A brief survey on discontinuous Galerkin methods in computational fluid dynamics[1]

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ABSTRACT

Discontinuous Galerkin (DG) methods combine features in finite element methods (weak formulation, finite dimensional solution and test function spaces) and in finite volume methods (numerical fluxes, nonlinear limiters) and are particularly suitable for simulating convection dominated problems, such as linear and nonlinear waves including shock waves. In this article we will give a brief survey of DG methods, emphasizing their applications in computational fluid dynamics (CFD). We will discuss essential ingredients and properties of DG methods, and will also give a few examples of recent developments of DG methods which have facilitated their applications in CFD.

1 Introduction

Discontinuous Galerkin (DG) methods belong to the class of finite element methods. They are based on weak formulations and with finite dimensional piecewise polynomial solution space and test function space. The main difference with traditional finite element methods is that the finite element function space corresponding to DG methods consists of piecewise polynomials (or other simple functions) *which are allowed to be completely discontinuous*

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across element interfaces. Since this leads to ambiguities at element interfaces, the technique from finite volume methodology, namely the choice of *numerical fluxes*, is introduced into the DG schemes. Another important technique from the finite volume methodology, namely the choice of nonlinear limiters to control spurious oscillations in the presence of strong discontinuities, is also introduced into the DG schemes. From this point of view, DG schemes can be considered as hybrid finite element and finite volume schemes.

The first DG method was introduced in 1973 by Reed and Hill in a Los Alamos technical report [53]. It solves the equations for neutron transport, which are time independent linear hyperbolic equations. A major development of the DG method was carried out by Cockburn et al. in a series of papers [18, 17, 16, 14, 19], in which the authors have established a framework to easily solve *nonlinear* time dependent hyperbolic equations, such as the Euler equations of compressible gas dynamics. The DG method of Cockburn et al. belongs to the class of method-of-lines, namely the DG discretization is used only for the spatial variables, and explicit, nonlinearly stable high order Runge-Kutta methods [60, 27] are used to discretize the time variable. The two techniques from the finite volume methodology mentioned above, namely the usage of exact or approximate Riemann solvers as interface fluxes and total variation bounded (TVB) nonlinear limiters [58] to achieve non-oscillatory properties for strong shocks, were also introduced into the DG method of Cockburn et al. At the beginning, applications of DG methods to computational fluid dynamics (CFD) were mainly for solving Euler equations of compressible gas dynamics. Later, the DG methodology was generalized to treat viscous terms as well and hence the DG schemes were designed to solve Navier-Stokes equations [2, 20]. Within the umbrella of CFD, the DG methods have also been applied to areas including aeroacoustics, granular flows, magneto-hydrodynamics, meteorology, modeling of shallow water, oceanography, transport of contaminant in porous media, turbulent flows, viscoelastic flows and weather forecasting, among many others. For earlier work on DG methods, we refer to the survey paper [15], and other papers in that Springer volume, which contains the conference proceedings of the First International Sym-

posium on Discontinuous Galerkin Methods held at Newport, Rhode Island in 1999. The lecture notes [13] is a good reference for many details, as well as the extensive review paper [21]. The review paper [65] covers the local DG method for partial differential equations (PDEs) containing higher order spatial derivatives, such as Navier-Stokes equations. More recently, there are three special journal issues devoted to the DG method [22, 24, 23], which contain many interesting papers on DG method in all aspects including algorithm design, analysis, implementation and applications. There are also a few recent books and lecture notes [30, 36, 40, 55, 59] on DG methods.

2 DG methods for hyperbolic conservation laws

In this section we describe briefly the main idea of DG methods for solving hyperbolic conservation laws. In CFD such equations include Euler equations of compressible gas dynamics, MHD equations, linearized Euler equations in aeroacoustics, shallow water equations, etc. We use the following one-dimensional scalar equation

$$u_t + f(u)_x = 0 \tag{2.1}$$

as an example.

Let us first settle on some notations. Assuming we are solving (2.1) in $x \in [0,1]$. We divide [0,1] into N cells

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1,$$

and denote

$$I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right), \quad x_j = \frac{1}{2}\left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right), \quad h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$$

as the cells, cell centers and cell lengths respectively. We also define $h = h_{\max} = \max_j h_j$ and $h_{\min} = \min_j h_j$, and we consider only regular meshes, that is $h_{\max} \le \lambda h_{\min}$ where $\lambda \ge 1$ is a constant during mesh refinement. Define the discontinuous Galerkin finite element space as

$$V_h^k = \{v : v|_{I_j} \in \mathcal{P}^k(I_j), j = 1, \cdots, N\}, \tag{2.2}$$

where $\mathcal{P}^k(I_j)$ denotes the space of polynomials in I_j of degree at most k. This polynomial degree k can actually change from cell to cell (p-adaptivity), but we assume it is a constant in this article for simplicity.

The semi-discrete DG method for solving (2.1) is defined as follows: find the unique function $u_h = u_h(t) \in V_h^k$ such that, for all test functions $v_h \in V_h^k$ and all $1 \le j \le N$, we have

$$\int_{I_j} (u_h)_t \, v_h dx - \int_{I_j} f(u_h)(v_h)_x dx + \hat{f}_{j+\frac{1}{2}}(v_h)^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}(v_h)^+_{j-\frac{1}{2}} = 0. \tag{2.3}$$

Here, $\hat{f}_{i+\frac{1}{2}}$ is the numerical flux, which is a single-valued function defined at the cell interfaces and in general depends on the values of the numerical solution u_h from both sides of the interface

$$\hat{f}_{i+\frac{1}{2}} = \hat{f}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t)). \tag{2.4}$$

We use the so-called monotone fluxes from finite volume schemes, which satisfy the following conditions:

- Consistency: $\hat{f}(u, u) = f(u)$;
- Continuity: $\hat{f}(u^-, u^+)$ is at least Lipschitz continuous with respect to both arguments u^- and u^+.
- Monotonicity: $\hat{f}(u^-, u^+)$ is a non-decreasing function of its first argument u^- and a non-increasing function of its second argument u^+. Symbolically $\hat{f}(\uparrow, \downarrow)$.

We refer to, e.g., [39] for more details about monotone fluxes.

Notice that the DG scheme (2.3) can also be written as

$$\int_{I_j} ((u_h)_t + f(u_h)_x) \, v_h dx + (\hat{f}_{j+\frac{1}{2}} - f((u_h)^-_{j+\frac{1}{2}})(v_h)^-_{j+\frac{1}{2}} - (\hat{f}_{j-\frac{1}{2}} - f((u_h)^+_{j-\frac{1}{2}})(v_h)^+_{j-\frac{1}{2}} = 0 \tag{2.5}$$

through integration by parts. Mathematically the two formulations (2.3) and (2.5) are equivalent, so users can choose to implement any one of them and will get the same result. However, if the integral terms in (2.3) and (2.5) are approximated by numerical quadrature

rules, then the two formulations may no longer be equivalent. Some of the variants of DG schemes, for example the CPR scheme [62], can be considered as scheme (2.5) with the integral term replaced by a numerical quadrature.

If (2.1) is a system of hyperbolic conservation laws, the formulation of the DG scheme is the same as in the scalar case, except that the numerical flux (2.4) is no longer a monotone flux but a flux based on an exact or approximate Riemann solver [61].

Since the DG scheme is defined in a weak form, its multi-dimensional version is similar to its one-dimensional version, with integration by parts replaced by Green's formulas. The numerical fluxes are still one-dimensional, along the normal direction of element interfaces. Elements can be of any shape, and mixed type meshes (e.g. both triangles and quadrilateral cells in 2D) can be easily accommodated. The method is thus easy to handle complicated geometry and boundary conditions.

Time discretization is usually through explicit Runge-Kutta methods, for example the total-variational-diminishing (TVD), or strong-stability-preserving (SSP) time discretizations [60, 27]. The fully discretized scheme is referred to as Runge-Kutta DG (RKDG) schemes.

The RKDG schemes are already energy stable [34, 31, 69], and can be used to solve equation (2.1) with smooth solutions or solutions with only weak discontinuities. However, for solutions with strong shocks, DG schemes will generate spurious oscillations which may lead to nonlinear instability and blow-ups of the code. In such cases, some form of nonlinear limiters would be needed. We will address this issue in Section 4.

The RKDG method is local, with communications only with immediate neighbors through the numerical fluxes, regardless of the order of accuracy of the scheme. This makes the implementation of the RKDG method highly efficient. It also makes the method easy for parallel implementation. The method can achieves almost 100% parallel efficiency for static meshes and over 80% parallel efficiency for dynamic load balancing with adaptive meshes [4, 54]. The DG method is also very friendly to the GPU environment [37].

3 DG methods for convection-diffusion equations

In this section we describe briefly the main idea of DG methods for solving convection-diffusion equations. In CFD such equations include Navier-Stokes equations of compressible gas dynamics, etc. We use the following one-dimensional scalar equation

$$u_t + f(u)_x = (a(u)u_x)_x \tag{3.1}$$

with $a(u) \geq 0$ as an example. We first discuss the local DG (LDG) method [20], for which we rewrite equation (3.1) as the following system

$$u_t + f(u)_x = (b(u)q)_x, \qquad q - B(u)_x = 0, \tag{3.2}$$

where

$$b(u) = \sqrt{a(u)}, \qquad B(u) = \int^u b(u)du. \tag{3.3}$$

The finite element space is still given by (2.2). The semi-discrete LDG scheme is defined as follows. Find $u_h, q_h \in V_h^k$ such that, for all test functions $v_h, p_h \in V_h^k$ and all $1 \leq i \leq N$, we have

$$\begin{gathered} \int_{I_i}(u_h)_t(v_h)dx - \int_{I_i}(f(u_h) - b(u_h)q_h)(v_h)_x dx \\ +(\hat{f} - \hat{b}\hat{q})_{i+\frac{1}{2}}(v_h)^-_{i+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{q})_{i-\frac{1}{2}}(v_h)^+_{i-\frac{1}{2}} = 0, \\ \int_{I_i} q_h p_h dx + \int_{I_i} B(u_h)(p_h)_x dx - \hat{B}_{i+\frac{1}{2}}(p_h)^-_{i+\frac{1}{2}} + \hat{B}_{i-\frac{1}{2}}(p_h)^+_{i-\frac{1}{2}} = 0. \end{gathered} \tag{3.4}$$

Here, all the "hat" terms are the numerical fluxes. We already know from Section 2 that the convection flux $\hat{f}$ should be chosen as a monotone flux. However, the upwinding principle is no longer a valid guiding principle for the design of the diffusion fluxes $\hat{b}$, $\hat{q}$ and $\hat{B}$. In [20], sufficient conditions for the choices of these diffusion fluxes to guarantee the stability of the scheme (3.4) are given. Here, we will only discuss a particularly attractive choice, called "alternating fluxes", defined as

$$\hat{b} = \frac{B(u_h^+) - B(u_h^-)}{u_h^+ - u_h^-}, \qquad \hat{q} = q_h^+, \qquad \hat{B} = B(u_h^-). \tag{3.5}$$

The important point is that $\hat{q}$ and $\hat{B}$ should be chosen from different directions. Thus, the choice

$$\hat{b} = \frac{B(u_h^+) - B(u_h^-)}{u_h^+ - u_h^-}, \qquad \hat{q} = q_h^-, \qquad \hat{B} = B(u_h^+)$$

is also fine.

Notice that, from the second equation in the scheme (3.4), we can solve q_h explicitly and locally (in cell I_i) in terms of u_h, by inverting the small mass matrix inside the cell I_i. This is why the method is referred to as the "local" discontinuous Galerkin method.

LDG method as defined above is energy stable [20]. Its multi-dimensional version is still similar to its one-dimensional version, and the advantages in local communication, easiness in handling complicated geometry and boundary conditions, and parallel efficiency, are still valid.

There are other types of DG approximations to the viscous terms, for example those in [2, 63, 1, 3, 45, 42, 43, 12].

4 Examples of recent developments on DG methods

In this section we give a few examples of recent developments on DG methods which are relevant to CFD.

4.1 Nonlinear limiters

The RKDG schemes for conservation laws defined in Section 2 are energy stable. However, for solving problems with strong discontinuities, the DG solution may generate spurious numerical oscillations. In practice, especially for nonlinear problems containing strong shocks, we often need to apply nonlinear limiters to control these oscillations. Most of the limiters studied in the literature come from the methodologies of finite volume high resolution schemes.

A limiter can be considered as a post-processor of the computed DG solution. In any cell which is deemed to contain a possible discontinuity (the so-called *troubled cells*), the

DG polynomial is replaced by a new polynomial of the same degree, while maintaining the original cell average for conservation. Different limiters compute this new polynomial in different fashions. The main idea is to require that the new polynomial is less oscillatory than the old one, and, if the solution in this cell happens to be smooth, then the new polynomial should have the same high order accuracy as the old one. Some of the limiters are applied to all cells, while they should take effect (change the polynomial in the cell) only in the cells near the discontinuities. The total variation diminishing (TVD) limiters [28] belong to this class. Unfortunately, such limiters tend to take effect also in some cells in which the solution is smooth, for example in cells near smooth extrema of the exact solution. Accuracy is therefore lost in such cells. The total variation bounded (TVB) limiters [58], applied to RKDG schemes in [17, 16, 14, 19], attempt to remove this difficulty and to ensure that the limiter takes effect only in cells near the discontinuities. The TVB limiters are widely used in applications, because of their simplicity in implementation. However, the TVB limiters involve a parameter M, related to the value of the second derivative of the exact solution near smooth extrema, which must be chosen by the user for different test cases. The moment-based limiter [4] and the improved moment limiter [8] also belong to this class, and they are specifically designed for DG methods and limit the moments of the polynomial sequentially, from the highest order moment downwards. Unfortunately, the moment-based limiters may also take effect in certain smooth cells, thereby destroying accuracy in these cells.

The limiters based on the weighted essentially non-oscillatory (WENO) methodology are designed with the objective of maintaining the high order accuracy even if they take effect in smooth cells. These limiters are based on the WENO methodology for finite volume schemes [44, 35], and involve nonlinear reconstructions of the polynomials in troubled cells using the information of neighboring cells. The WENO reconstructed polynomials have the same high order of accuracy as the original polynomials when the solution is smooth, and they are (essentially) non-oscillatory near discontinuities. Qiu and Shu [51] and Zhu et al.

[79] designed WENO limiters using the usual WENO reconstruction based on cell averages of neighboring cells as in [35, 32, 57], to reconstruct the values of the solutions at certain Gaussian quadrature points in the target cells, and then rebuild the solution polynomials from the original cell average and the reconstructed values at the Gaussian quadrature points through a numerical integration for the moments. This limiter needs to use the information from not only the immediate neighboring cells but also neighbors' neighbors, making it complicated to implement in multi-dimensions, especially for unstructured meshes [79, 32, 77]. It also destroys the local data structure of the base DG scheme (which needs only to communicate with immediate neighbors). The effort in [49, 50] attempts to construct Hermite type WENO approximations, which use the information of not only the cell averages but also the lower order moments such as slopes, to reduce the spread of reconstruction stencils. However for higher order methods the information of neighbors' neighbors is still needed.

More recently, Zhong and Shu [78] developed a new WENO limiting procedure for RKDG methods on structured meshes. The idea is to reconstruct the entire polynomial, instead of reconstructing point values or moments in the classical WENO reconstructions. That is, the entire reconstruction polynomial on the target cell is a convex combination of polynomials on this cell and its immediate neighboring cells, with suitable adjustments for conservation and with the nonlinear weights of the convex combination following the classical WENO procedure. The main advantage of this limiter is its simplicity in implementation, as it uses only the information from immediate neighbors and the linear weights are always positive. This simplicity is more prominent for multi-dimensional unstructured meshes, which is studied in [80] for two-dimensional unstructured triangular meshes. This new WENO limiter has also been designed for CPR type schemes [25].

The WENO limiters are typically applied only in designated "troubled cells", in order to save computational cost and to minimize the influence of accuracy in smooth regions. Therefore, a troubled cell indicator is needed, to correctly identify cells near discontinuities

in which the limiters should be applied. Qiu and Shu in [52] have compared several troubled cell indicators. In practice, the TVB indicator [58] and the KXRCF indicator [38] are often the best choices.

4.2 Bound preserving DG schemes

In many CFD problems the physical quantities have desired bounds which are satisfied by the exact solutions of the PDEs. For example, for two-dimensional incompressible Euler or Navier-Stokes equations written in a vorticity-streamfunction formulation, the vorticity satisfies a maximum principle. For Euler equations of compressible gas dynamics, density and pressure remain positive (non-negative) when the their initial values are positive. It would certainly be desirable if numerical solutions obey the same bounds. If the numerical solution goes out of the bounds because of spurious oscillations, it would either be non-physical (e.g. negative density, negative internal energy, a percentage of a component which goes below zero or above one), or worse still, it could lead to nonlinear instability and blowups of the code because the PDE becomes ill-posed (e.g. the Euler equations of compressible gas dynamics become ill-posed for negative density or pressure).

Not all limiters discussed in the previous section can enforce the bound-preserving property. When they do, they often degenerate the order of accuracy of the original scheme in smooth regions.

Recently, a general framework is established to preserve strict bounds (maximum principle for scalar problems and positivity of relevant quantities for scalar problems or systems), while maintaining provable high order accuracy of the original schemes. These techniques apply to multi-dimensions in general unstructured triangulations as well. See [70, 71, 75].

We will not repeat here the details of this general framework and refer the readers to the references. We will summarize here the main steps in this framework:

1. We first find a first order base DG scheme, using piecewise polynomials of degree zero (piecewise constants), which can be proved to be bound-preserving under certain CFL

conditions for Euler forward time discretization. Notice that a first order DG scheme is the same as a first order finite volume scheme.

For scalar hyperbolic conservation laws (2.1), the first order DG scheme using any monotone numerical flux would satisfy a maximum principle. For Euler equations of compressible gas dynamics, several first order schemes, including the Godunov scheme [26], the Lax-Friedrichs scheme [47, 71], the Harten-Lax-van Leer (HLLE) [29] scheme, and the Boltzmann type kinetic scheme [46], are positivity-preserving for density and pressure.

2. We then apply a simple scaling limiter to the high order DG solution at time level n. If the DG solution at time level n in cell I_j is a polynomial $p_j(x)$, we replace it by the limited polynomial $\tilde{p}_j(x)$ defined by

$$\tilde{p}_j(x) = \theta_j(p_j(x) - \bar{u}_j^n) + \bar{u}_j^n$$

where $\bar{u}_j^n$ is the cell average of $p_j(x)$, and

$$\theta_j = \min\left\{\left|\frac{M - \bar{u}_j^n}{M_j - \bar{u}_j^n}\right|, \left|\frac{m - \bar{u}_j^n}{m_j - \bar{u}_j^n}\right|, 1\right\},$$

with

$$M_j = \max_{x \in S_j} p_j(x), \qquad m_j = \min_{x \in S_j} p_j(x)$$

where S_j is the set of certain Legendre Gauss-Lobatto quadrature points of the cell I_j. Clearly, this limiter is just a simple scaling of the original polynomial around its average.

3. We then evolve the solution by Euler forward time discretization, or by TVD or SSP Runge-Kutta time discretization [60, 27].

We can see that this procedure is very simple and inexpensive to implement. The scaling limiter involves only evaluation of the DG polynomial at pre-determined quadrature points. The procedure can be applied in arbitrary triangular meshes. Amazingly, this simple process

leads to mathematically provable bound-preserving property without degenerating the high order accuracy of the base DG scheme.

For scalar nonlinear conservation laws, passive convection in a divergence-free velocity field, and 2D incompressible Euler equations in the vorticity-streamfunction formulation, high order DG schemes maintaining maximum principle have been designed in Zhang and Shu [70] and in Zhang, Xia and Shu [75].

For scalar nonlinear convection diffusion equations, second order DG schemes on unstructured triangulations maintaining maximum principle have been designed in Zhang, Zhang and Shu [76].

For Euler equations of gas dynamics, high order DG schemes maintaining positivity of density and pressure (or internal energy) have been designed in Zhang and Shu [71, 72, 73, 74] and in Zhang, Xia and Shu [75].

For shallow water equations, high order DG schemes maintaining non-negativity of water height have been designed in Xing, Zhang and Shu [64].

Positivity-preserving semi-Lagrangian DG schemes have been designed in Qiu and Shu [48] and in Rossmanith and Seal [56].

4.3 DG method for hyperbolic equations involving δ-functions

In a hyperbolic conservation law

$$\begin{array}{ll} u_t + f(u)_x = g(x,t), & (x,t) \in R \times (0,T], \\ u(x,0) = u_0(x), & x \in R, \end{array} \tag{4.1}$$

the initial condition u_0, or the source term $g(x,t)$, or the solution $u(x,t)$ may contain δ-singularities. Such singularities are more difficult to handle than discontinuities in the solutions. Many high order schemes would easily blow up in the presence of δ-function singularities, because of the severe oscillations leading to non-physical regimes (e.g. negative density). On the other hand, if one applies traditional limiters such as various slope limiters to enforce stability, the resolution of the δ-function singularities would be seriously

deteriorated. Resolution is also seriously affected by other commonly used strategies such as molifications by an approximate Gaussian to smear out the δ-function.

Since DG methods are based on weak formulations, they can be designed directly to handle δ-function singularities. Recently, we have designed and analyzed DG schemes for solving linear and nonlinear PDE models with δ-function singularities [67, 68]. For linear problems, we prove stability and high order error estimates in negative norms when the DG method is applied, and propose post-processing techniques to recover high order accuracy in strong norms away from these δ-function singularities. For nonlinear problems, such as Krause's consensus models [9] and pressureless Euler equations [10], an adequate design of bound preserving limiter, within the framework described in Section 4.2, to enforce the physical bounds without compromising resolution of δ-function singularities is shown to be crucial. With such limiters, high resolution and highly stable results can be obtained for such difficult nonlinear models.

4.4 DG method for Hamilton-Jacobi equations and nonlinear control

Time dependent Hamilton-Jacobi equations take the form

$$\varphi_t + H(\varphi_{x_1}, ..., \varphi_{x_d}) = 0, \qquad \varphi(x, 0) = \varphi^0(x), \tag{4.2}$$

where H is a Lipschitz continuous function. H could also depend on φ, x and t in some applications. Hamilton-Jacobi equations appear often in many applications. Examples in CFD include front propagation, level set methods, multi-material flows, and nonlinear control.

Since the spatial derivative operator is inside the nonlinear Hamiltonian H in equation (4.2), integration by parts leading to a weak formulation for the DG scheme cannot be directly performed. However, by exploring the strong relationship between the Hamilton-Jacobi equations and conservation laws, various formulations of DG methods have been designed in the literature.

The first attempt to design a DG method was based exactly on this relationship [33].

At least in one dimension, the viscosity solution of the Hamilton-Jacobi equation (4.2) is equivalent to the entropy solution of the conservation law (2.1), when we identify $\varphi_x = u$ and $H = f$. Therefore, a DG scheme for solving the conservation law (2.1) can be directly used to approximate the derivative of the viscosity solution of the Hamilton-Jacobi equation (4.2). The missing degree of freedom, to recover φ from φ_x, can be determined by

$$\int_{I_j} ((\varphi_h)_t + H(u_h))\, dx = 0,$$

which is an evolution equation for the cell average of φ when the approximation to its derivative $\varphi_x = u$ is known. This algorithm is well defined for one dimension. Additional complications exist for multi-dimensional cases, which can be handled either by a least square procedure [33] or by using locally curl-free DG spaces [41] (these two approaches are mathematically equivalent as shown in [41]).

Even though the DG schemes in [33, 41] are successful in approximating the Hamilton-Jacobi equation (4.2), it involves rewriting it as a conservation law satisfied by the derivatives of the solution φ. It is desirable to design a DG method which solves directly the solution φ to the Hamilton-Jacobi equation (4.2). The scheme of Cheng and Shu [11] serves this purpose. It starts from the alternative formulation of DG schemes given in (2.5) which does not require explicit integration by parts to the original PDE to write down the DG scheme. Further development and application of this method to problems in optimal control are given in [5, 6, 7]. Excellent simulation results are obtained for nonlinear control simulations in these references.

Another DG method which solves directly the Hamilton-Jacobi equations (4.2) is that of Yan and Osher [66]. This method is motivated by the local discontinuous Galerkin (LDG) method for solving second order partial differential equations [20].

5 Concluding remarks

In this short survey article we have given an overview of discontinuous Galerkin (DG) methods applied to computational fluid dynamics (CFD) problems. A few representative exam-

ples of recent developments are given to illustrate the vitality of this research area. The DG method has many advantages for large scale computing, especially for massively parallel (exascale) computing environments. Future research in DG methods might include study on reliable and efficient error indicators to guide *h-p* adaptivity, thus realizing the full power of DG methods for such adaptivity; more efficient and reliable limiters for solutions with strong shocks, which do not affect accuracy in smooth regions and do not affect convergence to steady state solutions; and DG methods with non-polynomial basis functions for multi-scale and turbulence modeling applications. It can be expected that such further research in DG methods will lead to the expansion of their application in CFD and other scientific and engineering areas.

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