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A Survey of High Order Schemes for the Shallow Water Equations

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Abstract. In this paper, we survey our recent work on designing high order positivity-preserving well-balanced finite difference and finite volume WENO (weighted essentially non-oscillatory) schemes, and discontinuous Galerkin finite element schemes for solving the shallow water equations with a non-flat bottom topography. These schemes are genuinely high order accurate in smooth regions for general solutions, are essentially non-oscillatory for general solutions with discontinuities, and at the same time they preserve exactly the water at rest or the more general moving water steady state solutions. A simple positivity-preserving limiter, valid under suitable CFL condition, has been introduced in one dimension and reformulated to two dimensions with triangular meshes, and we prove that the resulting schemes guarantee the positivity of the water depth.

Key Words: hyperbolic balance laws; WENO scheme; discontinuous Galerkin method; high order accuracy; source term; conservation laws; shallow water equation.

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1 Overview

Free surface flows have wide applications in ocean, environmental, hydraulic engineering and atmospheric modeling, with examples including the dam break and flooding problem, tidal flows in coastal water region, nearshore wave propagation with complex bathymetry structure, Tsunami wave propagation and ocean model. Three-dimensional Navier-Stokes equations can be used to simulate such flows directly. However, in the case where the horizontal length scale is much greater than the vertical length scale, one can average over the depth to get rid of the vertical direction and reduce the model into

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two-dimensional nonlinear shallow water equations (SWEs). SWEs play a critical role in the modeling and simulation of free surface flows in rivers and coastal areas, and can predict tides, storm surge levels and coastline changes from hurricanes and ocean currents. SWEs also arise in atmospheric flows, debris flows, and certain hydraulic structures like open channels and sedimentation tanks. SWEs take the form of non-homogeneous hyperbolic conservation laws with source terms modeling the effects of bathymetry and viscous friction on the bottom. In one space dimension, SWEs are defined as follows

$$\begin{cases} h_t+(hu)_x=0, \\ (hu)_t+\left(hu^2+\dfrac{1}{2}gh^2\right)_x=-ghb_x, \end{cases} \tag{1.1}$$

where h denotes the water height, u is the velocity of the fluid, b represents the bottom topography and g is the gravitational constant. In this paper, we will consider the variation of the bottom as the only source term, but other terms, such as a friction term or variations of the channel width, could also be added.

Due to the large scientific and engineering applications of the SWEs, research on effective and accurate numerical methods for their solutions has attracted great attention in the past two decades. Two types of difficulties are often encountered at the simulation of the SWEs, coming from the preservation of steady state solutions and the preservation of water height positivity. The first difficulty is related to the treatment of the source terms. An essential part for the SWEs and other conservation laws with source terms is that they often admit steady-state solutions in which the flux gradients are exactly balanced by the source terms. SWEs admit the general moving water equilibrium, given by

$$m:=hu=const \qquad \text{and} \qquad E:=\frac{1}{2}u^2+g(h+b)=const. \tag{1.2}$$

where m, E are the moving water equilibrium variables. People are often interested in the still water steady-state solution, which represents a still flat water surface, and referred as "lake at rest" solution:

$$u=v=0 \qquad \text{and} \qquad h+b=const. \tag{1.3}$$

Still water steady state (1.3) is simply a special case of the moving water steady state (1.2), when the velocity reduces to zero. Traditional numerical schemes with a straightforward handling of the source term cannot balance the effect of the source term and the flux, and usually fail to capture the steady state well. They will introduce spurious oscillations near the steady state. The well-balanced schemes are specially designed to preserve exactly these steady-state solutions up to machine error with relatively coarse meshes, and therefore it is desirable to design numerical methods which have the well-balanced property. The other major difficulty often encountered in the simulations of the SWEs is the appearance of dry regions in many engineering applications. Typical applications include the dam break problem, flood waves and run-up phenomena at a coast

with tsunamis being the most impressive example. Special attention needs to be paid near the dry/wet front to preserve the water height positivity, otherwise they may produce non-physical negative water height, which becomes problematic when calculating the eigenvalues $u \pm \sqrt{gh}$ to determine the time step size Δt, and renders the system not hyperbolic and not wellposed.

In the past two decades, many well-balanced numerical methods have been developed for the SWEs. The well-balanced property is often referred as "exact C-property", which means that the scheme is "exact" when applied to the stationary case (1.3). The concept was first proposed by Bermudez and Vazquez in [4], where they extended upwind methods to the SWEs with source terms. Following this pioneering work, many other schemes for the SWEs with such well-balanced property have been developed in the finite volume community. A quasi-steady wave propagation algorithm based on modified Riemann problems is presented in [2, 33, 34]. Another popular approach is to rewrite the equation in terms of the water surface instead of water height (also referred as the pre-balanced formulation), and well-balanced methods [31, 48, 69] can be designed based on such formulation. Well-balanced methods can also be derived utilizing the idea of hydrostatic reconstruction initially proposed in [1]. A kinetic approach to achieve well-balanced property has been shown in [44]. In the framework of residual distribution, simulation for the SWEs with well-balanced properties is shown in [46, 47]. For more related work, see also [17, 20, 22, 24, 25, 28, 35, 37, 49, 66].

Most of the works mentioned above are for numerical schemes of first or second order accuracy. In recent years, high-order accurate numerical schemes (with higher than second-order accuracy), have attracted increasing attention in many computational fields. They have been developed to reduce the number of computational cells and minimize the computational time to achieve the desired resolution. Some finite difference/volume weighted essentially non-oscillatory (WENO) schemes with well-balanced property have been designed for the SWEs recently. In [56,59], a special decomposition of the source term was introduced which leads to high order finite difference and finite volume well-balanced WENO methods. Hydrostatic reconstruction idea is extended to high order methods in [39, 60] with a careful high order approximation of the source term. Path-conservative methods for the non-conservative product are introduced in [13, 42] and extended to the SWEs. Other high order finite volume methods include [10–12]. Recently, finite element discontinuous Galerkin (DG) methods have attracted increasing attention in many computational fields. Several advantages of the DG method, including its high order accuracy, high parallel efficiency, flexibility for hp-adaptivity and arbitrary geometry and meshes, make it suited for the SWEs, see [16, 19, 23, 38, 51]. Several well-balanced DG methods have been proposed in the last few years, by the idea of special decomposition of the source term [59], hydrostatic reconstruction [18, 29, 60], and path-conservative [45].

The well-balanced methods mentioned above target to preserve the still water steady state (1.3). They cannot preserve the moving water steady state (1.2), and it is significantly more difficult to obtain well-balanced schemes for such equilibrium. In a recent pa-

per [63], several numerical examples are shown to demonstrate the advantage of moving-water well-balanced schemes over still-water well-balanced schemes for the SWEs. Those numerical examples clearly demonstrate the importance of utilizing moving-water well-balanced methods for solutions near a moving-water equilibrium. There have been a few attempts in developing well-balanced methods for the moving water equilibrium. A class of first order accurate flux-vector-splitting schemes based on the theory of nonconservative products was proposed in [24]. Well-balanced second order central schemes on staggered grids can be found in [49]. Numerical methods based on local subsonic steady state reconstruction, which are exactly well-balanced for subsonic moving equilibria, was shown in [7]. A few high order accurate well-balanced methods for the moving water equilibrium have been introduced recently. In [40], well-balanced finite volume weighted essentially non-oscillatory (WENO) methods are designed for arbitrary equilibria of the SWEs. The key component there is a special way to recover the moving water equilibrium and a well-balanced quadrature rule of the source term. Other high order well-balanced methods for the moving water equilibrium include the central WENO methods [50], path-conservative WENO methods [14] and DG methods [55].

The other difficulty in simulating the SWEs is related to the robustness of the numerical methods near the wet/dry front. This problem relates to the fact that there is no water in these areas, while the SWEs (1.1) are only defined in wet regions. Therefore we may need to deal with moving boundary problems. One could use the mesh adaption technique [6] which tracks the dry front by changing the meshes. It has the advantage in accuracy but is computationally expensive. A more popular approach is the thin layer technique, which maintains a very thin layer in dry elements and includes these dry elements in the computation. The difficulty then reduces to the issue of preserving the non-negativity of water height for the SWEs during the computation. Another related problem is the computation of velocity given height and discharge in the nearly dry region. One usually introduces a threshold on the velocity (or on the water height) to avoid extremely large velocity when $h \ll 1$. There have been a number of positivity-preserving schemes [1, 5, 8, 15, 21, 32, 35] in the finite volume framework. They usually rely on the positivity-preserving Riemann solver, for example the HLL solver [26]. Some positivity-preserving DG methods with P^1 polynomial spaces [9, 18, 30] have been developed in the past few years, mainly relying on modifying the slope to avoid negative values of water height. For both finite volume and DG methods, the issue of positivity-preserving property for high-order methods is non-trivial. Most existing high-order wetting and drying treatments are focused on post-processing reconstruction of the data obtained from the numerical solution at each time level. One example is to project the solution to a non-negative linear element in the cell near the wet/dry front. Even though the post-processing can bring the reconstruction to satisfy non-negative water height, this alone usually does *not* guarantee that the solution (e.g., cell average from a finite volume or DG scheme) at the next time step still maintains the non-negative water height property. If negative cell averages for the water height are obtained at the next time level, the positivity reconstruction post-processing will destroy the conservation. Recently, following the

general approach introduced in [67], a sufficient condition on the time step size to ensure the positivity of cell averages of water height, plus a simple positivity-preserving limiter, has been studied in [61] for the finite volume methods, and in [64,65] for the DG methods in one dimension and two dimensions with unstructured meshes.

In this paper, we survey the development of high order well-balanced positivity-preserving finite difference and finite volume WENO schemes and finite element DG methods developed by us recently in a series of papers [40, 41, 55–61, 64, 65], and comment on their applicability in different situations. In Section 2, we describe well-balanced finite difference WENO methods. High order well-balanced positivity-preserving finite volume WENO schemes and DG methods are discussed in Sections 3 and 4, respectively. Concluding remarks are given in Section 5. We do not present any numerical simulation results here, and simply refer to these papers for more details.

2 Finite difference methods

In this section, we focus on high-order accurate finite difference WENO methods for the SWEs, which are well-balanced for the still water at rest steady state (1.3). The main idea is to decompose the source term into a sum of two terms, and discretize each term independently using a finite difference formula consistent with the WENO approximation to the flux derivative terms in the conservation law.

For the ease of presentation, we denote the SWEs (1.1) by

$$U_t + f(U)_x = s(h,b)$$

where $U=(h,hu)^T$ with the superscript T denoting the transpose, $f(U)=(hu,hu^2+\frac{1}{2}gh^2)^T$ is the flux and $s(h,b)$ is the source term. In this paper, we mainly focus on the spatial discretization. Total variation diminishing (TVD) high-order Runge-Kutta time discretization [54] is usually used in practice for stability and to increase temporal accuracy. For example, the third order TVD Runge-Kutta method can be coupled with all the spatial discretization introduced in this paper:

$$\begin{aligned} U^{(1)} &= U^n + \Delta t \mathcal{F}(U^n) \\ U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}\left(U^{(1)} + \Delta t \mathcal{F}(U^{(1)})\right) \\ U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}\left(U^{(2)} + \Delta t \mathcal{F}(U^{(2)})\right), \end{aligned} \tag{2.1}$$

where $\mathcal{F}(U)$ is the spatial operator.

We first rewrite the SWEs by splitting the source term $-ghb_x$ into two terms $\left(gb^2/2\right)_x - g(h+b)b_x$. Hence the equations become

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = \left(\frac{1}{2}gb^2\right)_x - g(h+b)b_x. \end{cases} \tag{2.2}$$

One could also incorporate the source term $(gb^2/2)_x$ to the numerical flux on the left hand side. We will show that, if written in this form, any linear scheme is well-balanced for the still water at rest steady state (1.3). We define a linear finite difference operator D to be one satisfying $D(af_1+bf_2)=aD(f_1)+bD(f_2)$ for constants a, b and arbitrary grid functions f_1 and f_2. A scheme for (2.2) is said to be a linear scheme if all the spatial derivatives are approximated by linear finite difference operators. For any consistent linear scheme, the first equation $(hu)_x=0$ is satisfied exactly at the still water steady state (1.3). The second equation has the truncation error

$$D_1\left(hu^2+\frac{1}{2}gh^2\right)-D_2\left(\frac{1}{2}gb^2\right)+g(h+b)D_3(b),$$

where D_1, D_2 and D_3 are linear finite difference operators. We further restrict our attention to linear schemes which satisfy

$$D_1=D_2=D_3=D \tag{2.3}$$

for the still water steady state solutions. We can easily prove such linear scheme is well-balanced, and the detailed proof can be found in [56].

We now already have high-order well-balanced schemes for the SWEs. However, these schemes are linear, hence they will be oscillatory when the solution contains discontinuities. We would need to consider nonlinear schemes, for example, high-order finite difference WENO schemes [3, 27, 36]. Next, we will use the fifth order finite difference WENO scheme as an example to demonstrate the basic ideas. We will not give the details of the base WENO schemes, and refer to [27, 53] for such details.

We first consider the situation when the WENO scheme is used without the flux splitting and the local characteristic decomposition. The first equation in (2.2) does not cause a problem for the still water solution, as $hu=0$ and the consistent WENO approximation to $(hu)_x$ is exact. For the second equation in (2.2), there are three derivative terms, $\left(hu^2+gh^2/2\right)_x$, $\left(gb^2/2\right)_x$ and b_x, that must be approximated. The approximation to the flux derivative term $\left(hu^2+gh^2/2\right)_x$ proceeds as before using the WENO approximation. We notice that the WENO approximation to d_x where $d=hu^2+gh^2/2$ can be eventually written out as

$$d_x|_{x=x_i}\approx\sum_{k=-r}^{r}a_kd_{i+k}\equiv D_d(d)_i \tag{2.4}$$

where $r=3$ for the fifth order WENO approximation and the coefficients a_k depend nonlinearly on the smoothness indicators involving the grid function d. The key idea now is to use the difference operator D_d with $d=hu^2+gh^2/2$ *fixed*, namely to use the same coefficients a_k obtained through the smoothness indicators of $d=hu^2+gh^2/2$, and apply this difference operator D_d to approximate $\left(gb^2/2\right)_x$ and b_x in the source terms. Thus

$$\left(\frac{1}{2}gb^2\right)_x\Big|_{x=x_i}\approx\sum_{k=-r}^{r}a_k\left(\frac{1}{2}gb^2\right)_{i+k}\equiv D_d\left(\frac{1}{2}gb^2\right)_i;$$

$$b_x|_{x=x_i} \approx \sum_{k=-r}^{r} a_k b_{i+k} \equiv D_d(b)_i.$$

Clearly, the finite difference operator D_d, obtained from the fifth order WENO procedure, is a fifth order accurate approximation to the first derivative on any grid function, thus our approximation to the source terms is also fifth order accurate. Plus, the finite difference operator D_d, with the coefficients a_k based on the smoothness indicators of $d=hu^2+gh^2/2$ fixed, is a linear operator on any grid functions. Thus we can prove that component-wise WENO schemes, without the flux splitting or local characteristic decomposition, are well-balanced with the special handling of the source terms described above.

For the situation when the local characteristic decomposition or Lax-Friedrichs flux splitting is invoked in the WENO procedure, the same well-balanced property can be obtained when we apply the local characteristic decomposition or flux splitting to the source terms as well. One change in the Lax-Friedrichs flux is to replace the flux $f(U)$ defined by

$$f^{\pm}(U)=\frac{1}{2}\left[\begin{pmatrix} hu \\ hu^2+\frac{1}{2}gh^2 \end{pmatrix} \pm \alpha_i \begin{pmatrix} h \\ hu \end{pmatrix}\right] \tag{2.5}$$

to

$$f^{\pm}(U)=\frac{1}{2}\left[\begin{pmatrix} hu \\ hu^2+\frac{1}{2}gh^2 \end{pmatrix} \pm \alpha_i \begin{pmatrix} h+b \\ hu \end{pmatrix}\right]. \tag{2.6}$$

for the i-th characteristic field, where $\alpha_i=max_u|\lambda_i(u)|$ with $\lambda_i(u)$ being the i-th eigenvalue of the Jacobian $f'(U)$. This modification is to ensure that the effect of this viscosity term $\pm\alpha_i \begin{pmatrix} h+b \\ hu \end{pmatrix}$ towards the approximation of $f(U)_x$ is zero for the still water stationary solution (1.3), hence it will not violate the well-balanced properties. This modification is justified since b does not depend on the time t, hence the first equation in (1.1) can also be considered as an evolution equation for $h+b$ instead of for h. We refer to [56] for the details. Therefore, we have

Proposition 2.1. The WENO schemes as stated above are well-balanced for the still water steady state (1.3), and maintain their original high order accuracy.

Remark 2.1. One popular approach in designing well-balanced methods for the SWEs is to replace the water height h by the water surface $H:=h+b$ in the equations (1.1) [31,69], which leads to the pre-balanced formulation [48]

$$\begin{cases} H_t+(hu)_x=0, \\ (hu)_t+\left(\dfrac{(hu)^2}{H-b}+\dfrac{1}{2}g(H-b)^2\right)_x=-g(H-b)b_x=-gHb_x+\left(\dfrac{1}{2}gb^2\right)_x, \end{cases} \tag{2.7}$$

or equivalently,

$$\begin{cases} H_t+(hu)_x=0, \\ (hu)_t+\left(\dfrac{(hu)^2}{H-b}+\dfrac{1}{2}gH^2-gHb\right)_x=-gHb_x. \end{cases} \tag{2.8}$$

Although derived from totally different approaches, the special decomposition of the source term in (2.2) and the pre-balanced formulation share similar source terms structure. This also justifies the modification in the numerical flux from (2.5) to (2.6).

A major advantage of the high order finite difference WENO schemes is that it is straightforward to extend them to multiple space dimensions, by simply approximating each spatial derivative along the relevant coordinate. We can show that it is also straightforward to extend the well-balanced finite difference WENO schemes described above to two dimensions.

Remark 2.2. In [58,62], we have generalized these high order well-balanced finite difference WENO scheme, to solve a wider class of hyperbolic systems with separable source terms. This class of hyperbolic balance laws is quite broad, and includes the elastic wave equation, Euler equation with gravitational field, the hyperbolic model for a chemosensitive movement, the nozzle flow and a two phase flow model.

3 Finite volume methods

Finite volume schemes are very popular for solving hyperbolic conservation laws. They represent the underlying physics in a natural way. In this section, we recall the positivity-preserving high order well-balanced finite volume WENO schemes developed by us in [40,59–61]. We first present two approaches to achieve well-balanced methods for the still water steady state (1.3), followed by high order well-balanced methods for the moving water equilibrium (1.2). A simple positivity-preserving limiter will also be introduced to take care of numerical difficulty near the wet-dry front.

We first introduce some notations which will be used later. We discretize the computational domain into cells $I_j=[x_{j-1/2},x_{j+1/2}]$, and denote the size of the j-th cell by Δx_j and the maximum mesh size by $\Delta x=\max_j \triangle x_j$. In a finite volume scheme, our computational variables are $\overline{U}_j(t)$, which approximate the cell averages $\overline{U}(x_j,t)=\frac{1}{\Delta x_j}\int_{I_j}U(x,t)dx$. The conservative numerical scheme is given by

$$\frac{d}{dt}\overline{U}_j(t)+\frac{1}{\Delta x_j}\left(\hat{f}_{j+\frac{1}{2}}-\hat{f}_{j-\frac{1}{2}}\right)=\frac{1}{\Delta x_j}\int_{I_j}s(h,b)dx, \tag{3.1}$$

with $\hat{f}_{j+1/2}=F(U^-_{j+1/2},U^+_{j+1/2})$ being the numerical flux. $U^-_{j+1/2}$ and $U^+_{j+1/2}$, the high-order pointwise approximations to $U(x_{j+1/2},t)$ from left and right respectively, are computed through the neighboring cell average values $\overline{U}_j$ by a high-order WENO reconstruction procedure. Basically, for a $(2k-1)$-th order WENO scheme, we first compute k reconstructed boundary values $U^{(k),\pm}_{j+1/2}$ corresponding to different candidate stencils. Then by providing each value a weight which indicates the smoothness of the corresponding stencil, we define the $(2k-1)$-th order WENO reconstruction $U^{\pm}_{j+1/2}$ as a convex combination

of all these k reconstructed values. Eventually, the WENO reconstruction can be written out as:

$$U^+_{j+\frac{1}{2}} = \sum_{r=-k+1}^{k} w_r \overline{U}_{j+r}, \qquad U^-_{j+\frac{1}{2}} = \sum_{r=-k}^{k-1} \tilde{w}_r \overline{U}_{j+r}. \tag{3.2}$$

where $k=3$ for the fifth order WENO approximation and the coefficients w_r and $\tilde{w}_r$ depend nonlinearly on the smoothness indicators involving the cell average $\bar{u}$.

3.1 Well-balanced methods for the still water

In order to achieve the well-balanced property, we are interested in numerical methods which balance the numerical approximation of the flux and source term at the still water stationary solution (1.3). The key idea is to introduce high-order accurate numerical discretization of the source term, which mimics the approximation of the flux term, so that the exact balance between the source term and the flux can be achieved at the steady state numerically. Here we present two different approaches to achieve such goal. The first approach focus on a non-standard discretization of the source term, by following the idea of decomposing the source terms, as shown in Section 2. The second approach employs the idea of hydrostatic reconstruction [1] to modify the approximation of numerical flux and keep a simple source term approximation.

The main idea in Section 2 to design a well-balanced high-order finite difference WENO scheme is to decompose the source term into a sum of two terms in (2.2), each of which is discretized independently using a finite difference formula consistent with that of approximating the flux derivative terms in the conservation law. We follow a similar idea in the finite volume framework. After applying the WENO reconstruction on $\overline{U}_j$ to obtain $U^{\pm}_{j+1/2}$ in (3.2), we apply the same reconstruction to the function $(b(x),0)^T$, with coefficients computed from $(h,hu)^T$, to obtain $b^{\pm}_{j+1/2}$. It is easy to verify that $h^{\pm}_{j+1/2}+b^{\pm}_{j+1/2} = constant$ at the still water steady state (1.3). We now rewrite the integration of the source term as

$$\begin{aligned}
&\int_{I_j} -ghb_x dx = \int_{I_j} \left(\left(\frac{1}{2} g b^2 \right)_x - g(h+b) b_x \right) dx \\
&= \left(\frac{1}{2} g b^2 \right)(x_{j+\frac{1}{2}}) - \left(\frac{1}{2} g b^2 \right)(x_{j-\frac{1}{2}}) - g\overline{(h+b)}_j (b(x_{j+\frac{1}{2}}) - b(x_{j-\frac{1}{2}})) - \int_{I_j} g((h+b) - \overline{(h+b)}_j) b_x dx,
\end{aligned}$$

and approximate it numerically by

$$s_j = \frac{g}{2}\{b^2_{j+\frac{1}{2}}\} - \frac{g}{2}\{b^2_{j-\frac{1}{2}}\} - g\overline{(h+b)}_j(\{b_{j+\frac{1}{2}}\} - \{b_{j-\frac{1}{2}}\}) - \int_{I_j} g\left((h+b) - \overline{(h+b)}_j\right) b_x dx, \tag{3.3}$$

where the notation $\{\phi\}$ is defined as the average of $\phi^{\pm}$, and the last integral in (3.3) is approximated by a suitable high order Gaussian quadrature rule. The approximation of the values at those Gauss points are obtained by the WENO reconstruction procedure.

The semi-discrete form of the algorithm takes the form of

$$\frac{d}{dt}\overline{U}_j(t) = -\frac{1}{\triangle x_j}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}) + \frac{1}{\triangle x_j} S_j, \tag{3.4}$$

where $S_j = (0, s_j)^T$. If the Lax-Friedrichs numerical flux is used, we would need to modify it as in (2.6). It is easy to observe that high order accuracy is guaranteed for our scheme, and even if discontinuities exist in the solution, the non-oscillatory property is maintained. The well-balanced property for the still water (1.3) can be proved, and we refer to [59] for the details.

A different approach to achieve well-balanced property is to utilize the hydrostatic reconstruction idea in the numerical flux. As mentioned in [60], our well-balanced numerical scheme has the form

$$\frac{d}{dt}\overline{U}_j(t) + \frac{1}{\Delta x_j}\left(\hat{f}^l_{j+\frac{1}{2}} - \hat{f}^r_{j-\frac{1}{2}}\right) = \frac{1}{\Delta x_j}\int_{I_j} s(h,b)dx. \tag{3.5}$$

The left and right fluxes $\widehat{f}^l_{j+1/2}$ and $\widehat{f}^r_{j-1/2}$ are given by:

$$\begin{aligned}
\widehat{f}^l_{j+\frac{1}{2}} &= F(U^{*,-}_{j+\frac{1}{2}}, U^{*,+}_{j+\frac{1}{2}}) + \begin{pmatrix} 0 \\ \frac{g}{2}(h^-_{j+\frac{1}{2}})^2 - \frac{g}{2}(h^{*,-}_{j+\frac{1}{2}})^2 \end{pmatrix} \\
\widehat{f}^r_{j-\frac{1}{2}} &= F(U^{*,-}_{j-\frac{1}{2}}, U^{*,+}_{j-\frac{1}{2}}) + \begin{pmatrix} 0 \\ \frac{g}{2}(h^+_{j-\frac{1}{2}})^2 - \frac{g}{2}(h^{*,+}_{j-\frac{1}{2}})^2 \end{pmatrix}
\end{aligned} \tag{3.6}$$

with the left and right values of U^* defined as:

$$U^{*,\pm}_{j+\frac{1}{2}} = \begin{pmatrix} h^{*,\pm}_{j+\frac{1}{2}} \\ h^{*,\pm}_{j+\frac{1}{2}} u^{\pm}_{j+\frac{1}{2}} \end{pmatrix}, \tag{3.7}$$

$$h^{*,\pm}_{j+\frac{1}{2}} = \max\left(0, h^{\pm}_{j+\frac{1}{2}} + b^{\pm}_{j+\frac{1}{2}} - \max(b^+_{j+\frac{1}{2}}, b^-_{j+\frac{1}{2}})\right). \tag{3.8}$$

$b^{\pm}_{j+1/2}$ are constructed in the same way, i.e., we apply the same WENO reconstruction to the function $(b(x),0)^T$, with coefficients computed from $(h,hu)^T$, to obtain $b^{\pm}_{j+1/2}$. This ensures that $h^{\pm}_{j+1/2} + b^{\pm}_{j+1/2} = const$ if the still water $\bar{h}_j + \bar{b}_j = const$ is given.

Then, we use interpolation to obtain a high-order polynomial h_h (or b_h) on the cell I_j, based on the boundary values $h^+_{j-1/2}$, $h^-_{j+1/2}$ (or $b^+_{j-1/2}$, $b^-_{j+1/2}$) and several other neighboring boundary values. For example, we can use $h^-_{j+3/2}$, $h^-_{j+1/2}$, $h^+_{j-1/2}$ and $h^+_{j-3/2}$ to interpolate a third degree polynomial. Therefore, $\int_{I_j} s(h_h, b_h)dx$, a high-order approximation to the source term $\int_{I_j} s(h,b)dx$, can be exactly computed by a suitable Gauss quadrature. In order to obtain a $(2k-1)$-th order accurate method, h_h and b_h need to approximate h and b with $(k+1)$-th order accuracy. Combining these together, we have proven in [60] that the above methods (3.5) are actually well-balanced for the still water steady state (1.3) of the SWEs.

Remark 3.1. The well-balanced numerical scheme (3.5) can also be rewritten as

$$\frac{d}{dt}\overline{U}_j(t)+\frac{1}{\Delta x_j}\left(\hat{f}_{j+\frac{1}{2}}-\hat{f}_{j-\frac{1}{2}}\right)=\frac{1}{\Delta x_j}\int_{I_j}s(h,b)dx+\frac{1}{\Delta x_j}\left(\hat{f}_{j+\frac{1}{2}}-\hat{f}^l_{j-\frac{1}{2}}\right)-\frac{1}{\Delta x_j}\left(\hat{f}_{j+\frac{1}{2}}-\hat{f}^r_{j-\frac{1}{2}}\right).$$

We point out here that $\widehat{f}_{j+1/2}-\widehat{f}^l_{j+1/2}$ and $\widehat{f}_{j-1/2}-\widehat{f}^r_{j-1/2}$ are high order correction terms at the level of $O(\triangle x^{k+1})$ regardless of the smoothness of the solution U. Therefore, the scheme (3.5) is still a spatially $(k+1)$-th order conservative scheme and will converge to the weak solution.

3.2 Well-balanced methods for the moving water

The above approaches preserve the still water steady state (1.3). The more general moving water steady state of the SWEs take the form of (1.2). In this section, we generalize the approach to present a high order finite volume method which is well-balanced for the moving water (1.2).

In order to construct our well-balanced scheme, it is essential to transform the conservative variables $U:=(h,m)^T$ into the equilibrium variables $V:=(m,E)^T$ and vice versa. Given conservative variables U and a bottom function b, the energy E, and hence the equilibrium variables $V=V(U)$, can be easily computed by (1.2). The main difficulty comes from the inverse transform $U=U(V)$. As in [40], we first label the different flow regimes by defining the sign function

$$\sigma:=\text{sign}(Fr-1),\qquad Fr:=|u|/\sqrt{gh},\tag{3.9}$$

and a state is called sonic, sub- or supersonic if σ is zero, negative or positive. With given $V=(m,E)$, b and σ, one can recover the conservative variable h and establish the transform $U=U(V)$ in the following way. If $m=0$, the transformation is trivial. Assume m is a fixed nonzero parameter, we define the function φ by

$$\varphi(h):=\frac{m^2}{2h^2}+gh,\tag{3.10}$$

which achieves its minimum value $\varphi_0=3(g|m|)^{2/3}/2$ at $h_0=g^{-1/3}(|m|)^{2/3}$. Therefore, we have the following result (see [40] for details):

Lemma 3.1. *Let m be given, and suppose either $E=\varphi_0+gb$ if $\sigma=0$ or $E>\varphi_0+gb$ if $\sigma=\pm1$. Then there exists a unique solution $h=h(E,b,\sigma)$ to the equation $\varphi(h)=E-gb$, such that*

$$h<h_0\quad\text{for }\sigma=1,\qquad h=h_0\quad\text{for }\sigma=0,\qquad h>h_0\quad\text{for }\sigma=-1.\tag{3.11}$$

Given the set (E,b,σ) satisfying the condition in Lemma 3.1, it is straightforward to compute the solution h by Newton's method, and we refer to [40] for the strategy.

Assume the initial values $\overline{U}_j$ and $\bar{b}_j$ are given. At each time step, we first apply the WENO reconstruction procedure to the variables $\overline{U}_j$ to obtain $U^{\pm}_{j+1/2}$, and hence $V^{\pm}_{j+1/2}$.

The first difficulty encountered in designing well-balanced methods is the recovery of well-balanced states from the provided initial equilibrium data. Lets assume the initial data are in perfect equilibrium, i.e. $V(x)\equiv\overline{V}$ for some equilibrium state $\overline{V}$. However, the reconstructed cell boundary values $U^{\pm}_{j+1/2}$, or $V^{\pm}_{j+1/2}$ may not be in equilibrium any more. A strategy has been proposed in [40] to recover the well-balanced states. The main idea is, given cell averages $\overline{U}_j$ and a bottom function $b(x)$, we choose local reference values $\overline{V}_j$ of the equilibrium variables, which are defined implicitly by the requirement that

$$\frac{1}{\Delta x_j}\int_{I_j} U(\overline{V}_j,x)dx=\overline{U}_j. \tag{3.12}$$

$\overline{V}_j$ is chosen as the unique (see [40, Def.3.2]) local equilibrium such that the corresponding conserved variables $U(\overline{V}_j,b(x))$ have the same cell average $\overline{U}_j$. It was proven that, if the data $U(x)$ and $b(x)$ are in the equilibrium (i.e., $V(U(x),x)\equiv\overline{V}$ for all cells I_j), then the reference states $\overline{V}_j$ computed via (3.12) coincide with the true equilibrium $\overline{V}$. The WENO reconstruction is then completed by limiting the reconstruction $V^{\pm}_{j+1/2}$ with respect to the reference values $\overline{V}_j$:

$$\widetilde{V}^{\pm}_{i+\frac{1}{2}}=\lim(V^{\pm}_{i+\frac{1}{2}};\overline{V}_j,\overline{V}_{i\pm1}), \tag{3.13}$$

where we refer to [40, (3.16)] for the the limiter function lim. At the steady state where $V(x)\equiv\overline{V}$, the limited values (3.13) satisfy

$$\widetilde{V}^{\pm}_{i+\frac{1}{2}}=\widetilde{V}_j=\overline{V}_j=\overline{V}\quad\text{for all } i. \tag{3.14}$$

We can then update $\widetilde{U}^{\pm}_{j+1/2}$ by

$$\widetilde{U}^{\pm}_{j+\frac{1}{2}}:=U(\widetilde{V}^{\pm}_{j+\frac{1}{2}},b^{\pm}_{j+\frac{1}{2}},\sigma^{\pm}_{j+\frac{1}{2}}). \tag{3.15}$$

We would like to comment that, compared with the still water case, these extra difficulties in designing well-balanced schemes for moving water mainly comes from the nonlinearity of the equilibrium variables V.

Now we are ready to present well-balanced methods for the moving water steady state (1.2). The well-balanced methods take the form of (3.5), where the left and right fluxes $\widehat{f}^{l}_{j+1/2}$ and $\widehat{f}^{r}_{j-1/2}$ are given by:

$$\begin{aligned}\widehat{f}^{l}_{j+\frac{1}{2}} &= F(U^{*,-}_{j+\frac{1}{2}},U^{*,+}_{j+\frac{1}{2}})-f(U^{*,-}_{j+\frac{1}{2}})+f(\widetilde{U}^{-}_{j+\frac{1}{2}}),\\ \widehat{f}^{r}_{j-\frac{1}{2}} &= F(U^{*,-}_{j-\frac{1}{2}},U^{*,+}_{j-\frac{1}{2}})-f(U^{*,+}_{j-\frac{1}{2}})+f(\widetilde{U}^{+}_{j-\frac{1}{2}}),\end{aligned} \tag{3.16}$$

and the left and right values of U^* defined as:

$$U^{*,\pm}_{j+\frac{1}{2}}=U(\widetilde{V}^{\pm}_{j+\frac{1}{2}},b^{*,\pm}_{j+\frac{1}{2}},\sigma^{\pm}_{j+\frac{1}{2}}). \tag{3.17}$$

The intermediate values $b^{*,\pm}_{j+1/2}$ are required to satisfy

$$\min\{b^-_{j+\frac{1}{2}}, b^+_{j+\frac{1}{2}}\} \le b^{*,\pm}_{j+\frac{1}{2}} \le \max\{b^-_{j+\frac{1}{2}}, b^+_{j+\frac{1}{2}}\}. \tag{3.18}$$

In [1], Audusse et al. chose the maximum value in their well-balanced methods for the still water steady state, and our methods in Section 3.1 follows that choice. In [40,41], the minimum value is chosen, inspired by a simple example of supersonic flow. It was then commented in [41] that the optimal choice of $b^{*,\pm}_{j+1/2}$ remains an open problem. Recently, in the proof of the positivity preserving property, we found out in [55] that the optimal choice is to take the maximum value in the subsonic or sonic region, and take the minimum value in the supersonic region. See [55, Remark 3.1] for more explanations on the choice of $b^{*,\pm}_{j+1/2}$.

The last remaining piece is to define the source term approximation in (3.5). As explained in [40], some simple calculation leads to a second-order well-balanced approximation $s_j(\widetilde{U}^+_{j-1/2}, \widetilde{U}^-_{j+1/2}, b^+_{j-1/2}, b^-_{j+1/2})$, where

$$s_j(U_l, U_r, b_l, b_r) = -g\frac{h_l+h_r}{2}(b_r-b_l) + \frac{1}{4}(h_r-h_l)(u_r-u_l)^2. \tag{3.19}$$

For smooth flows, the cubic correction term on the right side of (3.19) is so small that it does not affect the order of the quadrature rule. In [40], we showed how to limit this term when the jumps h_r-h_l and u_r-u_l are no longer of the order of the gridsize, so that the resulting methods still satisfy the Lax-Wendroff theorem and converge to weak solutions. Using the idea of extrapolation [39,40], we can extend the source term approximation to any order of accuracy. A fourth-order method has been implemented and tested in [40]. We refer to that paper for the details of the proof of well-balanced properties for the moving water steady state.

3.3 Positivity-preserving limiters

In this section, we present a simple positivity-preserving limiter for the finite volume WENO methods for the SWEs (1.1) with dry areas. In the previous subsection, we discussed several different approaches to design well-balanced methods. The main difference of these approaches is how to approximate the source term and flux terms in the momentum equation, which has non-zero source term. Their discretizations to the mass equation are similar. The positivity-preserving property mainly relies on the numerical approximation to the mass equation, therefore, the positivity-preserving limiter discussed in this section can be applied to all these well-balanced methods.

We only consider the Euler forward in time (3.5) in this subsection. The same results can be generalized to TVD high-order Runge-Kutta [54] and multi-step [52] time discretizations since TVD time discretizations are convex combinations of the Euler forward operators. By plugging (3.7) and (3.6) into (3.5), the scheme satisfied by the cell averages of the water height in the well-balanced finite volume WENO methods (3.5) can be

written as

$$\overline{h}_j^{n+1} = \overline{h}_j^n - \lambda \left[\widehat{F}\left(h^{*,-}_{j+\frac{1}{2}}, u^-_{j+\frac{1}{2}}; h^{*,+}_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}} \right) - \widehat{F}\left(h^{*,-}_{j-\frac{1}{2}}, u^-_{j-\frac{1}{2}}; h^{*,+}_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}} \right) \right], \tag{3.20}$$

where $\lambda = \Delta x/\Delta t$, $h^{*,\pm}_{j+1/2}$ are defined in (3.8) and

$$\widehat{F}\left(h^{*,-}_{j+\frac{1}{2}}, u^-_{j+\frac{1}{2}}; h^{*,+}_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}} \right) = \frac{1}{2}\left(h^{*,-}_{j+\frac{1}{2}} u^-_{j+\frac{1}{2}} + h^{*,+}_{j+\frac{1}{2}} u^+_{j+\frac{1}{2}} - \alpha(h^{*,+}_{j+\frac{1}{2}} - h^{*,-}_{j+\frac{1}{2}}) \right). \tag{3.21}$$

Note that the well-balanced methods for moving water in Section 3.2 take the same form.

For the first order scheme with the well-balanced flux, we have the following lemma on its positivity.

Lemma 3.2. *Under the CFL condition $\lambda\alpha \le 1$, with $\alpha = \max(|u| + \sqrt{gh})$, consider the following scheme*

$$h_j^{n+1} = h_j^n - \lambda \left[\widehat{F}\left(h_j^{*,+}, u_j^n; h^{*,-}_{j+1}, u^n_{j+1} \right) - \widehat{F}\left(h^{*,+}_{j-1}, u^n_{j-1}; h_j^{*,-}, u_j^n \right) \right] \tag{3.22}$$

with $\widehat{F}$ the same as in (3.21) and

$$h_j^{*,+} = \max\left(0, h_j^n + b_j - \max(b_j, b_{j+1})\right), \qquad h_j^{*,-} = \max\left(0, h_j^n + b_j - \max(b_{j-1}, b_j)\right). \tag{3.23}$$

If h_j^n, $h^n_{j\pm1}$ are non-negative, then h_j^{n+1} is also non-negative.

We now consider the $(2k-1)$-th order scheme (3.20). For the ease of presentation, we consider a reconstructed polynomial $p_j(x)$ of degree $2k-2$, which satisfies

$$p_j(x_{j-\frac{1}{2}}) = h^+_{j-\frac{1}{2}}, \qquad p_j(x_{j+\frac{1}{2}}) = h^-_{j+\frac{1}{2}}, \qquad \frac{1}{\Delta x}\int_{I_j} p_j(x)dx = \overline{h}_j^n. \tag{3.24}$$

Moreover, $p_j(x)$ should be a $(2k-1)$-th order accurate approximation to the exact solution on I_j. As we will explain later, this polynomial only serves the theoretical purpose to understand the derivation of the limiter and will not need to be explicitly constructed in the implementation.

Let us introduce the N-point Legendre Gauss-Lobatto quadrature rule on the interval $I_j = [x_{j-1/2}, x_{j+1/2}]$, which is exact for the integral of polynomials of degree up to $2N-3$. We choose N such that $2N-3 \ge 2k-2$, therefore this N-point Gauss-Lobatto quadrature is exact for polynomial of degree $2k-2$. We denote these quadrature points on I_j as

$$S_j = \left\{ x_{j-\frac{1}{2}} = \widehat{x}_j^1, \widehat{x}_j^2, \cdots, \widehat{x}_j^{N-1}, \widehat{x}_j^N = x_{j+\frac{1}{2}} \right\}.$$

Let $\widehat{w}_r$ be the quadrature weights for the interval $[-1/2, 1/2]$ such that $\sum_{r=1}^N \widehat{w}_r = 1$. Since $p_j(x)$ is polynomial of degree $2k-2$ and this quadrature is exact, we have

$$\overline{h}_j^n = \frac{1}{\Delta x}\int_{I_j} p_j(x)dx = \sum_{r=1}^N \widehat{w}_r p_j(\widehat{x}_j^r) = \sum_{r=2}^{N-1} \widehat{w}_r p_j(\widehat{x}_j^r) + \widehat{w}_1 h^+_{j-\frac{1}{2}} + \widehat{w}_N h^-_{j+\frac{1}{2}}. \tag{3.25}$$

If we introduce the variable

$$\xi_j = \frac{1}{\sum_{r=2}^{N-1}\hat{w}_r}\sum_{t=2}^{N-1}\hat{w}_r p_j(\hat{x}_j^r) = \frac{\overline{h}_j^n - \hat{w}_1 h_{j-\frac{1}{2}}^+ - \hat{w}_N h_{j+\frac{1}{2}}^-}{1-\hat{w}_1-\hat{w}_N}, \tag{3.26}$$

we have

$$\overline{h}_j^n = (1-\hat{w}_1-\hat{w}_N)\xi_j + \hat{w}_1 h_{j-\frac{1}{2}}^+ + \hat{w}_N h_{j+\frac{1}{2}}^-. \tag{3.27}$$

Following the approaches in [43, 61, 65, 67], we have the following result.

Proposition 3.1. Consider the scheme (3.20) satisfied by the cell averages of the water height. Let ξ_j be defined in (3.26). If $h_{j-1/2}^{\pm}$, $h_{j+1/2}^{\pm}$ and ξ_j are all non-negative, then $\overline{h}_j^{n+1}$ is also non-negative under the CFL condition

$$\lambda\alpha \le \hat{w}_1. \tag{3.28}$$

This proposition tells us that for the scheme (3.20), we need to modify $p_j(x)$ (satisfying (3.24)) such that $p_j(x_{j\pm1/2})$ and ξ_j are all non-negative. At time level n, given $\overline{h}_j^n \ge 0$, we consider the following limiter on the piecewise polynomial $p_j(x)$ introduced in [67]. It is a linear scaling around the cell average:

$$\widetilde{p}_j(x) = \theta\left(p_j(x) - \overline{h}_j^n\right) + \overline{h}_j^n, \qquad \theta = \min\left\{1, \frac{\overline{h}_j^n}{\overline{h}_j^n - m_j}\right\}, \tag{3.29}$$

with

$$m_j = \min_{x\in I_j} p_j(x). \tag{3.30}$$

It is easy to observe that the conditions of Proposition 3.1 are satisfied with this limiter. Moreover, it can also be shown that this limiter does not destroy the high-order accuracy, and we refer to [67] for the detailed proof. Let $\widetilde{h}_{j-1/2}^+ = \widetilde{p}_j(x_{j-1/2})$, $\widetilde{h}_{j+1/2}^- = \widetilde{p}_j(x_{j+1/2})$, and define $\widetilde{h}_{j-1/2}^{*,+}$, $\widetilde{h}_{j+1/2}^{*,-}$ following (3.8). Then, the revised positivity-preserving version of the scheme (3.20) takes the form

$$\overline{h}_j^{n+1} = \overline{h}_j^n - \lambda\left[\widehat{F}\left(\widetilde{h}_{j+\frac{1}{2}}^{*,-}, u_{j+\frac{1}{2}}^-; \widetilde{h}_{j+\frac{1}{2}}^{*,+}, u_{j+\frac{1}{2}}^+\right) - \widehat{F}\left(\widetilde{h}_{j-\frac{1}{2}}^{*,-}, u_{j-\frac{1}{2}}^-; \widetilde{h}_{j-\frac{1}{2}}^{*,+}, u_{j-\frac{1}{2}}^+\right)\right]. \tag{3.31}$$

Notice that in (3.30) we need to evaluate the minimum of the reconstructed polynomial $p_j(x)$. We prefer to avoid an explicit construction of this additional reconstruction polynomial $p_j(x)$, and propose to replace (3.30) by

$$m_j = \min(h_{j-\frac{1}{2}}^+, h_{j+\frac{1}{2}}^-, \xi_j). \tag{3.32}$$

Since ξ_j can be computed by (3.26) easily, it is very easy to evaluate m_j. We have shown in [61] that the limiter (3.29) and (3.32) is a high-order accurate positivity-preserving limiter, and preserves the conservation of $p_j(x)$. We now have the following proposition.

Proposition 3.2. Consider the revised numerical scheme (3.31), with the positivity-preserving limiter (3.29), (3.32), i.e.

$$\widetilde{h}^+_{j-\frac{1}{2}} = \theta\left(h^+_{j-\frac{1}{2}} - \overline{h}^n_j\right) + \overline{h}^n_j, \qquad \widetilde{h}^-_{j+\frac{1}{2}} = \theta\left(h^-_{j+\frac{1}{2}} - \overline{h}^n_j\right) + \overline{h}^n_j, \tag{3.33}$$

with θ computed in (3.32). Suppose the well-balanced flux (3.7) is used, with $\widetilde{h}^{*,+}_{j-\frac{1}{2}}$, $\widetilde{h}^{*,-}_{j+\frac{1}{2}}$ computed following (3.8). This method is $(2k-1)$-th order accurate, positivity-preserving and conserves the mass, under the CFL condition (3.28). For a fifth-order WENO scheme with $k=3$, this CFL condition is $\lambda\alpha \le 1/12$.

We would like to mention that in wet regions, where m_j is $O(1)$ above zero, the limiter does not effect, i.e., $\widetilde{p}_j(x)=p_j(x)$. Therefore this positivity-preserving limiter is active only in the dry or nearly dry region. For high order time discretizations, we need to apply the limiter in each stage for a Runge-Kutta method or in each step for a multistep method. To be efficient, we could implement the time step restriction (3.28) only when a preliminary calculation to the next time step produces negative water height.

4 Finite element discontinuous Galerkin methods

Discontinuous Galerkin methods have been actively applied to hyperbolic conservation laws, especially the SWEs recently. In this section, we recall the positivity-preserving high order well-balanced finite element DG schemes developed by us in [55, 59, 60, 64, 65]. Similarly as in finite volume methods, we first present two approaches to achieve well-balanced methods for the still water steady state (1.3), followed by high order well-balanced methods for the moving water equilibrium (1.2). At the end, we introduce the simple positivity-preserving limiter, as well as the extension to unstructured triangular meshes.

We start by presenting the standard notations. In a high order DG method, we seek an approximation, still denoted by U for the ease of presentation with an abuse of notation, which belongs to the finite dimensional space

$$V_{\Delta x} = V^k_{\Delta x} \equiv \{w : w|_{I_j} \in P^k(I_j), j=1,...,J\}, \tag{4.1}$$

where $P^k(I)$ denotes the space of polynomials of degree at most k and J is the total number of computational cells. We project the bottom function b into the same space $V_{\Delta x}$, to obtain an approximation which is still denoted by b, again with an abuse of notation. The standard DG method is given by

$$\int_{I_j} \partial_t U v dx - \int_{I_j} f(U)\partial_x v dx + \widehat{f}_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} s(h,b) v dx, \tag{4.2}$$

where $v(x)$ is a test function from the test space $V_{\Delta x}$, $\widehat{f}_{j+1/2} = F(U(x^-_{j+1/2},t), U(x^+_{j+1/2},t))$ and $F(a_1,a_2)$ is a numerical flux.

4.1 Well-balanced methods for the still water

We are interested in preserving the still water stationary solution (1.3) exactly. Since there are a lot of similarities between finite volume and finite element methods, we follow the approaches in Section 3.1 and present two different ways to achieve such goal. The first approach focuses on a non-standard discretization of the source term, by following the idea of decomposing the source terms as shown in Section 2 and 3.1. The second approach employs the idea of hydrostatic reconstruction [1] to modify the approximation of the numerical flux and keep a simple source term approximation. We notice that the traditional DG methods are capable of maintaining certain steady states exactly, if a small modification on the numerical flux is provided. The computational cost to obtain such a well-balanced DG method is basically the same as the traditional DG method.

The main idea in the section 3.1 to design a well-balanced high-order finite volume WENO scheme is to decompose the source term into a sum of three terms in (3.3). Similarly, we decompose the integral of the source term on the right-hand side of (4.2) as:

$$\begin{aligned}\int_{I_j} -ghb_x v dx &= \int_{I_j} \left(\frac{1}{2}gb^2\right)_x v dx - g\overline{(h+b)_j}\int_{I_j} b_x v dx - \int_{I_j} g\left(h+b-\overline{(h+b)_j}\right) b_x v dx \\ &= \left(\frac{1}{2}gb^2\right)(x_{j+\frac{1}{2}})v(x^-_{j+\frac{1}{2}}) - \left(\frac{1}{2}gb^2\right)(x_{j-\frac{1}{2}})v(x^+_{j-\frac{1}{2}}) - \int_{I_j}\frac{1}{2}gb^2 v_x dx \\ &\quad - g\overline{(h+b)_j}\left(b(x_{j+\frac{1}{2}})v(x^-_{j+\frac{1}{2}}) - b(x_{j-\frac{1}{2}})v(x^+_{j-\frac{1}{2}}) - \int_{I_j} bv_x dx\right) \\ &\quad - \int_{I_j} g(h+b-\overline{(h+b)_j})b_x v dx, \end{aligned} \tag{4.3}$$

We then replace this source term with a high order approximation of it given by

$$\begin{aligned} s_j = &\left\{\frac{1}{2}gb^2_{j+\frac{1}{2}}\right\}v(x^-_{j+\frac{1}{2}}) - \left\{\frac{1}{2}gb^2_{j-\frac{1}{2}}\right\}v(x^+_{j-\frac{1}{2}}) - \int_{I_j}\frac{1}{2}gb^2 v_x dx \\ &- g\overline{(h+b)_j}\left(\{b_{j+\frac{1}{2}}\}v(x^-_{j+\frac{1}{2}}) - \{b_{j-\frac{1}{2}}\}v(x^+_{j-\frac{1}{2}}) - \int_{I_j} bv_x dx\right) - \int_{I_j} g(h+b-\overline{(h+b)_j})b_x v dx. \end{aligned} \tag{4.4}$$

where $\left\{gb^2_{j+1/2}/2\right\}$ and $\{b_{j+1/2}\}$ are approximations to $ghb^2/2$ and b at $x_{j+1/2}$, similarly as in (3.3). Combined with the semi-discrete form (4.2), this gives our well-balanced high order DG schemes. Usually, we perform the limiter on the function U after each Runge-Kutta stage. Note that the slope limiter procedure could destroy the preservation of still water steady state, since if the limiter is enacted, the resulting modified solution h may no longer satisfy $h+b=constant$. We therefore propose to first check whether any limiting is needed based on the function $h+b$ in each Runge-Kutta stage. If a certain cell is flagged by this procedure needing limiting, then the actual limiter is implemented on h, not on $h+b$, so that the slope limiter will not conflict with the well-balanced property. The well-balanced property for the still water (1.3) can be easily proved, and we refer to [59] for the details.

A different approach to achieve well-balanced property is to utilize the hydrostatic reconstruction idea in the numerical flux. As mentioned in [60], our well-balanced numerical scheme has the form:

$$\int_{I_j} \partial_t U v dx - \int_{I_j} f(U)\partial_x v dx + \widehat{f}^l_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}^r_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} s(h,b) v dx, \tag{4.5}$$

or equivalently,

$$\begin{aligned} &\int_{I_j} \partial_t U v dx - \int_{I_j} f(U)\partial_x v dx + \widehat{f}_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) \\ &\quad = \int_{I_j} s(h,b) v dx + (\widehat{f}_{j+\frac{1}{2}} - \widehat{f}^l_{j+\frac{1}{2}}) v(x^-_{j+\frac{1}{2}}) - (\widehat{f}_{j-\frac{1}{2}} - \widehat{f}^r_{j-\frac{1}{2}}) v(x^+_{j-\frac{1}{2}}). \end{aligned} \tag{4.6}$$

The left side of (4.6) is the traditional DG scheme, and the right side is our approximation to the source term. The design of the left flux $\widehat{f}^l_{j+1/2}$ and the right flux $\widehat{f}^r_{j-1/2}$ is the same as in the finite volume methods and take the form of (3.6).

We also require that all the integrals in formula (4.5) should be calculated exactly at the still water state. This can be easily achieved by using suitable Gauss-quadrature rules since the numerical solutions h, b and v are polynomials at the still water state in each cell I_j, hence $f(U)$ and $s(h,b)$ are both polynomials. We have proven in [60] that the above methods (4.5), combined with the choice of fluxes (3.6), are actually well-balanced for the still water steady state of the SWEs.

Remark 4.1. If we enforce the projection of the bottom bathymetry into the piecewise polynomial space to be continuous, i.e., $b^-_{j+1/2} = b^+_{j+1/2}$, (which can be done using the idea of essentially non-oscillatory (ENO) procedure to interpolate the polynomial based on the values $b_{j+1/2}$, see [60] for the details), we can show that the left and right fluxes reduces to the numerical flux of traditional DG methods:

$$\hat{f}^l_{j+\frac{1}{2}} = \hat{f}_{j+\frac{1}{2}}, \qquad \hat{f}^r_{j-\frac{1}{2}} = \hat{f}_{j-\frac{1}{2}}.$$

This make our well-balanced scheme (4.5) to be identical to the traditional DG methods without any modification. Unfortunately, although it works well for small perturbation solutions from still water for a smooth bottom, the numerical resolution for a discontinuous bottom is not ideal.

4.2 Well-balanced methods for the moving water

In this section, we design high-order finite element DG methods for the SWEs (1.1), with the objective to maintain the general moving steady state (1.2). The basic framework of the well-balanced scheme follows the one introduced in Section 4.1. However, extra attention is required to handle the flux and source term approximation due to the complexity of moving water equilibrium.

The main structure of well-balanced methods for moving water equilibrium (1.2) follows the one (4.5) for still water. The important ingredient in designing well-balanced methods is the approximation of the source term integral in (4.5), which will be explained below. As explained in Section 3.2, we could define the transformation between the conservative variables U into the equilibrium variables V, as well as the recovery of well-balanced states $\overline{V}_j = (\bar{m}_j, \bar{E}_j)$. The reference equilibrium values $\overline{V}_j$ lead to the reference equilibrium functions $U(\overline{V}_j, b(x))$. Since they may not be polynomials, we consider their L^2 projection into the finite element space $V_{\Delta x}$ which was introduced in (4.1), and denote it by

$$U_j^e(x) = (h_j^e(x), m_j^e(x)) = PU(\overline{V}_j, b(x)), \tag{4.7}$$

in each cell I_j, where P denotes the L^2 projection operator. Therefore, the numerical solutions U, which are piecewise polynomials, can be decomposed as

$$U = U^e + U^r, \tag{4.8}$$

where $U^r = U - U^e \in V_{\Delta x}$. The source term approximation now becomes

$$\int_{I_j} s(h,b)v dx = \int_{I_j} s(h^e,b)v dx + \int_{I_j} s(h^r,b)v dx, \tag{4.9}$$

since $s(h,b) = -ghb_x$ is linear with respect to h. The second term on the right hand side of (4.9) can be computed by the standard quadrature rule. Next, let us discuss how to approximate the first term numerically. Noticing the fact that $U(\overline{V}_j, b) = (h(\overline{V}_j, b), \bar{m}_j)^T$ is the equilibrium state, we have the relationship

$$\int_{I_j} s(h(\overline{V}_j,b),b)v dx = -\int_{I_j} f(U(\overline{V}_j,b))v_x dx + f(U(\overline{V}_j, b^-_{j+\frac{1}{2}}))v^-_{j+\frac{1}{2}} - f(U(\overline{V}_j, b^+_{j-\frac{1}{2}}))v^+_{j-\frac{1}{2}}.$$

Since U^e is the L^2 projection of $U(\overline{V}_j, b)$, we conclude that

$$\int_{I_j} s(h^e,b)v dx + O(\Delta x^{k+1}) = -\int_{I_j} f(U^e)v_x dx + f(U^{e,-}_{j+\frac{1}{2}})v^-_{j+\frac{1}{2}} - f(U^{e,+}_{j-\frac{1}{2}})v^+_{j-\frac{1}{2}}, \tag{4.10}$$

and can approximate the source term integral (4.9) by

$$\int_{I_j} s(h,b)v dx \approx -\int_{I_j} f(U^e)v_x dx + f(U^{e,-}_{j+\frac{1}{2}})v^-_{j+\frac{1}{2}} - f(U^{e,+}_{j-\frac{1}{2}})v^+_{j-\frac{1}{2}} + \int s(h^r,b)v dx. \tag{4.11}$$

Since U^e is always smooth inside a cell, the relation (4.10) is always true regardless of the smoothness of the solution U. Therefore, numerical methods with this source term approximation (4.11) will satisfy the Lax-Wendroff theorem and converge to the weak solution.

Next, we discuss an important and last piece of our method, namely the well-balanced numerical fluxes. They are computed by a generalized hydrostatic reconstruction. At

each time step t^n, one can compute the cell boundary values $U^{\pm}_{j+1/2}$ from the solution $U(x)$. But in the case of moving water equilibrium, suppose $U(x)$ are computed from the exact solution, these cell boundary values $U^{\pm}_{j+1/2}$ are not equal to the exact solution value at the same point, as $U(x)$ is the projection of the exact solution into the polynomial space and this projection does not preserve the equilibrium state. To overcome this problem, we redefine an updated boundary value as:

$$\tilde{U}^{\pm}_{j+\frac{1}{2}} = U(\overline{V}_j, b^{\pm}_{j+\frac{1}{2}}) + U^{r,\pm}_{j+\frac{1}{2}}, \tag{4.12}$$

where U^r is defined in (4.8). One can easily verify that $\tilde{U}^{\pm}_{j+1/2} = U(\overline{V}_j, b^{\pm}_{j+1/2})$ in the case of moving water equilibrium. We follow the idea of hydrostatic reconstruction to compute the numerical fluxes and define

$$\tilde{V}^{\pm}_{j+\frac{1}{2}} = V(\tilde{U}^{\pm}_{j+\frac{1}{2}}, b^{\pm}_{j+\frac{1}{2}}), \tag{4.13}$$

and

$$b^*_{j+\frac{1}{2}} = \begin{cases} \max(b^+_{j+\frac{1}{2}}, b^-_{j+\frac{1}{2}}), & \text{if } \sigma_{i+\frac{1}{2}} = -1, 0, \\ \min(b^+_{j+\frac{1}{2}}, b^-_{j+\frac{1}{2}}), & \text{if } \sigma_{i+\frac{1}{2}} = 1. \end{cases} \tag{4.14}$$

The cell boundary values (used to evaluate the numerical fluxes) are then defined by:

$$U^{*,\pm}_{j+\frac{1}{2}} = \left(\max\left(0, h(\tilde{V}^{\pm}_{j+\frac{1}{2}}, b^*_{j+\frac{1}{2}})\right), \tilde{m}^{\pm}_{j+\frac{1}{2}}\right)^T = \left(\max\left(0, h(\tilde{V}^{\pm}_{j+\frac{1}{2}}, b^*_{j+\frac{1}{2}})\right), m^{\pm}_{j+\frac{1}{2}}\right)^T, \tag{4.15}$$

as one can easily observe that $\tilde{m}^{\pm}_{j+1/2} = m^{\pm}_{j+1/2}$. At the end, the left and right fluxes $\widehat{f}^l_{j+1/2}$, $\widehat{f}^r_{j-1/2}$ are given by:

$$\begin{aligned} \widehat{f}^l_{j+\frac{1}{2}} &= F(U^{*,-}_{j+\frac{1}{2}}, U^{*,+}_{j+\frac{1}{2}}) + f(U^-_{j+\frac{1}{2}}) - f(U^{*,-}_{j+\frac{1}{2}}), \\ \widehat{f}^r_{j-\frac{1}{2}} &= F(U^{*,-}_{j-\frac{1}{2}}, U^{*,+}_{j-\frac{1}{2}}) + f(U^+_{j-\frac{1}{2}}) - f(U^{*,+}_{j-\frac{1}{2}}). \end{aligned} \tag{4.16}$$

This completes the well-balanced DG methods for the moving water (1.2), and we have proven their well-balanced property in [55].

4.3 Positivity-preserving limiters and extension to triangular unstructured meshes

In the previous sections, we have described our well-balanced methods in the one dimensional setting to illustrate the main idea. They also work on the two dimensional triangular unstructured meshes. In this section, we present a simple positivity-preserving limiter on triangular meshes, and couple it with well-balanced DG methods developed for the SWEs in Section 4.1. Once again, for the ease of presentation, Euler forward time

discretization will be discussed, but all the results hold for the TVD high order Runge-Kutta and multi-step time discretizations.

The two-dimensional shallow water equations take the form

$$\begin{cases} h_t+(hu)_x+(hv)_y=0 \\ (hu)_t+\left(hu^2+\frac{1}{2}gh^2\right)_x+(huv)_y=-ghb_x \\ (hv)_t+(huv)_x+\left(hv^2+\frac{1}{2}gh^2\right)_y=-ghb_y, \end{cases} \tag{4.17}$$

where $(u,v)^T$ is the velocity vector. Let $\mathcal{T}_\tau$ be a family of partitions of the computational domain Ω parameterized by $\tau>0$. For any triangle $K\in\mathcal{T}_\tau$, we define $\tau_K:=\text{diam}(K)$ and $\tau:=\max_{K\in\mathcal{T}_\tau}\tau_K$. For each edge e_K^i $(i=1,2,3)$ of K, we denote its length by l_K^i, and outward unit normal vector by ν_K^i. Let $K(i)$ be the neighboring triangle along the edge e_K^i and $|K|$ be the area of the triangle K. For the ease of presentation, we denote the shallow water equations (1.1) by

$$U_t+f(U)_x+g(U)_y=s(h,b), \quad \text{or} \quad U_t+\nabla\cdot\mathbf{F}(U)=s(h,b),$$

where $U=(h,hu,hv)^T$ with the superscript T denoting the transpose, $f(U)$, $g(U)$ or $\mathbf{F}(U)=(f(U),g(U))$ are the flux and $s(h,b)$ is the source term.

Let $\mathbf{x}$ denote (x,y), the DG scheme is given by

$$\iint_K \partial_t U w\,\mathrm{d}\mathbf{x}-\iint_K \mathbf{F}(U)\cdot\nabla w\,\mathrm{d}\mathbf{x}+\sum_{i=1}^3\int_{e_K^i}\widehat{\mathbf{F}}|_{e_K^i}\cdot\nu_K^i w\,\mathrm{d}s=\iint_K s(h,b)w\,\mathrm{d}\mathbf{x}, \tag{4.18}$$

where $w(\mathbf{x})$ is a test function, and the numerical flux $\widehat{\mathbf{F}}$ is defined by

$$\widehat{\mathbf{F}}|_{e_K^i}\cdot\nu_K^i=\mathcal{F}(U_i^{int(K)},U_i^{ext(K)},\nu_K^i). \tag{4.19}$$

where $U_i^{int(K)}$ and $U_i^{ext(K)}$ are the approximations to the values on the edge e_K^i obtained from the interior and the exterior of K. We could, for example, use the simple global Lax-Friedrichs flux

$$\mathcal{F}(a_1,a_2,\nu)=\frac{1}{2}[\mathbf{F}(a_1)\cdot\nu+\mathbf{F}(a_2)\cdot\nu-\alpha(a_2-a_1)], \quad \alpha=\max\left((|u|+\sqrt{gh},|v|+\sqrt{gh})\cdot\nu\right), \tag{4.20}$$

where the maximum is taken over the whole region.

For convenience, let $\mathcal{F}_1$ and $\widehat{\mathbf{F}}_1^*|_{e_K^i}\cdot\nu_K^i$ denote the first components of $\mathcal{F}$ and $\widehat{\mathbf{F}}^*|_{e_K^i}\cdot\nu_K^i$ respectively. Then $\widehat{\mathbf{F}}_1^*|_{e_K^i}\cdot\nu_K^i=\mathcal{F}_1(U_i^{*,int(K)},U_i^{*,ext(K)},\nu_K^i)$ by (3.6). Taking the test function as $w\equiv1$ in (4.18), we get the the scheme satisfied by the cell averages for the water height h:

$$\overline{h}_K^{n+1}=\overline{h}_K^n-\frac{\Delta t}{|K|}\sum_{i=1}^3\int_{e_K^i}\mathcal{F}_1(U_i^{*,int(K)},U_i^{*,ext(K)},\nu_K^i)\,\mathrm{d}s. \tag{4.21}$$

Suppose we use L-point Gaussian quadrature for the line integral in (4.21), and the subscript (i,β) will denote the point value at the β-th quadrature point of the i-th edge. Let w_β denote the Gauss quadrature weight on $[-1/2,1/2]$. Then (4.21) becomes

$$\overline{h}_K^{n+1}=\overline{h}_K^n-\frac{\Delta t}{|K|}\sum_{i=1}^{3}\sum_{\beta=1}^{L}\mathcal{F}_1(U_{i,\beta}^{*,int(K)},U_{i,\beta}^{*,ext(K)},\nu_K^i)w_\beta l_K^i. \tag{4.22}$$

To investigate the positivity of a high order scheme (4.22), we need to study its first order counterpart and have the following results:

Lemma 4.1. *Under the CFL condition* $\frac{\Delta t}{|K|}\alpha\sum_{i=1}^{3}l_K^i\le 1$, *with*

$$\alpha=\max\Big((|u|+\sqrt{gh},|v|+\sqrt{gh})\cdot\nu\Big), \tag{4.23}$$

if h_K^n *is non-negative for any* K, *then* h_K^{n+1} *is non-negative in the first order scheme*

$$h_K^{n+1}=h_K^n-\frac{\Delta t}{|K|}\sum_{i=1}^{3}\mathcal{F}_1(U_i^{*,int(K)},U_i^{*,ext(K)},\nu_K^i)l_K^i, \tag{4.24}$$

where

$$U_i^{*,int(K)}=\frac{h_K^{*,i}}{h_K^n}U_K^n,\quad U_i^{*,ext(K)}=\frac{h_{K(i)}^{*,i}}{h_{K(i)}^n}U_{K(i)}^n,$$

with

$$\begin{aligned}h_K^{*,i}&=\max\Big(0,h_K^n+b_K-\max(b_K,b_{K(i)})\Big),\\ h_{K(i)}^{*,i}&=\max\Big(0,h_{K(i)}^n+b_{K(i)}-\max(b_K,b_{K(i)})\Big).\end{aligned}$$

Following the approach in [64], we introduce a special quadrature rule satisfying: it is exact for integration of $h_K(x,y)$ on K; it include all L-point Gauss quadrature points for each edge e_K^i; and all the quadrature weights should be positive. This particular quadrature rule has been constructed by a transformation of the tensor product of M-point Gauss-Lobatto and L-point Gauss quadrature (see [64] for details). The set of quadrature points S_K for the P^2-DG method are shown in Fig 1 as an example. Again, we would like to comment that this special quadrature rule is used only for the purpose of introducing the positivity-preserving limiters, and we can use any quadrature rule in computing the integral in (4.18). We have the following results for high-order DG methods:

Proposition 4.1. For the scheme (4.22) to be positivity preserving, i.e., $\overline{h}_K^{n+1}\ge 0$, a sufficient condition is that $h_K(\mathbf{x})\ge 0,\forall\mathbf{x}\in S_K$ for all K, under the CFL condition

$$\alpha\frac{\Delta t}{|K|}\sum_{i=1}^{3}l_K^i\le\frac{2}{3}\widehat{w}_1. \tag{4.25}$$

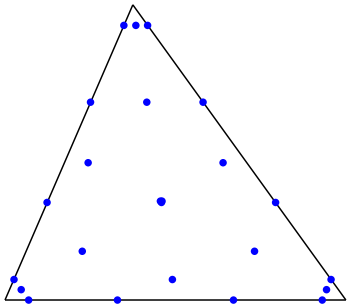

Figure 1: The quadrature points on a triangle for P^2 polynomials. There are 24 distinct points.

Here $h_K(\mathbf{x})$ denotes the polynomial for water height at time level n, $\widehat{w}_1$ is the quadrature weight of the M-point Gauss-Lobatto rule on $[-1/2,1/2]$ for the first quadrature point. For $k=2,3$, $\widehat{w}_1=1/6$ and for $k=4,5$, $\widehat{w}_1=1/12$.

At time level n, given the water height DG polynomial $h_K(\mathbf{x})$ with its cell average $\overline{h}_K^n \geq 0$, to enforce the sufficient condition $h_K(\mathbf{x}) \geq 0, \forall \mathbf{x} \in S_K$, the limiter in [64, 65, 67] can be used directly, i.e., replacing $h_K(\mathbf{x})$ by a linear scaling around the cell average:

$$\widetilde{h}_K(\mathbf{x}) = \theta_K(h_K(\mathbf{x}) - \overline{h}_K^n) + \overline{h}_K^n, \tag{4.26}$$

where $\theta_K \in [0,1]$ is determined by

$$\theta_K = \min_{\mathbf{x}\in S_K} \theta_{\mathbf{x}}, \quad \theta_{\mathbf{x}} = \min\left\{1, \frac{\overline{h}_K^n}{\overline{h}_K^n - h_K(\mathbf{x})}\right\}. \tag{4.27}$$

This limiter is conservative (the cell average of $\widetilde{p}_K$ is still $\overline{h}_K^n$), positivity-preserving ($\widetilde{h}_K(\mathbf{x}) \geq 0, \forall \mathbf{x} \in S_K$) and high order accurate.

Let $\overline{S}_K$ denote the set of the points in S_K that lie in the interior of the triangle K, and $\widetilde{S}_K$ be the points which lie on the edges of K. As mentioned in [61, 68], for those points in $\overline{S}_K$, instead of requiring $h_K(\mathbf{x}) \geq 0, \forall \mathbf{x} \in \overline{S}_K$, it suffices to require $\sum_{\mathbf{x}\in\overline{S}_K} h_K(\mathbf{x}) w_{\mathbf{x}} \geq 0$ to have positivity of $\overline{h}_K^{n+1}$. Notice that $\sum_{\mathbf{x}\in\overline{S}_K} h_K(\mathbf{x}) w_{\mathbf{x}} / \sum_{\mathbf{x}\in\overline{S}_K} w_{\mathbf{x}}$ is a convex combination of point values of $h_K(\mathbf{x})$, thus by the Mean Value Theorem, there exists some point $\mathbf{x}^* \in K$ such that

$$h_K(\mathbf{x}^*) = \frac{1}{\sum_{\mathbf{x}\in\overline{S}_K} w_{\mathbf{x}}} \sum_{\mathbf{x}\in\overline{S}_K} h_K(\mathbf{x}) w_{\mathbf{x}}.$$

An alternative limiter is to enforce this relaxed condition, and we can replace (4.27) with

$$\theta_K = \min\left\{\theta_{\mathbf{x}^*}, \min_{\mathbf{x}\in\widetilde{S}_K} \theta_{\mathbf{x}}\right\}, \quad \theta_{\mathbf{x}} = \min\left\{1, \frac{\overline{h}_K^n}{\overline{h}_K^n - h_K(\mathbf{x})}\right\}, \tag{4.28}$$

to save time in evaluating θ_K.

5 Conclusion remarks

In this paper we gave an overview of some recently developed high-order positivity-preserving well-balanced schemes, including finite difference, finite volume WENO schemes and finite element DG methods, for the SWEs. These schemes maintain well-balanced properties for certain steady state solutions, the genuine high order accuracy in smooth regions, and the robust simulation near the wet/dry front. The well-balanced approaches are quite general, and could be extended to other hyperbolic conservation laws with source terms, for example the Euler equations with a gravitational field [62]. The presentation and discussion of the construction principles should enable the reader to implement them and develop them further for an application at hand.

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