High order well-balanced WENO scheme for the gas dynamics equations under gravitational fields[1]

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Abstract

The gas dynamics equations, coupled with a static gravitational field, admit the hydrostatic balance where the flux produced by the pressure is exactly canceled by the gravitational source term. Many astrophysical problems involve the hydrodynamical evolution in a gravitational field, therefore it is essential to correctly capture the effect of gravitational force in the simulations. Improper treatment of the gravitational force can lead to a solution which either oscillates around the equilibrium, or deviates from the equilibrium after a long time run. In this paper we design high order well-balanced finite difference WENO schemes to this system, which can preserve the hydrostatic balance state exactly and at the same time can maintain genuine high order accuracy. Numerical tests are performed to verify high order accuracy, well-balanced property, and good resolution for smooth and discontinuous solutions. The main purpose of the well-balanced schemes designed in this paper is to well resolve small perturbations of the hydrostatic balance state on coarse meshes. The more difficult issue of convergence towards such hydrostatic balance state from an arbitrary initial condition is not addressed in this paper.

Keywords: Euler equations; well-balanced; WENO scheme; finite difference method; gravitational field

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1 Introduction

In recent years, research on well-balanced numerical methods for the hyperbolic equations with source terms (also referred as hyperbolic balance laws) has attracted much attention. In one space dimension, balance laws take the form of

$$U_t + f(U)_x = s(U, x), \tag{1.1}$$

where U is the solution vector with the corresponding flux $f(U)$, and $s(U, x)$ is the source term. They usually admit steady state solutions, in which the source term is exactly balanced by the flux gradient. A challenge in the numerical analysis of such balance laws is to maintain these steady states, and to compute their perturbations accurately. Indeed, if a scheme cannot balance the effects of convective fluxes and source term, it may introduce spurious oscillations near equilibria, unless the mesh size is extremely refined. To save the computational cost, well-balanced methods, which preserve exactly these steady-state solutions up to machine accuracy, are specially designed and work well on relatively coarse meshes. A typical example considered extensively in the literature for balance laws is the shallow water equation with a non-flat bottom topology. Many researchers have developed well-balanced methods for the shallow water equation using different approaches, see, e.g. [4, 1, 7, 2, 12, 21, 19] and the references therein.

Another important example of the hyperbolic balance laws is the gas dynamics system under gravitational field. Near the equilibrium state, there exists the hydrostatic balance where the flux produced by the pressure is canceled by the gravitational source term. Many astrophysical problems involve the hydrodynamical evolution in a gravitational field, therefore it is essential to correctly capture the effect of gravitational force in the simulations, especially if a long-time integration is involved, for example in modeling star and galaxy formation. Improper treatment of the gravitational force can lead to a solution which either oscillates around the equilibrium, or deviates from the equilibrium after a long time run. There have been several attempts in designing well-balanced methods for the gas dynam-

ics, which take care of the implementation of the gravitational field. LeVeque and Bale [8] proposed the quasi-steady wave-propagation methods for an ideal gas subject to a static gravitational field. A Riemann problem is introduced in the center of each grid cell such that the flux difference exactly cancels the source term. Zingale et al. [23] investigated the process of mapping an astrophysical initial model from a stellar evolution code onto the computational grid of an explicit code while maintaining hydrostatic equilibrium. A different strategy for the construction of well-balanced discretizations with respect to dominant hydrostatics has been proposed by Botta et al. [5] for the nearly hydrostatic flows belonging to a certain class of solutions. More recently, Xu and his collaborators [18, 22, 11] have extended the gas-kinetic scheme to the multidimensional gas dynamic equations to develop well-balanced numerical methods, where the gravitational potential was modeled as a piecewise step function with a potential jump at the cell interface.

Most of the works mentioned above are for numerical schemes of at most second order spatial accuracy. The main objective of this paper is to design a finite difference WENO scheme for the gas dynamic equations with gravitational source terms, which maintains the well-balanced property for the hydrostatic balance, and at the same time is genuinely high order accurate for the general solutions. We introduce a different treatment of the source term, which mimics the WENO approximation to the flux term, so that the exact balance between the source term and the flux can be achieved at the steady state. The proposed method is a generalization of the approach to develop well-balanced methods for the shallow water equations and other balanced laws in [19, 20]. The specific WENO scheme we use is the fifth order finite difference scheme introduced by Jiang and Shu [6]. It uses a convex combination of three candidate stencils, each producing a third order accurate flux, to obtain fifth order accuracy and an essentially nonoscillatory shock transition. Time discretization can be implemented by the TVD Range-Kutta method [16]. WENO schemes were first introduced by Liu, Osher and Chan in [10]. For more details of WENO schemes, we refer to [6, 14, 15].

The outline of the paper is as follows: In Section 2, the model and its steady state solutions are described. In Section 3, we develop the well-balanced finite difference WENO scheme, which at the same time is genuinely high order accurate for the general solutions of the gas dynamics equations. Extension to two dimensional problems is introduced in Section 4. In Section 5, we show selective numerical results in one and two dimensions to demonstrate the behavior of the proposed finite difference WENO methods, verifying high order accuracy, the well-balanced property, and good resolution for smooth and discontinuous solutions. Concluding remarks are given in Section 6.

2 The model

We consider the equations governing the conservation of mass, momentum and energy of an inviscid, non-heat conducting, isotropic fluid. These gas dynamic equations, coupled with a static gravitational potential, are given by

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2 + p)_x = -\rho\phi_x \\ &E_t + ((E+p)u)_x = -\rho u\phi_x, \end{aligned} \tag{2.1}$$

in one space dimension, where ρ denotes the fluid density, u is the velocity, p represents the pressure, and $E = \frac{1}{2}\rho u^2 + p/(\gamma - 1)$ is the non-gravitational energy which includes the kinetic and internal energy of the fluid. γ is the ratio of specific heats and $\phi = \phi(x)$ is the time independent gravitational potential.

2.1 Hydrostatic balance

For the static gravitational potential $\phi(x)$, we are interested in preserving the hydrostatic balance, which is a special steady state solution to (2.1)

$$\rho = \rho(x), \qquad u = 0, \qquad p_x = -\rho\phi_x, \tag{2.2}$$

with a constant temperature and zero velocity. For an ideal gas, we have the relation

$$p = \rho RT, \tag{2.3}$$

where R is the gas constant, and T is the temperature. Combined with the steady state equation $p_x = -\rho\phi_x$ from (2.2), it becomes

$$\rho(x) = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \tag{2.4}$$

which leads to the special steady state

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \qquad u = 0, \qquad p = RT\rho = RT\rho_0 \exp\left(-\frac{\phi}{RT}\right), \tag{2.5}$$

with constant temperature T.

The simplest and most commonly encountered case is the linear gravitational potential field: $d\phi/dx = g$, with the corresponding hydrostatic balance

$$\rho = \rho_0 \exp(-g\rho_0 x/p_0), \qquad u = 0, \qquad p = p_0 \exp(-g\rho_0 x/p_0). \tag{2.6}$$

3 Well-balanced finite difference WENO methods

In this section we design a genuinely high order accurate well-balanced WENO method for the gas dynamic equations (2.1) with gravitational source terms. The key idea is to discretize the source term by a finite difference WENO formula consistent with that for the flux. We will concentrate our discussion on the one-dimensional case. Generalization to the two dimensional problems will be presented in Section 4.

Well-balanced methods are specially designed to preserve exactly the steady-state solutions up to machine accuracy with relatively coarse meshes. High-order well-balanced WENO methods have been designed for the shallow-water equations by the same authors in [19]. The main idea there is to decompose the source term into a sum of two terms first, and discretize each of them independently using a finite difference formula consistent with that of approximating the flux derivative terms in the conservation law. The same technique

has been generalized to other hyperbolic balance laws in [20] and to hybrid WENO schemes in [9]. Similar idea has also been employed in designing well-balanced method for the Euler equation in [5], where the same discrete gradient operator has been used to approximate the pressure gradient and gravitational potential gradient. In this paper, we generalize this idea to the gas dynamic equations to design a genuinely high order well-balanced WENO method. We would like to preserve exactly the steady state solution satisfying (2.5). The special steady state of the form (2.6)

$$\rho = c\exp(-gx), \qquad u = 0, \qquad p = c\exp(-gx), \tag{3.1}$$

coupled with the linear gravitational potential field

$$\phi_x = g, \tag{3.2}$$

will be used as an example to illustrate the basic idea in this section. More general equilibrium (2.5) can be handled following the same technique, and will be explained briefly at the end of this section.

The first step in designing the well-balanced method is to rewrite the source term following the guideline presented in [20]. The gas dynamic equations (2.1) can be reformulated as

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0 \\
&(\rho u)_t + (\rho u^2 + p)_x = \rho\exp(gx)(\exp(-gx))_x \\
&E_t + ((E+p)u)_x = \rho u\exp(gx)(\exp(-gx))_x,
\end{aligned} \tag{3.3}$$

where the gravitational source $-\rho g$ is replaced by $\rho\exp(gx)(\exp(-gx))_x$, and $-\rho u g$ is treated in the same way. The main motivation of such a change is to let the source term and the corresponding flux term share similar form in the case of the steady state solution (3.2). As we will see below, this new form of the source term is crucial for the design of our well-balanced method. We refer to [20] for more motivation and explanation, as well as the extension to more general source terms.

For simplicity, we denote the one-dimensional Euler equations (3.3) by

$$U_t + f(U)_x = s(U, \phi),$$

where $U = (\rho, \rho u, E)^T$ with the superscript T denoting the transpose, $f(U)$ is the flux and $s(U, \phi) = (0, -\rho\phi_x, \rho u \phi_x)^T$ is the source term. We also assume the mesh is uniform with mesh size Δx. In a finite difference scheme, our computational variables are $U_j(t)$, which approximate the point value at x_j. The finite difference WENO scheme is given by

$$\frac{d}{dt}U_j(t) + \frac{1}{\Delta x_j}\left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right) = \hat{s}_j(U), \tag{3.4}$$

with $\hat{f}_{j+\frac{1}{2}}$ being the numerical flux, and $\hat{s}_j(U)$ being a high order approximation to the source term at the point x_j.

A well-balanced method is one that balances the flux and source term exactly, i.e. $\left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right)/\Delta x_j = \hat{s}_j(U)$, at the steady state solution (3.1). Following the ideas in [19], we will start by considering an identical linear finite difference operator for the flux derivative and the derivatives in the source terms. A linear finite difference operator D is defined to be one satisfying $D(ag_1 + bg_2) = aD(g_1) + bD(g_2)$ for constants a, b and arbitrary grid functions g_1 and g_2. A scheme for (3.3) is said to be a linear scheme if all the spatial derivatives are approximated by linear finite difference operators. At the steady state solution (3.1), for any consistent linear scheme, the first equation $(\rho u)_x = 0$ and the third equation $((E + p)u)_x = \rho u \exp(gx)(\exp(-gx))_x$ are both satisfied exactly since the velocity u equals to zero. The truncation error for the second equation takes the form of

$$D_1(\rho u^2 + p) - \rho \exp(gx) D_2(\exp(-gx)), \tag{3.5}$$

which reduces to

$$D_1(p) - \rho \exp(gx) D_2(\exp(-gx)),$$

where D_1 and D_2 are linear finite difference operators used to approximate the spatial derivatives. For any linear scheme which also satisfies that $D_1 = D_2$ for the steady state solution

(3.1), the truncation error for the second equation reduces to

$$D_1(p) - \rho \exp(gx) D_2(\exp(-gx)) = D_1(c \exp(-gx)) - c \exp(-gx) \exp(gx) D_2(\exp(-gx)) = 0,$$

at the steady state solution (3.1), which presents the desired well-balanced property. Therefore, we have

Proposition 3.1: For the gas dynamic equations (2.1) with the linear gravitational potential field (3.2), linear scheme which satisfies $D_1 = D_2$ for the steady state solution (3.1) are well-balanced, i.e. it can preserve these steady state solution exactly.

We now extend the well-balanced property to high order *nonlinear* finite difference WENO scheme [6, 3], in which the nonlinearity comes from the nonlinear weights and the smooth indicators. We would like to make minor modifications to the WENO scheme, so that the well-balanced property is maintained without affecting the accuracy and nonlinear stability.

To simplify the presentation, we first consider the finite difference WENO scheme without a flux splitting (e.g. the WENO-Roe scheme as described in [6]) or the local characteristic decomposition. At the steady state solution (3.1), the first and third equations in (3.3) can be satisfied exactly, since $u = 0$ and the WENO approximation to $(\rho u)_x$ and $((E + p)u)_x$ is consistent. There are two derivatives in the second momentum equation of (3.3): $(\rho u^2 + p)_x$ and $(\exp(-gx))_x$. The WENO approximation to the flux derivative term $(\rho u^2 + p)_x$ proceeds as usual. Notice that the WENO approximation to d_x where $d = \rho u^2 + p$ can be eventually written out as

$$d_x|_{x=x_j} \approx \sum_{k=-r}^{r} a_k d_{k+j} \equiv D_d(d)_j \tag{3.6}$$

where $r = 3$ for the fifth order WENO approximation and the coefficients a_k depend *nonlinearly* on the smoothness indicators involving the grid function d. For the other derivative $(\exp(-gx))_x$ in the source term, as explained in [20], the key idea now is to use the finite difference operator D_d with $d = \rho u^2 + p$ *fixed*, namely to use the same coefficients a_k obtained through the smoothness indicators of d in (3.6), and apply it to approximate $(\exp(-gx))_x$.

Thus

$$(\exp(-gx))_x|_{x=x_j} \approx \sum_{k=-r}^{r} a_k \exp(-gx_{k+j}) = D_d \left(\exp(-gx)\right)_j .$$

Clearly, with $d = f(u, x)$ being fixed, the finite difference operator D_d, defined from the high order WENO procedure, is a high order accurate *linear* approximation to the first derivative for any grid function. Therefore, the overall high order accuracy will not be affected, and the proof for Propositions 3.1 will go through. We can conclude that the high order finite difference WENO scheme as stated above, without the flux splitting or local characteristic decomposition, and with the special handling of the source terms described above, maintains exactly the steady state solution (3.1).

Now, we consider WENO scheme with the local characteristic decomposition, which is typically used in high order WENO scheme for system to obtain better non-oscillatory properties for strong discontinuities. To compute the numerical flux at $x_{i+\frac{1}{2}}$, we first compute an average state $u_{i+\frac{1}{2}}$ between u_i and u_{i+1}, using either the simple arithmetic mean or a Roe's average [13]. The local characteristic matrix R, consisting of the right eigenvectors of the Jacobian at $u_{i+\frac{1}{2}}$, is then fixed. The neighboring point values of the flux functions needed for computing the numerical flux is projected to the local characteristic fields determined by R^{-1}. The numerical fluxes are computed in the characteristic direction, and then projected back into the physical space by left multiplying with R, yielding finally the numerical fluxes in the physical space. In this process, we notice that (3.6) still holds, while d now becomes a vector grid function $(\rho u, \rho u^2 + p, (E + p)u)^T$, and a_k are 3×3 matrices depending nonlinearly on the smoothness indicators involving the grid function d. The key idea is still to use the difference operator D_d with $d = (\rho u, \rho u^2 + p, (E + p)u)^T$ fixed, and apply it to approximate $(0, \exp(-gx), \exp(-gx))_x^T$ in the source terms. The remaining arguments stay the same as above, and we can prove that after the incorporation of a local characteristic decomposition, the WENO scheme still maintains exactly the steady state solution.

At the end, WENO scheme with a Lax-Friedrichs flux splitting, such as the WENO-LF and WENO-LLF scheme described in [6], is considered. Here the flux $f(U)$ is split into

$f^+(U)$ and $f^-(U)$, defined by

$$f^{\pm}(U) = \frac{1}{2}\left[\begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix} \pm \alpha_i \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}\right], \tag{3.7}$$

for the i-th characteristic field, where $\alpha_i = \max_u |\lambda_i(u)|$ with $\lambda_i(u)$ being the ith eigenvalue of the Jacobian $f'(U)$, and the maximum is taken over either a local region (for WENO-LLF method) or a global region (WENO-LF). The artificial viscosity terms $\pm\alpha_i(\rho, \rho u, E)^T$ are important to provide a non-oscillatory solution, however they will destroy the well-balanced property. We now modify this flux splitting to

$$f^{\pm}(U) = \frac{1}{2}\left[\begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix} \pm \alpha_i' \begin{pmatrix} \rho \exp(gx) \\ \rho u \exp(gx) \\ E \exp(gx) \end{pmatrix}\right], \tag{3.8}$$

by replacing $\pm\alpha_i(\rho, \rho u, E)^T$ with $\pm\alpha_i'(\rho \exp(gx), \rho u \exp(gx), E \exp(gx))^T$. The coefficient α_i' is given by

$$\alpha_i' = \alpha_i \max_x \exp(-gx), \tag{3.9}$$

where the maximum is taken over a local or global region, in order to maintain enough artificial viscosity. This modification does not affect accuracy, which relies only on the fact $f(U) = f^+(U) + f^-(U)$. Our motivation of using $\pm\alpha_i'(\rho \exp(gx), \rho u \exp(gx), E \exp(gx))^T$ comes from the fact that they become constant vectors in the case of steady state solution (3.1). Thus, by the consistency of the WENO approximation, their contribution towards the numerical flux approximation is zero. The flux splitting WENO approximation in this situation becomes simply $f^{\pm}(U) = \frac{1}{2}f(U)$, hence the steady state solution is preserved as before, if we simply split the derivatives in the source term as:

$$\begin{pmatrix} 0 \\ \exp(-gx) \\ \exp(-gx) \end{pmatrix}_x = \frac{1}{2}\begin{pmatrix} 0 \\ \exp(-gx) \\ \exp(-gx) \end{pmatrix}_x + \frac{1}{2}\begin{pmatrix} 0 \\ \exp(-gx) \\ \exp(-gx) \end{pmatrix}_x, \tag{3.10}$$

and apply the same flux splitting WENO procedure to approximate them with the nonlinear coefficients a_k coming from the WENO approximations to $f^{\pm}(U)$ respectively. We have thus proved that

Proposition 3.2: The WENO-Roe, WENO-LF and WENO-LLF schemes as stated above are well-balanced for the steady state solution (3.1) and can maintain the original high order accuracy. □

At the end, we present how the high order well-balanced WENO method can be designed for the more general steady state (2.5). We first rewrite the gas dynamic equations (2.1) as

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0 \\
&(\rho u)_t + (\rho u^2 + p)_x = RT\rho \exp(\frac{\phi}{RT}) \left(\exp(-\frac{\phi}{RT})\right)_x \\
&E_t + ((E+p)u)_x = RT\rho u \exp(\frac{\phi}{RT}) \left(\exp(-\frac{\phi}{RT})\right)_x .
\end{aligned} \tag{3.11}$$

Following exactly the same technique as stated above, with the flux splitting (3.8) replaced by

$$f^{\pm}(U) = \frac{1}{2}\left[\begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix} \pm \alpha_i' \begin{pmatrix} \rho \exp(\frac{\phi}{RT}) \\ \rho u \exp(\frac{\phi}{RT}) \\ E \exp(\frac{\phi}{RT}) \end{pmatrix}\right], \tag{3.12}$$

and the source term splitting (3.10) replaced by

$$\begin{pmatrix} 0 \\ \exp(-\frac{\phi}{RT}) \\ \exp(-\frac{\phi}{RT}) \end{pmatrix}_x = \frac{1}{2}\begin{pmatrix} 0 \\ \exp(-\frac{\phi}{RT}) \\ \exp(-\frac{\phi}{RT}) \end{pmatrix}_x + \frac{1}{2}\begin{pmatrix} 0 \\ \exp(-\frac{\phi}{RT}) \\ \exp(-\frac{\phi}{RT}) \end{pmatrix}_x , \tag{3.13}$$

we can show that the resulting WENO method is both high order accurate and well-balanced.

Total variation diminishing (TVD) high order Runge-Kutta time discretization [16] is used in practice for stability and to increase temporal accuracy. For example, the third order TVD Runge-Kutta method is used in the simulation in this paper:

$$\begin{aligned}
U^{(1)} &= U^n + \Delta t \mathcal{F}(U^n) \\
U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}\left(U^{(1)} + \Delta t \mathcal{F}(U^{(1)})\right) \\
U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}\left(U^{(2)} + \Delta t \mathcal{F}(U^{(2)})\right),
\end{aligned} \tag{3.14}$$

where $\mathcal{F}(U)$ is the spatial operator.

4 Two-dimensional extension

In this section, we construct a well-balanced WENO scheme on rectangular meshes to solve the two-dimensional gas dynamic equations with the gravitational field, which take the form

$$\begin{aligned}
&\rho_t + (\rho u)_x + (\rho v)_y = 0 \\
&(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho\phi_x \\
&(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho\phi_y \\
&E_t + ((E+p)u)_x + ((E+p)v)_y = -\rho u\phi_x - \rho v\phi_y,
\end{aligned} \tag{4.1}$$

where (u, v) is the velocity of the fluid, and ρ, p, $\phi(x, y)$ follow the definitions below (2.1). $E = \frac{1}{2}\rho(u^2 + v^2) + p/(\gamma - 1)$ is the non-gravitational energy. The hydrostatic balance we are interested to preserve is the constant temperature and zero velocity steady state solution:

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \qquad u = v = 0, \qquad p = RT\rho = RT\rho_0 \exp\left(-\frac{\phi}{RT}\right), \tag{4.2}$$

and the steady state solution corresponding to the linear gravitational potential field: $d\phi/dx = g_1$ and $d\phi/dy = g_2$ takes the form

$$\rho = \rho_0 \exp(-\rho_0(g_1 x + g_2 y)/p_0), \qquad u = v = 0, \qquad p = p_0 \exp(-\rho_0(g_1 x + g_2 y)/p_0). \tag{4.3}$$

It is straightforward to extend the high order finite difference WENO scheme to multiple space dimensions, by simply approximating each spatial derivative along the relevant coordinate. This is one of its major advantages over the finite volume method. The conservative approximation to the derivative from point values is as simple in multi-dimensions as in one dimension. In fact, for fixed j, if we take $W(x) = f(U(x, y_j))$, then we only need to perform the one-dimensional WENO approximation to $W(x)$ to obtain an approximation to $W'(x_i) = f_x(U(x_i, y_j))$. See again [6, 14] for more details of finite difference WENO schemes in multi-dimensions.

Following the steps in designing a well-balanced WENO scheme for the one-dimensional gas dynamic equations in Section 3, we find out that it is also straightforward to extend

these results to two dimensions. For the steady state solution (4.2), we rewrite the source terms as

$$-\rho\phi_x = RT\rho\exp(\frac{\phi}{RT})\left(\exp(-\frac{\phi}{RT})\right)_x \tag{4.4}$$

$$-\rho\phi_y = RT\rho\exp(\frac{\phi}{RT})\left(\exp(-\frac{\phi}{RT})\right)_y \tag{4.5}$$

$$-\rho u\phi_x - \rho v\phi_y = RT\rho u\exp(\frac{\phi}{RT})\left(\exp(-\frac{\phi}{RT})\right)_x + RT\rho v\exp(\frac{\phi}{RT})\left(\exp(-\frac{\phi}{RT})\right)_y \tag{4.6}$$

in (4.1), and the one dimension procedure described in Section 3 is followed in each of the x and y directions. The residues are then summed up and the time discretization is still by a TVD Runge-Kutta method (3.14). All the desired properties proved in the one-dimensional case, such as high order accuracy and the well-balanced property, are still valid in the two dimensional case. We restricted our discussion in this section to two space dimensions only, although the algorithm can be easily designed for three dimension as well.

5 Numerical examples

In this section we present numerical results of our fifth order well-balanced finite difference WENO methods for the one- and two-dimensional gas dynamic equations with gravitational source terms. Time discretization is by the third order TVD Runge-Kutta time discretization (3.14). Unless otherwise specified, the CFL number is taken as 0.6.

5.1 Shock tube under gravitational field

The first test case is the standard Sod test, coupled with the gravitational field. Following the problem setup in [11], the computational domain is set as [0, 1], and the initial conditions are given by

$$\begin{aligned} &\rho = 1, && u = 0, && p = 1, && \text{if } x \le 0.5, \\ &\rho = 0.125, && u = 0, && p = 0.1, && \text{if } x \ge 0.5, \end{aligned}$$

The gravitational field ϕ takes a value of $g = \phi_x = 1$, and $\gamma = 1.4$. We compute this problem using our well-balanced finite difference WENO method with reflection boundary conditions and 100 uniform meshes. The solutions at time $t = 0.2$ are shown in Figure 5.1 for the density, velocity, pressure and the energy. We also plot a reference solution, computed by the traditional finite difference WENO method with much refined 2000 uniform meshes, in these figures to provide a comparison. Due to the gravitational force, the density distribution is pulling towards the left direction, and negative velocity appears in some regions. By comparing the results in these figures, we observe that this problem is well solved by the proposed numerical method on the relatively coarse mesh of 100 mesh points.

5.2 One dimensional isothermal equilibrium solution

This test case was first proposed by LeVeque and Bale [8] and later used in [18, 11], to demonstrate the capability of the proposed scheme for computations on the small perturbation of a steady state. The computational domain is set as [0,1]. We consider an ideal gas with $\gamma = 1.4$ and the linear gravitational field $\phi_x = g$, which stays at an isothermal equilibrium solution in the form of (2.6),

$$\rho_0(x) = p_0(x) = \exp(-x), \qquad \text{and} \qquad u_0(x) = 0. \tag{5.1}$$

The gravitational force, with $g = \phi_x = 1$, acts in the negative x direction.

5.2.1 Well-balanced test

We first show an example to demonstrate the well-balanced property of the scheme. The initial condition is taken as the steady state solution (5.1) which should be exactly preserved. We compute the solution until $t = 2$ using both 40 and 200 uniform mesh points. In order to demonstrate that the steady state is indeed maintained up to round-off error, we use single precision and double precision to perform the computation, and show the L^1 errors of ρ, ρu and E in Table 5.1. We can clearly see that the errors are at the level of round-off errors for different precisions, which verifies the well-balanced property.

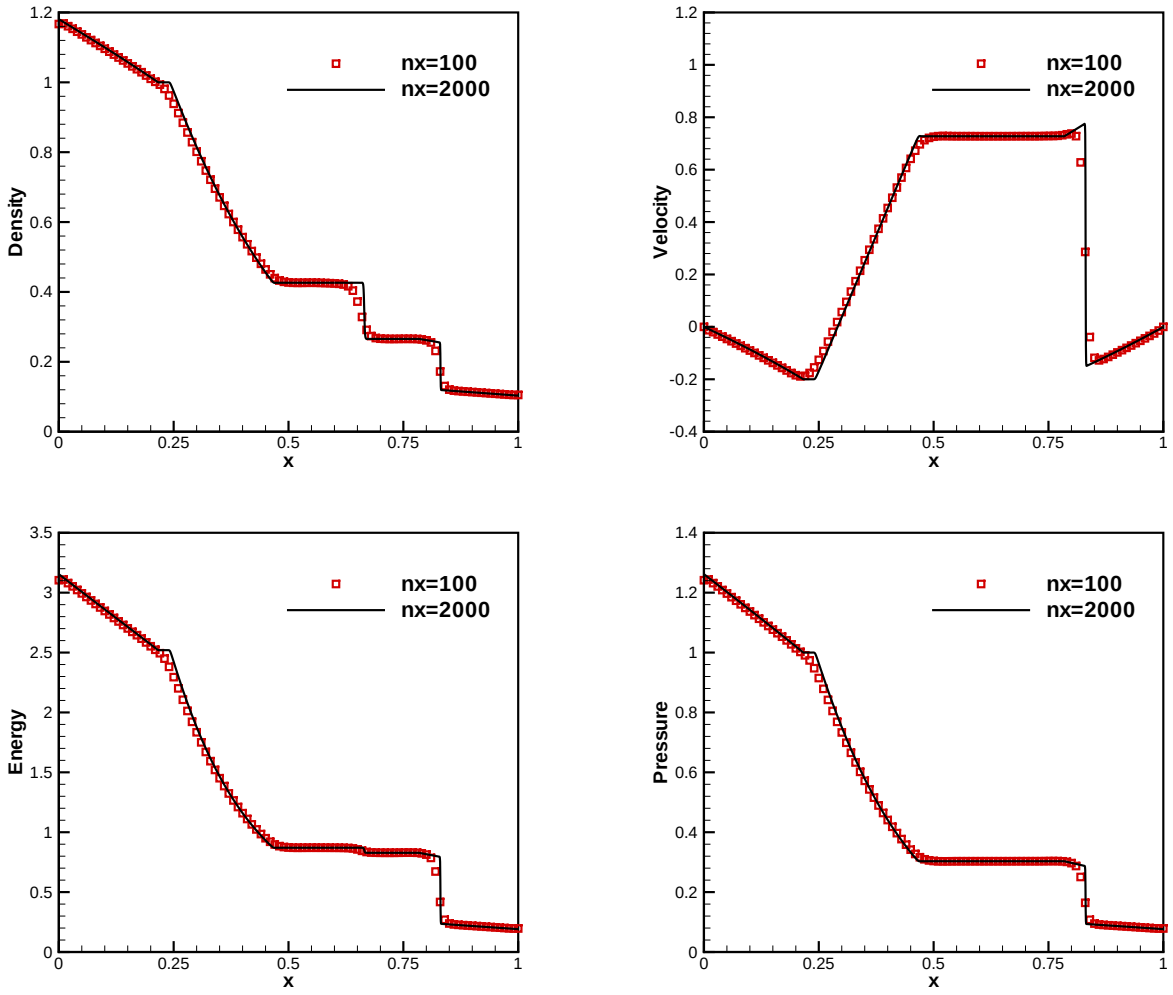

Figure 5.1: The numerical solutions of the Sod test under gravitational field in Section 5.1 at time $t = 0.2$. Top left: density distribution; Top right: velocity distribution; Bottom left: energy distribution; Bottom right: pressure distribution.

Table 5.1: L^1 error for different precisions for the steady state solution (5.1) in Section 5.2.

N	precision	ρ	ρu	E
40	single	2.57E-08	1.64E-08	1.01E-07
	double	4.63E-16	8.79E-17	4.46E-16
200	single	2.65E-07	8.32E-08	1.37E-07
	double	1.15E-15	1.43E-16	5.05E-16

5.2.2 Perturbation of the equilibrium solution

We now impose a small perturbation to the initial pressure state

$$p(x, t = 0) = p_0(x) + \eta \exp(-100(x - 0.5)^2), \tag{5.2}$$

where η is a non-zero perturbation constant. Two cases have been run: $\eta = 0.01$ and $\eta = 0.0001$. For small η, this disturbance should split into two waves, propagating left and right. We compute the solution until $t = 0.25$ with 200 grid points and simple transmissive boundary conditions. The results are shown in Figure 5.2. The initial pressure perturbation is included as the dashed line. For the purpose of comparison, we also plot a reference solution, obtained with 2000 points. The traditional non-well-balanced WENO method, with the straightforward calculation of the source terms, has been tested for these two cases, and their results are also included in Figure 5.2. We notice that, the traditional WENO methods are able to capture the big perturbation well, but do not perform well for the small perturbation test case with a coarse mesh of 200 mesh points. With the well-balanced technique, these small perturbations are well captured on this coarse mesh.

5.3 One dimensional gas falling into a fixed external potential

We consider the same gas dynamic equations with static gravitational potential (2.1) and a constant temperature and zero velocity steady state solution (2.5). The following steady state is taken from the paper by Slyz and Prendergast [17], and has also been considered in [18, 11].

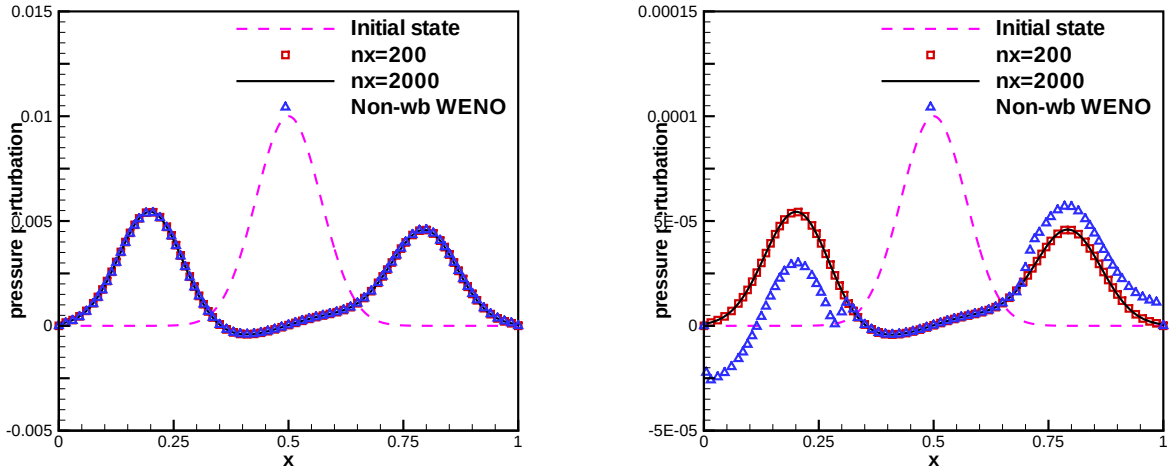

Figure 5.2: The pressure perturbation of a hydrostatic solution. The result of the well-balanced WENO method with 200 and 2000 grid points, and that of the non-well-balanced (denoted by non-wb) WENO method with 200 grid points. Left: $\eta = 0.01$ in (5.2); Right: $\eta = 0.0001$.

The gravitational potential has the form of a sine wave,

$$\phi(x) = -\phi_0 \frac{L}{2\pi} \sin \frac{2\pi x}{L}, \tag{5.3}$$

where L is the computational domain length and ϕ_0 is the amplitude. The steady state takes the form of

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \qquad u = 0, \qquad p = RT\rho_0 \exp\left(-\frac{\phi}{RT}\right), \tag{5.4}$$

with a constant temperature T.

5.3.1 Well-balanced test

We start with a test to verify the well-balanced property. Consider an ideal gas with $\gamma = 5/3$, and the parameters $\rho_0 = 1$, $R = 1$, $T = 0.6866$, $L = 64$, $\phi_0 = 0.02$ in (5.3) - (5.4). The initial condition is taken as the steady state (5.4), and we compute the solution until $t = 50$ using both 40 and 200 uniform mesh points. In order to demonstrate that the steady state is maintained up to round-off error, we use single precision and double precision to perform the computation, and show the L^1 errors of ρ, ρu and E in Table 5.2. We can easily observe that the errors are at the level of round-off errors for different precisions.

Table 5.2: L^1 error for different precisions for the steady state solution (5.4) in Section 5.3.

N	precision	ρ	ρu	E
40	single	2.03E-08	4.21E-08	3.63E-08
	double	3.76E-16	1.23E-16	2.33E-16
200	single	1.52E-07	9.63E-08	1.79E-07
	double	1.24E-15	1.93E-16	3.94E-16

5.3.2 Convergence test

In this test, we start from an initial state, which is a small perturbation of the steady state (5.4), and will eventually converge to an isothermal hydrostatic distribution after long time. The initial condition is given by

$$\rho = \rho_0 \exp\left(-\frac{\phi}{RT}\right), \qquad u = 0, \qquad p = RT\rho_0 \exp\left(-\frac{\phi}{RT}\right) + 0.001\exp(-10(x-32)^2), \tag{5.5}$$

where the parameters R, T et al. are given in Subsection 5.3.1. We run the simulation with 64 uniform mesh points for 1,000,000 time steps, and plot the numerical results at the final time in Figure 5.3. As a comparison, we also include the numerical results of the traditional non-well-balanced WENO method, where the constant temperature distribution of the steady state solution is not well-captured on this coarse mesh.

5.4 Testing the orders of accuracy

In this example we check the numerical orders of accuracy when the schemes are applied to the following two dimensional problem. Consider the equations (4.1) with a linear gravitational field $\phi_x = \phi_y = 1$. An exact solution takes the form of

$$\begin{aligned}
&\rho(x,y,t) = 1 + 0.2\sin(\pi(x+y-t(u_0+v_0))),\\
&u(x,y,t) = u_0, \qquad\qquad v(x,y,t) = v_0,\\
&p(x,y,t) = p_0 + t(u_0+v_0) - x - y + 0.2\cos(\pi(x+y-t(u_0+v_0)))/\pi,
\end{aligned}$$

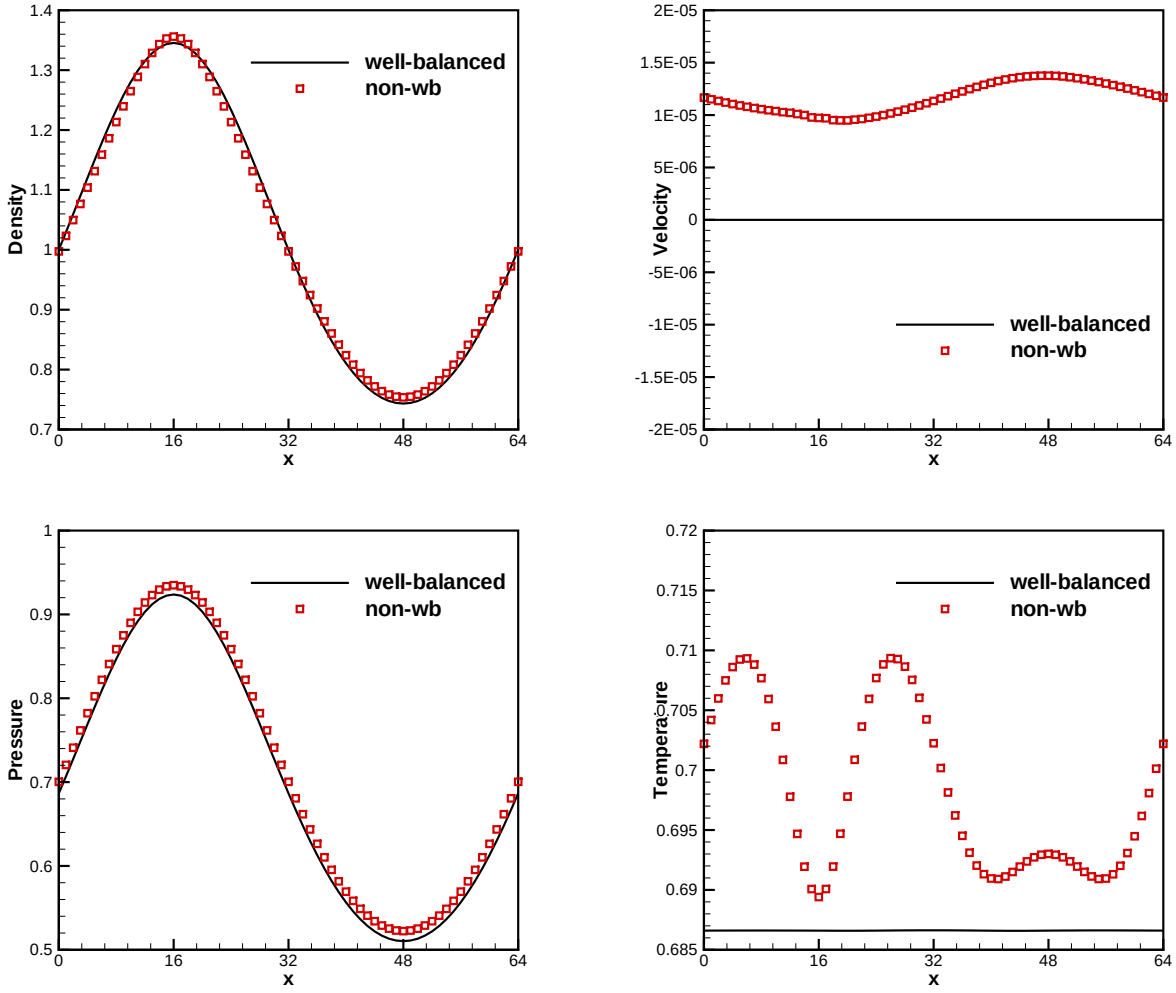

Figure 5.3: The numerical solutions of well-balanced method (solid line) and non-well-balanced method (square box, denoted by non-wb) for the convergence test in Section 5.3.2 after 1,000,000 time steps. Top left: density distribution; Top right: velocity distribution; Bottom left: pressure distribution; Bottom right: temperature distribution.

in the domain $[0, 2] \times [0, 2]$. $u_0 = v_0 = 1$ and $p_0 = 4.5$ are chosen in this test. The CFL condition is taken as 0.2 and the time step Δt is taken to be proportional to $(1/\Delta x + 1/\Delta y)^{-5/3}$, so that the temporal accuracy is of the same order as the spatial accuracy. The exact solutions are used as the boundary condition. We run the simulation until the stop time $t = 0.1$ and compute the numerical error. Table 5.3 contains the L^1 errors and orders of accuracy. We can clearly see that, in this two dimensional test case, the designed high order accuracy is achieved for the finite difference WENO scheme.

Table 5.3: L^1 errors and numerical orders of accuracy for the example in Section 5.4.

Number of cells	ρ		ρu		ρv		E	
	L^1 error	order	L^1 error	order	L^1 error	order	L^1 error	order
8×8	3.88E-03		3.66E-03		3.66E-03		4.83E-03	
16×16	5.06E-04	2.93	4.85E-04	2.91	4.85E-04	2.91	5.46E-04	3.14
32×32	2.75E-05	4.20	2.72E-05	4.15	2.72E-05	4.15	3.02E-05	4.17
64×64	1.15E-06	4.57	1.15E-06	4.56	1.15E-06	4.56	1.30E-06	4.53
128×128	4.33E-08	4.74	4.29E-08	4.73	4.29E-08	4.73	5.10E-08	4.68
256×256	1.54E-09	4.81	1.57E-09	4.78	1.57E-09	4.78	1.81E-09	4.81

5.5 Two dimensional isothermal equilibrium solution

The following test is used to demonstrate the well-balanced property and the capability of the proposed methods for capturing the small perturbation of an isothermal equilibrium solution in the two dimensional case. We set the computational domain as the unit square. Consider an ideal gas with $\gamma = 1.4$ and the linear gravitational field $\phi_x = \phi_y = g$. The isothermal equilibrium state under consideration takes the form of

$$\begin{aligned} \rho(x, y) &= \rho_0 \exp\left(-\frac{\rho_0 g}{p_0}(x + y)\right), \qquad u(x, y) = v(x, y) = 0, \\ p(x, y) &= p_0 \exp\left(-\frac{\rho_0 g}{p_0}(x + y)\right), \end{aligned} \tag{5.6}$$

with the parameters $\rho_0 = 1.21$, $p_0 = 1$ and $g = 1$.

5.5.1 Well-balanced test

The two dimensional well-balanced property is first tested. We take the initial condition as the steady state solution (5.6), and compute the solution until $t = 1$ using 50×50 uniform grid points. In order to demonstrate that the steady state is indeed maintained up to round-off error, we use single precision and double precision to perform the computation, and show the L^1 errors of ρ, ρu, ρv and E in Table 5.4. We can clearly see that the errors are at the level of round-off errors for different precisions, which verifies the well-balanced property.

Table 5.4: L^1 error for different precisions for the steady state solution (5.6) in Section 5.5.1.

precision	ρ	ρu	ρv	E
single	1.72E-08	3.31E-08	3.38E-08	3.42E-08
double	1.15E-16	9.46E-17	9.46E-17	1.38E-16

5.5.2 Perturbation of the equilibrium solution

Next, we study the capability of the proposed method for the small perturbation to the stationary state. Consider a small perturbation to the initial pressure state of (5.6),

$$p(x, y, t = 0) = p_0 \exp\left(-\frac{\rho_0 g}{p_0}(x + y)\right) + \eta \exp\left(-\frac{100\rho_0 g}{p_0}((x - 0.3)^2 + (y - 0.3)^2)\right), \quad (5.7)$$

centered at (0.3, 0.3), where η is a non-zero perturbation constant, and set as 0.001 in this test.

We compute the solution until $t = 0.15$ with 50×50 grid points and simple transmissive boundary conditions. The contour plot of the pressure perturbation is shown in the left plots of Figures 5.4 and 5.5. We also run this test with the traditional non-well-balanced WENO method, with the straightforward calculation of the source terms. Its result is shown in the right plots of Figures 5.4 and 5.5. The density perturbations are also shown in Figure 5.6. We notice that, the non-well-balanced WENO method is not capable of capturing such small perturbation on the coarse mesh, while the well-balanced one can resolve it very well.

We also run both methods with a refined 200×200 mesh, and the results are shown in Figure 5.7. As expected, the results of non-well-balanced WENO schemes improve, and the difference between well-balanced and non-well-balanced methods becomes smaller as the mesh is refined.

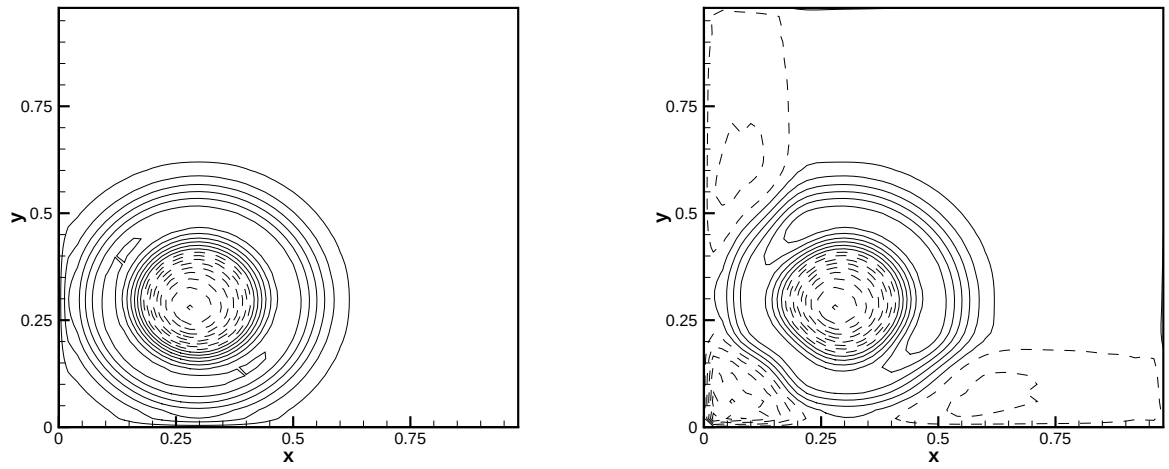

Figure 5.4: The contours of the pressure perturbation of a two dimensional hydrostatic solution in Section 5.5.2 at time $t = 0.15$ with 50×50 grid points. 20 uniformly spaced contour lines from -0.0003 to 0.0003. Left: results based on well-balanced scheme. Right: results based on non-well-balanced scheme.

6 Concluding remarks

In this paper we have constructed a well-balanced finite difference WENO scheme of arbitrary order of accuracy for the gas dynamics equations with a static gravitational field. Special source term discretization is introduced such that the resulting WENO scheme balances the zero-velocity and constant temperature steady state solutions to machine accuracy, and at the same time maintains the original high order accuracy and essentially nonoscillatory property for general solutions. Near the steady state solution, the well-balanced scheme is expected to generate accurate results, while far from the steady state it behaves similarly to the traditional WENO scheme. The main purpose of the well-balanced schemes designed in this paper is to well resolve small perturbations of the hydrostatic balance state

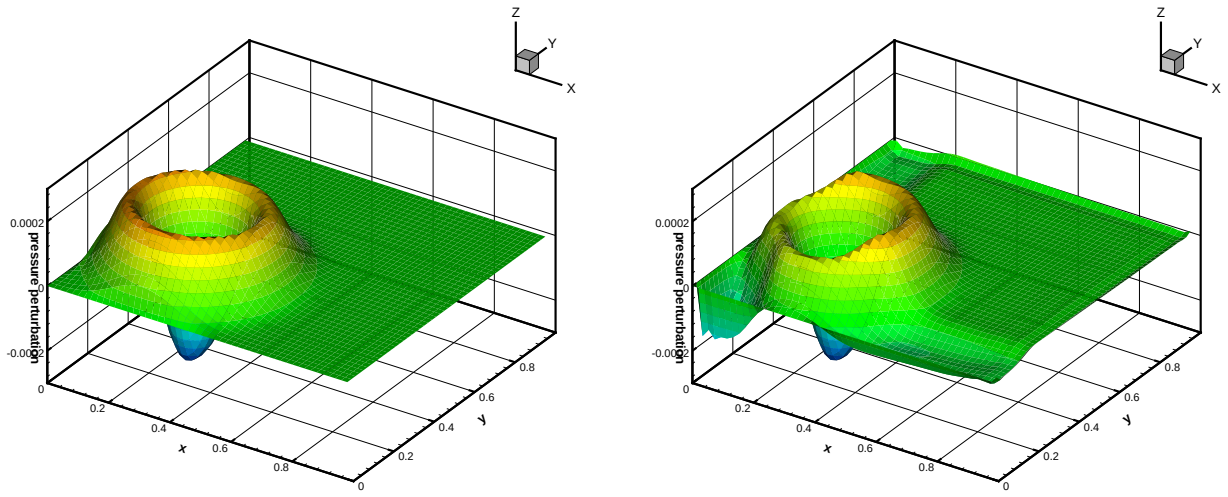

Figure 5.5: The 3D figure of the pressure perturbation of a two dimensional hydrostatic solution in Section 5.5.2 at time $t = 0.15$ with 50×50 grid points. Left: results based on well-balanced scheme. Right: results based on non-well-balanced scheme.

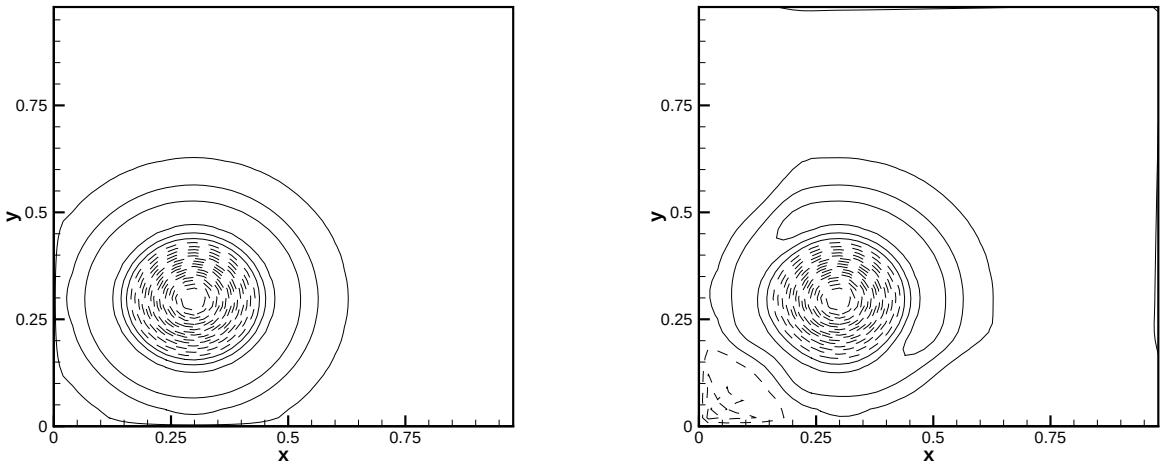

Figure 5.6: The contours of the density perturbation of a two dimensional hydrostatic solution in Section 5.5.2 at time $t = 0.15$ with 50×50 grid points. 20 uniformly spaced contour lines from -0.001 to 0.0002. Left: results based on well-balanced scheme. Right: results based on non-well-balanced scheme.

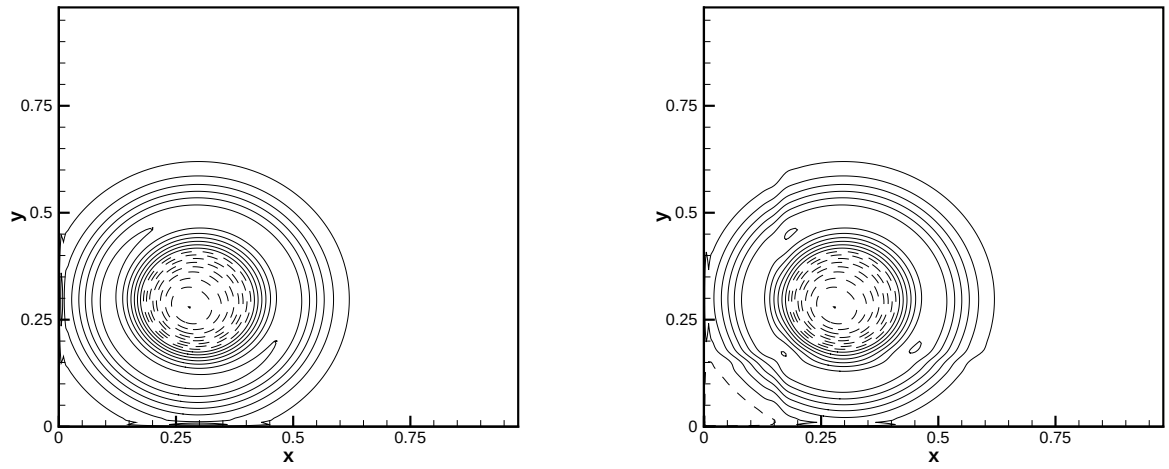

Figure 5.7: The contours of the pressure perturbation of a two dimensional hydrostatic solution in Section 5.5.2 at time $t = 0.15$ with 200×200 grid points. 20 uniformly spaced contour lines from -0.0003 to 0.0003. Left: results based on well-balanced scheme. Right: results based on non-well-balanced scheme.

on coarse meshes. The more difficult issue of convergence towards such hydrostatic balance state from an arbitrary initial condition is not addressed. Numerical examples are given to demonstrate the well-balanced property, accuracy, good capturing of the small perturbation to the steady-state solutions, and non-oscillatory shock resolution of the proposed numerical method. Ongoing work includes constructing well-balanced methods for more general steady state solutions. The main purpose of the well-balanced schemes designed in this paper is to well resolve small perturbations of the hydrostatic balance state on coarse meshes. The more difficult issue of convergence towards such hydrostatic balance state from an arbitrary initial condition is not addressed in this paper.

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