (x, t) \in Q_T = (-\pi, \pi) \times (0, T], \tag{5.2a}$$

$$U(x, 0) = \sin(x), \qquad x \in \Omega = (-\pi, \pi), \tag{5.2b}$$

TABLE 2
The second order scheme, $T = 10$, $k = 1$.

$d = 0.1$	$c = 1$		$c = 0.1$		$c = 0.01$	
N	L^2 error	order	L^2 error	order	L^2 error	order
40	2.89E-02	-	1.14E-03	-	1.10E-03	-
80	6.76E-03	2.09	2.73E-04	2.06	2.75E-04	1.99
160	1.69E-03	2.00	6.77E-05	2.01	6.84E-05	2.01
320	4.23E-04	2.00	1.71E-05	1.98	1.72E-05	1.99
640	1.06E-04	2.00	4.36E-06	1.98	4.32E-06	2.00

TABLE 3
The third order scheme, $T = 10$, $k = 2$.

$d = 0.1$	$c = 1$		$c = 0.1$		$c = 0.01$	
N	L^2 error	order	L^2 error	order	L^2 error	order
40	6.12E-04	-	1.53E-05	-	1.61E-05	-
80	7.80E-05	2.97	1.95E-06	2.97	1.90E-06	3.09
160	9.84E-06	2.99	2.44E-07	3.00	2.40E-07	2.99
320	1.24E-06	2.99	3.06E-08	3.00	3.04E-08	2.98
640	1.55E-07	3.00	3.80E-09	3.01	3.87E-09	2.97

where $g(x,t) = \frac{1}{2}e^{-2dt}\sin(2x)$. The exact solution of (5.2) is

$$U(x,t) = e^{-dt}\sin(x). \tag{5.3}$$

We list the L^2 errors and orders of accuracy for the schemes (3.13) and (3.16) for solving (5.2) on nonuniform meshes in Tables 4 and 5. The nonuniform meshes are obtained in the same way as before. We take $\tau = h$ in all the tests except for the case $d = 0.01$ in Tables 4 and 5, where we take $\tau = 0.3h$, because if larger τ is taken in this case, it will be beyond the maximum time step to ensure the stability when the mesh is not fine enough. We can again clearly observe the designed orders of accuracy from both tables.

TABLE 4
Burgers' equation. The second order scheme, $T = 10$, $k = 1$.

	$d = 1$		$d = 0.1$		$d = 0.01$	
N	L^2 error	order	L^2 error	order	L^2 error	order
40	8.77E-06	-	1.13E-03	-	2.23E-03	-
80	2.12E-07	5.37	2.56E-04	2.12	5.51E-04	2.02
160	5.21E-08	2.02	6.58E-05	1.96	1.40E-04	1.98
320	1.29E-08	2.01	1.69E-05	1.96	3.70E-05	1.92
640	3.22E-09	2.01	4.26E-06	1.99	9.71E-06	1.93

At the end, we would like to test the errors and orders of accuracy in time for the type of problems whose exact solution is exponentially increasing with respect to time. For this purpose, we consider the problem

$$U_t + U_x = dU_{xx} + g(x,t), \qquad (x,t) \in Q_T = (-\pi,\pi) \times (0,T], \tag{5.4a}$$

$$U(x,0) = \sin(x), \qquad x \in \Omega = (-\pi,\pi), \tag{5.4b}$$

TABLE 5
Burgers' equation. The third order scheme, $T = 10$, $k = 2$.

	$d = 1$		$d = 0.1$		$d = 0.01$	
N	L^2 error	order	L^2 error	order	L^2 error	order
40	7.35E-08	-	1.57E-05	-	3.39E-05	-
80	9.63E-09	2.93	1.97E-06	3.00	4.44E-06	2.93
160	1.23E-09	2.97	2.54E-07	2.96	5.75E-07	2.95
320	1.56E-10	2.98	3.06E-08	3.05	7.28E-08	2.98
640	1.96E-11	2.99	3.90E-09	2.97	9.25E-09	2.98

with the exact solution $U(x,t) = e^{dt}\sin(x)$, where $g(U) = e^{dt}(2d\sin(x) + \cos(x))$.

In order to test the orders of accuracy with respect to time, we take $N = 1280$ and use a higher order approximation in space, i.e, we use the space of piecewise polynomials of degree k for the k-th order time discretization, and we take proper time steps such that the temporal error is dominant. In Tables 6 and 7 , we list the L^2 errors and orders of accuracy for the schemes (3.13) and (3.16) for solving (5.4), with respect to time. Optimal orders of accuracy in time can be observed from both tables.

TABLE 6
Equation (5.4). The second order scheme, $T = 10$, $k = 2$.

	$d = 0.1$		$d = 0.5$		$d = 1$	
τ	L^2 error	order	L^2 error	order	L^2 error	order
0.2	4.56E-04	-	4.02E-01	-	1.62E+02	-
0.1	1.15E-04	1.98	1.03E-01	1.97	4.15E+01	1.96
0.05	2.89E-05	1.99	2.62E-02	1.98	1.05E+01	1.98
0.025	7.24E-06	2.00	6.53E-03	1.99	2.64E+00	1.99
0.0125	1.81E-06	2.00	1.64E-03	2.00	6.63E-01	2.00

TABLE 7
Equation (5.4). The third order scheme, $T = 10$, $k = 3$.

	$d = 0.1$		$d = 0.5$		$d = 1$	
τ	L^2 error	order	L^2 error	order	L^2 error	order
0.2	5.08E-05	-	7.06E-02	-	5.44E+01	-
0.1	6.41E-06	2.99	9.16E-03	2.95	7.15E+00	2.93
0.05	8.06E-07	2.99	1.17E-03	2.97	9.18E-01	2.96
0.025	1.01E-07	3.00	1.47E-04	2.99	1.16E-01	2.98
0.0125	1.26E-08	3.00	1.85E-05	2.99	1.47E-02	2.99

6. Concluding remarks. We consider several specific implicit-explicit Runge-Kutta time marching methods coupled with the LDG schemes for solving linear advection-diffusion problems with periodic boundary conditions. In these methods the diffusion terms are treated implicitly and the advection terms are treated explicitly. By establishing the important relationship between the numerical solution and its gradient, and with the aid of energy techniques, we prove that the corresponding

IMEX LDG schemes are stable under the time step restriction $\tau \leq \tau_0$, where the constant τ_0 only depends on the advection and diffusion coefficients and is independent of the mesh size h. We also present optimal error estimates in both space and time, under the same temporal condition $\tau \leq \tau_0$. The stability analysis and error estimates can be extended to convection-diffusion problems with a nonlinear convection part, and similar stability analysis and error estimates can also be carried out for multi-step IMEX LDG schemes, both of which constitute our current work. In the future, we would like to consider the IMEX LDG schemes for non-periodic boundary conditions, for which we expect stability to be similarly obtainable, but accurate numerical boundary conditions would require some work.

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