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Stability analysis and a priori error estimate of explicit Runge-Kutta discontinuous Galerkin methods for correlated random walk with density-dependent turning rates

Dedicated to Professor Shi Zhong-Ci on the Occasion of his 80th Birthday

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Abstract In this paper we analyze the explicit Runge-Kutta discontinuous Galerkin (RKDG) methods for the semilinear hyperbolic system of a correlated random walk model describing movement of animals and cells in biology. The RKDG methods use a third order explicit total-variation-diminishing Runge-Kutta (TVDRK3) time discretization and upwinding numerical fluxes. By using the energy method, under a standard CFL condition, we obtain L^2 stability for general solutions and a priori error estimates when the solutions are smooth enough. The theoretical results are proved for piecewise polynomials with any degree $k \geqslant 1$. Finally, since the solutions to this system are non-negative, we discuss a positivity-preserving limiter to preserve positivity without compromising accuracy. Numerical results are provided to demonstrate these RKDG methods.

Keywords Discontinuous Galerkin method, explicit Runge-Kutta method, stability, error estimates, correlated random walk, positivity-preserving

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1 Introduction

Aggregation and coordinated movement are common behaviors that can be observed in many species such as schools of fish and flocks of birds, as well as in some cell populations. These behaviors lead to a variety of forms and patterns and they serve different purposes. For example, the school of fish moving in a highly organized way is considered a strategy against predation [15], and under starvation conditions the cells aggregate and form stalks to become fruiting bodies [17]. There are two kinds of factors which play important roles in influencing these behaviors. One is external signals, such as chemicals, light, temperature and humidity. The other is interaction between individuals (self-organized movements) that causes group formation. In this paper we focus on models describing self-organized movements.

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For such movements, the simplest hyperbolic model is the classical Goldstein-Kac theory for correlated random walk [10, 14] when the turning rates are constants:

$$\begin{cases} u_t + \gamma u_x = \frac{\mu}{2}(v - u) \\ v_t - \gamma v_x = \frac{\mu}{2}(u - v) \end{cases} \tag{1.1}$$

This model describes two kinds of particles moving in opposite directions on a line, where $u(x,t)$ and $v(x,t)$ are the densities of left-moving and right-moving individuals respectively. The particles move in a constant speed γ and change their directions with a constant rate $\frac{\mu}{2}$. In stochastic processes this means that the probability they do not change their directions in time $[0,t)$ is $e^{-\frac{\mu t}{2}}$, where $\frac{\mu}{2}$ is the rate parameter of a Poisson process. Since most of biological populations live in limited areas, the mathematical models are often defined on bounded domains. Three types of boundary conditions (Dirichlet, Neumann and periodic) to (1.1) on bounded domains are discussed in [13]. It is pointed out that the solutions are as smooth as the initial conditions and singularities are transported along the characteristics for both Neumann and periodic boundary conditions, while for Dirichlet boundary conditions singularities disappear and the solutions are regularized in finite time.

Consider the total population density $p = u + v$ and the flux $q = u - v$. Then from (1.1) we obtain the following equations

$$\begin{cases} p_t + \gamma q_x = 0 \\ q_t + \gamma p_x = -\mu q \end{cases} \tag{1.2}$$

For periodic or Neumann boundary conditions the total density is preserved. Differentiating the equations of (1.2) with respect to x and t and eliminating $\frac{\partial^2 q}{\partial t \partial x}$ and $\frac{\partial q}{\partial x}$ (Kac's trick in [14]), we get the scalar telegraph equation

$$p_{tt} + \mu p_t = \gamma^2 p_{xx} \tag{1.3}$$

The relationships among (1.1), (1.2) and (1.3) are investigated in [11].

Since biological phenomena such as splitting, merging and population increase of groups are complicated, the assumption of a constant speed and constant turning rates may not always be true. Often, individuals in a group change their directions when interacting with their neighbors locally or globally. These interactions can be direct through neighbors' densities [7, 8, 12, 15, 17], or indirect through the chemicals produced by their neighbors [16]. With a constant speed and very simple turning rate functions, it is possible to find exact analytical solutions [12]. However, for general cases, only numerical and qualitative results (such as existence and asymptotic behavior) are shown [7, 8, 15, 17]. Numerical results in these works consider alignment, attraction and repulsion between individuals and obtain a variety of patterns by using first order upwind and second order Lax-Wendroff schemes. To improve on the efficiency and reliability of the numerical simulations, in this paper we develop high order accurate explicit Runge-Kutta discontinuous Galerkin (RKDG) methods to a semilinear hyperbolic model and provide theoretical supports to these methods including L^2 stability, a priori error estimates, and positivity-preserving properties.

The DG method was first introduced by Reed and Hill [18] in 1973 to solve the neutron transport equation, which is a linear hyperbolic conservation law. Later it was developed into RKDG methods by Cockburn et al [2–6]. They combine the DG discretization in space with explicit total variation diminishing (TVD) Runge-Kutta method [19] in time and successfully solve nonlinear conservation laws. The RKDG method has advantages of high-order accuracy, high parallel efficiency and the flexibility in handling complicated geometry. Stability results for RKDG method applied to linear hyperbolic equations are obtained in [21], and a priori error estimates for hyperbolic equations without the global source terms are given in [20, 21]. The analysis in this paper is generalization to the semilinear systems for correlated random walk.

This paper is organized as follows. In Section 2 we introduce our model and its properties including its energy-boundedness and positivity-preserving property. In Section 3 we introduce some notations and preliminary results to prepare for the analysis later. In Section 4 we first construct the semi-discrete DG scheme, and prove its L^2 stability. We then discuss third order TVD Runge-Kutta discontinuous Galerkin scheme and, using the energy method, we obtain L^2 stability and error estimates under suitable CFL conditions. The stability result holds for arbitrary solutions and the error estimates are obtained when the solutions are sufficiently smooth. In Section 5 we first discuss a first order upwind scheme to serve as building blocks for our higher order DG schemes, and prove its positivity-preserving property under a suitable CFL condition. We then discuss a positivity-preserving limiter to guarantee positivity of the numerical solution of higher order DG schemes without compromising its accuracy. In Section 6 we present some numerical results to demonstrate these numerical methods. Concluding remarks are given in Section 7. Some of the technical proofs for several lemmas are given in Section 8 which serves as an appendix.

2 The model and its properties

In this paper, we consider a correlated random walk model in [8]. It is a nonlocal one-dimensional hyperbolic system with a constant speed γ and density-dependent turning rate functions. The system is given as

$$\begin{cases} u_t + \gamma u_x = -\lambda_1 u + \lambda_2 v, \quad (x,t) \in \mathbb{R} \times [0,T] \\ v_t - \gamma v_x = \lambda_1 u - \lambda_2 v, \quad (x,t) \in \mathbb{R} \times [0,T] \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \mathbb{R} \end{cases} \tag{2.1}$$

We will study system (2.1) on interval $[0, L]$ with periodic boundary conditions

$$u(0,t) = u(L,t), \quad v(0,t) = v(L,t)$$

with the solution u, v extended periodically on $\mathbb{R}$ with period L. λ_1, λ_2 are the turning rate functions defined as follows

$$\lambda_1 = a_1 + a_2 f(y_1[u,v]) = a_1 + a_2 f(0) + a_2(f(y_1[u,v]) - f(0)) \tag{2.2a}$$
$$\lambda_2 = a_1 + a_2 f(y_2[u,v]) = a_1 + a_2 f(0) + a_2(f(y_2[u,v]) - f(0)) \tag{2.2b}$$

where a_1, a_2 are positive constants and $a_1 + a_2 f(0)$ is the autonomous turning rate, and $a_2(f(y_1[u,v]) - f(0))$ and $a_2(f(y_2[u,v]) - f(0))$ are the bias turning rates considered to be influenced by three social interactions: attraction ($y_{1,a}$, $y_{2,a}$), repulsion ($y_{1,r}$, $y_{2,r}$) and alignment ($y_{1,al}$, $y_{2,al}$).

$$\begin{aligned}
&f(y) = 0.5 + 0.5\tanh(y - y_0), \quad p = u + v,\\
&y_1[u,v] = y_{1,r}[u,v] - y_{1,a}[u,v] + y_{1,al}[u,v],\\
&y_2[u,v] = y_{2,r}[u,v] - y_{2,a}[u,v] + y_{2,al}[u,v],\\
&y_{1,r}[u,v] = q_r \int_0^\infty K_r(s)(p(x+s) - p(x-s))ds,\\
&y_{1,a}[u,v] = q_a \int_0^\infty K_a(s)(p(x+s) - p(x-s))ds,\\
&y_{1,al}[u,v] = q_{al} \int_0^\infty K_{al}(s)(v(x+s) - u(x-s))ds,\\
&y_{2,r}[u,v] = q_r \int_0^\infty K_r(s)(p(x-s) - p(x+s))ds,\\
&y_{2,a}[u,v] = q_a \int_0^\infty K_a(s)(p(x-s) - p(x+s))ds,\\
&y_{2,al}[u,v] = q_{al} \int_0^\infty K_{al}(s)(u(x-s) - v(x+s))ds,
\end{aligned}$$

$$K_i(s) = \frac{1}{\sqrt{2\pi m_i^2}} \exp\left(-(s-s_i)^2/(2m_i^2)\right), \quad i = r, a, al, \quad s \in [0, \infty)$$

Here the parameters are taken as in [7], listed in Table 1. We assume $L > 2s_i$ for $i = r, a, al$. We remark that this is just one of the models for the turning rate functions. The global Lipschitz continuity helps to simplify the analysis, however the analysis in this paper can be easily generalized also to models with only locally Lipschitz properties.

Table 1 List of the parameters in the model

Parameter	Description	Units	Fixed value
γ	Speed	L/T	No
a_1	Turning rate	1/T	No
a_2	Turning rate	1/T	No
y_0	Shift of the turning function	1	2
q_a	Magnitude of attraction	L/N	No
q_{al}	Magnitude of alignment	L/N	No
q_r	Magnitude of repulsion	L/N	No
s_a	Attraction range	L	1
s_{al}	Alignment range	L	0.5
s_r	Repulsion range	L	0.25
m_a	Width of attraction kernel	L	1/8
m_{al}	Width of alignment kernel	L	0.5/8
m_r	Width of repulsion kernel	L	0.25/8
A	Total population size	N	2
L	Domain size	L	10

2.1 Energy boundedness

By the expressions of λ_1 and λ_2 in (2.2) and the definition of the function $f(y)$, we clearly have

$$0 < a_1 \leqslant \lambda_i \leqslant a_1 + a_2, \qquad i = 1, 2 \tag{2.3}$$

Multiplying the two equations in (2.1) by u and v respectively, we obtain

$$\begin{aligned} uu_t + \gamma uu_x &= -\lambda_1 u^2 + \lambda_2 uv \\ vv_t - \gamma vv_x &= \lambda_1 uv - \lambda_2 v^2 \end{aligned}$$

Adding them up and then integrating on $[0, L]$, by using (2.3), Cauchy's inequality and the periodic boundary conditions, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L (u^2+v^2)dx &\leqslant \int_0^L (-\lambda_1 u^2 - \lambda_2 v^2 + (\lambda_1+\lambda_2)uv)dx \\ &\leqslant \frac{a_2}{2}\int_0^L (u^2+v^2)dx \end{aligned}$$

Through Gronwall's inequality we can get

$$\|u(\cdot,t)\|^2 + \|v(\cdot,t)\|^2 \leqslant e^{a_2 t}(\|u_0\|^2 + \|v_0\|^2) \tag{2.4}$$

which is the boundedness of the L^2 energy.

2.2 Positivity-preserving property

We will not discuss in detail the existence, uniqueness, smoothness, and positivity of solutions to (2.1), and refer to, e.g. [9]. For our purpose, the positivity-preserving property for the densities u and v is important, hence we will discuss this property through a first order upwind scheme in Section 5.

3 Preliminaries

We divide the domain [0,L] into n_x cells, which are $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $j = 1, \cdots, n_x$, $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{n_x+\frac{1}{2}} = L$. Denote $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $h = \max_j h_j$, $\rho = \min_j h_j$, and let $h \leqslant 1$. Assume the mesh is regular, i.e. there exists a positive constant ν such that

$$\nu h \leqslant \rho \leqslant h \tag{3.1}$$

We take the finite element space as

$$V_h = V_h^k = \{\varphi : \varphi|_{I_j} \in P^k(I_j); 1 \leqslant j \leqslant n_x\} \tag{3.2}$$

The L^2 norm in $L^2([0,L])$ in this paper is denoted by $\|\cdot\|_{L^2([0,L])}$, $\|\cdot\|_{L^2([0,L])} = \sqrt{(\cdot,\cdot)}$ where $(\cdot,\cdot)$ is the inner product on $L^2([0,L])$. For simplicity, we also write $\|\cdot\|$ instead of $\|\cdot\|_{L^2([0,L])}$. For any function $\phi \in V_h^k$ defined in (3.2), the jump at the element boundary point is denoted by $[\![\phi]\!] = \phi^+ - \phi^-$, and the L^2 norm on Γ_h, which is the union of all element boundary points, is denoted by

$$\|\phi\|_{\Gamma_h}^2 = \sum_j \left((\phi^+_{j+\frac{1}{2}})^2 + (\phi^-_{j+\frac{1}{2}})^2\right)$$

Here $\sum_j = \sum_{1\leqslant j\leqslant n_x}$.

Following [21], we introduce two notations to denote the DG spatial operators

$$B_j^1(\phi,\psi) = \int_{I_j} \gamma\phi\psi_x dx - \gamma\phi^-_{j+\frac{1}{2}}\psi^-_{j+\frac{1}{2}} + \gamma\phi^-_{j-\frac{1}{2}}\psi^+_{j-\frac{1}{2}}$$
$$B_j^2(\phi,\psi) = -\int_{I_j} \gamma\phi\psi_x dx + \gamma\phi^+_{j+\frac{1}{2}}\psi^-_{j+\frac{1}{2}} - \gamma\phi^+_{j-\frac{1}{2}}\psi^+_{j-\frac{1}{2}}$$

3.1 Gauss-Radau projection error

Suppose $\mathbb{R}_h$ is the Gauss-Radau projection into V_h^k, i.e. for any function w, the projection $\mathbb{R}_h w \in V_h^k$ satisfies

$$\int_{I_i} (\mathbb{R}_h w(x) - w(x))\varphi(x)dx = 0, \quad \forall\varphi \in P^{k-1}(I_i)$$

with a value assigned to an endpoint such as $\mathbb{R}_h w(x^-_{j+\frac{1}{2}}) = w(x_{j+\frac{1}{2}})$ or $\mathbb{R}_h w(x^+_{j-\frac{1}{2}}) = w(x_{j-\frac{1}{2}})$, then $\mathbb{R}_h w$ is unique in V_h^k.

For any function $w \in W_2^{k+1}([0,L])$, the projection error $\eta = w - \mathbb{R}_h w$ has following estimate [1].

$$\|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \leqslant C_1 h^{k+1} \tag{3.3}$$

Here C_1 is a positive constant independent of h.

3.2 Inverse inequalities

We list some inverse inequalities here. For more details one can refer to [1].

$$\|\phi_x\| \leqslant M_1 h^{-1}\|\phi\|, \qquad \forall\phi \in V_h; \tag{3.4}$$

Table 2 Inverse constants for piecewise polynomials of degree k

	k= 1	k= 2	k= 3
M_1	$\sqrt{12}\approx 3.46$	$\sqrt{60}\approx 7.75$	$\sqrt{2(45+\sqrt{1605})}\approx 13.04$
$(M_2)^2$	6	12	20

$$\|\phi\|_{\Gamma_h} \leqslant M_2 h^{-1/2}\|\phi\|, \qquad \forall \phi\in V_h. \tag{3.5}$$

The constants M_1 and M_2 are dependent of k but independent of ϕ and h. We denote

$$M=\max\{M_1,\ (M_2)^2\} \tag{3.6}$$

The values of M_1 and $(M_2)^2$ in Table 2 are cited from [21].

3.3 Properties of DG spatial operators

Denote

$$B^1(\phi,\psi)=\sum_j B_j^1(\phi,\psi)=\sum_j\left[\gamma\phi^-_{j+\frac12}[\![\psi]\!]_{j+\frac12}+\int_{I_j}\gamma\phi\psi_x dx\right]$$
$$B^2(\phi,\psi)=\sum_j B_j^2(\phi,\psi)=-\sum_j\left[\gamma\phi^+_{j+\frac12}[\![\psi]\!]_{j+\frac12}+\int_{I_j}\gamma\phi\psi_x dx\right]$$

Two lemmas are presented as follows. For more details we refer to [21].

Lemma 3.1. *$\forall\ \phi,\psi\in V_h$, we have*

$$|B^1(\phi,\psi)|\leqslant(\sqrt{2}+1)\gamma M h^{-1}\|\phi\|\ \|\psi\|,$$
$$|B^2(\phi,\psi)|\leqslant(\sqrt{2}+1)\gamma M h^{-1}\|\phi\|\ \|\psi\|$$

Lemma 3.2. *$\forall \phi,\psi\in V_h$, we have*

$$B^1(\phi,\psi)+B^1(\psi,\phi)=-\sum_j\gamma[\![\phi]\!]_{j+\frac12}[\![\psi]\!]_{j+\frac12},$$
$$B^2(\phi,\psi)+B^2(\psi,\phi)=-\sum_j\gamma[\![\phi]\!]_{j+\frac12}[\![\psi]\!]_{j+\frac12}$$

4 Stability and error estimates

4.1 Semi-discrete DG scheme

The semi-discrete DG scheme is a scheme of discretization in space by using discontinuous Galerkin method, but is kept continuous in time. The semi-discrete DG scheme of (2.1) is defined as follows. $\forall\varphi\in V_h$,

$$\int_{I_j}(u_h)_t\varphi dx=B_j^1(u_h,\varphi)-\int_{I_j}(\lambda_1)_h u_h\varphi dx+\int_{I_j}(\lambda_2)_h v_h\varphi dx \tag{4.1a}$$
$$\int_{I_j}(v_h)_t\varphi dx=B_j^2(v_h,\varphi)+\int_{I_j}(\lambda_1)_h u_h\varphi dx-\int_{I_j}(\lambda_2)_h v_h\varphi dx \tag{4.1b}$$

where $(\lambda_1)_h=\lambda_1(y_1[u_h,v_h])$, $(\lambda_2)_h=\lambda_2(y_2[u_h,v_h])$. The integrals in the numerical evaluation of $(\lambda_1)_h$ and $(\lambda_2)_h$ can be computed by a numerical quadrature of sufficient accuracy.

Taking $\varphi = u_h \in V_h$ in (4.1a) and $\varphi = v_h \in V_h$ in (4.1b), adding them up and summing them over j, we can get

$$\frac{1}{2}\frac{d}{dt}\int_0^L (u_h^2+v_h^2)dx = B^1(u_h,u_h)+B^2(v_h,v_h)+\int_0^L (-\lambda_1(y_1[u_h,v_h])u_h^2-\lambda_2(y_2[u_h,v_h])v_h^2 + (\lambda_1(y_1[u_h,v_h])+\lambda_2(y_2[u_h,v_h]))u_hv_h)dx$$

From Lemma 3.2, (2.3) and Cauchy's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L (u_h^2+v_h^2)dx \leqslant \frac{a_2}{2}\int_0^L (u_h^2+v_h^2)dx$$

Through Gronwall's inequality we then get

$$\|u_h(\cdot,t)\|^2+\|v_h(\cdot,t)\|^2 \leqslant e^{a_2 t}(\|u_h(0)\|^2+\|v_h(0)\|^2)$$

which is the the same boundedness of the L^2 energy as in the continuous case (2.4).

4.2 Fully discretized RKDG schemes: introduction

We now discuss fully discretized RKDG schemes using the DG method for space discretization and third order TVD Runge-Kutta method for time discretization. Assume the ODE system from the semidiscrete DG scheme is

$$u_t = L_h(u)$$

where L_h is the spatial DG operator independent of the partial derivative of u with respect to t. Then the third order TVD Runge-Kutta method [19], from time $n\tau$ to time $(n+1)\tau$, is

$$\begin{aligned} u^{(1)} &= u^n+\tau L_h(u^n)\\ u^{(2)} &= \frac{3}{4}u^n+\frac{1}{4}u^{(1)}+\frac{1}{4}\tau L_h(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n+\frac{2}{3}u^{(2)}+\frac{2}{3}\tau L_h(u^{(2)})\end{aligned} \tag{4.2}$$

4.3 Third order RKDG schemes

For $k=0$ the DG method is the same as the first order upwind method which we considered in Section 2.2. In the following we consider $k \geqslant 1$ only. Assume u_h^n, v_h^n are the numerical solutions at time $t = n\tau$. The scheme from time $n\tau$ to time $(n+1)\tau$ is defined as follows. $\forall \varphi \in V_h$, we have

$$\int_{I_j} u_h^{n,1}\varphi dx = \int_{I_j} u_h^n\varphi dx + \tau\Big[B_j^1(u_h^n,\varphi) - \int_{I_j}(\lambda_1)_h^n u_h^n\varphi dx + \int_{I_j}(\lambda_2)_h^n v_h^n\varphi dx\Big] \tag{4.3a}$$

$$\int_{I_j} v_h^{n,1}\varphi dx = \int_{I_j} v_h^n\varphi dx + \tau\Big[B_j^2(v_h^n,\varphi) + \int_{I_j}(\lambda_1)_h^n u_h^n\varphi dx - \int_{I_j}(\lambda_2)_h^n v_h^n\varphi dx\Big] \tag{4.3b}$$

$$\int_{I_j} u_h^{n,2}\varphi dx = \frac{3}{4}\int_{I_j} u_h^n\varphi dx + \frac{1}{4}\int_{I_j} u_h^{n,1}\varphi dx + \frac{\tau}{4}\Big[B_j^1(u_h^{n,1},\varphi) - \int_{I_j}(\lambda_1)_h^{n,1} u_h^{n,1}\varphi dx + \int_{I_j}(\lambda_2)_h^{n,1} v_h^{n,1}\varphi dx\Big] \tag{4.3c}$$

$$\int_{I_j} v_h^{n,2}\varphi dx = \frac{3}{4}\int_{I_j} v_h^n\varphi dx + \frac{1}{4}\int_{I_j} v_h^{n,1}\varphi dx + \frac{\tau}{4}\Big[B_j^2(v_h^{n,1},\varphi) + \int_{I_j}(\lambda_1)_h^{n,1} u_h^{n,1}\varphi dx - \int_{I_j}(\lambda_2)_h^{n,1} v_h^{n,1}\varphi dx\Big] \tag{4.3d}$$

$$\int_{I_j} u_h^{n+1}\varphi dx = \frac{1}{3}\int_{I_j} u_h^n\varphi dx + \frac{2}{3}\int_{I_j} u_h^{n,2}\varphi dx + \frac{2\tau}{3}\Big[B_j^1(u_h^{n,2},\varphi) - \int_{I_j}(\lambda_1)_h^{n,2} u_h^{n,2}\varphi dx$$

$$+\int_{I_j}(\lambda_2)_h^{n,2}v_h^{n,2}\varphi dx\Big] \tag{4.3e}$$

$$\int_{I_j}v_h^{n+1}\varphi dx=\frac{1}{3}\int_{I_j}v_h^{n}\varphi dx+\frac{2}{3}\int_{I_j}v_h^{n,2}\varphi dx+\frac{2\tau}{3}\Big[B_j^2(v_h^{n,2},\varphi)+\int_{I_j}(\lambda_1)_h^{n,2}u_h^{n,2}\varphi dx$$
$$-\int_{I_j}(\lambda_2)_h^{n,2}v_h^{n,2}\varphi dx\Big] \tag{4.3f}$$

Here $(\lambda_1)_h^{n,i}=\lambda_1(y_1[u_h^{n,i},v_h^{n,i}])$, $(\lambda_2)_h^{n,i}=\lambda_2(y_2[u_h^{n,i},v_h^{n,i}])$, $i=0,1,2$. $(\lambda_1)_h^{n,0}=(\lambda_1)_h^{n}$, $(\lambda_2)_h^{n,0}=(\lambda_2)_h^{n}$.

4.4 L^2 stability of the RKDG scheme (4.3)

Denote $\lambda=\gamma M\tau/h$ and assume τ/h is bounded. For simplicity, we take $\gamma\tau/h\leqslant 1$, and we can get $\lambda\leqslant M$ immediately.

Similar to [21], $\forall\phi\in V_h$, we define

$$\mathbb{D}_1(\phi)=\phi^{n,1}-\phi^n,\quad \mathbb{D}_2(\phi)=2\phi^{n,2}-\phi^{n,1}-\phi^n,\quad \mathbb{D}_3(\phi)=\phi^{n+1}-2\phi^{n,2}+\phi^n$$

We have three lemmas in the following. The proofs of them are given in the Appendix (Section 8).

Lemma 4.1.

$$\left|\int_0^L(\lambda_1)_h^{n,i}u_h^{n,i}\varphi dx\right|\leqslant(a_1+a_2)\|u_h^{n,i}\|\|\varphi\|,\quad i=0,1,2$$
$$\left|\int_0^L(\lambda_2)_h^{n,i}v_h^{n,i}\varphi dx\right|\leqslant(a_1+a_2)\|v_h^{n,i}\|\|\varphi\|,\quad i=0,1,2$$

Here $u_h^{n,0}=u_h^n, v_h^{n,0}=v_h^n$.

Lemma 4.2.

$$\|u_h^{n,1}\|\leqslant\alpha_1\|u_h^n\|+\alpha_2\|v_h^n\|,\quad \|v_h^{n,1}\|\leqslant\alpha_1\|v_h^n\|+\alpha_2\|u_h^n\|$$
$$\|u_h^{n,2}\|\leqslant C_3\|u_h^n\|+C_4\|v_h^n\|,\quad \|v_h^{n,2}\|\leqslant C_3\|v_h^n\|+C_4\|u_h^n\|$$
$$\|\mathbb{D}_1(u_h)\|\leqslant(\alpha_1+1)\|u_h^n\|+\alpha_2\|v_h^n\|,\quad \|\mathbb{D}_1(v_h)\|\leqslant(\alpha_1+1)\|v_h^n\|+\alpha_2\|u_h^n\|$$
$$\|\mathbb{D}_2(u_h)\|\leqslant C_5\|u_h^n\|+C_6\|v_h^n\|,\quad \|\mathbb{D}_2(v_h)\|\leqslant C_5\|v_h^n\|+C_6\|u_h^n\|$$

Here $\alpha_1=1+(\sqrt{2}+1)M+(a_1+a_2)/\gamma$, $\alpha_2=(a_1+a_2)/\gamma$, $C_3=\frac{3}{4}+\frac{1}{4}(\alpha_1^2+\alpha_2^2)$, $C_4=\frac{1}{2}\alpha_1\alpha_2$, $C_5=2C_3+\alpha_1+1$, $C_6=2C_4+\alpha_2$.

Lemma 4.3.

$$(\mathbb{D}_1(u_h),\varphi)\leqslant\tau B^1(u_h^n,\varphi)+(a_1+a_2)\tau(\|u_h^n\|+\|v_h^n\|)\|\varphi\| \tag{4.5a}$$
$$(\mathbb{D}_1(v_h),\varphi)\leqslant\tau B^2(v_h^n,\varphi)+(a_1+a_2)\tau(\|u_h^n\|+\|v_h^n\|)\|\varphi\| \tag{4.5b}$$
$$(\mathbb{D}_2(u_h),\varphi)\leqslant\frac{\tau}{2}B^1(\mathbb{D}_1(u_h),\varphi)+\frac{C_7\tau}{2}(\|u_h^n\|+\|v_h^n\|)\|\varphi\| \tag{4.5c}$$
$$(\mathbb{D}_2(v_h),\varphi)\leqslant\frac{\tau}{2}B^2(\mathbb{D}_1(v_h),\varphi)+\frac{C_7\tau}{2}(\|u_h^n\|+\|v_h^n\|)\|\varphi\| \tag{4.5d}$$
$$(\mathbb{D}_3(u_h),\varphi)\leqslant\frac{\tau}{3}B^1(\mathbb{D}_2(u_h),\varphi)+\frac{C_8\tau}{3}(\|u_h^n\|+\|v_h^n\|)\|\varphi\| \tag{4.5e}$$
$$(\mathbb{D}_3(v_h),\varphi)\leqslant\frac{\tau}{3}B^2(\mathbb{D}_2(v_h),\varphi)+\frac{C_8\tau}{3}(\|u_h^n\|+\|v_h^n\|)\|\varphi\| \tag{4.5f}$$

Here $C_7=(a_1+a_2)(\alpha_1+\alpha_2+1)$, $C_8=(a_1+a_2)(2C_3+2C_4+\alpha_1+\alpha_2+1)$.

We are now ready to prove the following theorem.

Theorem 4.4. *If $\gamma M\tau/h \leqslant 0.39$, where M is the constant defined in (3.6), then for the numerical solutions to scheme (4.3) we have, for any n,*

$$\|u_h^{n+1}\|^2 + \|v_h^{n+1}\|^2 \leqslant e^{C\tau}(\|u_h^n\|^2 + \|v_h^n\|^2),$$

where C is a constant independent of τ, h, u_h, v_h.

Proof. Taking $\varphi = u_h^n$ in (4.3a), $\varphi = v_h^n$ in (4.3b), $\varphi = 4u_h^{n,1}$ in (4.3c), $\varphi = 4v_h^{n,1}$ in (4.3d), $\varphi = 6u_h^{n,2}$ in (4.3e), $\varphi = 6v_h^{n,2}$ in (4.3f) and adding them up, then summing the result over j, we obtain

$$\int_0^L \mathbb{S}_1 dx = \tau(\mathbb{S}_2 + \mathbb{S}_3) \tag{4.6}$$

Here

$$\begin{aligned}\mathbb{S}_1 =& -2u_h^n u_h^{n,1} - (u_h^n)^2 + 4u_h^{n,1}u_h^{n,2} - (u_h^{n,1})^2 + 6u_h^{n,2}u_h^{n+1} - 2u_h^n u_h^{n,2} - 4(u_h^{n,2})^2\\ &- 2v_h^n v_h^{n,1} - (v_h^n)^2 + 4v_h^{n,1}v_h^{n,2} - (v_h^{n,1})^2 + 6v_h^{n,2}v_h^{n+1} - 2v_h^n v_h^{n,2} - 4(v_h^{n,2})^2\\ =&3((u_h^{n+1})^2 + (v_h^{n+1})^2) - 3((u_h^n)^2 + (v_h^n)^2) - \Big[(\mathbb{D}_2(u_h))^2 + (\mathbb{D}_2(v_h))^2\Big]\\ &- 3(\mathbb{D}_1 + \mathbb{D}_2 + \mathbb{D}_3)(u_h)\cdot\mathbb{D}_3(u_h) - 3(\mathbb{D}_1 + \mathbb{D}_2 + \mathbb{D}_3)(v_h)\cdot\mathbb{D}_3(v_h)\end{aligned}$$

$$\begin{aligned}\mathbb{S}_2 =&B^1(u_h^n, u_h^n) + B^2(v_h^n, v_h^n) + B^1(u_h^{n,1}, u_h^{n,1}) + B^2(v_h^{n,1}, v_h^{n,1}) + 4B^1(u_h^{n,2}, u_h^{n,2})\\ &+ 4B^2(v_h^{n,2}, v_h^{n,2})\end{aligned}$$

$$\begin{aligned}\mathbb{S}_3 =& -\int_0^L (\lambda_1)_h^n (u_h^n)^2 dx - \int_0^L (\lambda_2)_h^n (v_h^n)^2 dx + \int_0^L ((\lambda_1)_h^n + (\lambda_2)_h^n) u_h^n v_h^n dx\\ &- \int_0^L (\lambda_1)_h^{n,1} (u_h^{n,1})^2 dx - \int_0^L (\lambda_2)_h^{n,1} (v_h^{n,1})^2 dx + \int_0^L ((\lambda_1)_h^{n,1} + (\lambda_2)_h^{n,1}) u_h^{n,1} v_h^{n,1} dx\\ &- 4\int_0^L (\lambda_1)_h^{n,2} (u_h^{n,2})^2 dx - 4\int_0^L (\lambda_2)_h^{n,2} (v_h^{n,2})^2 dx + 4\int_0^L ((\lambda_1)_h^{n,2} + (\lambda_2)_h^{n,2}) u_h^{n,2} v_h^{n,2} dx\end{aligned}$$

We note that $\mathbb{S}_1$ and $\mathbb{S}_2$ are similar to those defined in [21], and $\mathbb{S}_3$ is from the source terms. Define

$$\begin{aligned}\Lambda_1 =&\|\mathbb{D}_2(u_h))\|^2 + \|\mathbb{D}_2(v_h))\|^2 + 3\big(\mathbb{D}_1(u_h), \mathbb{D}_3(u_h)\big) + 3\big(\mathbb{D}_1(v_h), \mathbb{D}_3(v_h)\big)\\ \Lambda_2 =&3\big(\mathbb{D}_2(u_h), \mathbb{D}_3(u_h)\big) + 3\big(\mathbb{D}_2(v_h), \mathbb{D}_3(v_h)\big)\\ \Lambda_3 =&3\big(\mathbb{D}_3(u_h), \mathbb{D}_3(u_h)\big) + 3\big(\mathbb{D}_3(v_h), \mathbb{D}_3(v_h)\big)\end{aligned}$$

Then (4.6) can be written as

$$3(\|u_h^{n+1}\|^2 + \|v_h^{n+1}\|^2) - 3(\|u_h^n\|^2 + \|v_h^n\|^2) = \Lambda_1 + \Lambda_2 + \Lambda_3 + \tau(\mathbb{S}_2 + \mathbb{S}_3) \tag{4.7}$$

From Lemma 4.3 we get

$$\begin{aligned}\Lambda_1 =& -\Big[\|\mathbb{D}_2(u_h)\|^2 + \|\mathbb{D}_2(v_h)\|^2\Big] + 2\Big[\big(\mathbb{D}_2(u_h), \mathbb{D}_2(u_h)\big) + \big(\mathbb{D}_2(v_h), \mathbb{D}_2(v_h)\big)\Big]\\ &+ 3\Big[\big(\mathbb{D}_1(u_h), \mathbb{D}_3(u_h)\big) + \big(\mathbb{D}_1(v_h), \mathbb{D}_3(v_h)\big)\Big]\\ \leqslant& -\Big[\|\mathbb{D}_2(u_h)\|^2 + \|\mathbb{D}_2(u_h)\|^2\Big] + \tau\big(B^1(\mathbb{D}_1(u_h), \mathbb{D}_2(u_h)) + B^1(\mathbb{D}_2(u_h), \mathbb{D}_1(u_h))\big)\\ &+ \tau\big(B^2(\mathbb{D}_1(v_h), \mathbb{D}_2(v_h)) + B^2(\mathbb{D}_2(v_h), \mathbb{D}_1(v_h))\big) + C_7\tau(\|u_h^n\| + \|v_h^n\|)\\ &(\|\mathbb{D}_2(u_h)\| + \|\mathbb{D}_2(v_h)\|) + C_8\tau(\|u_h^n\| + \|v_h^n\|)(\|\mathbb{D}_1(u_h)\| + \|\mathbb{D}_1(v_h)\|)\end{aligned}$$

From Lemma 3.2 and Cauchy's inequality, we get

$$\begin{aligned}
&\tau\big(B^1(\mathbb{D}_1(u_h),\mathbb{D}_2(u_h))+B^1(\mathbb{D}_2(u_h),\mathbb{D}_1(u_h))\big)+\tau\big(B^2(\mathbb{D}_1(v_h),\mathbb{D}_2(v_h))\\
&\quad+B^2(\mathbb{D}_2(v_h),\mathbb{D}_1(v_h))\big)\\
=&-\tau\gamma\sum_j[\![\mathbb{D}_1(u_h)]\!]_{j+\frac12}\cdot[\![\mathbb{D}_2(u_h)]\!]_{j+\frac12}-\tau\gamma\sum_j[\![\mathbb{D}_1(v_h)]\!]_{j+\frac12}\cdot[\![\mathbb{D}_2(v_h)]\!]_{j+\frac12}\\
\leqslant&\frac{\tau\gamma}{4}\sum_j\left([\![\mathbb{D}_1(u_h)]\!]^2_{j+\frac12}+[\![\mathbb{D}_1(v_h)]\!]^2_{j+\frac12}\right)+\tau\gamma\sum_j\left([\![\mathbb{D}_2(u_h)]\!]^2_{j+\frac12}+[\![\mathbb{D}_2(v_h)]\!]^2_{j+\frac12}\right)
\end{aligned}$$

From Lemma 4.2 and Cauchy's inequality, we get

$$\begin{aligned}
&C_7\tau(\|u_h^n\|+\|v_h^n\|)(\|\mathbb{D}_2(u_h)\|+\|\mathbb{D}_2(v_h)\|)+C_8\tau(\|u_h^n\|+\|v_h^n\|)(\|\mathbb{D}_1(u_h)\|+\|\mathbb{D}_1(v_h)\|)\\
\leqslant&(2(C_5+C_6)C_7+2(\alpha_1+\alpha_2+1)C_8)\tau(\|u_h^n\|^2+\|v_h^n\|^2)
\end{aligned}$$

Define $C_9=2(C_5+C_6)C_7+2(\alpha_1+\alpha_2+1)C_8$. We get

$$\begin{aligned}
\Lambda_1\leqslant&-\Big[\|\mathbb{D}_2(u_h)\|^2+\|\mathbb{D}_2(v_h)\|^2\Big]+\frac{\tau\gamma}{4}\sum_j\left([\![\mathbb{D}_1(u_h)]\!]^2_{j+\frac12}+[\![\mathbb{D}_1(v_h)]\!]^2_{j+\frac12}\right)\\
&+\tau\gamma\sum_j\left([\![\mathbb{D}_2(u_h)]\!]^2_{j+\frac12}+[\![\mathbb{D}_2(v_h)]\!]^2_{j+\frac12}\right)+C_9\tau(\|u_h^n\|^2+\|v_h^n\|^2)
\end{aligned}\tag{4.8}$$

From Lemma 3.2, Lemma 4.2, Lemma 4.3 and Cauchy's inequality, we have

$$\Lambda_2\leqslant-\frac{\tau\gamma}{2}\sum_j\left([\![\mathbb{D}_2(u_h)]\!]^2_{j+\frac12}+[\![\mathbb{D}_2(v_h)]\!]^2_{j+\frac12}\right)+C_{10}\tau(\|u_h^n\|^2+\|v_h^n\|^2)\tag{4.9}$$

where $C_{10}=2(C_5+C_6)C_8$.

From Lemma 4.3, Lemma 3.1, Cauchy's inequality and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|\mathbb{D}_3(u_h)\|^2+\|\mathbb{D}_3(v_h)\|^2\leqslant&\frac{\tau}{3}\big(B^1(\mathbb{D}_2(u_h),\mathbb{D}_3(u_h))+B^2(\mathbb{D}_2(v_h),\mathbb{D}_3(v_h))\big)\\
&+\frac{C_8\tau}{3}(\|u_h^n\|+\|v_h^n\|)(\|\mathbb{D}_3(u_h)\|+\|\mathbb{D}_3(v_h)\|)\\
\leqslant&\frac{\sqrt2+1}{3}\tau\gamma M/h\big(\|\mathbb{D}_2(u_h)\|\|\mathbb{D}_3(u_h)\|+\|\mathbb{D}_2(v_h)\|\|\mathbb{D}_3(v_h)\|\big)\\
&+\frac{2C_8}{3}\tau(\|u_h^n\|^2+\|v_h^n\|^2)^{1/2}(\|\mathbb{D}_3(u_h)\|^2+\|\mathbb{D}_3(v_h)\|^2)^{1/2}\\
\leqslant&\frac{\sqrt2+1}{3}\lambda\big(\|\mathbb{D}_2(u_h)\|^2+\|\mathbb{D}_2(v_h)\|^2\big)^{1/2}\big(\|\mathbb{D}_3(u_h)\|^2+\|\mathbb{D}_3(v_h)\|^2\big)^{1/2}\\
&+\frac{2C_8}{3}\tau(\|u_h^n\|^2+\|v_h^n\|^2)^{1/2}(\|\mathbb{D}_3(u_h)\|^2+\|\mathbb{D}_3(v_h)\|^2)^{1/2}
\end{aligned}$$

When $\|\mathbb{D}_3(u_h)\|^2+\|\mathbb{D}_3(v_h)\|^2\neq0$, we have

$$\begin{aligned}
\big(\|\mathbb{D}_3(u_h)\|^2+\|\mathbb{D}_3(v_h)\|^2\big)^{1/2}\leqslant&\frac{\sqrt2+1}{3}\lambda\big(\|\mathbb{D}_2(u_h)\|^2+\|\mathbb{D}_2(v_h)\|^2\big)^{1/2}\\
&+\frac{2C_8}{3}\tau(\|u_h^n\|^2+\|v_h^n\|^2)^{1/2}
\end{aligned}$$

Clearly, this inequality holds for $\|\mathbb{D}_3(u_h)\|^2+\|\mathbb{D}_3(v_h)\|^2=0$ as well.

Through Cauchy's inequality and $\tau\leqslant1/\gamma$, we get

$$\begin{aligned}
\Lambda_3=&3\Big[\big(\|\mathbb{D}_3(u_h)\|^2+\|\mathbb{D}_3(v_h)\|^2\big)^{1/2}\Big]^2\\
\leqslant&3\times2\Big[\frac{2\sqrt2+3}{9}\lambda^2\big(\|\mathbb{D}_2(u_h)\|^2+\|\mathbb{D}_2(v_h)\|^2\big)+\big(\frac{2C_8}{3}\tau\big)^2(\|u_h^n\|^2+\|v_h^n\|^2)\Big]
\end{aligned}$$

$$\leqslant \sigma\lambda^2\big(\|\mathbb{D}_2(u_h)\|^2 + \|\mathbb{D}_2(v_h)\|^2\big) + \frac{8(C_8)^2}{3\gamma}\tau(\|u_h^n\|^2 + \|v_h^n\|^2) \tag{4.10}$$

Here $\sigma = (4\sqrt{2}+6)/3$.

From (4.8), (4.9) and (4.10), we have

$$\begin{aligned}\Lambda_1 + \Lambda_2 + \Lambda_3 \leqslant &(\sigma\lambda^2 - 1)\big(\|\mathbb{D}_2(u_h)\|^2 + \|\mathbb{D}_2(v_h)\|^2\big) + C_{11}\tau(\|u_h^n\|^2 + \|v_h^n\|^2)\\ &+ \frac{\tau\gamma}{4}\sum_j \big([\![\mathbb{D}_1(u_h)]\!]^2_{j+\frac12} + [\![\mathbb{D}_1(v_h)]\!]^2_{j+\frac12}\big) + \frac{\tau\gamma}{2}\sum_j \big([\![\mathbb{D}_2(u_h)]\!]^2_{j+\frac12}\\ &+ [\![\mathbb{D}_2(v_h)]\!]^2_{j+\frac12}\big)\end{aligned}$$

Here $C_{11} = C_9 + C_{10} + \frac{8(C_8)^2}{3\gamma}$.

Through Cauchy's inequality we get

$$\begin{aligned}\frac{\tau\gamma}{4}\sum_j \big([\![\mathbb{D}_1(u_h)]\!]^2_{j+\frac12} + [\![\mathbb{D}_1(v_h)]\!]^2_{j+\frac12}\big) \leqslant &\frac{\tau\gamma}{2}\sum_j \big([\![u_h^n]\!]^2_{j+\frac12} + [\![v_h^n]\!]^2_{j+\frac12} + [\![u_h^{n,1}]\!]^2_{j+\frac12}\\ &+ [\![v_h^{n,1}]\!]^2_{j+\frac12}\big)\end{aligned}$$

Through Cauchy's inequality and the inverse inequality (3.5) we get

$$\begin{aligned}\frac{\tau\gamma}{2}\sum_j \big([\![\mathbb{D}_2(u_h)]\!]^2_{j+\frac12} + [\![\mathbb{D}_2(v_h)]\!]^2_{j+\frac12}\big) \leqslant &\tau\gamma\big(\|\mathbb{D}_2(u_h)\|^2_{\Gamma_h} + \|\mathbb{D}_2(v_h)\|^2_{\Gamma_h}\big)\\ \leqslant &\lambda\big(\|\mathbb{D}_2(u_h)\|^2 + \|\mathbb{D}_2(v_h)\|^2\big)\end{aligned}$$

We therefore have

$$\begin{aligned}\Lambda_1 + \Lambda_2 + \Lambda_3 \leqslant &(\sigma\lambda^2 + \lambda - 1)\big(\|\mathbb{D}_2(u_h)\|^2 + \|\mathbb{D}_2(v_h)\|^2\big) + C_{11}\tau(\|u_h^n\|^2 + \|v_h^n\|^2)\\ &+ \frac{\tau\gamma}{2}\sum_j \big([\![u_h^n]\!]^2_{j+\frac12} + [\![v_h^n]\!]^2_{j+\frac12} + [\![u_h^{n,1}]\!]^2_{j+\frac12} + [\![v_h^{n,1}]\!]^2_{j+\frac12}\big)\end{aligned} \tag{4.11}$$

From Lemma 3.2, we get

$$\begin{aligned}\mathbb{S}_2 = &-\frac{\gamma}{2}\sum_j \Big[[\![u_h^n]\!]^2_{j+\frac12} + [\![v_h^n]\!]^2_{j+\frac12} + [\![u_h^{n,1}]\!]^2_{j+\frac12} + [\![v_h^{n,1}]\!]^2_{j+\frac12}\\ &+ 4[\![u_h^{n,2}]\!]^2_{j+\frac12} + 4[\![v_h^{n,2}]\!]^2_{j+\frac12}\Big]\end{aligned} \tag{4.12}$$

From Lemma 4.1, Lemma 4.2 and Cauchy's inequality, we have

$$\begin{aligned}\mathbb{S}_3 \leqslant &\frac{a_2}{2}(\|u_h^n\|^2 + \|v_h^n\|^2 + \|u_h^{n,1}\|^2 + \|v_h^{n,1}\|^2 + 4\|u_h^{n,2}\|^2 + 4\|v_h^{n,2}\|^2)\\ \leqslant &C_{12}(\|u_h^n\|^2 + \|v_h^n\|^2)\end{aligned} \tag{4.13}$$

Here $C_{12} = a_2((\alpha_1+\alpha_2)^2 + 4(C_3+C_4)^2 + 1)/2$.

When $\lambda \leqslant \frac{-3+\sqrt{3(27+16\sqrt{2})}}{2(6+4\sqrt{2})} \approx 0.39$, we have $\sigma\lambda^2 + \lambda - 1 \leqslant 0$. Hence from (4.11), (4.12) and (4.13) we can get

$$\Lambda_1 + \Lambda_2 + \Lambda_3 + \tau(\mathbb{S}_2 + \mathbb{S}_3) \leqslant (C_{11} + C_{12})\tau(\|u_h^n\|^2 + \|v_h^n\|^2) \tag{4.14}$$

Through (4.7) and (4.14), we get

$$\begin{aligned}\|u_h^{n+1}\|^2 + \|v_h^{n+1}\|^2 \leqslant &(1 + C_{13}\tau)(\|u_h^n\|^2 + \|v_h^n\|^2)\\ \leqslant &e^{C_{13}\tau}(\|u_h^n\|^2 + \|v_h^n\|^2)\end{aligned} \tag{4.15}$$

Here $C_{13} = (C_{11} + C_{12})/3$ is independent of τ, h. Thus we have obtained the L^2 stability of the scheme (4.3).

□

4.5 Error estimates

For smooth solutions we can obtain the following a priori error estimates.

Theorem 4.5. *Assume* $uu, v \in W^4_\infty([0,T]; H^{k+1}([0,L]))$ *are solutions to (2.1) and* u_h, v_h *are the numerical solutions to the scheme (4.3). Let* V_h *be the finite element space defined in (3.2). If* $\gamma M\tau/h \leqslant 0.39$*, where* M *is the constant defined in (3.6), then we have the following error estimate*

$$\max_{n\tau\leqslant T}\left(\|u^n - u^n_h\| + \|v^n - v^n_h\|\right) \leqslant C(h^{k+1} + \tau^3)$$

where C *is a constant independent of* τ, h, u_h, v_h.

First we introduce reference functions which are parallel to $u_h^{n,1}$, $v_h^{n,1}$, $u_h^{n,2}$, $v_h^{n,2}$, u_h^{n+1}, v_h^{n+1}.

$$\begin{aligned}
u^{n,1} =& u^n - \tau\gamma u^n_x + \tau\big(-\lambda^n_1 u^n + \lambda^n_2 v^n\big)\\
v^{n,1} =& v^n + \tau\gamma v^n_x + \tau\big(\lambda^n_1 u^n - \lambda^n_2 v^n\big)\\
u^{n,2} =& \frac{3}{4}u^n + \frac{1}{4}u^{n,1} - \frac{\tau}{4}\gamma u^{n,1}_x + \frac{\tau}{4}\big(-\lambda^{n,1}_1 u^{n,1} + \lambda^{n,1}_2 v^{n,1}\big)\\
v^{n,2} =& \frac{3}{4}v^n + \frac{1}{4}v^{n,1} + \frac{\tau}{4}\gamma v^{n,1}_x + \frac{\tau}{4}\big(\lambda^{n,1}_1 u^{n,1} - \lambda^{n,1}_2 v^{n,1}\big)\\
u^{n+1} =& \frac{1}{3}u^n + \frac{2}{3}u^{n,2} - \frac{2\tau}{3}\gamma u^{n,2}_x + \frac{2\tau}{3}\big(-\lambda^{n,2}_1 u^{n,2} + \lambda^{n,2}_2 v^{n,2}\big) + \mathcal{E}_1\\
v^{n+1} =& \frac{1}{3}v^n + \frac{2}{3}v^{n,2} + \frac{2\tau}{3}\gamma v^{n,2}_x + \frac{2\tau}{3}\big(\lambda^{n,2}_1 u^{n,2} - \lambda^{n,2}_2 v^{n,2}\big) + \mathcal{E}_2
\end{aligned}$$

Here $\lambda^{n,i}_1 = \lambda_1(y_1[u^{n,i}, v^{n,i}])$, $\lambda^{n,i}_2 = \lambda_2(y_2[u^{n,i}, v^{n,i}])$. We have the estimates for $\mathcal{E}_1$ and $\mathcal{E}_2$ due to the properties of the third order Runge-Kutta method:

$$\|\mathcal{E}_1(x,t)\| \leqslant C_{14}\tau^4, \quad \|\mathcal{E}_2(x,t)\| \leqslant C_{14}\tau^4 \tag{4.16}$$

Here C_{14} depends on u, v and their partial derivatives. We require $\|u_{tttt}\|$ and $\|v_{tttt}\|$ to be bounded uniformly in $[0,T]$. For more details we refer to [20].

Multiplying these equations with the test function $\varphi \in V_h$, then integrating them on I_j, we get

$$\int_{I_j} u^{n,1}\varphi dx = \int_{I_j} u^n\varphi dx + \tau\Big[B^1_j(u^n,\varphi) - \int_{I_j}\lambda^n_1 u^n\varphi dx + \int_{I_j}\lambda^n_2 v^n\varphi dx\Big] \tag{4.17a}$$

$$\int_{I_j} v^{n,1}\varphi dx = \int_{I_j} v^n\varphi dx + \tau\Big[B^2_j(v^n,\varphi) + \int_{I_j}\lambda^n_1 u^n\varphi dx - \int_{I_j}\lambda^n_2 v^n\varphi dx\Big] \tag{4.17b}$$

$$\begin{aligned}\int_{I_j} u^{n,2}\varphi dx =& \frac{3}{4}\int_{I_j} u^n\varphi dx + \frac{1}{4}\int_{I_j} u^{n,1}\varphi dx + \frac{\tau}{4}\Big[B^1_j(u^{n,1},\varphi) - \int_{I_j}\lambda^{n,1}_1 u^{n,1}\varphi dx\\
&+ \int_{I_j}\lambda^{n,1}_2 v^{n,1}\varphi dx\Big]\end{aligned} \tag{4.17c}$$

$$\begin{aligned}\int_{I_j} v^{n,2}\varphi dx =& \frac{3}{4}\int_{I_j} v^n\varphi dx + \frac{1}{4}\int_{I_j} v^{n,1}\varphi dx + \frac{\tau}{4}\Big[B^2_j(v^{n,1},\varphi) + \int_{I_j}\lambda^{n,1}_1 u^{n,1}\varphi dx\\
&- \int_{I_j}\lambda^{n,1}_2 v^{n,1}\varphi dx\Big]\end{aligned} \tag{4.17d}$$

$$\begin{aligned}\int_{I_j} u^{n+1}\varphi dx =& \frac{1}{3}\int_{I_j} u^n\varphi dx + \frac{2}{3}\int_{I_j} u^{n,2}\varphi dx + \frac{2\tau}{3}\Big[B^1_j(u^{n,2},\varphi) - \int_{I_j}\lambda^{n,2}_1 u^{n,2}\varphi dx\\
&+ \int_{I_j}\lambda^{n,2}_2 v^{n,2}\varphi dx\Big] + \int_{I_j}\mathcal{E}_1\varphi dx\end{aligned} \tag{4.17e}$$

$$\begin{aligned}\int_{I_j} v^{n+1}\varphi dx =& \frac{1}{3}\int_{I_j} v^n\varphi dx + \frac{2}{3}\int_{I_j} v^{n,2}\varphi dx + \frac{2\tau}{3}\Big[B^2_j(v^{n,2},\varphi) + \int_{I_j}\lambda^{n,2}_1 u^{n,2}\varphi dx\\
&- \int_{I_j}\lambda^{n,2}_2 v^{n,2}\varphi dx\Big] + \int_{I_j}\mathcal{E}_2\varphi dx\end{aligned} \tag{4.17f}$$

For any function w, define

$$\xi^{n,j}(w)=\mathbb{R}_h w^{n,j}-w_h^{n,j},\quad \eta^{n,j}(w)=\mathbb{R}_h w^{n,j}-w^{n,j},\quad e^{n,j}(w)=\xi^{n,j}(w)-\eta^{n,j}(w)$$

$j=0,1,2,3$. Here $\xi^{n,0}(w)=\xi^{n}(w),\xi^{n,3}(w)=\xi^{n+1}(w)$, $\eta^{n,0}(w)=\eta^{n}(w),\eta^{n,3}(w)=\eta^{n+1}(w)$, $e^{n,0}(w)=e^{n}(w),e^{n,3}(w)=e^{n+1}(w)$. When $w\in W_2^{k+1}([0,L])$, we have the estimate of η in (3.3). A result cited from [21] is the following. When $w_t\in W_2^{k+1}([0,L])$ for any $t\in[0,T]$,

$$\|\sum_{0\leqslant i\leqslant 3} d_j\eta^{n,i}(w)\|\leqslant C_{15}h^{k+1}\tau,\quad \forall\sum_{0\leqslant j\leqslant 3}d_j=0,\quad \forall n\tau\leqslant T. \tag{4.18}$$

Here C_{15} is independent of h.

Subtracting (4.17) from (4.3), we can get

$$\begin{aligned}\int_{I_j}\xi^{n,1}(u)\varphi dx=&\int_{I_j}\xi^{n}(u)\varphi dx+\int_{I_j}(\eta^{n,1}(u)-\eta^{n}(u))\varphi dx+\tau\Big[B_j^1(\xi^n(u),\varphi)\\&-\int_{I_j}(\lambda_1^n u^n-(\lambda_1)_h^n u_h^n)\varphi dx+\int_{I_j}(\lambda_2^n v^n-(\lambda_2)_h^n v_h^n)\varphi dx\Big]\end{aligned} \tag{4.19a}$$

$$\begin{aligned}\int_{I_j}\xi^{n,1}(v)\varphi dx=&\int_{I_j}\xi^{n}(v)\varphi dx+\int_{I_j}(\eta^{n,1}(v)-\eta^{n}(v))\varphi dx+\tau\Big[B_j^2(\xi^n(v),\varphi)\\&+\int_{I_j}(\lambda_1^n u^n-(\lambda_1)_h^n u_h^n)\varphi dx-\int_{I_j}(\lambda_2^n v^n-(\lambda_2)_h^n v_h^n)\varphi dx\Big]\end{aligned} \tag{4.19b}$$

$$\begin{aligned}\int_{I_j}\xi^{n,2}(u)\varphi dx=&\frac34\int_{I_j}\xi^{n}(u)\varphi dx+\frac14\int_{I_j}\xi^{n,1}(u)\varphi dx+\int_{I_j}(\eta^{n,2}(u)-\frac34\eta^{n}(u)-\frac14\eta^{n,1}(u))\varphi dx\\&+\frac{\tau}{4}\Big[B_j^1(\xi^{n,1}(u),\varphi)-\int_{I_j}(\lambda_1^{n,1}u^{n,1}-(\lambda_1)_h^{n,1}u_h^{n,1})\varphi dx\\&+\int_{I_j}(\lambda_2^{n,1}v^{n,1}-(\lambda_2)_h^{n,1}v_h^{n,1})\varphi dx\Big]\end{aligned} \tag{4.19c}$$

$$\begin{aligned}\int_{I_j}\xi^{n,2}(v)\varphi dx=&\frac34\int_{I_j}\xi^{n}(v)\varphi dx+\frac14\int_{I_j}\xi^{n,1}(v)\varphi dx+\int_{I_j}(\eta^{n,2}(v)-\frac34\eta^{n}(v)-\frac14\eta^{n,1}(v))\varphi dx\\&+\frac{\tau}{4}\Big[B_j^2(\xi^{n,1}(v),\varphi)+\int_{I_j}(\lambda_1^{n,1}u^{n,1}-(\lambda_1)_h^{n,1}u_h^{n,1})\varphi dx\\&-\int_{I_j}(\lambda_2^{n,1}v^{n,1}-(\lambda_2)_h^{n,1}v_h^{n,1})\varphi dx\Big]\end{aligned} \tag{4.19d}$$

$$\begin{aligned}\int_{I_j}\xi^{n+1}(u)\varphi dx=&\frac13\int_{I_j}\xi^{n}(u)\varphi dx+\frac23\int_{I_j}\xi^{n,2}(u)\varphi dx+\int_{I_j}(\eta^{n+1}(u)-\frac13\eta^{n}(u)+\frac23\eta^{n,2}(u))\varphi dx\\&+\frac{2\tau}{3}\Big[B_j^1(\xi^{n,2}(u),\varphi)-\int_{I_j}(\lambda_1^{n,2}u^{n,2}-(\lambda_1)_h^{n,2}u_h^{n,2})\varphi dx\\&+\int_{I_j}(\lambda_2^{n,2}v^{n,2}-(\lambda_2)_h^{n,2}v_h^{n,2})\varphi dx\Big]+\int_{I_j}\mathcal{E}_1\varphi dx\end{aligned} \tag{4.19e}$$

$$\begin{aligned}\int_{I_j}\xi^{n+1}(v)\varphi dx=&\frac13\int_{I_j}\xi^{n}(v)\varphi dx+\frac23\int_{I_j}\xi^{n,2}(v)\varphi dx+\int_{I_j}(\eta^{n+1}(v)-\frac13\eta^{n}(v)-\frac23\eta^{n,2}(v))\varphi dx\\&+\frac{2\tau}{3}\Big[B_j^2(\xi^{n,2}(v),\varphi)+\int_{I_j}(\lambda_1^{n,2}u^{n,2}-(\lambda_1)_h^{n,2}u_h^{n,2})\varphi dx\\&-\int_{I_j}(\lambda_2^{n,2}v^{n,2}-(\lambda_2)_h^{n,2}v_h^{n,2})\varphi dx\Big]+\int_{I_j}\mathcal{E}_2\varphi dx\end{aligned} \tag{4.19f}$$

The proofs of following three Lemmas are given in the Appendix (Section 8).

Lemma 4.6.

$$\left|\int_0^L(\lambda_1^{n,i}u^{n,i}-(\lambda_1)_h^{n,i}u_h^{n,i})\varphi dx\right|\leqslant\qquad C_{16}(\|\xi^{n,i}(u)\|+\|\xi^{n,i}(v)\|)\|\varphi\|+2C_{16}C_1h^{k+1}\|\varphi\|,$$

$$\left|\int_0^L (\lambda_2^{n,i} v^{n,i} - (\lambda_2)_h^{n,i} v_h^{n,i})\varphi dx\right| \leqslant \qquad C_{16}(\|\xi^{n,i}(u)\| + \|\xi^{n,i}(v)\|)\|\varphi\| + 2C_{16}C_1 h^{k+1}\|\varphi\|.$$

$i = 0,1,2$. *Here* $C_{16} = a_2 M_T(q_r C_r + q_a C_a + q_{al}C_{al}/2) + a_1 + a_2$, $\|u^{n,i}\|, \|v^{n,i}\| \leqslant M_T$, $\forall t \in [0,T]$, $i = 0,1,2$. $\left(\int_0^L (K_r(s))^2 ds\right)^{1/2} \leqslant C_r$, $\left(\int_0^L (K_a(s))^2 ds\right)^{1/2} \leqslant C_a$, $\left(\int_0^L (K_{al}(s))^2 ds\right)^{1/2} \leqslant C_{al}$.

Lemma 4.7.

$$\begin{aligned}
\|\xi^{n,1}(u)\| &\leqslant \beta_1\|\xi^n(u)\| + \beta_2\|\xi^n(v)\| + \beta_3 h^{k+1}\tau\\
\|\xi^{n,1}(v)\| &\leqslant \beta_1\|\xi^n(v)\| + \beta_2\|\xi^n(u)\| + \beta_3 h^{k+1}\tau\\
\|\xi^{n,2}(u)\| &\leqslant C_{17}\|\xi^n(u)\| + C_{18}\|\xi^n(v)\| + C_{19} h^{k+1}\tau\\
\|\xi^{n,2}(v)\| &\leqslant C_{17}\|\xi^n(v)\| + C_{18}\|\xi^n(u)\| + C_{19} h^{k+1}\tau\\
\|\mathbb{D}_1(\xi(u))\| &\leqslant (\beta_1+1)\|\xi^n(u)\| + \beta_2\|\xi^n(v)\| + \beta_3 h^{k+1}\tau\\
\|\mathbb{D}_1(\xi(v))\| &\leqslant (\beta_1+1)\|\xi^n(v)\| + \beta_2\|\xi^n(u)\| + \beta_3 h^{k+1}\tau\\
\|\mathbb{D}_2(\xi(u))\| &\leqslant C_{20}\|\xi^n(u)\| + C_{21}\|\xi^n(v)\| + C_{22} h^{k+1}\tau\\
\|\mathbb{D}_2(\xi(v))\| &\leqslant C_{20}\|\xi^n(v)\| + C_{21}\|\xi^n(u)\| + C_{22} h^{k+1}\tau
\end{aligned}$$

Here $\beta_1 = 1+(\sqrt{2}+1)M+2C_{16}/\gamma$, $\beta_2 = 2C_{16}/\gamma$, $\beta_3 = C_{15}+4C_{16}C_1$, $C_{17} = \frac{3}{4}+\frac{1}{4}(\beta_1^2+\beta_2^2)$, $C_{18} = \frac{1}{2}\beta_1\beta_2$, $C_{19} = \frac{1}{4}(\beta_1+\beta_2+1)\beta_3$, $C_{20} = 2C_{17}+\beta_1+1$, $C_{21} = 2C_{18}+\beta_2$, $C_{22} = 2C_{19}+\beta_3$.

Lemma 4.8.

$$\begin{aligned}
(\mathbb{D}_1(\xi(u)),\varphi) &\leqslant \tau B^1(\xi^n(u),\varphi) + 2C_{16}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)\|\varphi\| + \beta_3 h^{k+1}\tau\|\varphi\|\\
(\mathbb{D}_1(\xi(v)),\varphi) &\leqslant \tau B^2(\xi^n(v),\varphi) + 2C_{16}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)\|\varphi\| + \beta_3 h^{k+1}\tau\|\varphi\|\\
(\mathbb{D}_2(\xi(u)),\varphi) &\leqslant \frac{\tau}{2}B^1(\mathbb{D}_1(\xi(u)),\varphi) + \frac{C_{23}\tau}{2}(\|\xi^n(u)\| + \|\xi^n(v)\|)\|\varphi\|\\
&\quad + C_{24}h^{k+1}\tau\|\varphi\|\\
(\mathbb{D}_2(\xi(v)),\varphi) &\leqslant \frac{\tau}{2}B^2(\mathbb{D}_1(\xi(v)),\varphi) + \frac{C_{23}\tau}{2}(\|\xi^n(u)\| + \|\xi^n(v)\|)\|\varphi\|\\
&\quad + C_{24}h^{k+1}\tau\|\varphi\|\\
(\mathbb{D}_3(\xi(u)),\varphi) &\leqslant \frac{\tau}{3}B^1(\mathbb{D}_2(\xi(u)),\varphi) + \frac{C_{25}\tau}{3}(\|\xi^n(u)\| + \|\xi^n(v)\|)\|\varphi\|\\
&\quad + C_{26}h^{k+1}\tau\|\varphi\| + C_{14}\tau^4\|\varphi\|\\
(\mathbb{D}_3(\xi(v)),\varphi) &\leqslant \frac{\tau}{3}B^2(\mathbb{D}_2(\xi(v)),\varphi) + \frac{C_{25}\tau}{3}(\|\xi^n(u)\| + \|\xi^n(v)\|)\|\varphi\|\\
&\quad + C_{26}h^{k+1}\tau\|\varphi\| + C_{14}\tau^4\|\varphi\|
\end{aligned}$$

Here $C_{23} = 2C_{16}(\beta_1+\beta_2+1)$, $C_{24} = 2C_{16}\beta_3/\gamma + \beta_3$, $C_{25} = C_{23} + 8C_{16}(C_{17}+C_{18})$, $C_{26} = 4C_{16}((\beta_3 + 2C_{19})/\gamma + 4C_1)/3 + C_{15}$.

We are now ready to prove Theorem 4.5.

Proof. Taking $\varphi = \xi^n(u)$ in (4.19a), $\varphi = \xi^n(v)$ in (4.19b), $\varphi = 4\xi^{n,1}(u)$ in (4.19c), $\varphi = 4\xi^{n,1}(v)$ in (4.19d), $\varphi = 6\xi^{n,2}(u)$ in (4.19e), $\varphi = 6\xi^{n,2}(v)$ in (4.19f), summing them over j and adding them up, we can get

$$\int_0^1 \hat{\mathbb{S}}_1 dx = \tau(\hat{\mathbb{S}}_2 + \hat{\mathbb{S}}_3) + \hat{\mathbb{S}}_4 \tag{4.21}$$

Here

$$
\begin{aligned}
\hat{\mathbb{S}}_1 =& -2\xi^n(u)\xi^{n,1}(u) - (\xi^n(u))^2 + 4\xi^{n,1}(u)\xi^{n,2}(u) - (\xi^{n,1}(u))^2 + 6\xi^{n,2}(u)\xi^{n+1}(u) \\
&- 2\xi^n(u)\xi^{n,2}(u) - 4(\xi^{n,2}(u))^2 - 2\xi^n(v)\xi^{n,1}(v) - (\xi^n(v))^2 + 4\xi^{n,1}(v)\xi^{n,2}(v) \\
&- (\xi^{n,1}(v))^2 + 6\xi^{n,2}(v)\xi^{n+1}(v) - 2\xi^n(v)\xi^{n,2}(v) - 4(\xi^{n,2}(v))^2 \\
=& 3((\xi^{n+1}(u))^2 + (\xi^{n+1}(v))^2) - 3((\xi^n(u))^2 + (\xi^n(v))^2) - \big((\mathbb{D}_2(\xi(u)))^2 + (\mathbb{D}_2(\xi(v)))^2\big) \\
&- 3(\mathbb{D}_1 + \mathbb{D}_2 + \mathbb{D}_3)(\xi(u)) \cdot \mathbb{D}_3(\xi(u)) - 3(\mathbb{D}_1 + \mathbb{D}_2 + \mathbb{D}_3)(\xi(v)) \cdot \mathbb{D}_3(\xi(v))
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbb{S}}_2 =& B^1(\xi^n(u), \xi^n(u)) + B^2(\xi^n(v), \xi^n(v)) + B^1(\xi^{n,1}(u), \xi^{n,1}(u)) + B^2(\xi^{n,1}(v), \xi^{n,1}(v)) \\
&+ 4B^1(\xi^{n,2}(u), \xi^{n,2}(u)) + 4B^2(\xi^{n,2}(u), \xi^{n,2}(u))
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbb{S}}_3 =& -\int_0^L (\lambda_1^n u^n - (\lambda_1)_h^n u_h^n)\xi^n(u)dx + \int_0^L (\lambda_2^n v^n - (\lambda_2)_h^n v_h^n)\xi^n(u)dx \\
&+ \int_0^L (\lambda_1^n u^n - (\lambda_1)_h^n u_h^n)\xi^n(v)dx - \int_0^L (\lambda_2^n v^n - (\lambda_2)_h^n v_h^n)\xi^n(v)dx \\
&- \int_0^L (\lambda_1^{n,1} u^{n,1} - (\lambda_1)_h^{n,1} u_h^{n,1})\xi^{n,1}(u)dx + \int_0^L (\lambda_2^{n,1} v^{n,1} - (\lambda_2)_h^{n,1} v_h^{n,1})\xi^{n,1}(u)dx \\
&+ \int_0^L (\lambda_1^{n,1} u^{n,1} - (\lambda_1)_h^{n,1} u_h^{n,1})\xi^{n,1}(v)dx - \int_0^L (\lambda_2^{n,1} v^{n,1} - (\lambda_2)_h^{n,1} v_h^{n,1})\xi^{n,1}(v)dx \\
&- 4\int_0^L (\lambda_1^{n,2} u^{n,2} - (\lambda_1)_h^{n,2} u_h^{n,2})\xi^{n,2}(u)dx + 4\int_0^L (\lambda_2^{n,2} v^{n,2} - (\lambda_2)_h^{n,2} v_h^{n,2})\xi^{n,2}(u)dx \\
&+ 4\int_0^L (\lambda_1^{n,2} u^{n,2} - (\lambda_1)_h^{n,2} u_h^{n,2})\xi^{n,2}(v)dx - 4\int_0^L (\lambda_2^{n,2} v^{n,2} - (\lambda_2)_h^{n,2} v_h^{n,2})\xi^{n,2}(v)dx
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbb{S}}_4 =& \big(\eta^{n,1}(u) - \eta^n(u), \xi^n(u)\big) + \big(\eta^{n,1}(v) - \eta^n(v), \xi^n(v)\big) + \big(4\eta^{n,2}(u) - 3\eta^n(u) \\
&- \eta^{n,1}(u), \xi^{n,1}(u)\big) + \big(4\eta^{n,2}(v) - 3\eta^n(v) - \eta^{n,1}(v), \xi^{n,1}(v)\big) + 2\big(3\eta^{n+1}(u) - \\
&\eta^n(u) - 2\eta^{n,2}(u), \xi^{n,2}(u)\big) + 2\big(3\eta^{n+1}(v) - \eta^n(v) - 2\eta^{n,2}(vu), \xi^{n,2}(v)\big) \\
&+ 6\int_0^L \mathcal{E}_1 \xi^{n,2}(u)dx + 6\int_0^L \mathcal{E}_2 \xi^{n,2}(v)dx
\end{aligned}
$$

Denote

$$
\begin{aligned}
\hat{\Lambda}_1 =& \Big[\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2\Big] + 3\big(\mathbb{D}_1(\xi(u)), \mathbb{D}_3(\xi(u))\big) + 3\big(\mathbb{D}_1(\xi(v)), \mathbb{D}_3(\xi(v))\big) \\
\hat{\Lambda}_2 =& 3\big(\mathbb{D}_2(\xi(u)), \mathbb{D}_3(\xi(u))\big) + 3\big(\mathbb{D}_2(\xi(v)), \mathbb{D}_3(\xi(v))\big) \\
\hat{\Lambda}_3 =& 3\big(\mathbb{D}_3(\xi(u)), \mathbb{D}_3(\xi(u))\big) + 3\big(\mathbb{D}_3(\xi(v)), \mathbb{D}_3(\xi(v))\big)
\end{aligned}
$$

Then (4.21) can be written as

$$
3(\|\xi^{n+1}(u)\|^2 + \|\xi^{n+1}(v)\|^2) - 3(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) = \hat{\Lambda}_1 + \hat{\Lambda}_2 + \hat{\Lambda}_3 + \tau(\hat{\mathbb{S}}_2 + \hat{\mathbb{S}}_3) + \hat{\mathbb{S}}_4 \tag{4.22}
$$

Using Lemma 3.2 and Lemma 4.8, we get

$$
\begin{aligned}
\hat{\Lambda}_1 \leqslant& -\Big[\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2\Big] + \frac{\tau\gamma}{4}\sum_j \big([\![\mathbb{D}_1(\xi(u))]\!]_{j+\frac{1}{2}}^2 + [\![\mathbb{D}_1(\xi(v))]\!]_{j+\frac{1}{2}}^2\big) \\
&+ \tau\gamma\sum_j \big([\![\mathbb{D}_2(\xi(u))]\!]_{j+\frac{1}{2}}^2 + [\![\mathbb{D}_2(\xi(v))]\!]_{j+\frac{1}{2}}^2\big) + C_{23}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|) \\
&(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|) + 2C_{24}h^{k+1}\tau(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|)
\end{aligned}
$$

$$+ C_{25}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)(\|\mathbb{D}_1(\xi(u))\| + \|\mathbb{D}_1(\xi(v))\|) + 3C_{26}h^{k+1}\tau(\|\mathbb{D}_1(\xi(u))\| + \|\mathbb{D}_1(\xi(v))\|) + 3C_{14}\tau^4(\|\mathbb{D}_1(\xi(u))\| + \|\mathbb{D}_1(\xi(v))\|)$$

From Lemma (4.7), $\tau \leqslant 1/\gamma$ and Cauchy's inequality, we have

$$\begin{aligned}
&C_{23}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|)\\
\leqslant& C_{23}(C_{20} + C_{21})\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)^2 + \frac{2C_{22}C_{23}}{\gamma}h^{k+1}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)\\
\leqslant& 2C_{23}(C_{20} + C_{21})\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + \frac{C_{22}C_{23}}{\gamma}(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2 + 2h^{2k+2}\tau)\\
\leqslant& C_{27}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{28}h^{2k+2}\tau
\end{aligned}$$

Here $C_{27} = 2(C_{20} + C_{21})C_{23} + C_{22}C_{23}/\gamma$, $C_{28} = 2C_{22}C_{23}/\gamma$.

Similarly we have

$$\begin{aligned}
&2C_{24}h^{k+1}\tau(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|) \leqslant C_{29}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{30}h^{2k+2}\tau\\
&C_{25}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)(\|\mathbb{D}_1(\xi(u))\| + \|\mathbb{D}_1(\xi(v))\|) \leqslant C_{31}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{32}h^{2k+2}\tau\\
&3C_{26}h^{k+1}\tau(\|\mathbb{D}_1(\xi(u))\| + \|\mathbb{D}_1(\xi(v))\|) \leqslant C_{33}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{34}h^{2k+2}\tau\\
&3C_{14}\tau^4(\|\mathbb{D}_1(\xi(u))\| + \|\mathbb{D}_1(\xi(v))\|) \leqslant C_{35}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{36}h^{k+1}\tau^4 + C_{37}\tau^7
\end{aligned}$$

Here $C_{29} = (C_{20}+C_{21})C_{24}$, $C_{30} = 2C_{29}+4C_{22}C_{24}/\gamma$, $C_{31} = 2(\beta_1+\beta_2+1)C_{25}+\beta_3C_{25}/\gamma$, $C_{32} = 2\beta_3C_{25}/\gamma$, $C_{33} = \frac{3}{2}(\beta_1+\beta_2+1)C_{26}$, $C_{34} = 2C_{33}+6\beta_3C_{26}/\gamma$, $C_{35} = \frac{3}{2}(\beta_1+\beta_2+1)C_{14}$, $C_{36} = 6\beta_3C_{14}/\gamma$, $C_{37} = 2C_{35}$.

So we get

$$\begin{aligned}
\hat{\Lambda}_1 \leqslant& -\left[\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2\right] + \frac{\tau\gamma}{4}\sum_j \left([\![\mathbb{D}_1(\xi(u))]\!]^2_{j+\frac{1}{2}} + [\![\mathbb{D}_1(\xi(v))]\!]^2_{j+\frac{1}{2}}\right)\\
&+ \tau\gamma\sum_j \left([\![\mathbb{D}_2(\xi(u))]\!]^2_{j+\frac{1}{2}} + [\![\mathbb{D}_2(\xi(v))]\!]^2_{j+\frac{1}{2}}\right) + C_{38}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2)\\
&+ C_{39}h^{2k+2}\tau + C_{36}h^{k+1}\tau^4 + C_{37}\tau^7
\end{aligned} \tag{4.23}$$

Here $C_{38} = C_{27} + C_{29} + C_{31} + C_{33} + C_{35}$, $C_{39} = C_{28} + C_{30} + C_{32} + C_{34}$.

Using Lemma 3.2 and Lemma 4.8, we get

$$\begin{aligned}
\hat{\Lambda}_2 \leqslant& -\frac{\tau\gamma}{2}\sum_j \left([\![\mathbb{D}_2(\xi(u))]\!]^2_{j+\frac{1}{2}} + [\![\mathbb{D}_2(\xi(v))]\!]^2_{j+\frac{1}{2}}\right) + C_{25}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)\\
&(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|) + 3C_{26}h^{k+1}\tau(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|)\\
&+ 3C_{14}\tau^4(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|)
\end{aligned}$$

From Lemma (4.7), $\tau \leqslant 1/\gamma$ and Cauchy's inequality, we have

$$\begin{aligned}
&C_{25}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|) \leqslant C_{40}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{41}h^{2k+2}\tau\\
&3C_{26}h^{k+1}\tau(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|) \leqslant C_{42}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{43}h^{2k+2}\tau\\
&3C_{14}\tau^4(\|\mathbb{D}_2(\xi(u))\| + \|\mathbb{D}_2(\xi(v))\|) \leqslant C_{44}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{45}h^{k+1}\tau^4 + C_{46}\tau^7
\end{aligned}$$

Here $C_{40} = 2(C_{20} + C_{21})C_{25} + C_{22}C_{25}/\gamma$, $C_{41} = 2C_{22}C_{25}/\gamma$, $C_{42} = \frac{3}{2}(C_{20} + C_{21})C_{26}$, $C_{43} = 2C_{42} + 6C_{22}C_{26}/\gamma$, $C_{44} = \frac{3}{2}C_{14}(C_{20} + C_{21})$, $C_{45} = 6C_{14}C_{22}/\gamma$, $C_{46} = 2C_{44}$.

So we get

$$\begin{aligned}
\hat{\Lambda}_2 \leqslant& -\frac{\tau\gamma}{2}\sum_j \left([\![\mathbb{D}_2(\xi(u))]\!]^2_{j+\frac{1}{2}} + [\![\mathbb{D}_2(\xi(v))]\!]^2_{j+\frac{1}{2}}\right) + C_{47}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{48}h^{2k+2}\tau\\
&+ C_{45}h^{k+1}\tau^4 + C_{46}\tau^7
\end{aligned} \tag{4.24}$$

Here $C_{47} = C_{40} + C_{42} + C_{44}$, $C_{48} = C_{41} + C_{43}$.

From Lemma (3.1), Lemma (4.8), Cauchy's inequality and Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\big(\mathbb{D}_3(\xi(u)), \mathbb{D}_3(\xi(u))\big) + \big(\mathbb{D}_3(\xi(v)), \mathbb{D}_3(\xi(v))\big)\\ \leqslant&\frac{\sqrt{2}+1}{3}\lambda(\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2)^{1/2}(\|\mathbb{D}_3(\xi(u))\|^2 + \|\mathbb{D}_3(\xi(v))\|^2)^{1/2}\\ &+\sqrt{2}(\frac{C_{25}}{3}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|) + C_{26}h^{k+1}\tau + C_{14}\tau^4)(\|\mathbb{D}_3(\xi(u))\|^2 + \|\mathbb{D}_3(\xi(v))\|^2)^{1/2}\end{aligned}$$

When $\|\mathbb{D}_3(\xi(u))\|^2 + \|\mathbb{D}_3(\xi(v))\|^2 \neq 0$, we have

$$\begin{aligned}(\|\mathbb{D}_3(\xi(u))\|^2 + \|\mathbb{D}_3(\xi(v))\|^2)^{1/2} \leqslant& \frac{\sqrt{2}+1}{3}\lambda(\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2)^{1/2}\\ &+\sqrt{2}(\frac{C_{25}}{3}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|) + C_{26}h^{k+1}\tau + C_{14}\tau^4)\end{aligned}$$

which also holds for $\|\mathbb{D}_3(\xi(u))\|^2 + \|\mathbb{D}_3(\xi(v))\|^2 = 0$.

Using Cauchy's inequality and $\tau \leqslant 1/\gamma$, we get

$$\begin{aligned}\hat{\Lambda}_3 \leqslant&3 \times 2\Big[\Big(\frac{\sqrt{2}+1}{3}\lambda(\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2)^{1/2}\Big)^2 + \Big(\sqrt{2}(\frac{C_{25}}{3}\tau(\|\xi^n(u)\|\\ &+\|\xi^n(v)\|) + C_{26}h^{k+1}\tau + C_{14}\tau^4)\Big)^2\Big]\\ \leqslant&\sigma\lambda^2\Big[\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2\Big] + C_{49}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{50}h^{2k+2}\tau + C_{51}\tau^7\end{aligned}\tag{4.25}$$

Here $C_{49} = 8(C_{25})^2/\gamma$, $C_{50} = 36(C_{26})^2/\gamma$, $C_{51} = 36(C_{14})^2/\gamma$.

From (4.23), (4.24) and (4.25), we can get

$$\begin{aligned}\hat{\Lambda}_1 + \hat{\Lambda}_2 + \hat{\Lambda}_3 \leqslant&(\sigma\lambda^2 - 1)\Big[\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2\Big] + \frac{\tau\gamma}{4}\sum_j\big([\![\mathbb{D}_1(\xi(u))]\!]^2_{j+\frac12}\\ &+[\![\mathbb{D}_1(\xi(v))]\!]^2_{j+\frac12}\big) + \frac{\tau\gamma}{2}\sum_j\big([\![\mathbb{D}_2(\xi(u))]\!]^2_{j+\frac12} + [\![\mathbb{D}_2(\xi(v))]\!]^2_{j+\frac12}\big)\\ &+C_{52}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{53}h^{2k+2}\tau + C_{54}h^{k+1}\tau^4 + C_{55}\tau^7\end{aligned}$$

Here $C_{52} = C_{38} + C_{47} + C_{49}$, $C_{53} = C_{39} + C_{48} + C_{50}$, $C_{54} = C_{36} + C_{45}$, $C_{55} = C_{37} + C_{46} + C_{51}$.

Through Cauchy's inequality, we have

$$\begin{aligned}&\frac{\tau\gamma}{4}\sum_j\big([\![\mathbb{D}_1(\xi(u))]\!]^2_{j+\frac12} + [\![\mathbb{D}_1(\xi(v))]\!]^2_{j+\frac12}\big)\\ \leqslant&\frac{\tau\gamma}{2}\sum_j\Big[[\![\xi^n(u)]\!]^2_{j+\frac12} + [\![\xi^n(v)]\!]^2_{j+\frac12} + [\![\xi^{n,1}(u)]\!]^2_{j+\frac12} + [\![\xi^{n,1}(v)]\!]^2_{j+\frac12}\Big]\end{aligned}$$

Using Cauchy's inequality and the inverse inequality (3.5), we have

$$\begin{aligned}\frac{\tau\gamma}{2}\sum_j\big([\![\mathbb{D}_2(\xi(u))]\!]^2_{j+\frac12} + [\![\mathbb{D}_2(\xi(v))]\!]^2_{j+\frac12}\big) \leqslant&\tau\gamma(\|\mathbb{D}_2(\xi(u))\|^2_{\Gamma_h} + \|\mathbb{D}_2(\xi(v))\|^2_{\Gamma_h})\\ \leqslant&\lambda(\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2)\end{aligned}$$

So we have

$$\begin{aligned}\hat{\Lambda}_1 + \hat{\Lambda}_2 + \hat{\Lambda}_3 \leqslant&(\sigma\lambda^2 + \lambda - 1)\Big[\|\mathbb{D}_2(\xi(u))\|^2 + \|\mathbb{D}_2(\xi(v))\|^2\Big] + \frac{\tau\gamma}{2}\sum_j\Big[[\![\xi^n(u)]\!]^2_{j+\frac12}\\ &+[\![\xi^n(v)]\!]^2_{j+\frac12} + [\![\xi^{n,1}(u)]\!]^2_{j+\frac12} + [\![\xi^{n,1}(v)]\!]^2_{j+\frac12}\Big] + C_{52}\tau(\|\xi^n(u)\|^2\\ &+\|\xi^n(v)\|^2) + C_{53}h^{2k+2}\tau + C_{54}h^{k+1}\tau^4 + C_{55}\tau^7\end{aligned}\tag{4.26}$$

From Lemma 3.2 we get

$$\hat{\mathbb{S}}_2 = -\frac{\gamma}{2}\sum_j \Big[[\![\xi^n(u)]\!]^2_{j+\frac12} + [\![\xi^n(v)]\!]^2_{j+\frac12} + [\![\xi^{n,1}(u)]\!]^2_{j+\frac12} + [\![\xi^{n,1}(v)]\!]^2_{j+\frac12} + 4[\![\xi^{n,2}(u)]\!]^2_{j+\frac12} + 4[\![\xi^{n,2}(v)]\!]^2_{j+\frac12} \Big] \tag{4.27}$$

From Lemma 4.6, we get

$$\begin{aligned}\tau\hat{\mathbb{S}}_3 \leqslant & 2C_{16}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)^2 + 4C_{16}C_1h^{k+1}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)\\ & + 2C_{16}\tau(\|\xi^{n,1}(u)\| + \|\xi^{n,1}(v)\|)^2 + 4C_{16}C_1h^{k+1}\tau(\|\xi^{n,1}(u)\| + \|\xi^{n,1}(v)\|)\\ & + 8C_{16}\tau(\|\xi^{n,2}(u)\| + \|\xi^{n,2}(v)\|)^2 + 16C_{16}C_1h^{k+1}\tau(\|\xi^{n,2}(u)\| + \|\xi^{n,2}(v)\|)\end{aligned}$$

By Lemma (4.7), $\tau \leqslant 1/\gamma$ and Cauchy's inequality, we have

$$\begin{aligned}& 2C_{16}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)^2 + 4C_{16}C_1h^{k+1}\tau(\|\xi^n(u)\| + \|\xi^n(v)\|)\\ \leqslant & C_{56}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{57}h^{2k+2}\tau,\\ & 2C_{16}\tau(\|\xi^{n,1}(u)\| + \|\xi^{n,1}(v)\|)^2 + 4C_{16}C_1h^{k+1}\tau(\|\xi^{n,1}(u)\| + \|\xi^{n,1}(v)\|)\\ \leqslant & C_{58}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{59}h^{2k+2}\tau,\\ & 8C_{16}\tau(\|\xi^{n,2}(u)\| + \|\xi^{n,2}(v)\|)^2 + 16C_{16}C_1h^{k+1}\tau(\|\xi^{n,2}(u)\| + \|\xi^{n,2}(v)\|)\\ \leqslant & C_{60}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{61}h^{2k+2}\tau.\end{aligned}$$

Here $C_{56} = 2C_{16}(2 + C_1)$, $C_{57} = 4C_{16}C_1$, $C_{58} = 2C_{16}(2(\beta_1 + \beta_2)^2 + 2(\beta_1 + \beta_2)\beta_3/\gamma + (\beta_1 + \beta_2)C_1)$, $C_{59} = 4C_{16}(2(\beta_1 + \beta_2)\beta_3/\gamma + 2(\beta_3/\gamma)^2 + (\beta_1 + \beta_2)C_1 + 2\beta_3C_1/\gamma)$, $C_{60} = 8C_{16}(2(C_{17} + C_{18})^2 + 2(C_{17} + C_{18})C_{19}/\gamma + (C_{17} + C_{18})C_1)$, $C_{61} = 16C_{16}(2(C_{17} + C_{18})C_{19}/\gamma + 2(C_{19}/\gamma)^2 + (C_{17} + C_{18})C_1 + 2C_1C_{19}/\gamma)$.

So we have

$$\tau\hat{\mathbb{S}}_3 \leqslant C_{62}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{63}h^{2k+2}\tau \tag{4.28}$$

where $C_{62} = C_{56} + C_{58} + C_{60}$, $C_{63} = C_{57} + C_{59} + C_{61}$.

From (4.16), (4.18), Cauchy's inequality, $\tau \leqslant 1/\gamma$ and Lemma 4.7 we get

$$\begin{aligned}\hat{\mathbb{S}}_4 \leqslant & C_{15}h^{k+1}\tau(\|\xi^n(u)\| + \|\xi^n(v)\| + \|\xi^{n,1}(u)\| + \|\xi^{n,1}(v)\| + 2\|\xi^{n,2}(u)\| + 2\|\xi^{n,2}(v)\|)\\ & + 6C_{14}\tau^4(\|\xi^{n,2}(u)\| + \|\xi^{n,2}(v)\|)\\ \leqslant & C_{64}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{65}h^{2k+2}\tau + C_{66}h^{k+1}\tau^4 + C_{67}\tau^7\end{aligned} \tag{4.29}$$

Here $C_{64} = C_{15}(C_{17} + C_{18} + \frac12(\beta_1 + \beta_2 + 1)) + 3C_{14}(C_{17} + C_{18})$, $C_{65} = C_{15}(2(C_{17} + C_{18}) + \beta_1 + \beta_2 + 1 + (2\beta_3 + 4C_{19})/\gamma)$, $C_{66} = 12C_{14}C_{19}/\gamma$, $C_{67} = 6C_{14}(C_{17} + C_{18})$.

From (4.26), (4.27), (4.28), (4.29) and let $\lambda \leqslant 0.39$ such that $\sigma\lambda^2 + \lambda - 1 \leqslant 0$, we obtain

$$\hat{\Lambda}_1 + \hat{\Lambda}_2 + \hat{\Lambda}_3 + \tau(\hat{\mathbb{S}}_2 + \hat{\mathbb{S}}_3) + \hat{\mathbb{S}}_4 \leqslant C_{68}\tau(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{69}(h^{k+1} + \tau^3)^2\tau \tag{4.30}$$

Here $C_{68} = C_{52} + C_{62} + C_{64}$, $C_{69} = \max\{C_{53} + C_{63} + C_{65}, \frac12(C_{54} + C_{66}), C_{55} + C_{67}\}$.

From (4.22) and (4.30), we can get

$$\begin{aligned}\|\xi^{n+1}(u)\|^2 + \|\xi^{n+1}(v)\|^2 \leqslant & (1 + C_{68}\tau/3)(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{69}(h^{k+1} + \tau^3)^2\tau\\ \leqslant & e^{C_{68}\tau/3}(\|\xi^n(u)\|^2 + \|\xi^n(v)\|^2) + C_{69}(h^{k+1} + \tau^3)^2\tau\\ \leqslant & e^{2C_{68}\tau/3}(\|\xi^{n-1}(u)\|^2 + \|\xi^{n-1}(v)\|^2) + C_{69}(h^{k+1} + \tau^3)^2\tau(1 + e^{C_{48}\tau})\\ & \cdots\\ \leqslant & e^{C_{68}(n+1)\tau/3}(\|\xi^0(u)\|^2 + \|\xi^0(v)\|^2) + C_{69}(h^{k+1} + \tau^3)^2\tau\frac{1 - e^{C_{68}(n+1)\tau/3}}{1 - e^{C_{68}\tau/3}}\end{aligned}$$

By Taylor expansion and L^2 projection error estimates, we have

$$\frac{e^{C_{68}\tau/3}-1}{\tau} \geqslant C_{68}/3$$
$$\|\xi^0(u)\|^2+\|\xi^0(v)\|^2 \leqslant C_{70}h^{2k+2}$$

Here C_{70} is independent of h. From $(n+1)\tau \leqslant T$, we get

$$\|\xi^{n+1}(u)\|^2+\|\xi^{n+1}(v)\|^2 \leqslant C_{71}(h^{k+1}+\tau^3)^2$$

Here $C_{71}=C_{70}e^{C_{68}T/3}+3C_{69}(e^{C_{68}T/3}-1)/C_{68}$.

Therefore finally we get

$$\|e^{n+1}(u)\|^2+\|e^{n+1}(v)\|^2 \leqslant (C_{71}+2C_1)(h^{k+1}+\tau^3)^2$$

where $C_{71}+2C_1$ is independent of τ, h, u_h, v_h. This finishes the proof.

□

5 Positivity-preserving RKDG schemes

The positivity-preserving property for the densities u and v is important, and we would like to have this property for our DG schemes. We start this section with a discussion of this property for a first order upwind scheme.

5.1 First order upwind scheme

Lemma 5.1. *If the initial condition $u_0(x)$, $v_0(x)$ are nonnegative, then the first order upwind scheme*

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\gamma\frac{u_j^n-u_{j-1}^n}{h}=-(\lambda_1)_j^n u_j^n+(\lambda_2)_j^n v_j^n \tag{5.1a}$$
$$\frac{v_j^{n+1}-v_j^n}{\tau}-\gamma\frac{v_{j+1}^n-v_j^n}{h}=(\lambda_1)_j^n u_j^n-(\lambda_2)_j^n v_j^n \tag{5.1b}$$

where u_j^n and v_j^n are approximations to the solution $u(x_j,t^n)$ and $v(x_j,t^n)$ at the grid point $x_j=jh$ and time level $t^n=n\tau$, $(\lambda_1)_j^n=\lambda_1(y_1[u_j^n,v_j^n])$, $(\lambda_2)_j^n=\lambda_2(y_2[u_j^n,v_j^n])$, can maintain positivity under the time step restriction

$$\tau \leqslant 1/(a_1+a_2+\gamma/h). \tag{5.2}$$

The integrals in the numerical evaluation of $(\lambda_1)_j^n$ and $(\lambda_2)_j^n$ are obtained by trapezoidal rules.

Proof. Denote $\kappa=\tau\gamma/h$. We can rewrite (5.1a) as

$$\begin{aligned} u_j^{n+1} =& u_j^n-\kappa(u_j^n-u_{j-1}^n)-\tau(\lambda_1)_j^n u_j^n+(\lambda_2)_j^n v_j^n \\ =& \Big(1-\kappa-\tau(\lambda_1)_j^n\Big)u_j^n+\kappa u_{j-1}^n+\tau(\lambda_2)_j^n v_j^n \end{aligned}$$

By (2.3) and the time step restriction (5.2), we have

$$1-\kappa-\tau(\lambda_1)_j^n \geqslant 0, \qquad \kappa \geqslant 0, \qquad \tau(\lambda_2)_j^n \geqslant 0.$$

Therefore, $u_i^n, v_i^n \geqslant 0, \forall i$ implies $u_j^{n+1} \geqslant 0$. Similarly we can prove $v_j^{n+1} \geqslant 0$. The positivity is thus preserved. □

5.2 High order DG schemes

In [22], a simple scaling limiter is shown to be able to make the numerical solution satisfying the maximum principle while maintaining the original high order accuracy of DG or finite volume schemes for solving scalar conservation laws $u_t + u_x = 0$. The result holds for both Euler forward time discretizations and for TVD Runge-Kutta methods which are convex combinations of Euler forward steps, such as the third order TVD Runge-Kutta method (4.2).

Assume $m = \min u_0(x)$, $M = \max u_0(x)$, then for scalar conservation laws, the entropy solution satisfies $m \leqslant u(x,t) \leqslant M$. Assume $p_j(x)$ is the numerical DG solution on $I_j = [x_{j-1/2}, x_{j+1/2}]$, which is a polynomial of degree k. Denote by $\bar{p}_j$ the cell average of $p_j(x)$. If $m \leqslant \bar{p}_j \leqslant M$, then the scaling limiter can be defined to replace $p_j(x)$ with $\tilde{p}_j(x)$ where

$$\tilde{p}_j(x) = \theta p_j(x) + (1-\theta)\bar{p}_j(x), \quad \theta = \min\left\{\left|\frac{M-\bar{p}_j}{M_j-\bar{p}_j}\right|, \left|\frac{m-\bar{p}_j}{m_j-\bar{p}_j}\right|, 1\right\} \tag{5.3}$$

Here $m_j = \min\limits_{x\in S_j} p_j(x)$, $M_j = \max\limits_{x\in S_j} p_j(x)$, and $S_j = \{x_{\alpha,j}\}_{\alpha=1}^N$ are the Gauss-Lobatto quadrature points on I_j, with the requirement that the Gauss-Lobatto quadrature is exact for all polynomials of degree k, i.e. $2N-3 \geqslant k$. As we can see, the limiter does not change the cell averages. It is proved in [22] that, under suitable CFL condition, the numerical solution that is modified by the scaling limiter (5.3) will guarantee $m \leqslant \bar{p}_j \leqslant M$ for all time steps, will satisfy the maximum principle at the points in S_j for all time steps, and will maintain the original high order accuracy of the unlimited scheme. If the exact solution of the PDE has only one bound, for example the lower bound zero for the density and pressure of compressible Euler equations, then a similar scaling limiter, with the term involving the irrelevant bound M and M_j removed from the list of the minimum in the definition of θ in (5.3), can maintain this one bound while keeping the order of accuracy. See for example [23] for the positivity-preserving DG scheme for compressible Euler equations. In the following we will adapt this strategy to obtain positivity-preserving DG schemes for the system (2.1).

We convert $u_h^{n,i}, v_h^{n,i}$ to $\tilde{u}_h^{n,i}, \tilde{v}_h^{n,i}$ by the positivity-preserving limiter

$$\tilde{u}_h^{n,i} = \theta_{1,i,j} u_h^{n,i} + (1-\theta_{1,i,j})\bar{u}_h^{n,i}, \qquad \theta_{1,i,j} = \min\left\{\left|\frac{m-\bar{u}_h^{n,i}}{m_{1,i,j}-\bar{u}_h^{n,i}}\right|, 1\right\} \tag{5.4a}$$

$$\tilde{v}_h^{n,i} = \theta_{2,i,j} v_h^{n,i} + (1-\theta_{2,i,j})\bar{v}_h^{n,i}, \qquad \theta_{2,i,j} = \min\left\{\left|\frac{m-\bar{v}_h^{n,i}}{m_{2,i,j}-\bar{v}_h^{n,i}}\right|, 1\right\} \tag{5.4b}$$

Here $m = 0$, $m_{1,i,j} = \min\limits_{x\in S_j} u_h^{n,i}(x)$, $m_{2,i,j} = \min\limits_{x\in S_j} v_h^{n,i}(x)$, $i = 0,1,2$, and $j = 1,\cdots,n_x$.

As a result, the scheme (4.3) is modified as follows.

$$\tilde{u}_h^n = \theta_{1,0,j} u_h^n + (1-\theta_{1,0,j})\bar{u}_h^n, \quad \tilde{v}_h^n = \theta_{2,0,j} v_h^n + (1-\theta_{2,0,j})\bar{v}_h^n \tag{5.5a}$$

$$\int_{I_j} u_h^{n,1}\varphi dx = \int_{I_j} \tilde{u}_h^n \varphi dx + \tau\Big[B_j^1(\tilde{u}_h^n,\varphi) - \int_{I_j} (\tilde{\lambda}_1)_h^n \tilde{u}_h^n \varphi dx + \int_{I_j} (\tilde{\lambda}_2)_h^n \tilde{v}_h^n \varphi dx\Big] \tag{5.5b}$$

$$\int_{I_j} v_h^{n,1}\varphi dx = \int_{I_j} \tilde{v}_h^n \varphi dx + \tau\Big[B_j^2(\tilde{v}_h^n,\varphi) + \int_{I_j} (\tilde{\lambda}_1)_h^n \tilde{u}_h^n \varphi dx - \int_{I_j} (\tilde{\lambda}_2)_h^n \tilde{v}_h^n \varphi dx\Big] \tag{5.5c}$$

$$\tilde{u}_h^{n,1} = \theta_{1,1,j} u_h^{n,1} + (1-\theta_{1,1,j})\bar{u}_h^{n,1}, \quad \tilde{v}_h^{n,1} = \theta_{2,1,j} v_h^{n,1} + (1-\theta_{2,1,j})\bar{v}_h^{n,1} \tag{5.5d}$$

$$\begin{aligned}\int_{I_j} u_h^{n,2}\varphi dx =& \frac{3}{4}\int_{I_j} \tilde{u}_h^n \varphi dx + \frac{1}{4}\int_{I_j} \tilde{u}_h^{n,1}\varphi dx + \frac{\tau}{4}\Big[B_j^1(\tilde{u}_h^{n,1},\varphi) - \int_{I_j} (\tilde{\lambda}_1)_h^{n,1} \tilde{u}_h^{n,1}\varphi dx \\ &+ \int_{I_j} (\tilde{\lambda}_2)_h^{n,1} \tilde{v}_h^{n,1}\varphi dx\Big]\end{aligned} \tag{5.5e}$$

$$\begin{aligned}\int_{I_j} v_h^{n,2}\varphi dx =& \frac{3}{4}\int_{I_j} \tilde{v}_h^n \varphi dx + \frac{1}{4}\int_{I_j} \tilde{v}_h^{n,1}\varphi dx + \frac{\tau}{4}\Big[B_j^2(\tilde{v}_h^{n,1},\varphi) + \int_{I_j} (\tilde{\lambda}_1)_h^{n,1} \tilde{u}_h^{n,1}\varphi dx \\ &- \int_{I_j} (\tilde{\lambda}_2)_h^{n,1} \tilde{v}_h^{n,1}\varphi dx\Big]\end{aligned} \tag{5.5f}$$

$$\tilde{u}_h^{n,2} = \theta_{1,2,j} u_h^{n,1} + (1-\theta_{1,2,j})\bar{u}_h^{n,2}, \quad \tilde{v}_h^{n,2} = \theta_{2,2,j} v_h^{n,2} + (1-\theta_{2,2,j})\bar{v}_h^{n,2} \tag{5.5g}$$

$$\begin{aligned}\int_{I_j} u_h^{n+1}\varphi dx =& \frac{1}{3}\int_{I_j} \tilde{u}_h^n \varphi dx + \frac{2}{3}\int_{I_j} \tilde{u}_h^{n,2}\varphi dx + \frac{2\tau}{3}\Big[B_j^1(\tilde{u}_h^{n,2},\varphi) - \int_{I_j}(\tilde{\lambda}_1)_h^{n,2}\tilde{u}_h^{n,2}\varphi dx \\ &+ \int_{I_j}(\tilde{\lambda}_2)_h^{n,2}\tilde{v}_h^{n,2}\varphi dx\Big]\end{aligned} \tag{5.5h}$$

$$\begin{aligned}\int_{I_j} v_h^{n+1}\varphi dx =& \frac{1}{3}\int_{I_j} \tilde{v}_h^n \varphi dx + \frac{2}{3}\int_{I_j} \tilde{v}_h^{n,2}\varphi dx + \frac{2\tau}{3}\Big[B_j^2(\tilde{v}_h^{n,2},\varphi) + \int_{I_j}(\tilde{\lambda}_1)_h^{n,2}\tilde{u}_h^{n,2}\varphi dx \\ &- \int_{I_j}(\tilde{\lambda}_2)_h^{n,2}\tilde{v}_h^{n,2}\varphi dx\Big]\end{aligned} \tag{5.5i}$$

Here $(\tilde{\lambda}_1)_h^{n,i} = \lambda_1(y_1[\tilde{u}_h^{n,i},\tilde{v}_h^{n,i}])$, $(\tilde{\lambda}_2)_h^{n,i} = \lambda_2(y_2[\tilde{u}_h^{n,i},\tilde{v}_h^{n,i}])$, $i = 0,1,2$.

We have the following result for the positivity-preserving property of the limited RKDG scheme.

Lemma 5.2. *The modified scheme (5.5) is positivity-preserving under the time step restriction*

$$\tau \leqslant \frac{\omega_N}{\frac{\gamma}{\nu h} + (a_1 + a_2)} \tag{5.6}$$

Here, $\{\omega_\alpha\}_{\alpha=1}^N > 0$ *are the weights of the Gauss-Lobatto quadrature for the interval* $[-1/2, 1/2]$. *The integrals for the source term are computed also by the same Gauss-Lobatto quadrature.*

Proof. Assume $\tilde{u}_h^n(x_{\alpha,j}) \geqslant 0$, $\tilde{v}_h^n(x_{\alpha,j}) \geqslant 0$, and therefore also $\bar{u}_h^n \geqslant 0$, $\bar{v}_h^n \geqslant 0$. Taking $\varphi = 1/h_j$ in (5.5b), and using the Gauss-Lobatto quadrature, (2.3) and (3.1), we get

$$\begin{aligned}\bar{u}_h^{n,1} =& \bar{\tilde{u}}_h^n + \frac{\tau\gamma}{h_j}(-\tilde{u}_h^n(x_{j+1/2}^-) + \tilde{u}_h^n(x_{j-1/2}^-)) - \frac{\tau}{h_j}\int_{I_j}(\tilde{\lambda}_1)_h^n \tilde{u}_h^n dx + \frac{\tau}{h_j}\int_{I_j}(\tilde{\lambda}_2)_h^n \tilde{v}_h^n dx \\ \geqslant& \sum_{\alpha=1}^N \omega_\alpha \tilde{u}_h^n(x_{\alpha,j}) - \frac{\tau\gamma}{\nu h}\tilde{u}_h^n(x_{j+1/2}^-) + \frac{\tau\gamma}{h}\tilde{u}_h^n(x_{j-1/2}^-) - (a_1+a_2)\tau\sum_{\alpha=1}^N \omega_\alpha \tilde{u}_h^n(x_{\alpha,j}) \\ &+ a_1\tau\sum_{\alpha=1}^N \omega_\alpha \tilde{v}_h^n(x_{\alpha,j}) \\ =& \sum_{\alpha=1}^{N-1}(\omega_\alpha - (a_1+a_2)\tau)\tilde{u}_h^n(x_{\alpha,j}) + (\omega_N - \frac{\tau\gamma}{\nu h} - (a_1+a_2)\tau)\tilde{u}_h^n(x_{j+1/2}^-) \\ &+ \frac{\tau\gamma}{h}\tilde{u}_h^n(x_{j-1/2}^-) + a_1\tau\sum_{\alpha=1}^N \omega_\alpha \tilde{v}_h^n(x_{\alpha,j})\end{aligned}$$

Notice that $\omega_1 = \omega_N = \min_{1\leqslant\alpha\leqslant N}\omega_\alpha$, hence, under the condition (5.6), we have

$$\omega_\alpha - (a_1+a_2)\tau \geqslant 0, \quad \alpha = 1,\cdots,N-1, \qquad \omega_N - \frac{\tau\gamma}{\nu h} - (a_1+a_2)\tau \geqslant 0.$$

Therefore, since $\tilde{u}_h^n(x_{\alpha,j}) \geqslant 0$, $\tilde{v}_h^n(x_{\alpha,j}) \geqslant 0$, $\forall \alpha, j$, $a_1 \geqslant 0$, $\{\omega_\alpha\}_{\alpha=1}^N > 0$ and $\tau\gamma/h \geqslant 0$, we get $\bar{u}_h^{n,1} \geqslant 0$ and $\bar{v}_h^{n,1} \geqslant 0$. This is the crucial conclusion needed in the positivity-preserving limiter [22]. The scaling limiter (5.5d), which does not change the cell averages of $u_h^{n,1}$ and $v_h^{n,1}$, further guarantees $\tilde{u}_h^{n,1}(x_{\alpha,j}) \geqslant 0$, $\tilde{v}_h^{n,1}(x_{\alpha,j}) \geqslant 0$, $\forall \alpha, j$. Similar arguments then apply to the remaining Runge-Kutta stages for $\tilde{u}_h^{n,i}$ and $\tilde{v}_h^{n,i}$ with $i = 2,3$, hence finishing the proof of the positivity-preserving property of the limited RKDG scheme.

□

We remark that the same proof as in [22] ensures that the scaling limiter does not destroy the original high order accuracy of the DG scheme.

6 Numerical results

In this section we present some numerical tests using the first order upwind scheme (5.1) and third order RKDG method (4.3). In the tests we use uniform mesh and periodic boundary conditions. From Table 1, we have $s_r = 0.25$, $s_a = 1$, $s_{al} = 0.5$ and $m_i = s_i/8$ (i=r, a, al), $L = 10$. We also take $\gamma = 0.1$, $a_1 = 0.2$, $a_2 = 0.9$ during the experiments. The infinite integrals are approximated by finite integrals on $[0, 2s_i]$, with $i = r, a, al$, for the reason that the mass of the kernel functions mostly concentrate on $[0, 2s_i]$. In fact, we have

$$\int_{2s_i}^{\infty} K_i(s)ds \leqslant 2 \times 10^{-15}, \quad i = r, a, al.$$

In the scheme (4.3), y_1, y_2 are computed by the Gauss-Lobatto quadrature with four points. In scheme (5.1), y_1, y_2 are computed by the trapezoidal rule. The CFL number is 0.6 for the scheme (5.1) and we take τ and h satisfying the CFL condition in Theorem 4.5 for the scheme (4.3).

Example 1. Firstly we will test the system without the source term

$$\begin{cases} u_t + \gamma u_x = 0, \quad (x,t) \in [0,L] \times [0,T] \\ v_t - \gamma v_x = 0, \quad (x,t) \in [0,L] \times [0,T] \\ u_0(x) = 1 + \sin(2\pi x), \qquad v_0(x) = 1 + \cos(2\pi x), \qquad x \in [0,L] \end{cases} \tag{6.1}$$

with periodic boundary conditions

$$u(0,t) = u(L,t), \quad v(0,t) = v(L,t).$$

Since the source term is removed, we can find its exact solutions $u(x,t) = 1 + \sin(2\pi(x - \gamma t))$ and $v(x,t) = 1 + \cos(2\pi(x + \gamma t))$, which indicates $u(x,t) \geqslant 0$ and $v(x,t) \geqslant 0$. We will show the results of both RKDG scheme and positivity preserving RKDG scheme.

Take $L = 10$, $k = 2$, $t = 1$ and the same CFL condition as in Theorem 4.2. The errors and orders of accuracy are listed in Tables 3.

Table 3. Errors and orders of accuracy and average percentage of cells affected by the positivity-preserving limiter for the equations (6.1) without source terms.

	without positivity-preserving limiter				with positivity-preserving limiter				
n	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order	percentage
20	1.32E-01	–	3.62E-01	–	1.91E-01	–	4.38E-01	–	35.7%
40	2.11E-02	2.64	4.86E-02	2.90	2.87E-02	2.73	9.89E-02	2.15	15.4%
80	2.67E-03	2.98	8.29E-03	2.55	3.29E-03	3.12	1.23E-02	3.00	10.0%
160	3.34E-04	3.00	1.00E-03	3.05	3.42E-04	3.27	1.00E-03	3.62	2.50%
320	4.16E-05	3.00	1.26E-04	2.99	4.17E-05	3.03	1.26E-04	2.99	1.52%
640	5.21E-06	3.00	1.58E-05	3.00	5.21E-06	3.00	1.58E-05	3.00	0.78%

Then we will modify the system (2.1) with an additional source term so that we have an explicit exact solution to test accuracy. Denote

$$g(x,t) = \lambda_1(y_1[u_0(x - \gamma t), v_0(x + \gamma t)])u_0(x - \gamma t) - \lambda_2(y_2[u_0(x - \gamma t), v_0(x + \gamma t)])v_0(x + \gamma t)$$

where

$$u_0(x) = 1 + \sin(2\pi x), \qquad v_0(x) = 1 + \cos(2\pi x).$$

We consider the modified system

$$\begin{cases} u_t + \gamma u_x = -\lambda_1 u + \lambda_2 v + g, & (x,t) \in [0,L] \times [0,T] \\ v_t - \gamma v_x = \lambda_1 u - \lambda_2 v - g, & (x,t) \in [0,L] \times [0,T] \\ u(x,0) = 1 + \sin(2\pi x), \quad v(x,0) = 1 + \cos(2\pi x), & x \in [0,L] \end{cases} \tag{6.2}$$

with periodic boundary conditions

$$u(0,t) = u(L,t), \quad v(0,t) = v(L,t)$$

It is easy to verify that the exact solution is given by $u(x,t) = 1 + \sin(2\pi(x - \gamma t))$ and $v(x,t) = 1 + \cos(2\pi(x + \gamma t))$, which satisfies $u(x,t) \geqslant 0$ and $v(x,t) \geqslant 0$. Consider this system on $[0,10]$. Notice that $\min\limits_{(x,t)\in[0,L]\times[0,T]} u(x,t) = 0$ and $\min\limits_{(x,t)\in[0,L]\times[0,T]} v(x,t) = 0$, we also expect this to be a stringent test case for the positivity-preserving limiter to maintain accuracy. We report numerical results both with the positivity-preserving limiter and without it. Set the parameters $q_r = 0.5$, $q_a = 1.6$, $q_{al} = 2$ as Pattern 5 in [8]. Take $k = 2$, $t = 1$ and the same CFL condition as in Theorem 4.2. The errors and orders of accuracy are listed in Tables 4.

Table 4. Errors and orders of accuracy and average percentage of cells affected by the positivity-preserving limiter for the equations (6.2) with source terms.

	without positivity-preserving limiter				with positivity-preserving limiter				
n	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order	percentage
20	1.55E-01	–	4.50E-01	–	2.12E-01	–	5.24E-01	–	71.4%
40	1.99E-02	2.96	4.76E-02	3.24	2.64E-02	3.00	8.72E-02	2.59	15.4%
80	2.70E-03	2.88	8.33E-03	2.52	3.02E-03	3.13	1.10E-02	2.99	8.00%
160	3.33E-04	3.02	1.00E-03	3.05	3.40E-04	3.15	1.01E-03	3.44	2.75%
320	4.16E-05	3.00	1.26E-04	2.99	4.17E-05	3.03	1.26E-04	3.00	1.52%
640	5.20E-06	3.00	1.58E-05	3.00	5.21E-06	3.00	1.58E-05	3.00	0.79%

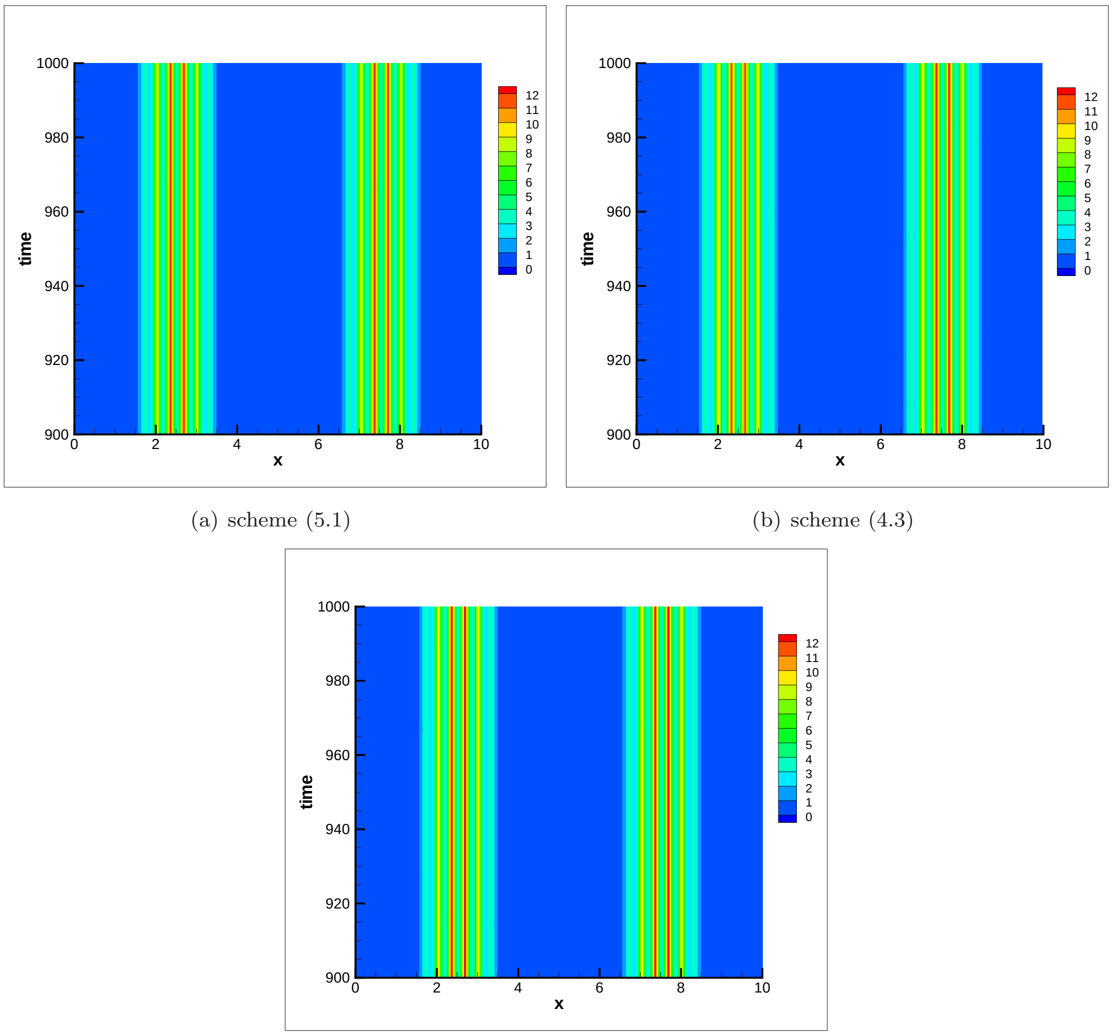
We can clearly see that the algorithm achieves its designed third order accuracy either without or with the positivity-preserving limiter from these two tests. The last column in Table 4 and Table 3 records the average (over all time steps) percentage of cells in which the positivity-preserving limiter takes effect. We can see that the limiter has indeed taken effect although the original high order accuracy is maintained.

Example 2. We take the parameters $q_r = 2.4$, $q_a = 2$, $q_{al} = 0$ in system (2.1) as Pattern 1 in [8]. The initial conditions are taken from a small random perturbation of amplitude 0.01 of spatially homogeneous steady states, which are $(u,v) = (u^*, u^{**})$, u^*, u^{**} are constants satisfying $u^* + u^{**} = A$, where A is the total population density. For $q_{al} = 0$, we have only one steady state $(u,v) = (A/2, A/2)$. For $q_{al} \neq 0$, the system (2.1) can have one, three or five solutions. These solutions are obtained by the steady state equation from (2.1),

$$-u^*(a_1 + a_2 f(Aq_{al} - 2u^* q_{al} - y_0)) + (A - u^*)(a_1 + a_2 f(-Aq_{al} + 2u^* q_{al} - y_0)) = 0.$$

From Table 1 we have $A = 2$ and we can see that $(u,v) = (1,1)$ is the homogeneous steady state no matter what value q_{al} takes. In the following numerical tests we take $(u^*, u^{**}) = (1,1)$. We generate random perturbation data satisfying the above requirements and use them as our initial conditions. We have tested several initial data and have observed no significant differences other than a shift among them, so we report the result of only one choice of the initial data. The solution evolves into stationary pulses (i.e. $u_t = v_t = 0$), in which high density subgroups emerge. In Figure 1 we plot the total density

$p = u + v$ from time $t = 900$ to $t = 1000$ using scheme (5.1) with $n_x = 600$ and scheme (4.3) with $n_x = 200$ and $k = 2$. Note that we have used the same number of degree of freedom for both schemes. In Figure 2 and 3 we plot $u + v$, u, v obtained by these two schemes at the time $t = 1000$. We find the numerical solutions converge when refining the mesh for the scheme (5.1), starting at around $n_x = 1600$. Here we choose numerical solution of $n_x = 2000$ as our converged solution.



(a) scheme (5.1)

(b) scheme (4.3)

(c) converged solution

Figure 1 *Stationary pulses, $u + v$ from time $t = 900$ to $t = 1000$ of Example 2.*

In Figure 1, we can see that the numerical solutions are stationary for both schemes (5.1) and (4.3). In Figure 2 we can see that both numerical solutions generated by scheme (5.1) and scheme (4.3) perform well comparing to the converged solution. In Figure 3, the converged solution shows that $u = v$ at time $t = 1000$. In Figure 3(a)-(d), u and v generated by the higher order scheme (4.3) almost overlap with each other, while u and v generated by scheme (5.1) show a slight translation, when the same number of degree of freedom is used.

Example 3. We take the parameters $q_r = 0.5$, $q_a = 1.6$, $q_{al} = 2$ in system (2.1) as Pattern 5 in [8]. The initial conditions here satisfy the requirements described in Example 2. In Figure 4 we plot the total density $p = u + v$ from time $t = 1150$ to $t = 1300$ using scheme (5.1) with $n_x = 600$ and scheme (4.3) with $n_x = 200$, $k = 2$. In Figure 5 we plot $u + v$ obtained by these two schemes at the time $t = 1300$. Here the numerical solution of $n_x = 2000$ obtained by the scheme (5.1) is still taken as our converged

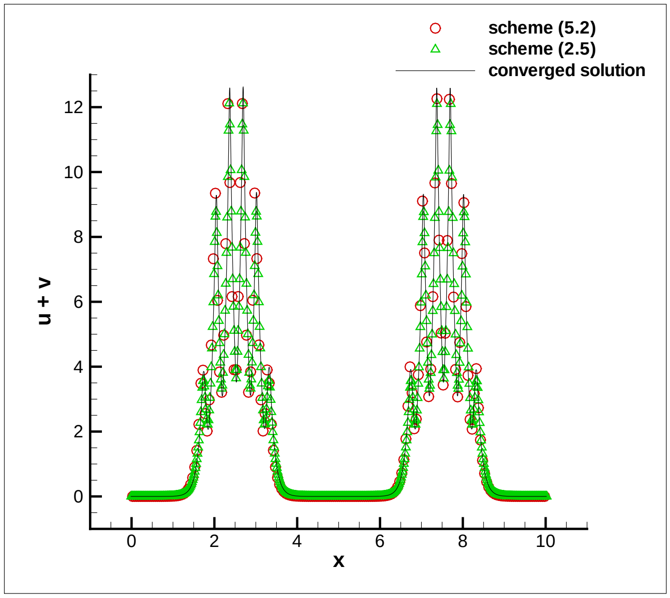


Figure 2 *Stationary pulses,* $u+v$ *at time* $t=1000$ *of Example 2.*

solution.

In Figure 4 we can see that the numerical solutions are traveling for both schemes (5.1) and (4.3). In Figure 5 we can see that the numerical solution obtained by the higher order scheme (4.3) is closer to the converged solution than the lower order scheme (5.1), for the same number of degree of freedom.

7 Concluding remarks

In this paper we present an analysis of the RKDG scheme for a nonlocal hyperbolic system, describing a correlated random walk model with density dependent turning rates. We construct a first order upwind scheme and prove its positivity-preserving property under a suitable time step restriction. We then describe the semi-discrete DG scheme and prove its L^2 stability. This is followed by the analysis on the fully discretized RKDG method using the third order TVD Runge-Kutta time discretization and discontinuous Galerkin spatial discretization with arbitrary polynomial degree $k \geqslant 1$. By generalizing the energy method in [21] to our current semilinear system, we obtain L^2 stability for general solutions and optimal a priori L^2 error estimates when the solutions are smooth enough under a suitable CFL condition. Finally, we discuss a positivity-preserving limiter which guarantees positivity of the solution without compromising the accuracy of the RKDG scheme. Numerical results are presented to demonstrate that the RKDG method performs well in several test problems.

8 Appendix

We collect the proof of some of the technical lemmas in this section, which serves as an appendix.

8.1 The proof of Lemma 4.1

Proof. From Hölder's inequality and (2.3), we obtain

$$\left|\int_0^L (\lambda_1)_h^{n,i} u_h^{n,i} \varphi dx\right| \leqslant (a_1+a_2)\|u_h^{n,i}\|\|\varphi\|, \quad i=0,1,2$$

Similarly we can get

$$\left|\int_0^L (\lambda_2)_h^{n,i} v_h^{n,i} \varphi dx\right| \leqslant (a_1+a_2)\|v_h^{n,i}\|\|\varphi\|, \quad i=0,1,2$$

This finishes the proof. □

8.2 The proof of Lemma 4.2

Proof. Taking $\varphi = u_h^{n,1}$ in (4.3a) and summing it over j, we can get

$$\begin{aligned}\|u_h^{n,1}\|^2 = &\int_0^L u_h^n u_h^{n,1} dx + \tau\Big[B^1(u_h^n, u_h^{n,1}) - \int_0^L (\lambda_1)_h^n u_h^n u_h^{n,1} dx \\ &+ \int_0^L (\lambda_2)_h^n v_h^n u_h^{n,1} dx\Big]\end{aligned}$$

From Hölder's inequality, Lemma 3.1 and Lemma 4.1, we can obtain

$$\|u_h^{n,1}\|^2 \leqslant \|u_h^n\|\|u_h^{n,1}\| + (\sqrt{2}+1)\lambda\|u_h^n\|\|u_h^{n,1}\| + (a_1+a_2)\tau(\|u_h^n\|+\|v_h^n\|)\|u_h^{n,1}\|$$

If $\|u_h^{n,1}\| \neq 0$, through $\tau \leqslant 1/\gamma$ and $0 < \lambda \leqslant M$ we have

$$\|u_h^{n,1}\| \leqslant \alpha_1\|u_h^n\| + \alpha_2\|v_h^n\|$$

Here $\alpha_1 = 1+(\sqrt{2}+1)M+(a_1+a_2)/\gamma$, $\alpha_2 = (a_1+a_2)/\gamma$. If $\|u_h^{n,1}\| = 0$, the above inequality also holds. Similarly we have

$$\begin{aligned}\|v_h^{n,1}\| &\leqslant \alpha_1\|v_h^n\| + \alpha_2\|u_h^n\| \\ \|u_h^{n,2}\| &\leqslant \frac{3}{4}\|u_h^n\| + \frac{1}{4}(\alpha_1\|u_h^{n,1}\| + \alpha_2\|v_h^{n,1}\|) \\ &\leqslant \Big(\frac{3}{4}+\frac{1}{4}(\alpha_1^2+\alpha_2^2)\Big)\|u_h^n\| + \frac{1}{2}\alpha_1\alpha_2\|v_h^n\| \\ \|v_h^{n,2}\| &\leqslant \Big(\frac{3}{4}+\frac{1}{4}(\alpha_1^2+\alpha_2^2)\Big)\|v_h^n\| + \frac{1}{2}\alpha_1\alpha_2\|u_h^n\|\end{aligned}$$

Define $C_3 = \frac{3}{4}+\frac{1}{4}(\alpha_1^2+\alpha_2^2)$, $C_4 = \frac{1}{2}\alpha_1\alpha_2$. From the triangle inequality and above inequalities, we get

$$\begin{aligned}\|\mathbb{D}_1(u_h)\| &\leqslant (\alpha_1+1)\|u_h^n\| + \alpha_2\|v_h^n\| \\ \|\mathbb{D}_1(v_h)\| &\leqslant (\alpha_1+1)\|v_h^n\| + \alpha_2\|u_h^n\| \\ \|\mathbb{D}_2(u_h)\| &\leqslant C_5\|u_h^n\| + C_6\|v_h^n\| \\ \|\mathbb{D}_2(v_h)\| &\leqslant C_5\|v_h^n\| + C_6\|u_h^n\|\end{aligned}$$

Here $C_5 = 2C_3+\alpha_1+1$, $C_6 = 2C_4+\alpha_2$. Then we get the desired results. □

8.3 The proof of Lemma 4.3

Proof. Summing (4.3a) over j, we can get

$$(D_1(u_h), \varphi) = \tau\Big[B^1(u_h^n, \varphi) - \int_0^L (\lambda_1)_h^n u_h^n \varphi dx + \int_0^L (\lambda_2)_h^n v_h^n \varphi dx\Big]$$

From Lemma 4.1, we get (4.5a).

Subtracting $\frac{1}{2}\times(4.3a)$ from $2\times(4.3c)$, then summing it over j , we get

$$(D_2(u_h), \varphi) = \frac{\tau}{2}\Big[B^1(D_1(u_h), \varphi) - \int_0^L (\lambda_1)_h^{n,1} u_h^{n,1} \varphi dx + \int_0^L (\lambda_2)_h^{n,1} v_h^{n,1} \varphi dx$$

$$+ \int_0^L (\lambda_1)_h^n u_h^n \varphi dx - \int_0^L (\lambda_2)_h^n v_h^n \varphi dx\Big]$$

From Lemma 4.1 and Lemma 4.2, we get

$$\begin{aligned}(D_2(u_h),\varphi) \leqslant &\frac{\tau}{2}B^1(D_1(u_h),\varphi) + \frac{\tau}{2}(a_1+a_2)(\|u_h^n\| + \|v_h^n\| + \|u_h^{n,1}\| + \|v_h^{n,1}\|)\|\varphi\| \\ \leqslant &\frac{\tau}{2}B^1(D_1(u_h),\varphi) + \frac{\tau}{2}(a_1+a_2)(\alpha_1+\alpha_2+1)(\|u_h^n\| + \|v_h^n\|)\|\varphi\|\end{aligned}$$

Define $C_7 = (a_1+a_2)(\alpha_1+\alpha_2+1)$, we get (4.5c).

Subtracting $\frac{4}{3}\times(4.3c)$ and $\frac{1}{3}\times(4.3a)$ from (4.3e), then summing it over j, we get

$$\begin{aligned}(D_3(u_h),\varphi) =&\frac{\tau}{3}\Big[B^1(D_2(u_h),\varphi) - 2\int_0^L (\lambda_1)_h^{n,2} u_h^{n,2}\varphi dx + 2\int_0^L (\lambda_2)_h^{n,2} v_h^{n,2}\varphi dx \\ &+ \int_0^L (\lambda_1)_h^{n,1} u_h^{n,1}\varphi dx - \int_0^L (\lambda_2)_h^{n,1} v_h^{n,1}\varphi dx + \int_0^L (\lambda_1)_h^{n} u_h^{n}\varphi dx \\ &- \int_0^L (\lambda_1)_h^{n} v_h^{n}\varphi dx\Big]\end{aligned}$$

From Lemma 4.1 and Lemma 4.2, we get

$$\begin{aligned}(D_3(u_h),\varphi) \leqslant &\frac{\tau}{3}B^1(D_2(u_h),\varphi) + \frac{(a_1+a_2)\tau}{3}(2\|u_h^{n,2}\| + 2\|v_h^{n,2}\| + \|u_h^{n,1}\| + \|v_h^{n,1}\| \\ &+ \|u_h^n\| + \|v_h^n\|)\|\varphi\| \\ \leqslant &\frac{\tau}{3}B^1(D_2(u_h),\varphi) + \frac{C_8\tau}{3}(\|u_h^n\| + \|v_h^n\|)\|\varphi\|\end{aligned}$$

where $C_8 = (a_1+a_2)(2C_3+2C_4+\alpha_1+\alpha_2+1)$. We have therefore obtained (4.5e). Similarly we can get (4.5b), (4.5d) and (4.5f). □

8.4 The proof of Lemma 4.6

Proof. The function $f(y) = 0.5 + 0.5\tanh(y-y_0)$ is Lipschitz continuous in $(-\infty,+\infty)$, i.e.

$$|f(x)-f(y)| \leqslant \frac{1}{2}|x-y|, \quad x,y \in (-\infty,+\infty) \tag{8.1}$$

Denote $p^{n,i} = u^{n,i} + v^{n,i}$, $p_h^{n,i} = u_h^{n,i} + v_h^{n,i}$. From periodic conditions, we have

$$\begin{aligned}&y_{1,r}[u^{n,i},v^{n,i}] - y_{1,r}[u_h^{n,i},v_h^{n,i}] \\ =&q_r\int_0^\infty K_r(s)(p^{n,i}(x+s) - p^{n,i}(x-s) - p_h^{n,i}(x+s) + p_h^{n,i}(x-s))ds \\ =&q_r\sum_{m=0}^\infty\int_{mL}^{(m+1)L} K_r(s)(p^{n,i}(x+s) - p^{n,i}(x-s) - p_h^{n,i}(x+s) + p_h^{n,i}(x-s))ds \\ =&q_r\sum_{m=0}^\infty\int_0^L K_r(s+mL)(p^{n,i}(x+s) - p^{n,i}(x-s) - p_h^{n,i}(x+s) + p_h^{n,i}(x-s))ds\end{aligned}$$

Since we assume $L > 2s_r$, we have $\sum\limits_{m=0}^\infty K_r(s+mL) \leqslant 2K_r(s)$, $s\in[0,L]$.

From Hölder's inequality and triangle inequality, we get

$$\Big|y_{1,r}[u^{n,i},v^{n,i}] - y_{1,r}[u_h^{n,i},v_h^{n,i}]\Big| \leqslant q_r\Big\|\sum_{m=0}^\infty K_r(s+mL)\Big\|(2\|u^{n,i}-u_h^{n,i}\| + 2\|v^{n,i}-v_h^{n,i}\|)$$

$$\leqslant 4q_rC_r(\|\xi^{n,i}(u)\|+\|\eta^{n,i}(u)\|+\|\xi^{n,i}(v)\|+\|\eta^{n,i}(v)\|)$$

Here $\left(\int_0^L K_r(s)^2ds\right)^{1/2}\leqslant C_r$.

Similarly we have

$$\begin{aligned}
\left|y_{1,a}[u^{n,i},v^{n,i}]-y_{1,a}[u_h^{n,i},v_h^{n,i}]\right| &\leqslant 4q_aC_a(\|\xi^{n,i}(u)\|+\|\eta^{n,i}(u)\|+\|\xi^{n,i}(v)\|+\|\eta^{n,i}(v)\|)\\
\left|y_{1,al}[u^{n,i},v^{n,i}]-y_{1,al}[u_h^{n,i},v_h^{n,i}]\right| &\leqslant 2q_{al}C_{al}(\|\xi^{n,i}(u)\|+\|\eta^{n,i}(u)\|+\|\xi^{n,i}(v)\|+\|\eta^{n,i}(v)\|)\\
\left|y_{2,r}[u^{n,i},v^{n,i}]-y_{2,r}[u_h^{n,i},v_h^{n,i}]\right| &\leqslant 4q_rC_r(\|\xi^{n,i}(u)\|+\|\eta^{n,i}(u)\|+\|\xi^{n,i}(v)\|+\|\eta^{n,i}(v)\|)\\
\left|y_{2,a}[u^{n,i},v^{n,i}]-y_{2,a}[u_h^{n,i},v_h^{n,i}]\right| &\leqslant 4q_aC_a(\|\xi^{n,i}(u)\|+\|\eta^{n,i}(u)\|+\|\xi^{n,i}(v)\|+\|\eta^{n,i}(v)\|)\\
\left|y_{2,al}[u^{n,i},v^{n,i}]-y_{2,al}[u_h^{n,i},v_h^{n,i}]\right| &\leqslant 2q_{al}C_{al}(\|\xi^{n,i}(u)\|+\|\eta^{n,i}(u)\|+\|\xi^{n,i}(v)\|+\|\eta^{n,i}(v)\|)
\end{aligned}$$

Here $\left(\int_0^L K_a(s)^2ds\right)^{1/2}\leqslant C_a$, $\left(\int_0^L K_{al}(s)^2ds\right)^{1/2}\leqslant C_{al}$. So we get

$$\begin{aligned}
&\left|y_1[u^{n,i},v^{n,i}]-y_1[u_h^{n,i},v_h^{n,i}]\right|\\
\leqslant&\left|y_{1,r}[u^{n,i},v^{n,i}]-y_{1,r}[u_h^{n,i},v_h^{n,i}]\right|+\left|y_{1,a}[u^{n,i},v^{n,i}]-y_{1,a}[u_h^{n,i},v_h^{n,i}]\right|\\
&+\left|y_{1,al}[u^{n,i},v^{n,i}]-y_{1,al}[u_h^{n,i},v_h^{n,i}]\right|+\left|y_{2,r}[u^{n,i},v^{n,i}]-y_{2,r}[u_h^{n,i},v_h^{n,i}]\right|\\
&+\left|y_{2,a}[u^{n,i},v^{n,i}]-y_{2,a}[u_h^{n,i},v_h^{n,i}]\right|+\left|y_{2,al}[u^{n,i},v^{n,i}]-y_{2,al}[u_h^{n,i},v_h^{n,i}]\right|\\
\leqslant&2(2q_rC_r+2q_aC_a+q_{al}C_{al})(\|\xi^{n,i}(u)\|+\|\xi^{n,i}(v)\|+\|\eta^{n,i}(u)\|+\|\eta^{n,i}(v)\|)
\end{aligned}$$

From (8.1), (2.3), triangle inequality and the above inequality, we get

$$\begin{aligned}
&\left|\lambda_1(y_1[u^{n,i},v^{n,i}])u^{n,i}-\lambda_1(y_1[u_h^{n,i},v_h^{n,i}])u_h^{n,i}\right|\\
\leqslant&\left|(\lambda_1(y_1[u^{n,i},v^{n,i}])-\lambda_1(y_1[u_h^{n,i},v_h^{n,i}]))u^{n,i}\right|+\left|\lambda_1(y_1[u_h^{n,i},v_h^{n,i}])(u^{n,i}-u_h^{n,i})\right|\\
\leqslant&\frac{a_2|u^{n,i}|}{2}\left|y_1[u^{n,i},v^{n,i}]-y_1[u_h^{n,i},v_h^{n,i}]\right|+(a_1+a_2)|u^{n,i}-u_h^{n,i}|\\
\leqslant&a_2(2q_rC_r+2q_aC_a+q_{al}C_{al})(\|\xi^{n,i}(u)\|+\|\xi^{n,i}(v)\|+\|\eta^{n,i}(u)\|+\|\eta^{n,i}(v)\|)|u^{n,i}|\\
&+(a_1+a_2)(|\xi^{n,i}(u)|+|\eta^{n,i}(u)|)
\end{aligned}\tag{8.2}$$

In the following we will estimate $u^{n,i},v^{n,i}$ in L^2 norm, $i=0,1,2$.

From the definitions of $u^{n,1},v^{n,1}$, $\tau\leqslant 1/\gamma$ and (2.3), we have

$$\begin{aligned}
\|u^{n,1}\|&\leqslant\|u^n\|+\|u_x^n\|+\frac{a_1+a_2}{\gamma}(\|u^n\|+\|v^n\|),\\
\|v^{n,1}\|&\leqslant\|v^n\|+\|v_x^n\|+\frac{a_1+a_2}{\gamma}(\|u^n\|+\|v^n\|)
\end{aligned}$$

Before estimating $u^{n,2},v^{n,2}$, consider $\frac{\partial}{\partial x}y_1[u^n,v^n]$ first. The interchanging of the derivative and the integral in the following are based on $u,v\in W_2^{k+1}([0,L])$.

$$\begin{aligned}
\frac{\partial}{\partial x}y_{1,r}[u^n,v^n]&=\frac{\partial}{\partial x}q_r\int_0^\infty K_r(s)(p^n(x+s)-p^n(x-s))ds\\
&=\frac{\partial}{\partial x}q_r\int_0^L\left(\sum_{m=0}^\infty K_r(s+mL)\right)(p_x^n(x+s)-p_x^n(x-s))ds\\
&=q_r\int_0^L\left(\sum_{m=0}^\infty K_r(s+mL)\right)(p_x^n(x+s)-p_x^n(x-s))ds
\end{aligned}$$

From Hölder's inequality and periodic boundary conditions, we have

$$\left|\frac{\partial}{\partial x} y_{1,r}[u^n, v^n]\right| \leqslant 4q_r C_r(\|u_x^n\| + \|v_x^n\|)$$

So we have

$$\left|\frac{\partial}{\partial x} y_1[u^n, v^n]\right| \leqslant 2(2q_r C_r + 2q_a C_a + q_{al} C_{al})(\|u_x^n\| + \|v_x^n\|)$$

From (2.3), triangle inequality and periodic boundary conditions, we have

$$\begin{aligned}\|\big(\lambda_1(y_1[u^n, v^n])u^n\big)_x\| \leqslant& \left\|a_2 f'(y_1[u^n, v^n]) \left(\frac{\partial}{\partial x} y_1[u^n, v^n]\right) u^n\right\| + (a_1 + a_2)\|u_x^n\| \\ \leqslant& a_2(2q_r C_r + 2q_a C_a + q_{al} C_{al})(\|u_x^n\| + \|v_x^n\|)\|u^n\| + (a_1 + a_2)\|u_x^n\|\end{aligned}$$

Here $|f'| \leqslant 1/2$. Similarly we have

$$\|\big(\lambda_2(y_2[u^n, v^n])v^n\big)_x\| \leqslant a_2(2q_r C_r + 2q_a C_a + q_{al} C_{al})(\|u_x^n\| + \|v_x^n\|)\|v^n\| + (a_1 + a_2)\|v_x^n\|$$

From above inequalities, the definitions of $u^{n,1}$, $\tau \leqslant 1/\gamma$ and triangle inequality, we have

$$\begin{aligned}\|u_x^{n,1}\| \leqslant& \|u_x^n\| + \|u_{xx}^n\| + \frac{1}{\gamma}\|\big(\lambda_1(y_1[u^n, v^n])u^n\big)_x\| + \frac{1}{\gamma}\|\big(\lambda_2(y_2[u^n, v^n])v^n\big)_x\| \\ \leqslant& \|u_x^n\| + \|u_{xx}^n\| + \frac{a_2}{\gamma}(2q_r C_r + 2q_a C_a + q_{al} C_{al})(\|u_x^n\| + \|v_x^n\|)(\|u^n\| + \|v^n\|) \\ &+ \frac{a_1 + a_2}{\gamma}(\|u_x^n\| + \|v_x^n\|)\end{aligned}$$

From above inequality, the definitions of $u^{n,2}$, $\tau \leqslant 1/\gamma$, (2.3) and triangle inequality, we have

$$\begin{aligned}\|u^{n,2}\| \leqslant& \frac{3}{4}\|u^n\| + \frac{1}{4}\|u^{n,1}\| + \frac{1}{4}\|u_x^{n,1}\| + \frac{a_1 + a_2}{4\gamma}(\|u^{n,1}\| + \|v^{n,1}\|) \\ \leqslant& \frac{3}{4}\|u^n\| + \frac{1}{4}\|u^{n,1}\| + \frac{1}{4}\|u_x^n\| + \frac{1}{4}\|u_{xx}^n\| + \frac{a_1 + a_2}{4\gamma}(\|u^{n,1}\| + \|v^{n,1}\|) \\ &+ \frac{a_2}{4\gamma}(2q_r C_r + 2q_a C_a + q_{al} C_{al})(\|u_x^n\| + \|v_x^n\|)(\|u^n\| + \|v^n\|) \\ &+ \frac{a_1 + a_2}{4\gamma}(\|u_x^n\| + \|v_x^n\|)\end{aligned}$$

Similarly we have

$$\begin{aligned}\|v^{n,2}\| \leqslant& \frac{3}{4}\|v^n\| + \frac{1}{4}\|v^{n,1}\| + \frac{1}{4}\|v_x^n\| + \frac{1}{4}\|v_{xx}^n\| + \frac{a_1 + a_2}{4\gamma}(\|u^{n,1}\| + \|v^{n,1}\|) \\ &+ \frac{a_2}{4\gamma}(2q_r C_r + 2q_a C_a + q_{al} C_{al})(\|u_x^n\| + \|v_x^n\|)(\|u^n\| + \|v^n\|) \\ &+ \frac{a_1 + a_2}{4\gamma}(\|u_x^n\| + \|v_x^n\|)\end{aligned}$$

From $u, v \in W_2^{k+1}([0, L])$ for all $t \in [0, T]$, we have the boundedness of $\|u^{n,i}\|, \|v^{n,i}\| < \infty$, $i = r, a, al$. We set M_T as their upper bound, i.e.

$$\|u^{n,i}\|, \|v^{n,i}\| \leqslant M_T, \qquad i = 0, 1, 2. \tag{8.3}$$

From (8.2), triangle inequality, Hölder's inequality, (8.3) and (3.3), we get

$$\left|\int_0^L (\lambda_1(y_1[u^{n,i}, v^{n,i}])u^{n,i} - \lambda_1(y_1[u_h^{n,i}, v_h^{n,i}])u_h^{n,i})\varphi dx\right|$$

$$\leqslant C_{16}(\|\xi^{n,i}(u)\| + \|\xi^{n,i}(v)\|)\|\varphi\| + 2C_{16}C_1h^{k+1}\|\varphi\|, \quad i = 0, 1, 2.$$

Here $C_{16} = a_2M_T(2q_rC_r + 2q_aC_a + q_{al}C_{al}) + a_1 + a_2$. Similarly we can get

$$\left|\int_0^L (\lambda_2(y_2[u^{n,i}, v^{n,i}])u^{n,i} - \lambda_2(y_2[u_h^{n,i}, v_h^{n,i}])u_h^{n,i})\varphi dx\right|$$
$$\leqslant C_{16}(\|\xi^{n,i}(u)\| + \|\xi^{n,i}(v)\|)\|\varphi\| + 2C_{16}C_1h^{k+1}\|\varphi\|, \quad i = 0, 1, 2.$$

□

8.5 The proof of Lemma 4.7

Proof. Taking $\varphi = \xi^{n,1}(u)$ in (4.19a) and summing it over j, we can get

$$\begin{aligned}\|\xi^{n,1}(u)\|^2 = &\int_0^L \xi^n(u)\xi^{n,1}(u)dx + \int_0^L (\eta^{n,1}(u) - \eta^n(u))\xi^{n,1}(u)dx + \tau\Big[B^1(\xi^n(u), \xi^{n,1}(u)) \\ &- \int_0^L (\lambda_1^n u^n - (\lambda_1)_h^n u_h^n)\varphi dx + \int_0^L (\lambda_2^n v^n - (\lambda_2)_h^n v_h^n)\varphi dx\Big]\end{aligned}$$

From Hölder's inequality, (4.18), Lemma 3.1 and Lemma 4.6, we can obtain

$$\begin{aligned}\|\xi^{n,1}(u)\|^2 \leqslant &(1 + (\sqrt{2} + 1)\lambda + 2C_{16}\tau)\|\xi^n(u)\|\|\xi^{n,1}(u)\| + 2C_{16}\tau\|\xi^n(v)\|\|\xi^{n,1}(u)\| \\ &+ (4C_1C_{16} + C_{15})h^{k+1}\tau\|\xi^{n,1}(u)\|\end{aligned}$$

When $\|\xi^{n,1}(u)\| \neq 0$, from $\tau \leqslant 1/\gamma$ and $\lambda \leqslant M$, we have

$$\|\xi^{n,1}(u)\| \leqslant \beta_1\|\xi^n(u)\| + \beta_2\|\xi^n(v)\| + \beta_3h^{k+1}\tau$$

Here $\beta_1 = 1 + (\sqrt{2} + 1)M + 2C_{16}/\gamma$, $\beta_2 = 2C_{16}/\gamma$, $\beta_3 = 4C_1C_{16} + C_{15}$. The inequality also holds for $\|\xi^{n,1}(u)\| = 0$.

Similarly we can obtain

$$\begin{aligned}\|\xi^{n,1}(v)\| &\leqslant \beta_1\|\xi^n(v)\| + \beta_2\|\xi^n(u)\| + \beta_3h^{k+1}\tau \\ \|\xi^{n,2}(u)\| &\leqslant \left(\frac{3}{4} + \frac{1}{4}(\beta_1^2 + \beta_2^2)\right)\|\xi^n(u)\| + \frac{1}{2}\beta_1\beta_2\|\xi^n(v)\| \\ &\quad + \frac{1}{4}(\beta_1 + \beta_2 + 1)\beta_3h^{k+1}\tau \\ \|\xi^{n,2}(v)\| &\leqslant \left(\frac{3}{4} + \frac{1}{4}(\beta_1^2 + \beta_2^2)\right)\|\xi^n(v)\| + \frac{1}{2}\beta_1\beta_2\|\xi^n(u)\| \\ &\quad + \frac{1}{4}(\beta_1 + \beta_2 + 1)\beta_3h^{k+1}\tau\end{aligned}$$

Define $C_{17} = \frac{3}{4} + \frac{1}{4}(\beta_1^2 + \beta_2^2)$, $C_{18} = \frac{1}{2}\beta_1\beta_2$, $C_{19} = \frac{1}{4}(\beta_1 + \beta_2 + 1)\beta_3$. From triangle inequality and above inequalities we get

$$\begin{aligned}\|\mathbb{D}_1(\xi(u))\| &\leqslant (\beta_1 + 1)\|\xi^n(u)\| + \beta_2\|\xi^n(v)\| + \beta_3h^{k+1}\tau \\ \|\mathbb{D}_1(\xi(v))\| &\leqslant (\beta_1 + 1)\|\xi^n(v)\| + \beta_2\|\xi^n(u)\| + \beta_3h^{k+1}\tau \\ \|\mathbb{D}_2(\xi(u))\| &\leqslant (2C_{17} + \beta_1 + 1)\|\xi^n(u)\| + (2C_{18} + \beta_2)\|\xi^n(v)\| \\ &\quad + (2C_{19} + \beta_3)h^{k+1}\tau \\ \|\mathbb{D}_2(\xi(v))\| &\leqslant (2C_{17} + \beta_1 + 1)\|\xi^n(v)\| + (2C_{18} + \beta_2)\|\xi^n(u)\| \\ &\quad + (2C_{19} + \beta_3)h^{k+1}\tau\end{aligned}$$

Define $C_{20} = 2C_{17} + \beta_1 + 1$, $C_{21} = 2C_{18} + \beta_2$, $C_{22} = 2C_{19} + \beta_3$. Then we get the desired results. □

8.6 The proof of Lemma 4.8

Proof. Summing (4.19a) over j and using Hölder's inequality, (4.18) and Lemma 4.6, we get

$$(\mathbb{D}_1(\xi(u)),\varphi)\leqslant\tau B^1(\xi^n(u),\varphi)+2C_{16}\tau(\|\xi^n(u)\|+\|\xi^n(v)\|)\|\varphi\|+\beta_3h^{k+1}\tau$$

Similarly we can get

$$(\mathbb{D}_1(\xi(v)),\varphi)\leqslant\tau B^2(\xi^n(v),\varphi)+2C_{16}\tau(\|\xi^n(u)\|+\|\xi^n(v)\|)\|\varphi\|+\beta_3h^{k+1}\tau$$

Subtracting $\frac{1}{2}\times(4.19a)$ from $2\times(4.19c)$, then summing it over j and using Hölder's inequality, (4.18), Lemma 4.6, Lemma 4.7 and $\tau\leqslant 1/\gamma$, we get

$$(\mathbb{D}_2(\xi(u)),\varphi)\leqslant\frac{\tau}{2}B^1(\mathbb{D}_1(\xi^n(u)),\varphi)+\frac{C_{23}\tau}{2}(\|\xi^n(u)\|+\|\xi^n(v)\|)\|\varphi\|+C_{24}h^{k+1}\tau$$

Here $C_{23}=2C_{16}(\beta_1+\beta_2+1)$, $C_{24}=2C_{16}\beta_3/\gamma+\beta_3$.

Similarly we have

$$(\mathbb{D}_2(\xi(v)),\varphi)\leqslant\frac{\tau}{2}B^2(\mathbb{D}_1(\xi^n(v)),\varphi)+\frac{C_{23}\tau}{2}(\|\xi^n(u)\|+\|\xi^n(v)\|)\|\varphi\|+C_{24}h^{k+1}\tau$$

Subtracting $\frac{4}{3}\times(4.3c)$ and $\frac{1}{3}\times(4.3a)$ from $(4.3e)$, then summing it over j and using Hölder's inequality, (4.16), (4.18), Lemma 4.6, Lemma 4.7 and $\tau\leqslant 1/\gamma$, we get

$$\begin{aligned}(\mathbb{D}_3(\xi(u)),\varphi)\leqslant&\frac{\tau}{3}B^1(\xi^n(u),\varphi)+\frac{C_{25}\tau}{3}(\|\xi^n(u)\|+\|\xi^n(v)\|)\|\varphi\|\\&+C_{26}h^{k+1}\tau\|\varphi\|+C_{14}\tau^4\|\varphi\|\end{aligned}$$

Here $C_{25}=2C_{16}(1+\beta_1+\beta_2+2(C_{17}+C_{18}))$, $C_{26}=4C_{16}((\beta_3+2C_{19})/\gamma+4C_1)/3+C_{15}$. Similarly we can get

$$\begin{aligned}(\mathbb{D}_3(\xi(v)),\varphi)\leqslant&\frac{\tau}{3}B^2(\xi^n(v),\varphi)+\frac{C_{25}\tau}{3}(\|\xi^n(u)\|+\|\xi^n(v)\|)\|\varphi\|\\&+C_{26}h^{k+1}\tau\|\varphi\|+C_{14}\tau^4\|\varphi\|\end{aligned}$$

Then we get the desired results. □

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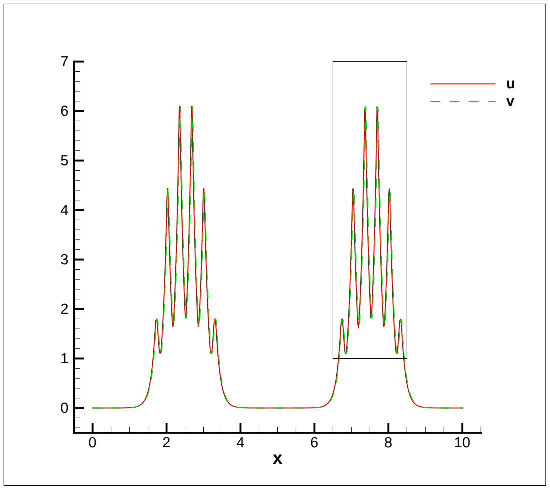
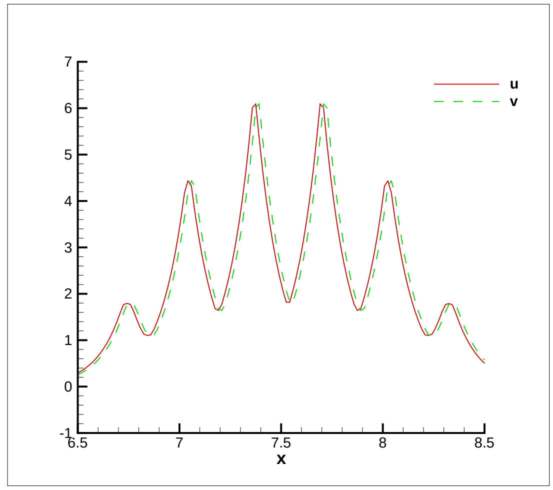
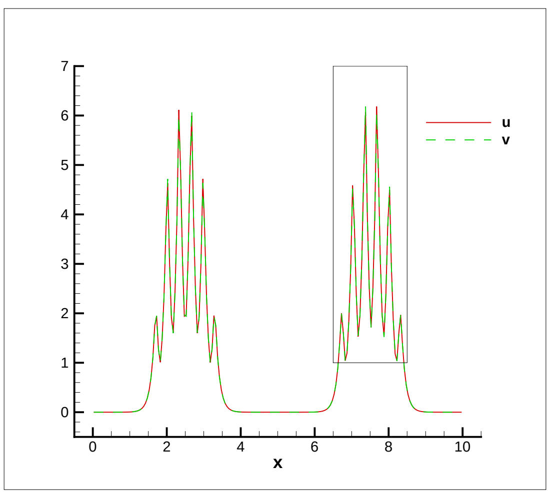


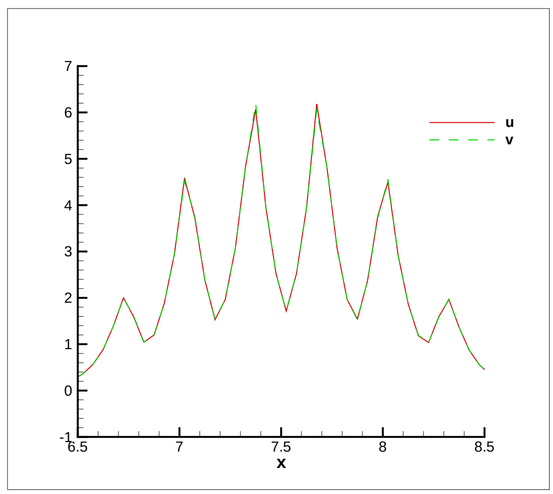
(a) scheme (5.1)



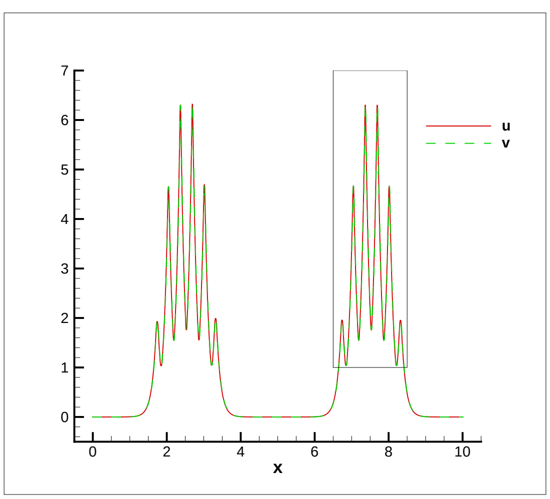
(b) Enlarged view inside the rectangle in fig.(a)



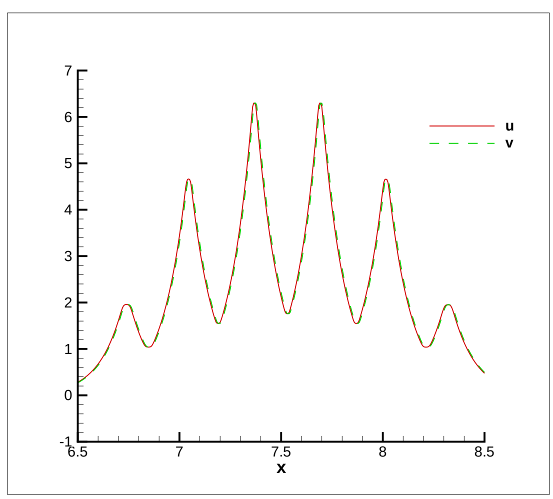
(c) scheme (4.3)



(d) Enlarged view inside the rectangle in fig.(c)

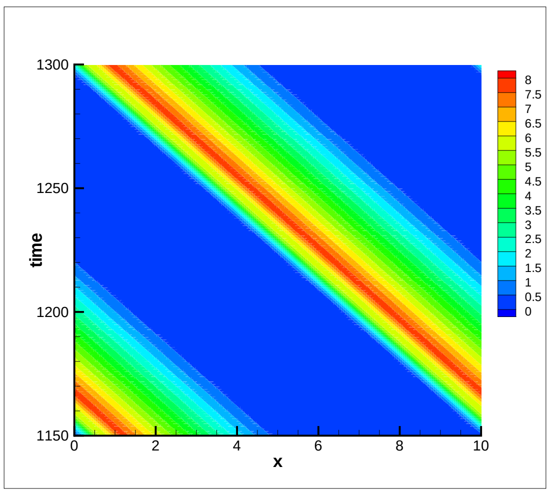


(e) converged solution

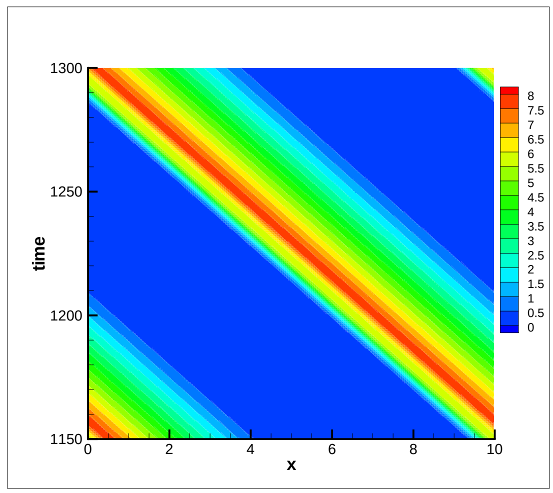


(f) Enlarged view inside the rectangle in fig.(e)

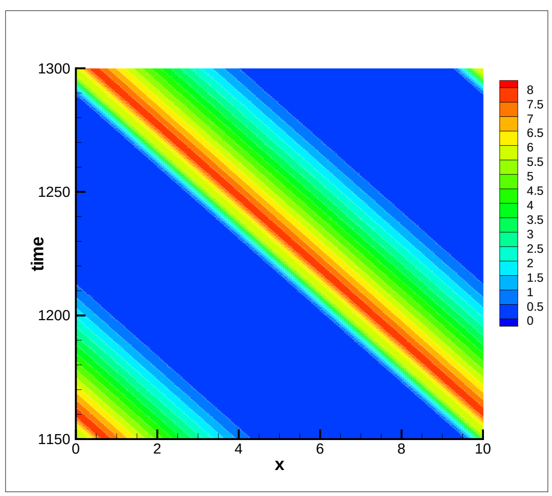
Figure 3   *Stationary pulses,* $u + v$ *at time* $t = 1000$ *of Example 2.*



(a) scheme (5.1)



(b) scheme (4.3)



(c) converged solution

Figure 4 *Traveling pulses,* $u + v$ *from time* $t = 1150$ *to* $t = 1300$ *of Example 3.*

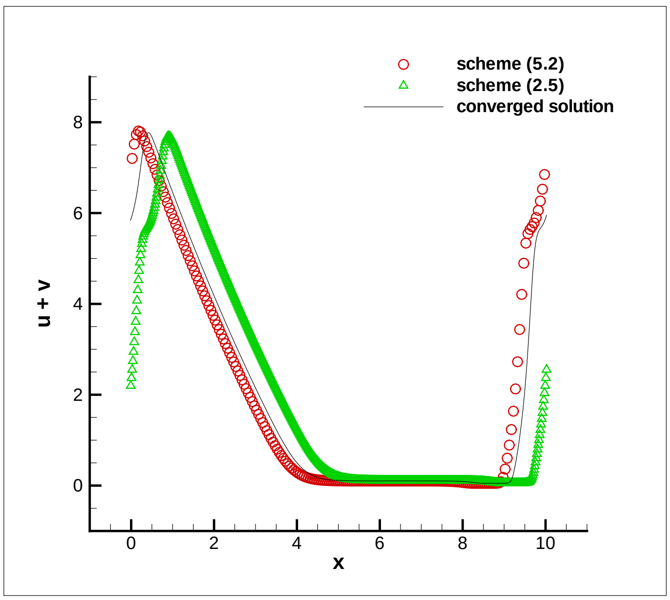


Figure 5 *Traveling pulses,* $u + v$ *at time* $t = 1300$ *of Example 3.*