

## A PRIORI ERROR ESTIMATES FOR SEMI-DISCRETE DISCONTINUOUS GALERKIN METHODS SOLVING NONLINEAR HAMILTON-JACOBI EQUATIONS WITH SMOOTH SOLUTIONS

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**Abstract.** In this paper, we provide a priori  $L^2$  error estimates for the semi-discrete discontinuous Galerkin method [3] and the local discontinuous Galerkin method [22] for one- and two-dimensional nonlinear Hamilton-Jacobi equations with smooth solutions. With a special Gauss-Radau projection, the optimal error estimates on rectangular meshes are obtained.

**Key words.** Hamilton-Jacobi equations, discontinuous Galerkin method, local discontinuous Galerkin method, a priori error estimates.

### 1. Introduction

In this paper, we are interested in the a priori  $L^2$  error estimates of the semi-discrete discontinuous Galerkin (DG) and local discontinuous Galerkin (LDG) methods for smooth solutions of nonlinear Hamilton-Jacobi (HJ) equations in the one-dimensional case

$$(1) \quad \phi_t + H(\phi_x, x) = 0, \quad \phi(x, 0) = \phi^0(x)$$

and in the two-dimensional case:

$$(2) \quad \phi_t + H(\phi_x, \phi_y, x, y) = 0, \quad \phi(x, y, 0) = \phi^0(x, y).$$

The Hamiltonian  $H$  is assumed to be a smooth function of all the arguments. When there is no ambiguity, we also take the concise notation  $H(\phi_x) = H(\phi_x, x)$  and  $H(\phi_x, \phi_y) = H(\phi_x, \phi_y, x, y)$ .

The DG method is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution in the spatial variables. It can be discretized in time by the explicit and nonlinearly stable high order Runge-Kutta time discretization [20], resulting in the so-called RKDG method. The RKDG method was first developed for nonlinear hyperbolic conservation laws by Cockburn et al. in [8, 7, 5, 9]. Later it was generalized to the LDG method for solving convection-diffusion equations by Cockburn and Shu [10].

The time-dependent Hamilton-Jacobi (HJ) equations (1) and (2) are closely related to the conservation laws. In the one-dimensional case, they are equivalent if one takes the spatial derivative in (1) and writes out the equation satisfied by  $u = \phi_x$ . It is thus not surprisingly that many successful numerical methods for

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the conservation laws have been adapted to solve the Hamilton-Jacobi equations. For finite difference schemes, the high order essentially non-oscillatory (ENO) and weighted ENO (WENO) schemes [18, 14, 25] are such examples. However, it is less straightforward to adapt DG schemes to solve the Hamilton-Jacobi equations, since the nonlinear Hamiltonian  $H$  prevents a direct integration by parts. Hu and Shu developed a DG scheme [13] for solving the nonlinear Hamilton-Jacobi equations, which is based on the Runge-Kutta discontinuous Galerkin (RKDG) method for solving conservation laws. They first solve the conservation law equation satisfied by  $u = \phi_x$  with the standard DG method, which can determine  $\phi$  for each element up to a constant, and then the missing constant is obtained by integration either in time or from the boundary. In two dimensions, this scheme involves a least square procedure to obtain  $\phi$  from the numerical approximations of  $u = \phi_x$  and  $v = \phi_y$ , as they may not satisfy the compatibility condition  $u_y = v_x = \phi_{xy}$ . Later, Li and Shu [16] reinterpreted the method in [13] by using a curl-free subspace for the discontinuous Galerkin method in the two-dimensional case to avoid the least squares procedure. The two algorithms in [13] and [16] are mathematically equivalent, however the latter avoids the least square procedure and also uses a smaller finite element space, resulting in a significant simplification in implementation with a reduced cost. The DG scheme in [13] achieves the optimal  $k$ -th order of accuracy for  $u = \phi_x$  (and also  $v = \phi_y$  in two dimensions), however the optimal  $(k + 1)$ -th order accuracy for  $\phi$  is not always observed numerically when  $k$ -th degree piecewise polynomial space is used. For the one-dimensional case, the error estimates for conservation laws in [23, 21, 24] can be directly applied, yielding  $k$ -th order error accuracy for the upwind fluxes and  $(k - \frac{1}{2})$ -th order error accuracy for general numerical fluxes for the derivative  $u = \phi_x$  when  $k$ -th degree piecewise polynomial space is used. For the two-dimensional case, we can follow the a priori error estimates for  $u = \phi_x$  and  $v = \phi_y$  in the DG curl-free subspace, however only  $(k - \frac{1}{2})$ -th order accuracy can be obtained either for the upwind fluxes or for general fluxes, since the special projections need for the optimal error estimates in two dimensions cannot be defined in the curl-free subspace.

More recently, Cheng and Shu in [3] proposed a DG method for directly solving Hamilton-Jacobi equations without going through the derivatives  $u = \phi_x$  and  $v = \phi_y$ . Also, Yan and Osher [22] designed a direct LDG method for solving Hamilton-Jacobi equations. Numerically, optimal order error accuracy has been observed for both of these two methods. For linear Hamiltonians, the DG and LDG methods in [3] and [22] are equivalent to those for solving conservation laws, hence stability and error estimates can be obtained following the techniques for conservation laws. However, for nonlinear Hamiltonians, the methods in [3] and [22] are distinct from the DG methods for conservation laws. In this paper, we follow and generalize the techniques in [23, 21, 24] to obtain a priori  $L^2$  error estimates for the DG and LDG methods in [3] and [22] for directly solving nonlinear Hamilton-Jacobi equations with smooth solutions.

The paper is organized as follows. In Section 2, we introduce notations, definitions and auxiliary results used later in this paper. In Section 3, we obtain a priori error estimates for the one-dimensional Hamilton-Jacobi equations. In Section 4,

we follow the same line as the one-dimensional case to obtain a priori error estimates for the two-dimensional Hamilton-Jacobi equations. Concluding remarks are given in Section 5.

## 2. Notations, definitions and auxiliary results

In this section, we follow [21, 24] to first introduce notations and definitions to be used later in this paper and also present some auxiliary results. We use a special Gauss-Radau projection as in [24], and present certain interpolation and inverse properties for the finite element spaces that will be used in the error analysis.

### 2.1. Basic notations.

**2.1.1. One-dimensional case.** We consider the one-dimensional Hamilton-Jacobi equation in the interval  $I = (0, 1)$ , which is divided into  $N$  cells as  $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots < x_{N+\frac{1}{2}} = 1$ . We set  $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ ,  $\bar{I}_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ ,  $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$ ,  $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$  and  $\bar{I}_{j+\frac{1}{2}} = [x_j, x_{j+1}]$ , for  $j = 1, \dots, N$ , and we define the quantities

$$(3) \quad h = \max_{1 \leq j \leq N} h_j \quad \text{and} \quad \rho = \min_{1 \leq j \leq N} h_j$$

We assume the mesh is regular, namely the ratio of  $h$  over  $\rho$  stays bounded by a fixed positive constant  $\nu^{-1}$  during mesh refinements, that is  $\nu h \leq \rho \leq h$ . The piecewise-polynomial approximation space is

$$(4) \quad V_h = \{v : v|_{I_j} \in P^k(I_j), j = 1, \dots, N\}$$

where  $P^k(I_j)$  denotes all polynomials of degree at most  $k$  on  $I_j$ .

The numerical solution is denoted by  $\phi_h \in V_h$ . As usual, we denote by  $(\phi_h)_{j+\frac{1}{2}}^+$  and  $(\phi_h)_{j+\frac{1}{2}}^-$  the values of  $\phi_h$  at  $x_{j+\frac{1}{2}}$  from the right cell  $I_{j+1}$  and the left cell  $I_j$ , respectively. We also use the notations  $[\phi_h] = \phi_h^+ - \phi_h^-$  and  $\bar{\phi}_h = \frac{1}{2}(\phi_h^+ + \phi_h^-)$  to denote the jump and the mean of the function  $\phi_h$  at each element boundary point, respectively. Finally, we denote by  $H_1(\phi_x, x) = \frac{\partial H}{\partial \phi_x}(\phi_x, x)$  and  $H_{11}(\phi_x, x) = \frac{\partial^2 H}{\partial^2 \phi_x}(\phi_x, x)$ , the first and second derivatives of  $H$  with respect to its first argument, respectively.

**2.1.2. Two-dimensional case.** We consider the two-dimensional Hamilton-Jacobi equation on the domain  $\Omega$ . For a rectangular partition of  $I \times J = [0, L_x] \times [0, L_y]$ , we denote the mesh by  $I_i \times J_j$  with  $I_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$  and  $J_j = (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ , for  $i = 1, \dots, N_x$  and  $j = 1, \dots, N_y$ . We set  $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$  and  $y_j = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}})$ . The cell lengths are denoted by  $h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$  and  $h_j^y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$  with  $h^x = \max_{1 \leq i \leq N_x} h_i^x$ ,  $h^y = \max_{1 \leq j \leq N_y} h_j^y$  and  $h = \max(h^x, h^y)$ . We also assume the mesh is regular as in the one-dimensional case.

We define the space  $Z_h$  as the space of tensor product piecewise polynomials of degree at most  $k$  in each variable on every element, i.e.

$$(5) \quad Z_h = \{v : v \in Q^k(I_i \times J_j), \forall (x, y) \in I_i \times J_j, i = 1, \dots, N_x, j = 1, \dots, N_y\}$$

where  $Q^k$  is the space of tensor products of one-dimensional polynomials of degree up to  $k$ .

We denote by  $(\phi_h)_{i+\frac{1}{2},y}^+$  and  $(\phi_h)_{i+\frac{1}{2},y}^-$  the values of  $\phi_h$  at  $(x_{i+\frac{1}{2}}, y)$  from the right cell  $I_{i+1} \times J_j$  and from the left cell  $I_i \times J_j$ , respectively, when  $y \in J_j$  on all vertical edges. Also  $[\phi_h]_{i+\frac{1}{2},y} = (\phi_h)_{i+\frac{1}{2},y}^+ - (\phi_h)_{i+\frac{1}{2},y}^-$  and  $(\bar{\phi}_h)_{i+\frac{1}{2},y} = \frac{1}{2}((\phi_h)_{i+\frac{1}{2},y}^+ + (\phi_h)_{i+\frac{1}{2},y}^-)$  denote the jump and the mean of the function  $\phi_h$  at  $(x_{i+\frac{1}{2}}, y)$  when  $y \in J_j$ . Similarly, we can define  $(\phi_h)_{x,j+\frac{1}{2}}^+, (\phi_h)_{x,j+\frac{1}{2}}^-, [\phi_h]_{x,j+\frac{1}{2}}$  and  $(\bar{\phi}_h)_{x,j+\frac{1}{2}}$ . We denote by  $H_1(\phi_x, \phi_y, x, y) = \frac{\partial H}{\partial \phi_x}(\phi_x, \phi_y, x, y)$  and  $H_{11}(\phi_x, \phi_y, x, y) = \frac{\partial^2 H}{\partial^2 \phi_x}(\phi_x, \phi_y, x, y)$  the first and second derivatives of  $H$  with respect to its first argument, respectively. Similarly, we define  $H_2(\phi_x, \phi_y, x, y) = \frac{\partial H}{\partial \phi_y}(\phi_x, \phi_y, x, y)$  and  $H_{22}(\phi_x, \phi_y, x, y) = \frac{\partial^2 H}{\partial^2 \phi_y}(\phi_x, \phi_y, x, y)$ . The mixed derivative is  $H_{12}(\phi_x, \phi_y, x, y) = \frac{\partial^2 H}{\partial \phi_x \partial \phi_y}(\phi_x, \phi_y, x, y)$ .

For an arbitrary unstructured triangulation, let  $\mathbb{T}_h$  denote a tessellation of  $\Omega$  with shape-regular elements  $K$ . Let  $\Gamma_h$  denotes the union of the boundary faces of elements  $K \in \mathbb{T}_h$ , i.e.,  $\Gamma_h = \cup_{K \in \mathbb{T}_h} \partial K$ , and  $\Gamma_0 = \Gamma_h \setminus \partial \Omega$ . Let  $P^k(K)$  be the space of polynomials of degree at most  $k \geq 0$  on  $K \in \mathbb{T}_h$ . We denote the finite element space by

$$(6) \quad W_h = \{v : v \in P^k(K) \text{ for } (x, y) \in K, \forall K \in \mathbb{T}_h\}$$

Let  $e$  be an edge shared by the elements  $K$  and  $K'$ . Define the unit normal vectors  $n$  and  $n'$  on  $e$  pointing exterior to  $K$  and  $K'$ , respectively. If  $\phi_h$  is a function on  $K$  and  $K'$ , but possibly discontinuous across  $e$ , let  $(\phi_h)^{int_K}$  denote  $((\phi_h)|_K)|_e$  and  $(\phi_h)^{ext_K}$  denote  $((\phi_h)|_{K'})|_e$ .

## 2.2. Projection and interpolation properties.

**2.2.1. One-dimensional case.** We will consider the Gauss-Radau projection  $\mathbb{R}_h$  that projects  $\phi(x, t) \in L^2(0, 1)$  into the finite element space  $V_h$ , which depends on the function  $\phi(x, t)$  itself and is defined in each element as:

$$(7) \quad \mathbb{R}_h = \begin{cases} \mathbb{R}_h^+ & \text{if } H_1(\partial_x \phi) > 0 \text{ in the element } I_j, \\ \mathbb{R}_h^- & \text{if } H_1(\partial_x \phi) < 0 \text{ in the element } I_j, \\ \mathbb{P}_h & \text{if } H_1(\partial_x \phi) \text{ changes its sign on the element } \bar{I}_j. \end{cases}$$

where the  $L^2$ -projection  $\mathbb{P}_h$  on the element  $I_j$  is,

$$(8) \quad \int_{I_j} (\mathbb{P}_h \phi(x) - \phi(x)) v_h(x) dx = 0, \forall v_h \in P^k(I_j),$$

and the projection  $\mathbb{R}_h^\pm$  on the element  $I_j$  is,

$$(9) \quad \int_{I_j} (\mathbb{R}_h^- \phi(x) - \phi(x)) v_h(x) dx = 0, \forall v_h \in P^{k-1}(I_j),$$

with  $\mathbb{R}_h^- \phi(x_{j-\frac{1}{2}}^+) - \phi(x_{j-\frac{1}{2}}^+) = 0$ ,

$$(10) \quad \int_{I_j} (\mathbb{R}_h^+ \phi(x) - \phi(x)) v_h(x) dx = 0, \forall v_h \in P^{k-1}(I_j),$$

with  $\mathbb{R}_h^+ \phi(x_{j+\frac{1}{2}}^-) - \phi(x_{j+\frac{1}{2}}^-) = 0$ .

Denote by  $\eta = \mathbb{R}_h \phi - \phi$  the projection error, by a standard scaling argument, for this projection, we can obtain [2, 4, 15]

$$(11) \quad \|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \leq Ch^{k+1}$$

and it follows from Sobolev's inequality that

$$(12) \quad \|\eta\|_\infty \leq Ch^{k+\frac{1}{2}}.$$

The positive constant  $C$ , solely depending on  $\phi$ , is independent of  $h$ .  $\|\cdot\|$  and  $\|\cdot\|_\infty$  are the usual  $L^2$ -norm and  $L^\infty$ -norm in Sobolev spaces, respectively.  $\Gamma_h$  is the union of all element interface points, and the  $L^2$ -norm on  $\Gamma_h$  is defined by

$$(13) \quad \|\eta\|_{\Gamma_h} = \left[ \sum_{1 \leq j \leq N} \left( (\eta_{j+\frac{1}{2}}^+)^2 + (\eta_{j+\frac{1}{2}}^-)^2 \right) \right]^{1/2}$$

From the projection (7), for each partition  $\{I_j\}$  of  $I$ , we can accumulate the elements with the same projection into three classes, i.e.,  $S_1$  denotes the class of cells with projection  $\mathbb{R}_h^+$ ,  $S_2$  denotes the class of cells with projection  $\mathbb{R}_h^-$  and  $S_3$  denotes the class of cells with projection  $\mathbb{P}_h$ .

**2.2.2. Two-dimensional case.** For two-dimensional problems on Cartesian meshes, we also use the Gauss-Radau projection, which is the tensor product of the projections in the one-dimensional case. On a rectangle  $I \times J = [0, L_x] \times [0, L_y]$ , the projection is defined to be

$$(14) \quad \mathcal{P}\phi = (\mathbb{R}_h)_x \otimes (\mathbb{R}_h)_y \phi$$

where the subscripts indicate the application of the one-dimensional projection  $\mathbb{R}_h$  with respect to the corresponding variable.

Denote by  $\eta = \mathcal{P}\phi - \phi$  the projection error, there holds the similar approximation results [6, 12]

$$(15) \quad \|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \leq Ch^{k+1}$$

where  $\Gamma_h$  denotes the set of intercell boundaries of all elements  $I_i \times J_j$ .

**2.3. Notations for different constants.** We will adopt the following convention for different constants. These constants may have a different value in each occurrence.

We will denote by  $C$  a positive constant independent of  $h$ , which may depend on the exact solution of the PDE. Especially, in the following we will denote by  $\mathbb{C}$  a positive constant only for the a priori assumption. For problems considered in this paper, the exact solution is assumed to be smooth with periodic or compactly supported boundary condition. Also,  $0 \leq t \leq T$  for a fixed  $T$ . Therefore, the exact solution is always bounded.

**2.4. Inverse properties.** Finally, we list some inverse properties of the finite element space  $V_h$ . For any  $v_h \in V_h$ , there exists a positive constant  $C$ , independent of  $h$ , such that

(16)

$$(i) \|(v_h)_x\| \leq Ch^{-1}\|v_h\|, \quad (ii) \|v_h\|_{\Gamma_h} \leq Ch^{-\frac{1}{2}}\|v_h\|, \quad (iii) \|v_h\|_{\infty} \leq Ch^{-\frac{n}{2}}\|v_h\|$$

where  $n = 1$  or  $2$  is the spatial dimension. More details of the inverse properties can be found in [4].

### 3. Error estimates for the HJ equations

#### 3.1. One-dimensional case.

**3.1.1. DG scheme for directly solving the HJ equations.** In this section, we describe the DG scheme in [3] for directly solving the one-dimensional Hamilton-Jacobi equations. Here the DG scheme is formulated as: find  $\phi_h(x, t) \in V_h$ , such that for any  $v_h \in V_h$ ,

$$\begin{aligned} & \int_{I_j} (\partial_t \phi_h(x, t) + H(\partial_x \phi_h(x, t), x)) v_h(x) dx \\ & + \frac{1}{2} \left( \min_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x \phi_h, x_{j+\frac{1}{2}}) - \left| \min_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x \phi_h, x_{j+\frac{1}{2}}) \right| \right) [\phi_h]_{j+\frac{1}{2}} (v_h)_{j+\frac{1}{2}}^- \\ & + \frac{1}{2} \left( \max_{x \in I_{j-\frac{1}{2}}} H_1(\partial_x \phi_h, x_{j-\frac{1}{2}}) + \left| \max_{x \in I_{j-\frac{1}{2}}} H_1(\partial_x \phi_h, x_{j-\frac{1}{2}}) \right| \right) [\phi_h]_{j-\frac{1}{2}} (v_h)_{j-\frac{1}{2}}^+ \\ (17) \quad & = 0, \quad j = 1, \dots, N \end{aligned}$$

Noticing that, when taking the maximum or minimum, the scheme needs the reconstructed information of  $\partial_x \phi_h$  on the cells  $\bar{I}_{j-\frac{1}{2}}$  and  $\bar{I}_{j+\frac{1}{2}}$ , which contains the points  $x_{j-\frac{1}{2}}$  and  $x_{j+\frac{1}{2}}$  respectively, where the numerical solution  $\phi_h(x, t)$  is discontinuous. A polynomial  $w_{j+\frac{1}{2}}(x) \in P^{2k+1}$  is defined on  $I_j \cup I_{j+1}$ , such that

$$(18) \quad \int_{I_j} \phi_h v dx = \int_{I_j} w_{j+\frac{1}{2}} v dx$$

for any  $v \in P^k$  on  $I_j$ , and

$$(19) \quad \int_{I_{j+1}} \phi_h v dx = \int_{I_{j+1}} w_{j+\frac{1}{2}} v dx$$

for any  $v \in P^k$  on  $I_{j+1}$ . Then we use  $\partial_x \phi_h = \partial_x w_{j+\frac{1}{2}}$  on  $\bar{I}_{j+\frac{1}{2}}$  when taking the maximum or minimum in the scheme (17).

**3.1.2. LDG scheme for directly solving the HJ equations.** In this section, we describe the LDG scheme in [22] for directly solving the Hamilton-Jacobi equations. The LDG scheme is defined as follows: find  $\phi_h \in V_h$  such that for all test function  $u \in V_h$ , we have

$$(20) \quad \int_{I_j} (\phi_h)_t u dx + \int_{I_j} \hat{H}(p_1, p_2) u dx = 0, \quad j = 1, \dots, N$$

where  $p_1$  and  $p_2$  are two auxiliary variables to approximate  $\phi_x$  and  $\hat{H}(p_1, p_2)$  is a monotone consistent numerical Hamiltonian [11] chosen to approximate  $H(\phi_x)$ . For any test function  $v \in V_h$ ,  $p_1 \in V_h$  is obtained by solving the following right upwind DG scheme

$$(21) \quad \int_{I_j} p_1 v dx + \int_{I_j} \phi_h v_x dx - (\phi_h)_{j+\frac{1}{2}}^+ v_{j+\frac{1}{2}}^- + (\phi_h)_{j-\frac{1}{2}}^+ v_{j-\frac{1}{2}}^+ = 0$$

and for any test function  $w \in V_h$ ,  $p_2 \in V_h$  is obtained by solving the following left upwind DG scheme

$$(22) \quad \int_{I_j} p_2 w dx + \int_{I_j} \phi_h w_x dx - (\phi_h)_{j+\frac{1}{2}}^- w_{j+\frac{1}{2}}^- + (\phi_h)_{j-\frac{1}{2}}^- w_{j-\frac{1}{2}}^+ = 0.$$

We can use the simple Lax-Friedrichs numerical Hamiltonian

$$(23) \quad \hat{H}(p_1, p_2) = H\left(\frac{p_1 + p_2}{2}\right) - \frac{1}{2}\alpha(p_1 - p_2)$$

with  $\alpha = \max_{p \in D} \left| \frac{\partial H(p)}{\partial p} \right|$ . When  $D$  is taken as a local domain, which is evaluated locally as  $D = [\min(p_1, p_2), \max(p_1, p_2)]|_{I_j}$ , it is called a local Lax-Friedrichs Hamiltonian. With  $D$  taken as a global domain, namely  $D$  is evaluated over the whole computational domain and defined as  $D = [\min(p_1, p_2), \max(p_1, p_2)]|_{\Omega}$ , it is called a global Lax-Friedrichs Hamiltonian.

**3.1.3. The main results.** We state the main error estimates of the semi-discrete DG scheme (17) and the semi-discrete LDG scheme (20)-(22). Detailed proof will be given in the subsequent subsections.

**Theorem 3.1.** *Let  $\phi$  be the exact solution of the problem (1), which is sufficiently smooth with bounded derivatives, and assume  $H(\partial_x \phi, x) \in C^2$ . Let  $\phi_h$  be the numerical solution of the semi-discrete DG scheme (17) or the semi-discrete LDG scheme (20)-(22), and denote the corresponding numerical error by  $e_\phi = \phi - \phi_h$ . For regular triangulations of  $I = (0, 1)$ , if the finite element space  $V_h$  is the piecewise polynomial space of degree  $k \geq 2$ , then for small enough  $h$ , there holds the following optimal error estimate*

$$(24) \quad \|\phi - \phi_h\| \leq Ch^{k+1}$$

where the positive constant  $C$  depends on the final time  $T$ ,  $k$ ,  $\|\phi\|_{L^\infty((0,T), H^{k+1}(I))}$  and the bounds on the  $m$ -th derivatives of  $H(\partial_x \phi, x)$  with respect to its first argument,  $m = 1, 2$ .  $\|\phi\|_{L^\infty((0,T), H^{k+1}(I))}$  is the maximum over  $0 \leq t \leq T$  of the standard Sobolev  $(k+1)$ -norm in space.

**3.1.4. Proof of Theorem 3.1 for the DG scheme.** The cell error equation for this scheme is

$$\begin{aligned}
& \int_{I_j} \partial_t(\phi(x, t) - \phi_h(x, t))v_h(x)dx + \int_{I_j} (H(\partial_x\phi(x, t), x) - H(\partial_x\phi_h(x, t), x))v_h(x)dx \\
& - \frac{1}{2} \left( \min_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) - \left| \min_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) \right| \right) [\phi_h]_{j+\frac{1}{2}} (v_h)_{j+\frac{1}{2}}^- \\
& - \frac{1}{2} \left( \max_{x \in I_{j-\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j-\frac{1}{2}}) + \left| \max_{x \in I_{j-\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j-\frac{1}{2}}) \right| \right) [\phi_h]_{j-\frac{1}{2}} (v_h)_{j-\frac{1}{2}}^+ \\
(25) \quad & = 0
\end{aligned}$$

We take the Taylor expansion on  $H(\partial_x\phi_h, x)$  with respect to its first argument

$$(26) \quad H(\partial_x\phi_h, x) = H(\partial_x\phi, x) - H_1(\partial_x\phi, x)\partial_x(\phi - \phi_h) + \frac{1}{2}\bar{H}_{11}(\partial_x(\phi - \phi_h))^2$$

where the second derivative  $\bar{H}_{11}$  is evaluated at some point between  $\partial_x\phi$  and  $\partial_x\phi_h$ . We would like to adopt the following a priori assumption

$$(27) \quad \|\partial_x e_\phi\|_\infty = \|\partial_x(\phi - \phi_h)\|_\infty \leq \mathbb{C}h.$$

The reasonableness of this a priori assumption will be justified later. Denoting  $\phi - \phi_h = e_\phi = (\mathbb{R}_h\phi - \phi_h) - (\mathbb{R}_h\phi - \phi) = \xi - \eta$  and taking the test function  $v_h = \xi$ , we obtain

$$\begin{aligned}
& \int_{I_j} \xi_t \xi dx + \int_{I_j} H_1 \xi \partial_x \xi dx + (H_{1min})_{j+\frac{1}{2}} [\xi]_{j+\frac{1}{2}} (\xi)_{j+\frac{1}{2}}^- + (H_{1max})_{j-\frac{1}{2}} [\xi]_{j-\frac{1}{2}} (\xi)_{j-\frac{1}{2}}^+ \\
& = \int_{I_j} \eta_t \xi dx + \int_{I_j} H_1 \xi \partial_x \eta dx + (H_{1min})_{j+\frac{1}{2}} [\eta]_{j+\frac{1}{2}} (\xi)_{j+\frac{1}{2}}^- \\
(28) \quad & + (H_{1max})_{j-\frac{1}{2}} [\eta]_{j-\frac{1}{2}} (\xi)_{j-\frac{1}{2}}^+ + \frac{1}{2} \int_{I_j} \xi \bar{H}_{11} (\partial_x(e_\phi))^2 dx
\end{aligned}$$

where we have taken the short-hand notations

$$(29) \quad H_1 = H_1(\partial_x\phi, x)$$

$$(30) \quad (H_{1min})_{j+\frac{1}{2}} = \frac{1}{2} \left( \min_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) - \left| \min_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) \right| \right)$$

$$(31) \quad (H_{1max})_{j+\frac{1}{2}} = \frac{1}{2} \left( \max_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) + \left| \max_{x \in I_{j+\frac{1}{2}}} H_1(\partial_x\phi_h, x_{j+\frac{1}{2}}) \right| \right)$$

when there is no confusion, and  $\bar{H}_{11}$  is a mean value from the Taylor expansion. For the second term on the left side of (28), we have

$$(32) \quad \int_{I_j} H_1 \xi \partial_x \xi dx = -\frac{1}{2} \int_{I_j} \xi^2 \partial_x H_1 dx + \frac{1}{2} \left( (H_1 \xi^2)_{j+\frac{1}{2}}^- - (H_1 \xi^2)_{j-\frac{1}{2}}^+ \right)$$



and for the second term on the right side of (28), we have

$$\begin{aligned}
 \int_{I_j} H_1 \xi \partial_x \eta dx &= - \int_{I_j} (\xi \partial_x H_1 + H_1 \partial_x \xi) \eta dx + (H_1 \eta \xi)_{j+\frac{1}{2}}^- - (H_1 \eta \xi)_{j-\frac{1}{2}}^+ \\
 &= - \int_{I_j} \xi \eta \partial_x H_1 dx - \int_{I_j} (H_1(\partial_x \phi, x) - H_1(\partial_x \phi, x)|_{x=x_j}) \eta \partial_x \xi dx \\
 (33) \quad &+ (H_1 \eta \xi)_{j+\frac{1}{2}}^- - (H_1 \eta \xi)_{j-\frac{1}{2}}^+
 \end{aligned}$$

where we have used the property that  $\mathbb{R}_h \phi - \phi$  is locally orthogonal to all polynomials of degree up to  $k - 1$ , so

$$(34) \quad \int_{I_j} H_1(\partial_x \phi, x)|_{x=x_j} \eta \partial_x \xi dx = 0.$$

Now (28) can be rewritten as

$$\begin{aligned}
 \int_{I_j} \xi_t \xi dx &= \int_{I_j} \eta_t \xi dx + \frac{1}{2} \int_{I_j} \xi^2 \partial_x H_1 dx - \int_{I_j} \xi \eta \partial_x H_1 dx \\
 &- \int_{I_j} (H_1(\partial_x \phi, x) - H_1(\partial_x \phi, x)|_{x=x_j}) \eta \partial_x \xi dx - \frac{1}{2} (H_1 \xi^2)_{j+\frac{1}{2}}^- \\
 &+ (H_1 \eta \xi)_{j+\frac{1}{2}}^- - (H_{1min})_{j+\frac{1}{2}} [\xi]_{j+\frac{1}{2}} (\xi)_{j+\frac{1}{2}}^- + (H_{1min})_{j+\frac{1}{2}} [\eta]_{j+\frac{1}{2}} (\xi)_{j+\frac{1}{2}}^- \\
 &+ \frac{1}{2} (H_1 \xi^2)_{j-\frac{1}{2}}^+ - (H_1 \eta \xi)_{j-\frac{1}{2}}^+ - (H_{1max})_{j-\frac{1}{2}} [\xi]_{j-\frac{1}{2}} (\xi)_{j-\frac{1}{2}}^+ \\
 (35) \quad &+ (H_{1max})_{j-\frac{1}{2}} [\eta]_{j-\frac{1}{2}} (\xi)_{j-\frac{1}{2}}^+ + \frac{1}{2} \int_{I_j} \xi \bar{H}_{11} (\partial_x (e_\phi))^2 dx.
 \end{aligned}$$

Summing over  $j$  from  $j_1$  to  $j_2$ , the error equation (35) becomes

$$\begin{aligned}
 \sum_{j=j_1}^{j_2} \int_{I_j} \xi_t \xi dx &= \sum_{j=j_1}^{j_2} \int_{I_j} \eta_t \xi dx + \sum_{j=j_1}^{j_2} \frac{1}{2} \int_{I_j} \xi^2 \partial_x H_1 dx - \sum_{j=j_1}^{j_2} \int_{I_j} \xi \eta \partial_x H_1 dx \\
 &- \sum_{j=j_1}^{j_2} \int_{I_j} (H_1(\partial_x \phi, x) - H_1(\partial_x \phi, x)|_{x=x_j}) \eta \partial_x \xi dx \\
 &+ \sum_{j=j_1}^{j_2} \left( -\frac{1}{2} (H_1 \xi^2)^- + (H_1 \eta \xi)^- - H_{1min} [\xi] \xi^- + H_{1min} [\eta] \xi^- \right)_{j+\frac{1}{2}} \\
 &+ \sum_{j=j_1}^{j_2} \left( \frac{1}{2} (H_1 \xi^2)^+ - (H_1 \eta \xi)^+ - H_{1max} [\xi] \xi^+ + H_{1max} [\eta] \xi^+ \right)_{j-\frac{1}{2}} \\
 (36) \quad &+ \sum_{j=j_1}^{j_2} \frac{1}{2} \int_{I_j} \xi \bar{H}_{11} (\partial_x (e_\phi))^2 dx.
 \end{aligned}$$

Since  $H_1(\partial_x \phi, x)$  is continuous with respect to its first argument, we have

$$(37) \quad \max_j \left| H_1(\partial_x \phi, x) - H_1(\partial_x \phi, x)|_{x=x_j} \right|_{x \in I_j} \leq Ch$$

then for the first two lines of the right side of (36), we obtain

$$\begin{aligned}
& \sum_{j=j_1}^{j_2} \int_{I_j} \eta_t \xi dx + \sum_{j=j_1}^{j_2} \frac{1}{2} \int_{I_j} \xi^2 \partial_x H_1 dx - \sum_{j=j_1}^{j_2} \int_{I_j} \xi \eta \partial_x H_1 dx \\
& - \sum_{j=j_1}^{j_2} \int_{I_j} (H_1(\partial_x \phi, x) - H_1(\partial_x \phi, x) \Big|_{x=x_j}) \eta \partial_x \xi dx \\
(38) \quad & \leq C \sum_{j=j_1}^{j_2} \left( \int_{I_j} |\eta_t \xi| dx + \int_{I_j} \xi^2 dx + \int_{I_j} |\xi \eta| dx + h \int_{I_j} |\eta \partial_x \xi| dx \right).
\end{aligned}$$

By the a priori assumption (27), for the last line of the right side of (36), we obtain

$$\begin{aligned}
& \sum_{j=j_1}^{j_2} \frac{1}{2} \int_{I_j} \xi \bar{H}_{11} (\partial_x (e_\phi))^2 dx \leq C \|\partial_x e_\phi\|_\infty \sum_j \frac{1}{2} \int_{I_j} |\xi \partial_x (\xi - \eta)| dx \\
(39) \quad & \leq Ch \sum_{j=j_1}^{j_2} \left( \int_{I_j} |\xi \partial_x \xi| dx + \int_{I_j} |\xi \partial_x \eta| dx \right).
\end{aligned}$$

Now for the boundary terms of the third and fourth lines on the right side of (36), denoted by  $T_{bry}$ , first we have that  $H_1(\partial_x \phi, x)$  is continuous at each cell boundary, which is  $H_1 = (H_1)^- = (H_1)^+$ . If  $H_1(\partial_x \phi, x)$  changes its sign in the cell  $\bar{I}_j$  of  $S_3$ , then it has at least one zero point  $x^*$  in the cell  $\bar{I}_j$ . By taking a Taylor expansion at this zero point  $x^*$ , we easily obtain that  $|H_1(\partial_x \phi, x)| \leq Ch$  in such  $\bar{I}_j$ .

If we assume  $\{I_j\}_{j=j_1}^{j_2}$  is a sequence of maximum consecutive cells in  $S_1$  and  $\bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$ , then in the interior open set  $I_s$ , we have  $H_1 > 0$ , so that  $H_{1min} = 0$  and  $\max_{j_1 \leq j \leq j_2} |H_1 - H_{1max}|_{j+\frac{1}{2}} \leq Ch$  (The proof of this inequality is given in the appendix). The projection is  $\mathbb{R}_h^+$ . The two boundary points of  $\bar{I}_s$ ,  $x_{j_1-\frac{1}{2}}$  and  $x_{j_2+\frac{1}{2}}$ , should belong to the cells  $\bar{I}_{j_1-1}$  and  $\bar{I}_{j_2+1}$  of  $S_3$  respectively, otherwise we have  $I_s = I = (0, 1)$ . In the following, we only consider the more general case that  $\bar{I}_{j_1-1} \in S_3$  and  $\bar{I}_{j_2+1} \in S_3$ . In this case, we have  $\left| H_1(\partial_x \phi, x) \Big|_{x=x_{j_1-\frac{1}{2}}} \right| \leq Ch$  and

$$\begin{aligned}
 & \left| H_1(\partial_x \phi, x) \Big|_{x=x_{j_2+\frac{1}{2}}} \right| \leq Ch, \text{ and we can obtain} \\
 T_{bry} &= \sum_{j=j_1}^{j_2-1} \left( H_1 \left( -\frac{1}{2}(\xi^2)^- + \frac{1}{2}(\xi^2)^+ - [\xi]\xi^+ \right) \right)_{j+\frac{1}{2}} \\
 & \quad + \sum_{j=j_1}^{j_2-1} \left( H_1 \left( (\eta\xi)^- - (\eta\xi)^+ + [\eta]\xi^+ \right) \right)_{j+\frac{1}{2}} \\
 & \quad + (H_1)_{j_2+\frac{1}{2}} \left( -\frac{1}{2}(\xi^2)^- + (\eta\xi)^- \right)_{j_2+\frac{1}{2}} \\
 & \quad + (H_1)_{j_1-\frac{1}{2}} \left( \frac{1}{2}(\xi^2)^+ - (\eta\xi)^+ - [\xi]\xi^+ + [\eta]\xi^+ \right)_{j_1-\frac{1}{2}} \\
 & \quad + \sum_{j=j_1}^{j_2} \left( (H_{1max} - H_1) \left( -[\xi]\xi^+ + [\eta]\xi^+ \right) \right)_{j+\frac{1}{2}} \\
 &= - \sum_{j=j_1}^{j_2-1} \frac{1}{2} \left( H_1 \left( \xi^+ - \xi^- \right)^2 \right)_{j+\frac{1}{2}} + \sum_{j=j_1}^{j_2-1} \left( H_1 \left( \eta^- \xi^- - \eta^+ \xi^+ \right) \right)_{j+\frac{1}{2}} \\
 & \quad + (H_1)_{j_2+\frac{1}{2}} \left( -\frac{1}{2}(\xi^2)^- + (\eta\xi)^- \right)_{j_2+\frac{1}{2}} \\
 & \quad + (H_1)_{j_1-\frac{1}{2}} \left( \frac{1}{2}(\xi^2)^+ - (\eta\xi)^+ - [\xi]\xi^+ + [\eta]\xi^+ \right)_{j_1-\frac{1}{2}} \\
 & \quad + \sum_{j=j_1}^{j_2} \left( (H_{1max} - H_1) \left( -[\xi]\xi^+ + [\eta]\xi^+ \right) \right)_{j+\frac{1}{2}} \\
 (40) \quad & \leq Ch \sum_{j=j_1-1}^{j_2} \left( (\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2 \right)_{j+\frac{1}{2}}
 \end{aligned}$$

where for the last inequality in (40), we have used the special interpolating property of the projection of  $\mathbb{R}_h^+$ .

Combining (36), (38), (39) and (40), we have

$$\begin{aligned}
 \sum_{j=j_1}^{j_2} \int_{I_j} \xi_t \xi dx &\leq C \sum_{j=j_1}^{j_2} \left( \int_{I_j} |\eta_t \xi| dx + \int_{I_j} \xi^2 dx + \int_{I_j} |\xi \eta| dx \right) \\
 & \quad + Ch \sum_{j=j_1}^{j_2} \left( \int_{I_j} |\eta \partial_x \xi| dx + \int_{I_j} |\xi \partial_x \xi| dx + \int_{I_j} |\xi \partial_x \eta| dx \right) \\
 (41) \quad & \quad + Ch \sum_{j=j_1-1}^{j_2} \left( (\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2 \right)_{j+\frac{1}{2}}.
 \end{aligned}$$

Similarly, if  $\{I_j\}_{j=j_1}^{j_2}$  is a sequence of consecutive cells in  $S_2$ , and the two boundary points  $x_{j_1-\frac{1}{2}}$  and  $x_{j_2+\frac{1}{2}}$  of  $\bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$  also belong to the cells  $\bar{I}_{j_1-1}$  and

$\bar{I}_{j_2+1}$  of  $S_3$  respectively, then in  $I_s$ , we have  $H_1 < 0$ , so that  $H_{1max} = 0$  and  $\max_j |H_1 - H_{1min}|_{j+\frac{1}{2}} \leq Ch$  (similar to the proof in the appendix), and the projection is  $\mathbb{R}_h = \mathbb{R}_h^-$ . With the special property of  $\mathbb{R}_h^-$ , we have the same results as (41).

Now if  $\{I_j\}_{j=j_1}^{j_2}$  is a sequence of consecutive cells in  $S_3$ , the estimates of (38) and (39) are still the same. For the boundary term, first we have

$$(42) \quad \left| H_1(\partial_x \phi, x) \Big|_{x=x_{j+\frac{1}{2}}} \right| \leq Ch, \text{ for } j = j_1 - 1, \dots, j_2.$$

and so  $|H_{1min}| \leq Ch$  and  $|H_{1max}| \leq Ch$  in each  $\bar{I}_{j+\frac{1}{2}}$  ( $j_1 \leq j \leq j_2$ ) (also similar to the proof in the appendix). We can then obtain

$$(43) \quad \begin{aligned} T_{bry} &= \sum_{j=j_1}^{j_2} \left( -\frac{1}{2}(H_1\xi^2)^- + (H_1\eta\xi)^- - H_{1min}[\xi]\xi^- + H_{1min}[\eta]\xi^- \right)_{j+\frac{1}{2}} \\ &\quad + \sum_{j=j_1}^{j_2} \left( \frac{1}{2}(H_1\xi^2)^+ - (H_1\eta\xi)^+ - H_{1max}[\xi]\xi^+ + H_{1max}[\eta]\xi^+ \right)_{j-\frac{1}{2}} \\ &\leq Ch \sum_{j=j_1-1}^{j_2} \left( (\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2 \right)_{j+\frac{1}{2}}. \end{aligned}$$

In this case, we also have the same estimate as (41).

Since each case in the classes of  $S_1$ ,  $S_2$  and  $S_3$  has the same estimate (41), summing over all the sequences of consecutive cells in the three nonoverlapping classes, we have

$$(44) \quad \begin{aligned} \frac{d}{dt} \|\xi\|^2 &\leq C \left( \int_0^1 (|\eta_t \xi| + \xi^2 + |\xi \eta|) dx \right) \\ &\quad + Ch \left( \int_0^1 (|\eta \partial_x \xi| + |\xi \partial_x \xi| + |\xi \partial_x \eta|) dx \right) \\ &\quad + Ch \sum_j \left( (\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2 \right)_{j+\frac{1}{2}} \end{aligned}$$

using Schwarz inequality, the inverse property (16) and the interpolating property (11), we have

$$(45) \quad \begin{aligned} \frac{d}{dt} \|\xi\|^2 &\leq C(\|\xi\|^2 + \|\eta_t\|^2 + \|\eta\|^2) + Ch^2(\|\partial_x \xi\|^2 + \|\partial_x \eta\|^2) + Ch(\|\xi\|_{\Gamma_h}^2 + \|\eta\|_{\Gamma_h}^2) \\ &\leq C(\|\xi\|^2 + h^{2k+2}) \end{aligned}$$

applying the Gronwall's inequality and the triangle inequality, we get the optimal error estimate for the DG scheme.

**Remark 3.1.** *When summing over all the sequences for (41), each boundary term has been counted at most twice. This does not affect the optimal error estimate. This remark also applied for the following LDG scheme.*

**3.1.5. Proof of Theorem 3.1 for the LDG scheme.** In this subsection, we give the proof for the local Lax-Friedrichs LDG scheme of Theorem 3.1. We use  $\alpha_j$  to denote the local  $\alpha$  in the cell  $I_j$ . We also would like to make an a priori assumption that, for small enough  $h$ , there holds

$$(46) \quad \left\| \partial_x \phi - \frac{p_1 + p_2}{2} \right\|_{\infty} \leq Ch$$

and leave its justification to the next subsection. Denoting  $\phi - \phi_h = e_\phi = (\mathbb{R}_h \phi - \phi_h) - (\mathbb{R}_h \phi - \phi) = \xi - \eta$  and  $z_h = \partial_x \phi_h - \frac{p_1 + p_2}{2} \in V_h$ , for the term  $z_h$ , we have

**Lemma 3.2.** *With the interpolation property (11), we have*

$$(47) \quad \|z_h\| \leq Ch^{-1}(\|\xi\| + h^{k+1}).$$

**Proof.** *We have*

$$(48) \quad \int_{I_j} \partial_x \phi_h \zeta dx + \int_{I_j} \phi_h \zeta_x dx - (\phi_h)_{j+\frac{1}{2}}^- \zeta_{j+\frac{1}{2}}^- + (\phi_h)_{j-\frac{1}{2}}^+ \zeta_{j-\frac{1}{2}}^+ = 0.$$

Taking  $v = w$  in (21) and  $\zeta = w$  in (48), and combining Eqns. (21), (22) and (48), we obtain

$$(49) \quad \int_{I_j} z_h w dx + \frac{1}{2} [\phi_h]_{j+\frac{1}{2}} w_{j+\frac{1}{2}}^- + \frac{1}{2} [\phi_h]_{j-\frac{1}{2}} w_{j-\frac{1}{2}}^+ = 0.$$

Summing over  $j$ , we get

$$(50) \quad \sum_j \int_{I_j} z_h w dx = - \sum_j ([\phi_h] \bar{w})_{j+\frac{1}{2}}.$$

Taking  $w = z_h$ , from the inverse property (16), the equation becomes

$$(51) \quad \begin{aligned} \|z_h\|^2 &= - \sum_j ([\phi_h] \bar{z}_h)_{j+\frac{1}{2}} \\ &= \sum_j ([\phi - \phi_h] \bar{z}_h)_{j+\frac{1}{2}} \\ &\leq C \|\xi - \eta\|_{\Gamma_h} \|z_h\|_{\Gamma_h} \\ &\leq Ch^{-1}(\|\xi\| + h^{k+1}) \|z_h\| \end{aligned}$$

so we have  $\|z_h\| \leq Ch^{-1}(\|\xi\| + h^{k+1})$ .

Now we are going to follow the main procedure as the proof for the DG scheme to get the error estimates for the LDG scheme. We first assume  $\{I_j\}_{j=j_1}^{j_2}$  is a sequence of consecutive cells in  $S_1$ , with the two boundary points  $x_{j_1-\frac{1}{2}}$  and  $x_{j_2+\frac{1}{2}}$  of  $\bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$  belonging to the cells  $\bar{I}_{j_1-1}$  and  $\bar{I}_{j_2+1}$  of  $S_3$  respectively. Then we have  $H_1 > 0$  in  $I_s$ , so that  $\|\alpha_j - H_1\|_{\infty} \leq Ch$  in each cell  $I_j \subset I_s$ , and we have the

following cell error equations:

$$(52) \quad \int_{I_j} (\phi - \phi_h)_t u dx + \int_{I_j} (H(\partial_x \phi) - \hat{H}(p_1, p_2)) u dx = 0,$$

$$(53) \quad \int_{I_j} (\partial_x \phi - p_2) w dx + \int_{I_j} (\phi - \phi_h) w_x dx - (\phi - \phi_h)_{j+\frac{1}{2}}^- w_{j+\frac{1}{2}}^- + (\phi - \phi_h)_{j-\frac{1}{2}}^- w_{j-\frac{1}{2}}^+ = 0.$$

By taking a Taylor expansion for  $H(\frac{p_1+p_2}{2})$ , we have

$$(54) \quad H\left(\frac{p_1+p_2}{2}\right) = H(\partial_x \phi) - H_1(\partial_x \phi) \left(\partial_x \phi - \frac{p_1+p_2}{2}\right) + \frac{1}{2} \bar{H}_{11} \left(\partial_x \phi - \frac{p_1+p_2}{2}\right)^2.$$

Taking  $u = \xi$  and  $w = -\alpha_j \xi$ , and adding (53) to (52), we obtain

$$(55) \quad \begin{aligned} & \int_{I_j} \xi_t \xi dx + \int_{I_j} \xi H_1 \left(\partial_x \phi - \frac{p_1+p_2}{2}\right) dx + \frac{1}{2} \alpha_j \int_{I_j} \xi (p_1 - p_2) dx \\ & \quad - \alpha_j \int_{I_j} \xi (\partial_x \phi - p_2) dx - \alpha_j \int_{I_j} \xi_x \xi dx + \alpha_j (\xi_{j+\frac{1}{2}}^- \xi_{j+\frac{1}{2}}^- - \xi_{j-\frac{1}{2}}^- \xi_{j-\frac{1}{2}}^+) \\ & = \int_{I_j} \eta_t \xi dx + \int_{I_j} \frac{1}{2} \xi \bar{H}_{11} \left(\partial_x \phi - \frac{p_1+p_2}{2}\right)^2 dx - \alpha_j \int_{I_j} \xi_x \eta dx \\ & \quad + \alpha_j (\eta_{j+\frac{1}{2}}^- \xi_{j+\frac{1}{2}}^- - \eta_{j-\frac{1}{2}}^- \xi_{j-\frac{1}{2}}^+). \end{aligned}$$

Here  $H_1$  is a short-hand notation for  $H_1(\partial_x \phi)$ . For the left hand side (LHS) of (55), we have

$$(56) \quad \begin{aligned} LHS & = \int_{I_j} \xi_t \xi dx - \int_{I_j} \xi (\alpha_j - H_1) \left(\partial_x \phi - \frac{p_1+p_2}{2}\right) dx \\ & \quad + \frac{1}{2} \alpha_j (\xi_{j+\frac{1}{2}}^- \xi_{j+\frac{1}{2}}^- - 2\xi_{j-\frac{1}{2}}^- \xi_{j-\frac{1}{2}}^+ + \xi_{j-\frac{1}{2}}^+ \xi_{j-\frac{1}{2}}^+). \end{aligned}$$

Since  $H_1 > 0$ , we have  $\mathbb{R}_h = \mathbb{R}_h^+$ . Applying the special property of  $\mathbb{R}_h^+$ , the right hand side (RHS) of (55) is

$$(57) \quad RHS = \int_{I_j} \eta_t \xi dx + \int_{I_j} \frac{1}{2} \xi \bar{H}_{11} \left(\partial_x \phi - \frac{p_1+p_2}{2}\right)^2 dx$$

We now sum over  $j$  for the LHS and RHS of (55) from  $j_1$  to  $j_2$ . Since  $I_{j_1-1} \in S_3$  and  $I_{j_2+1} \in S_3$ , in the adjacent cells, we have  $|\alpha_{j_2}| \leq Ch$  and  $|\alpha_{j_1}| \leq Ch$ . Combining

with (46), we obtain

$$\begin{aligned}
 \sum_{j=j_1}^{j_2} \int_{I_j} \xi_t \xi dx &= \sum_{j=j_1}^{j_2} \int_{I_j} \xi (\alpha_j - H_1) \left( \partial_x \phi - \frac{p_1 + p_2}{2} \right) dx - \sum_{j=j_1}^{j_2-1} \frac{1}{2} \alpha_j (\xi^- - \xi^+)_{j+\frac{1}{2}}^2 \\
 &\quad + \frac{1}{2} \alpha_{j_2} \xi_{j_2+\frac{1}{2}}^- \xi_{j_2+\frac{1}{2}}^- - \alpha_{j_1} \xi_{j_1-\frac{1}{2}}^- \xi_{j_1-\frac{1}{2}}^+ + \frac{1}{2} \alpha_{j_1} \xi_{j_1-\frac{1}{2}}^+ \xi_{j_1-\frac{1}{2}}^+ \\
 &\quad + \sum_{j=j_1}^{j_2} \int_{I_j} \eta_t \xi dx + \sum_{j=j_1}^{j_2} \int_{I_j} \frac{1}{2} \xi \bar{H}_{11} \left( \partial_x \phi - \frac{p_1 + p_2}{2} \right)^2 dx \\
 &\leq \sum_{j=j_1}^{j_2} \int_{I_j} |\alpha_j - H_1| |\xi (\partial_x (\phi - \phi_h) + z_h)| dx + \frac{1}{2} \alpha_{j_2} \xi_{j_2+\frac{1}{2}}^- \xi_{j_2+\frac{1}{2}}^- \\
 &\quad - \alpha_{j_1} \xi_{j_1-\frac{1}{2}}^- \xi_{j_1-\frac{1}{2}}^+ + \frac{1}{2} \alpha_{j_1} \xi_{j_1-\frac{1}{2}}^+ \xi_{j_1-\frac{1}{2}}^+ + \sum_{j=j_1}^{j_2} \int_{I_j} \eta_t \xi dx \\
 &\quad + C \left\| \partial_x \phi - \frac{p_1 + p_2}{2} \right\|_\infty \sum_{j=j_1}^{j_2} \int_{I_j} |\xi (\partial_x (\phi - \phi_h) + z_h)| dx \\
 &\leq Ch \left( \sum_{j=j_1}^{j_2} \int_{I_j} |\xi (\partial_x \xi + \partial_x \eta + z_h)| dx + (\xi_{j_2+\frac{1}{2}}^-)^2 + (\xi_{j_1-\frac{1}{2}}^-)^2 + (\xi_{j_1-\frac{1}{2}}^+)^2 \right) \\
 (58) \quad &\quad + \sum_{j=j_1}^{j_2} \int_{I_j} |\eta_t \xi| dx.
 \end{aligned}$$

Similarly, if  $\{I_j\}_{j=j_1}^{j_2}$  is a sequence of consecutive cells in  $S_2$ , with the two boundary points  $x_{j_1-\frac{1}{2}}$  and  $x_{j_2+\frac{1}{2}}$  of  $\bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$  belonging to the cells  $\bar{I}_{j_1-1}$  and  $\bar{I}_{j_2+1}$  of  $S_3$  respectively, then we have  $H_1 < 0$  in  $I_s$ , so that  $\|\alpha_j + H_1\|_\infty \leq Ch$  in each cell  $I_j \subset I_s$ , and the cell error equations are (52) and the following:

$$(59) \quad \int_{I_j} (\partial_x \phi - p_1) v dx + \int_{I_j} (\phi - \phi_h) v_x dx - (\phi - \phi_h)_{j+\frac{1}{2}}^- v_{j+\frac{1}{2}}^+ + (\phi - \phi_h)_{j-\frac{1}{2}}^+ v_{j-\frac{1}{2}}^+ = 0.$$

With the special property of  $\mathbb{R}_h^-$ , we can get the similar estimate as in (58), that is

$$\begin{aligned}
 \sum_{j=j_1}^{j_2} \int_{I_j} \xi_t \xi dx &\leq Ch \left( \sum_{j=j_1}^{j_2} \int_{I_j} |\xi (\partial_x \xi + \partial_x \eta + z_h)| dx + (\xi_{j_2+\frac{1}{2}}^-)^2 + (\xi_{j_2+\frac{1}{2}}^+)^2 + (\xi_{j_1-\frac{1}{2}}^+)^2 \right) \\
 (60) \quad &\quad + \sum_{j=j_1}^{j_2} \int_{I_j} |\eta_t \xi| dx.
 \end{aligned}$$

Now if  $\{I_j\}_{j=j_1}^{j_2}$  is a sequence of consecutive cells in  $S_3$ , in each cell  $I_j \subset \bar{I}_s = \cup_{j=j_1}^{j_2} \bar{I}_j$ , we directly have  $|H_1(\partial_x \phi)| \leq Ch$  and  $\alpha_j \leq Ch$ . In this case, the cell error equations are (52), (53) and (59). By taking  $u = 2\xi$ ,  $v = \alpha_j \xi$ ,  $w = -\alpha_j \xi$ , and

summing over  $j$  from  $j_1$  to  $j_2$ , we can obtain that

$$\begin{aligned}
2 \sum_{j=j_1}^{j_2} \int_{I_j} \xi_t \xi dx &= 2 \sum_{j=j_1}^{j_2} \int_{I_j} \eta_t \xi dx - 2 \sum_{j=j_1}^{j_2} \int_{I_j} \xi H_1 \left( \partial_x \phi - \frac{p_1 + p_2}{2} \right) dx \\
&\quad + \sum_{j=j_1}^{j_2} \alpha_j \left( ([\xi - \eta] \xi^+)_{j-\frac{1}{2}} - ([\xi - \eta] \xi^-)_{j+\frac{1}{2}} \right) \\
&\quad + \sum_{j=j_1}^{j_2} \int_{I_j} \xi \bar{H}_{11} \left( \partial_x \phi - \frac{p_1 + p_2}{2} \right)^2 dx \\
&\leq 2 \sum_{j=j_1}^{j_2} \int_{I_j} |\eta_t \xi| dx + 2 \sum_{j=j_1}^{j_2} \int_{I_j} |H_1| |\xi (\partial_x (\phi - \phi_h) + z_h)| dx \\
&\quad + \sum_{j=j_1}^{j_2} \alpha_j \left| ([\xi - \eta] \xi^+)_{j-\frac{1}{2}} - ([\xi - \eta] \xi^-)_{j+\frac{1}{2}} \right| \\
&\quad + C \left\| \partial_x \phi - \frac{p_1 + p_2}{2} \right\|_\infty \sum_j \int_{I_j} |\xi (\partial_x (\phi - \phi_h) + z_h)| dx \\
&\leq 2 \sum_{j=j_1}^{j_2} \int_{I_j} |\eta_t \xi| dx + Ch \left( \sum_{j=j_1}^{j_2} \int_{I_j} |\xi (\partial_x \xi + \partial_x \eta + z_h)| dx \right. \\
(61) \quad &\quad \left. + \sum_{j=j_1-1}^{j_2} ((\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2)_{j+\frac{1}{2}} \right).
\end{aligned}$$

Combining (58), (60) and (61), for each sequence of consecutive cells in the classes of  $S_1$ ,  $S_2$  and  $S_3$ , we have

$$\begin{aligned}
\sum_{j=j_1}^{j_2} \int_{I_j} \xi_t \xi dx &\leq \sum_{j=j_1}^{j_2} \int_{I_j} |\eta_t \xi| dx + Ch \left( \sum_{j=j_1}^{j_2} \int_{I_j} |\xi (\partial_x \xi + \partial_x \eta + z_h)| dx \right. \\
(62) \quad &\quad \left. + \sum_{j=j_1-1}^{j_2} ((\xi^+)^2 + (\xi^-)^2 + (\eta^+)^2 + (\eta^-)^2)_{j+\frac{1}{2}} \right).
\end{aligned}$$

Summing over all the sequences in the three nonoverlapping classes, we have

$$\begin{aligned}
\frac{d}{dt} \|\xi\|^2 &\leq \int_0^1 |\eta_t \xi| dx + Ch \left( \int_0^1 |\xi (\partial_x \xi + \partial_x \eta + z_h)| dx \right. \\
(63) \quad &\quad \left. + \|\xi\|_{\Gamma_h}^2 + \|\eta\|_{\Gamma_h}^2 \right).
\end{aligned}$$

From the inverse property (16), the interpolation property (11) and the estimate for the term  $z_h$  (47), and using Schwarz inequality, (63) can be rewritten as

$$\begin{aligned}
\frac{d}{dt} \|\xi\|^2 &\leq C(\|\xi\|^2 + \|\eta_t\|^2) + Ch^2(\|\partial_x \xi\|^2 + \|\partial_x \eta\|^2 + \|z_h\|^2) + Ch(\|\xi\|_{\Gamma_h}^2 + \|\eta\|_{\Gamma_h}^2) \\
(64) \quad &\leq C(\|\xi\|^2 + h^{2k+2}).
\end{aligned}$$



Applying the Gronwall's inequality and the triangle inequality, we finally get the optimal error estimate for the LDG scheme.

**Remark 3.2.** *For the LDG scheme, we can only prove the optimal error estimate for the local, not the global, Lax-Friedrichs numerical Hamiltonian. This is because in the upwind case  $H_1(\partial_x \phi) > 0$ , we need  $\hat{H}(p_2, p_1) = H(p_2) + O(h)(p_2 - p_1)$ . Besides the local Lax-Friedrichs Hamiltonian, other purely upwind numerical Hamiltonians, such as the Godunov [18] and Osher-Sethian [19] numerical Hamiltonians, would also yield optimal error estimates.*

**3.1.6. Justification of the a priori assumption.** Now, to complete the proof of Theorem 3.1, we follow [21, 24] to verify the a priori assumption (27) and (46). For (27), we have

$$(65) \quad \|\partial_x(\phi - \phi_h)\|_\infty \leq \|\partial_x \xi\|_\infty + \|\partial_x \eta\|_\infty \leq C(h^{-\frac{3}{2}}\|\xi\| + h^{k-\frac{1}{2}})$$

and for (46), from the inverse property (16) and (47), we have

$$(66) \quad \begin{aligned} \|\partial_x \phi - \frac{p_1 + p_2}{2}\|_\infty &\leq \|\partial_x(\phi - \phi_h) + z_h\|_\infty \\ &\leq \|\partial_x \xi\|_\infty + \|\partial_x \eta\|_\infty + \|z_h\|_\infty \\ &\leq C(h^{-\frac{3}{2}}\|\xi\| + h^{k-\frac{1}{2}} + h^{-\frac{1}{2}}\|z_h\|) \\ &\leq C(h^{-\frac{3}{2}}\|\xi\| + h^{k-\frac{1}{2}}) \end{aligned}$$

Here  $\|\partial_x \eta\|_\infty \leq Ch^{k-\frac{1}{2}}$  can be referred to [1, 4].

From  $\|\xi\| \leq Ch^{k+1}$  and  $k \geq 2$ , we can get

$$(67) \quad \|\partial_x(\phi - \phi_h)\|_\infty \leq Ch^{k-\frac{1}{2}} \leq C_0 h^{\frac{3}{2}}$$

where the positive constant  $C_0$  depends on  $T$  but is independent of  $h$ . Certainly there exists a constant  $h_0 > 0$  such that  $C_0 h^{\frac{1}{2}} < \mathbb{C}$ , and consequently  $\|\partial_x(\phi - \phi_h)\| \leq \mathbb{C}h$  if  $h \leq h_0$ . Thus the a priori assumptions (27) and (46) are justified.

## 3.2. Two-dimensional case.

**3.2.1. DG scheme for directly solving HJ equations.** The DG scheme for directly solving the two-dimensional HJ equations in [3] is defined on rectangular

meshes as: find  $\phi_h(x, y, t) \in Z_h$ , such that for any test function  $v_h \in Z_h$ ,

$$\begin{aligned}
& \int_{I_{i,j}} (\partial_t \phi_h(x, y, t) + H(\partial_x \phi_h(x, y, t), \partial_y \phi_h(x, y, t), x, y)) v_h(x, y) dx dy \\
& + \int_{J_j} H_{1min}(x_{i+\frac{1}{2}}, y) [\phi_h](x_{i+\frac{1}{2}}, y) v_h(x_{i+\frac{1}{2}}^-, y) dy \\
& + \int_{J_j} H_{1max}(x_{i-\frac{1}{2}}, y) [\phi_h](x_{i-\frac{1}{2}}, y) v_h(x_{i-\frac{1}{2}}^+, y) dy \\
& + \int_{I_i} H_{2min}(x, y_{j+\frac{1}{2}}) [\phi_h](x, y_{j+\frac{1}{2}}) v_h(x, y_{j+\frac{1}{2}}^-) dy \\
& + \int_{I_i} H_{2max}(x, y_{j-\frac{1}{2}}) [\phi_h](x, y_{j-\frac{1}{2}}) v_h(x, y_{j-\frac{1}{2}}^+) dy \\
(68) \quad & = 0
\end{aligned}$$

where

$$\begin{aligned}
H_{1min}(x_{i+\frac{1}{2}}, y) &= \frac{1}{2} \left( \min_{x \in I_{i+\frac{1}{2}}} H_1(\partial_x \phi_h, \overline{\partial_y \phi_h}, x_{i+\frac{1}{2}}, y) - \left| \min_{x \in I_{i+\frac{1}{2}}} H_1(\partial_x \phi_h, \overline{\partial_y \phi_h}, x_{i+\frac{1}{2}}, y) \right| \right) \\
H_{1max}(x_{i+\frac{1}{2}}, y) &= \frac{1}{2} \left( \max_{x \in I_{i+\frac{1}{2}}} H_1(\partial_x \phi_h, \overline{\partial_y \phi_h}, x_{i+\frac{1}{2}}, y) + \left| \max_{x \in I_{i+\frac{1}{2}}} H_1(\partial_x \phi_h, \overline{\partial_y \phi_h}, x_{i+\frac{1}{2}}, y) \right| \right) \\
H_{2min}(x, y_{j+\frac{1}{2}}) &= \frac{1}{2} \left( \min_{y \in J_{j+\frac{1}{2}}} H_2(\overline{\partial_x \phi_h}, \partial_y \phi_h, x, y_{j+\frac{1}{2}}) - \left| \min_{y \in J_{j+\frac{1}{2}}} H_2(\overline{\partial_x \phi_h}, \partial_y \phi_h, x, y_{j+\frac{1}{2}}) \right| \right) \\
H_{2max}(x, y_{j+\frac{1}{2}}) &= \frac{1}{2} \left( \max_{y \in J_{j+\frac{1}{2}}} H_2(\overline{\partial_x \phi_h}, \partial_y \phi_h, x, y_{j+\frac{1}{2}}) + \left| \max_{y \in J_{j+\frac{1}{2}}} H_2(\overline{\partial_x \phi_h}, \partial_y \phi_h, x, y_{j+\frac{1}{2}}) \right| \right)
\end{aligned}$$

and in these formulae, we define

$$\overline{\partial_x \phi_h} = \frac{1}{2} ((\partial_x \phi_h)^+ + (\partial_x \phi_h)^-), \quad \overline{\partial_y \phi_h} = \frac{1}{2} ((\partial_y \phi_h)^+ + (\partial_y \phi_h)^-).$$

On the interfaces of the cells, along the normal direction, the reconstructed information of the partial derivatives as in the one-dimensional case is used, and tangential to the interface, the average of the partial derivatives from the two neighboring cells is used. The reconstruction process is the same as that in the one-dimensional case, except that we need to fix  $x$  or  $y$ , then perform the reconstruction on the other spatial variable.

**3.2.2. LDG scheme for directly solving HJ equations.** The LDG scheme for directly solving the two-dimensional HJ equations in [22] defined on arbitrary triangulation is: find  $\phi_h \in W_h$ , such that for any test function  $u \in W_h$ , we have

$$(69) \quad \int_K (\phi_h)_t u dx dy + \int_K \hat{H}(p_1, p_2, q_1, q_2) u dx dy = 0.$$

The variables  $p_1$  and  $p_2$  are used to approximate  $\phi_x$ , and similar to the 1D case,  $p_1$  and  $p_2$  are obtained by solving two simple upwind DG schemes, that is: find

$p_1 \in W_h$  and  $p_2 \in W_h$ , such that for any test functions  $v_1 \in W_h$  and  $v_2 \in W_h$ , we have

$$(70) \quad \begin{cases} \int_K p_1 v_1 dx dy + \int_K \phi_h(v_1)_x dx dy - \int_{\partial K} (\phi_h)^+ n_x v_1^{int_K} ds = 0 \\ \int_K p_2 v_2 dx dy + \int_K \phi_h(v_2)_x dx dy - \int_{\partial K} (\phi_h)^- n_x v_2^{int_K} ds = 0 \end{cases}$$

with

$$(\phi_h)^+ = \begin{cases} (\phi_h)^{ext_K}, & \text{if } n_x \geq 0 \\ (\phi_h)^{int_K}, & \text{else} \end{cases}$$

and

$$(\phi_h)^- = \begin{cases} (\phi_h)^{int_K}, & \text{if } n_x \geq 0 \\ (\phi_h)^{ext_K}, & \text{else} \end{cases}$$

where  $\partial K$  is the boundary of element  $K$ ,  $n = (n_x, n_y)$  is the outward unit normal for element  $K$  along the element boundary  $\partial K$ . Here  $(\phi_h)^{int_K}$  denotes the value of  $\phi_h$  on  $\partial K$  evaluated from inside the element  $K$ , and correspondingly  $(\phi_h)^{ext_K}$  denotes the value of  $\phi_h$  on  $\partial K$  evaluated from the outside element  $K$  (inside the neighboring element  $K'$  which shares the same edge with  $K$ ).

Two other variables  $q_1$  and  $q_2$  are used to approximate  $\phi_y$  and are obtained by solving the following two upwind DG schemes: find  $q_1 \in W_h$  and  $q_2 \in W_h$ , such that for any test functions  $w_1 \in W_h$  and  $w_2 \in W_h$ , we have

$$(71) \quad \begin{cases} \int_K q_1 w_1 dx dy + \int_K \phi_h(w_1)_y dx dy - \int_{\partial K} (\phi_h)^+ n_y w_1^{int_K} ds = 0, \\ \int_K q_2 w_2 dx dy + \int_K \phi_h(w_2)_y dx dy - \int_{\partial K} (\phi_h)^- n_y w_2^{int_K} ds = 0 \end{cases}$$

with

$$(\phi_h)^+ = \begin{cases} (\phi_h)^{ext_K}, & \text{if } n_y \geq 0 \\ (\phi_h)^{int_K}, & \text{else} \end{cases}$$

and

$$(\phi_h)^- = \begin{cases} (\phi_h)^{int_K}, & \text{if } n_y \geq 0 \\ (\phi_h)^{ext_K}, & \text{else} \end{cases}$$

The Lax-Friedrichs type numerical Hamiltonian can be used, which is defined to be

$$(72) \quad \hat{H}(p_1, p_2, q_1, q_2) = H\left(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) - \frac{1}{2}\alpha(p_1 - p_2) - \frac{1}{2}\beta(q_1 - q_2)$$

with

$$(73) \quad \alpha = \max_{p \in D, q \in E} \left| \frac{\partial H(p, q)}{\partial p} \right|, \quad \beta = \max_{p \in D, q \in E} \left| \frac{\partial H(p, q)}{\partial q} \right|$$

here  $H(u, v)$  is a short-hand notation for  $H(u, v, x, y)$ . With  $D = [\min(p_1, p_2), \max(p_1, p_2)]|_K$  and  $E = [\min(q_1, q_2), \max(q_1, q_2)]|_K$ , it is called the local Lax-Friedrichs Hamiltonian. With  $D = [\min(p_1, p_2), \max(p_1, p_2)]|_\Omega$  and  $E = [\min(q_1, q_2), \max(q_1, q_2)]|_\Omega$ , then it is called the global Lax-Friedrichs Hamiltonian.

The definition of the LDG scheme on rectangular meshes is that we only need to replace  $W_h$  with  $Z_h$ . In this case, our domain is  $I \times J$  and each element  $K = I_{i,j}$ .

**3.2.3. The main results.** In this section, we state the main error estimate results for the semi-discrete DG scheme (68) and the semi-discrete LDG scheme (69)-(71) on Cartesian meshes.

**Theorem 3.3.** *Let  $\phi$  be the exact solution of the problem (2), which is sufficiently smooth with bounded derivatives, and assume  $H(u, v, x, y) \in C^2$ . Let  $\phi_h$  be the numerical solution of the semi-discrete DG scheme (68) or the semi-discrete LDG scheme (69), (70) and (71). Denote the corresponding numerical error as  $e_\phi = \phi - \phi_h$ . For a rectangular triangulation of  $I \times J$ , if the finite element space  $Z_h$  is the piecewise tensor product polynomials with degree  $k \geq 3$ , then for small enough  $h$ , there holds the following error estimates*

$$(74) \quad \|\phi - \phi_h\| \leq Ch^{k+1}$$

where the positive constant  $C$  depends on the final time  $T$ ,  $k$ ,  $\|\phi\|_{L^\infty((0,T),H^{k+1}(I \times J))}$  and the bounds on the  $m$ -th derivatives of  $H(\partial_x \phi, \partial_y \phi, x, y)$  with respect to its first and second arguments for  $m = 1, 2$ .  $\|\phi\|_{L^\infty((0,T),H^{k+1}(I \times J))}$  is the maximum over  $0 \leq t \leq T$  of the standard Sobolev  $(k+1)$ -norm in space.

**Remark 3.3.** *Notice that our proof for the optimal error estimate works only for the finite element space  $Z_h$ , not for the usual  $k$ -th degree polynomial space  $W_h$ . This is because the main technique is the special tensor product projection. However, numerical examples in [22] do verify the optimal order of accuracy for the LDG scheme defined on  $W_h$ .*

**3.2.4. Sketch of the proof for the DG scheme.** First we would like to adopt an a priori assumption for the two-dimensional DG scheme, that is

$$(75) \quad \|\partial_x(\phi - \phi_h)\|_\infty \leq Ch, \quad \|\partial_y(\phi - \phi_h)\|_\infty \leq Ch.$$

The cell error equation for the DG scheme is

$$(76) \quad \begin{aligned} & \int_{I_{i,j}} (\phi - \phi_h)_t v_h dx dy + \int_{I_{i,j}} (H(\partial_x \phi, \partial_y \phi, x, y) - H(\partial_x \phi_h, \partial_y \phi_h, x, y)) v_h dx dy \\ & \quad - \int_{J_j} H_{1min}(x_{i+\frac{1}{2}}, y) [\phi_h](x_{i+\frac{1}{2}}, y) v_h(x_{i+\frac{1}{2}}^-, y) dy \\ & \quad - \int_{J_j} H_{1max}(x_{i-\frac{1}{2}}, y) [\phi_h](x_{i-\frac{1}{2}}, y) v_h(x_{i-\frac{1}{2}}^+, y) dy \\ & \quad - \int_{I_i} H_{2min}(x, y_{j+\frac{1}{2}}) [\phi_h](x, y_{j+\frac{1}{2}}) v_h(x, y_{j+\frac{1}{2}}^-) dx \\ & \quad - \int_{I_i} H_{2max}(x, y_{j-\frac{1}{2}}) [\phi_h](x, y_{j-\frac{1}{2}}) v_h(x, y_{j-\frac{1}{2}}^+) dx \\ & = 0. \end{aligned}$$

By taking a Taylor expansion

$$(77) \quad \begin{aligned} & H(\partial_x \phi, \partial_y \phi, x, y) - H(\partial_x \phi_h, \partial_y \phi_h, x, y) = H_1 \partial_x(\phi - \phi_h) + H_2 \partial_y(\phi - \phi_h) \\ & - \frac{1}{2} (\bar{H}_{11}(\partial_x(\phi - \phi_h))^2 + 2\bar{H}_{12} \partial_x(\phi - \phi_h) \partial_y(\phi - \phi_h) + \bar{H}_{22}(\partial_y(\phi - \phi_h))^2) \end{aligned}$$

we have

$$(78) \quad \int_{I_{i,j}} (\phi - \phi_h)_t v_h dx dy + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 = 0$$

where

$$\begin{aligned} \mathcal{J}_1 &= \int_{I_{i,j}} (H_1 \partial_x(\phi - \phi_h) - \frac{1}{2} \bar{H}_{11}(\partial_x(\phi - \phi_h))^2) v_h dx dy \\ &\quad - \int_{J_j} H_{1min}(x_{i+\frac{1}{2}}, y) [\phi_h](x_{i+\frac{1}{2}}, y) v_h(x_{i+\frac{1}{2}}^-, y) dy \\ &\quad - \int_{J_j} H_{1max}(x_{i-\frac{1}{2}}, y) [\phi_h](x_{i-\frac{1}{2}}, y) v_h(x_{i-\frac{1}{2}}^+, y) dy \\ \mathcal{J}_2 &= \int_{I_{i,j}} (H_2 \partial_y(\phi - \phi_h) - \frac{1}{2} \bar{H}_{22}(\partial_y(\phi - \phi_h))^2) v_h dx dy \\ &\quad - \int_{I_i} H_{2min}(x, y_{j+\frac{1}{2}}) [\phi_h](x, y_{j+\frac{1}{2}}) v_h(x, y_{j+\frac{1}{2}}^-) dx \\ &\quad - \int_{I_i} H_{2max}(x, y_{j-\frac{1}{2}}) [\phi_h](x, y_{j-\frac{1}{2}}) v_h(x, y_{j-\frac{1}{2}}^+) dx \\ \mathcal{J}_3 &= - \int_{I_{i,j}} \bar{H}_{12} \partial_x(\phi - \phi_h) \partial_y(\phi - \phi_h) v_h dx dy. \end{aligned}$$

$\mathcal{J}_1$  and  $\mathcal{J}_2$  can be estimated similarly as in the one-dimensional case, and  $\mathcal{J}_3$  can be estimated similarly as in (39). The optimal error estimate of the DG scheme for the two-dimensional case can then be obtained.

**3.2.5. Sketch of the proof for the LDG scheme.** In this section, we also give a sketch of the proof for the local Lax-Friedrichs LDG scheme on Cartesian meshes. An a priori assumption we adopt for the two-dimensional LDG scheme is

$$(79) \quad \|\partial_x \phi - \frac{p_1 + p_2}{2}\|_\infty \leq Ch, \quad \|\partial_y \phi - \frac{q_1 + q_2}{2}\|_\infty \leq Ch$$

The cell error equation for the LDG scheme is

$$(80) \quad \begin{aligned} & \int_{I_{i,j}} (\phi - \phi_h)_t u dx dy + \int_{I_{i,j}} (H(\partial_x \phi, \partial_y \phi) - H(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2})) u dx dy \\ & + \frac{1}{2} \alpha \int_{I_{i,j}} (p_1 - p_2) u dx dy + \frac{1}{2} \beta \int_{I_{i,j}} (q_1 - q_2) u dx dy = 0. \end{aligned}$$

By taking a Taylor expansion

$$\begin{aligned} H(\partial_x \phi, \partial_y \phi) - H\left(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) &= H_1\left(\partial_x \phi - \frac{p_1 + p_2}{2}\right) + H_2\left(\partial_y \phi - \frac{q_1 + q_2}{2}\right) \\ &- \frac{1}{2} \left( \bar{H}_{11}\left(\partial_x \phi - \frac{p_1 + p_2}{2}\right)^2 - 2\bar{H}_{12}\left(\partial_x \phi - \frac{p_1 + p_2}{2}\right)\left(\partial_y \phi - \frac{q_1 + q_2}{2}\right) + \bar{H}_{22}\left(\partial_y \phi - \frac{q_1 + q_2}{2}\right)^2 \right) \end{aligned}$$

we have

$$(81) \quad \int_{I_{i,j}} (\phi - \phi_h)_t u dx dy + \mathcal{J}_4 + \mathcal{J}_5 + \mathcal{J}_6 = 0$$

where

$$\begin{aligned} \mathcal{J}_4 &= \int_{I_{i,j}} u \left( H_1\left(\partial_x \phi - \frac{p_1 + p_2}{2}\right) - \frac{1}{2} \bar{H}_{11}\left(\partial_x \phi - \frac{p_1 + p_2}{2}\right)^2 \right) dx dy \\ &\quad + \frac{1}{2} \alpha \int_{I_{i,j}} (p_1 - p_2) u dx dy \\ \mathcal{J}_5 &= \int_{I_{i,j}} u \left( H_2\left(\partial_y \phi - \frac{q_1 + q_2}{2}\right) - \frac{1}{2} \bar{H}_{22}\left(\partial_y \phi - \frac{q_1 + q_2}{2}\right)^2 \right) dx dy \\ &\quad + \frac{1}{2} \beta \int_{I_{i,j}} (q_1 - q_2) u dx dy \\ \mathcal{J}_6 &= - \int_{I_{i,j}} \bar{H}_{12}\left(\partial_x \phi - \frac{p_1 + p_2}{2}\right)\left(\partial_y \phi - \frac{q_1 + q_2}{2}\right) u dx dy. \end{aligned}$$

With (70) and (71),  $\mathcal{J}_4$  and  $\mathcal{J}_5$  can be estimated similarly as in the one-dimensional case, and  $\mathcal{J}_6$  can be estimated similarly as the term  $\int_{I_{i,j}} \bar{H}_{11}\left(\partial_x \phi - \frac{p_1 + p_2}{2}\right)^2 u dx dy$ . The optimal error estimate of the LDG scheme for the two-dimensional case is thus obtained.

**Remark 3.4.** For the two-dimensional LDG scheme on a rectangular mesh, another more general definition for the local Lax-Friedrichs Hamiltonian in (72) and (73) is that [18],  $D = [\min(p_1, p_2), \max(p_1, p_2)]|_{I_i}$  but  $E = [\min(q_1, q_2), \max(q_1, q_2)]|_{J_j}$  for  $\alpha$ , and  $D = [\min(p_1, p_2), \max(p_1, p_2)]|_I$  but  $E = [\min(q_1, q_2), \max(q_1, q_2)]|_{J_j}$  for  $\beta$ , this Hamiltonian is monotone. However, our estimates for the two-dimensional case only works for the non monotone local Lax-Friedrichs Hamiltonian defined in (72) and (73), since we need  $|H_1 - \alpha| \leq Ch$  and  $|H_2 - \beta| \leq Ch$  in each cell  $I_{ij}$ , with the  $\alpha$  and  $\beta$  contained in (80). But the monotone Godunov [18] and Osher-Sethian [19] numerical Hamiltonian, which do not contain  $\alpha$  and  $\beta$  in (80), would also yield optimal error estimates for the two-dimensional case.

**3.2.6. Justification of the a priori assumption.** For the two-dimensional case, the a priori assumptions (75) and (79) can be similarly verified as in the one-dimensional case. We need the restriction  $k \geq 3$  in Theorem 3.3, due to the different bounds in (16). We omit the detailed proof here.

#### 4. Concluding remarks

In this paper, we have obtained the optimal a priori  $L^2$  error estimates for the semi-discrete DG scheme and the semi-discrete LDG scheme for directly solving

the Hamilton-Jacobi equations in one- and two-dimensional cases on rectangular meshes. By using the regular  $L^2$  projection rather than the special projections, we can prove sub-optimal a priori  $L^2$  error estimates (half an order lower) for the same schemes on arbitrary triangulations in two dimensions. The proof follows the same lines and hence is not given in this paper. The sub-optimal a priori  $L^2$  error estimates can also be obtained for the central DG scheme defined in [17] along the same lines.

As is well known, the viscosity solution to the Hamilton-Jacobi equation is generically only Lipschitz continuous, with possible discontinuous derivatives. Convergence and error estimates of the DG and LDG schemes for such cases are much more difficult to analyze and are worthy of future investigation.

## 5. Appendix

In this appendix, we are going to prove  $\max_{j_1 \leq j \leq j_2} |H_1 - H_{1max}|_{j+\frac{1}{2}} \leq Ch$  in Section 3.1.4.  $H_1 = H_1(\partial_x \phi) > 0$  and  $H_{1max} = \max_{x \in \bar{I}_{j+\frac{1}{2}}} H_1(\partial_x \phi_h)$ , we omit the index  $x$  here. Although we use  $\phi_h$  in the definition of  $H_{1max}$ , from (18) and (19),  $\phi_h$  is actually the polynomial  $w_{j+\frac{1}{2}}$ , which leads to the proof of the inequality nontrivial.  $H_{1max}(\partial_x w_{j+\frac{1}{2}})$  is actually the function  $H_1(\partial_x w_{j+\frac{1}{2}})$  evaluated at some point  $x^*$  inside the cell  $\bar{I}_{j+\frac{1}{2}}$ , which we denote to be  $H_1((\partial_x w_{j+\frac{1}{2}})^*)$ .

First, we can define a similar projection  $P\phi \in P^{2k+1}(I_j \cup I_{j+1})$  as  $w_{j+\frac{1}{2}}$  but corresponding to the exact smooth solution  $\phi$ , which is

$$(A.1) \quad \int_{I_s} \phi \zeta dx = \int_{I_s} P\phi \zeta dx,$$

for any  $\zeta \in P^k(I_s)$ ,  $s = j, j+1$ . And from [4], we have the error estimate that

$$(A.2) \quad \|\partial_x(\phi - P\phi)\|_{L^\infty(I_j \cup I_{j+1})} \leq Ch^{k-\frac{1}{2}}.$$

We adopt the a priori assumption similar to (27) in this section, that is

$$(A.3) \quad \|\phi - \phi_h\| \leq Ch^{\frac{5}{2}}$$

the justification is easily followed from Section 3.1.6, since we have  $k \geq 2$ .

In the following, we are going to prove that  $\|P\phi - w_{j+\frac{1}{2}}\|_{L^2(I_j \cup I_{j+1})} \leq C\|\phi - \phi_h\|$ . From (18), (19) and (A.1), we have

$$(A.4) \quad \int_{I_s} (P\phi - w_{j+\frac{1}{2}}) \zeta dx = \int_{I_s} (\phi - \phi_h) \zeta dx,$$

for any  $\zeta \in P^k(I_s)$ ,  $s = j, j+1$ . Denote  $u = P\phi - w_{j+\frac{1}{2}} \in P^{2k+1}(I_j \cup I_{j+1})$ , we can define the  $L^2$ -projection  $u_1 \in P^k(I_j)$  of  $u$  in the cell  $I_j$  and  $u_2 \in P^k(I_{j+1})$  of  $u$  in the cell  $I_{j+1}$ , which satisfy

$$(A.5) \quad \int_{I_j} u_1^2 dx = \int_{I_j} u_1 u dx$$

and

$$(A.6) \quad \int_{I_{j+1}} u_2^2 dx = \int_{I_{j+1}} u_2 u dx$$

we have  $\|u_1\|_{L^2(I_j)} \leq \|u\|_{L^2(I_j)}$  and  $\|u_2\|_{L^2(I_{j+1})} \leq \|u\|_{L^2(I_{j+1})}$ . However, conversely, we also have

$$(A.7) \quad \|u\|_{L^2(I_j \cup I_{j+1})} \leq C(\|u_1\|_{L^2(I_j)} + \|u_2\|_{L^2(I_{j+1})}).$$

We can first prove (A.7) on a standard cell  $I = (0, 1)$ , we define

$$\begin{aligned} V &= P^{2k+1}(I) = \text{Span}\{1, x, x^2, \dots, x^{2k+1}\}, \\ V_1 &= \text{Span}\{1, x^2, x^4, \dots, x^{2k}\}, \\ V_2 &= \text{Span}\{x, x^3, x^5, \dots, x^{2k+1}\}, \\ W &= P^k(I) = \text{Span}\{1, x, x^2, \dots, x^k\}. \end{aligned}$$

that is  $V = V_1 + V_2$ . For any  $v \in V$ , we have  $v = v_1 + v_2$  where  $v_1 \in V_1$  and  $v_2 \in V_2$ . Now we perform a  $L^2$ -projection  $P$  from  $V_1$  to  $W$ , which is

$$(A.8) \quad \int_0^1 v_1 \zeta dx = \int_0^1 w_1 \zeta dx$$

for any  $\zeta \in W$ , here  $w_1 = Pv_1$ . Using the basis of  $V_1$  and  $W$ , (A.8) can be written in the form  $A_1 \tilde{v}_1 = A \tilde{w}_1$ , here  $\tilde{v}_1 = (\tilde{v}_1^0, \tilde{v}_1^2, \dots, \tilde{v}_1^{2k})$  and  $\tilde{w}_1 = (\tilde{w}_1^0, \tilde{w}_1^1, \dots, \tilde{w}_1^k)$  are the coefficients of  $v_1$  and  $w_1$  with the polynomial basis in the corresponding space,  $A_1$  is a constant matrix which can be easily found to be invertible, and  $A$  is the mass matrix in  $W$ . So we have  $\tilde{v}_1 = (A_1)^{-1} A \tilde{w}_1$ , which means that

$$(A.9) \quad \|v_1\|_{L^2(V_1)}^2 \leq C \sum_{l=0}^k (\tilde{v}_1^l)^2 \leq C |(A_1)^{-1} A|^2 \sum_{l=0}^k (\tilde{w}_1^l)^2 \leq C \|w_1\|_{L^2(W)}^2$$

Similarly, for  $v_2 \in V_2$ , we have the  $L^2$ -projection  $w_2 = Pv_2$  in  $W$ , such that  $\|v_2\|_{L^2(V_2)} \leq C \|w_2\|_{L^2(W)}$ , combining with (A.9), we have

$$(A.10) \quad \|v\|_{L^2(V)} = \|v_1 + v_2\|_{L^2(V)} \leq \|v_1\|_{L^2(V_1)} + \|v_2\|_{L^2(V_2)} \leq C(\|w_1\|_{L^2(W)} + \|w_2\|_{L^2(W)})$$

using the scaling argument, where  $v$ ,  $w_1$  and  $w_2$  corresponding to  $u$ ,  $u_1$  and  $u_2$  respectively, we obtain (A.7).

In (A.4), taking  $\zeta = u_1$  in  $I_j$  and  $\zeta = u_2$  in  $I_{j+1}$ , combining (A.5) and (A.6), we obtain

$$(A.11) \quad \begin{aligned} \int_{I_j} u_1^2 dx + \int_{I_{j+1}} u_2^2 dx &= \int_{I_j} u_1 u dx + \int_{I_{j+1}} u_2 u dx \\ &= \int_{I_j} (\phi - \phi_h) u_1 dx + \int_{I_{j+1}} (\phi - \phi_h) u_2 dx \\ &\leq \|\phi - \phi_h\|_{L^2(I_j)} \|u_1\|_{L^2(I_j)} + \|\phi - \phi_h\|_{L^2(I_{j+1})} \|u_2\|_{L^2(I_{j+1})} \\ &\leq \|\phi - \phi_h\| \|u\|_{L^2(I_j \cup I_{j+1})} \end{aligned}$$



from (A.7) and (A.11), we obtain  $\|P\phi - w_{j+\frac{1}{2}}\|_{L^2(I_j \cup I_{j+1})} \leq C\|\phi - \phi_h\|$ . Then with the inverse property (16) and the a priori assumption (A.3), we have

$$\begin{aligned}
 \|\partial_x(P\phi - w_{j+\frac{1}{2}})\|_{L^\infty(I_j \cup I_{j+1})} &\leq Ch^{-\frac{3}{2}}\|P\phi - w_{j+\frac{1}{2}}\|_{L^2(I_j \cup I_{j+1})} \\
 &\leq Ch^{-\frac{3}{2}}\|\phi - \phi_h\| \\
 \text{(A.12)} \qquad \qquad \qquad &\leq Ch
 \end{aligned}$$

Now we can get our result from

$$\begin{aligned}
 |H_1(\partial_x\phi) - H_{1max}(\partial_x w_{j+\frac{1}{2}})| &\leq |H_1(\partial_x\phi) - H_1(\partial_x(P\phi))| \\
 &\quad + |H_1(\partial_x(P\phi)) - H_1(\partial_x w_{j+\frac{1}{2}})| \\
 &\quad + |H_1(\partial_x w_{j+\frac{1}{2}}) - H_1((\partial_x w_{j+\frac{1}{2}})^*)| \\
 &\leq C(|\partial_x(\phi - P\phi)| + |\partial_x(P\phi - w_{j+\frac{1}{2}})| \\
 &\quad + |\partial_x w_{j+\frac{1}{2}} - (\partial_x w_{j+\frac{1}{2}})^*|) \\
 &\leq C(h^{k-\frac{1}{2}} + h + h) \\
 \text{(A.13)} \qquad \qquad \qquad &\leq Ch
 \end{aligned}$$

here  $k \geq 2$  and taking the maximum over  $j_1 \leq j \leq j_2$  in (A.13), the inequality  $\max_{j_1 \leq j \leq j_2} |H_1 - H_{1max}|_{j+\frac{1}{2}} \leq Ch$  has been obtained.

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