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SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHODS FOR SCALAR NONLINEAR CONSERVATION LAWS IN ONE SPACE DIMENSION*

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Abstract. In this paper, an analysis of the superconvergence property of the semidiscrete discontinuous Galerkin (DG) method applied to one-dimensional time-dependent nonlinear scalar conservation laws is carried out. We prove that the error between the DG solution and a particular projection of the exact solution achieves $\left(k+\frac{3}{2}\right)$th order superconvergence when upwind fluxes are used. The results hold true for arbitrary nonuniform regular meshes and for piecewise polynomials of degree k ($k \geq 1$), under the condition that $|f'(u)|$ possesses a uniform positive lower bound. Numerical experiments are provided to show that the superconvergence property actually holds true for nonlinear conservation laws with general flux functions, indicating that the restriction on $f(u)$ is artificial.

Key words. discontinuous Galerkin method, superconvergence, upwind flux, error estimates

AMS subject classifications. 65M60, 65N12

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1. Introduction. In this paper, we investigate the superconvergence of the semidiscrete discontinuous Galerkin (DG) method applied to one-dimensional scalar conservation laws

$$u_t + f(u)_x = g(x,t), \tag{1.1a}$$

$$u(x,0) = u_0(x), \tag{1.1b}$$

where $g(x,t)$ and $u_0(x)$ are smooth functions. We assume that the nonlinear flux function $f(u)$ is sufficiently smooth with respect to the variable u, for example, $f \in C^3$ is enough. For the sake of simplicity and easy presentation, we only consider the periodic or compactly supported boundary conditions. We show the superconvergence property of the DG solutions toward a particular projection of the exact solution when the upwind fluxes are used.

We would like to mention related theoretical results including stability analysis and error estimates of the DG methods for conservation laws. For smooth solutions of linear conservation laws, optimal a priori error estimates of order $k+1$ for one-dimensional and some multidimensional cases [13, 16, 8] and a suboptimal L^2 error estimate of order $k+\frac{1}{2}$ for arbitrary meshes [12] are obtained for steady-state solution or for the space-time DG discretization. Here and in what follows, k is the polynomial degree of the finite element space. The results in [12] are later proved to be sharp

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for the most general situation [15]. For nonsmooth solutions of nonlinear conservation laws, Jiang and Shu [11] proved a cell entropy inequality for the semidiscrete DG method as well as for some fully discrete DG scheme with implicit time discretizations such as the backward Euler and Crank–Nicholson algorithms. This trivially implies the nonlinear L^2 stability of the method even without limiters. More recently, the L^2 stability of the fully discrete Runge–Kutta DG (RKDG) method with explicit third order total variation diminishing (TVD) Runge–Kutta time-marching method is given for the linear time-dependent conservation laws with possibly discontinuous solutions [18]. Also in [17, 18], suboptimal a priori error estimates for general monotone numerical fluxes and optimal error estimates for upwind numerical fluxes are obtained for the RKDG schemes to sufficiently smooth solutions of nonlinear conservation laws.

Let us now mention in particular the superconvergence results of the DG methods in the literature. In [1, 2], Adjerid and others showed that the DG solution is superconvergent at Radau points for solving ordinary differential equations and steady-state nonlinear hyperbolic problems. In [5], Cheng and Shu proved superconvergence of order $k+\frac{3}{2}$ of the DG solution toward a particular projection of the exact solution for linear conservation law when upwind numerical fluxes are used in the framework of Fourier analysis. The proof in [5] works only for piecewise linear polynomials on uniform meshes with periodic boundary conditions. The results were later improved, using a different technique, in [6] for arbitrary nonuniform regular meshes and schemes of any order. The objective of this paper is to study the superconvergence property of the DG method for time-dependent nonlinear scalar conservation laws, extending the results in [6] for linear problems. Superconvergence of order $k+\frac{3}{2}$ is proved for smooth solutions of nonlinear conservation laws when upwind numerical fluxes are used, under the condition that $|f'(u)|$ has a uniform positive lower bound, i.e., either $f'(u(x,t)) \geq \delta > 0$ or $f'(u(x,t)) \leq -\delta < 0$ for all $(x,t) \in I \times [0,T]$. Let us emphasize that this restriction is artificial due to the limitation of the technical proof; the superconvergence property actually holds true for nonlinear conservation laws with general flux functions; see numerical results in section 4 below and also in [5]. As far as we know, this is the first superconvergence proof for DG methods applied to time-dependent nonlinear hyperbolic equations. The generalization from linear problems in [6] to the nonlinear case in this paper involves several technical difficulties. For example, one of the most essential points is how to obtain a sharp bound for the time derivative of the error, e_t; see Appendix A.2 for a detailed proof. The main tool employed in this paper is an energy analysis. To deal with the nonlinearity of the flux, Taylor expansion and an a priori assumption about the numerical solution are used.

Besides the theoretical interest in knowing the faster convergence rate demonstrated by the superconvergence result, the superconvergence result can also be used to provide good error indicators which are useful for adaptive computation, although this aspect is not pursued in this paper. Another useful consequence of the superconvergence property with linear growth in time is that it implies the error of the DG scheme will not grow for fine grids over a long time periods; see [5] and [14] for this excellent long time behavior of the error.

This paper is organized as follows. In section 2, we review the DG scheme for conservation laws. In section 3, we present some preliminaries about the discontinuous finite element space, state the main results, and then display the main proofs. In section 4, various numerical experiments are shown to verify the theoretical results. Concluding remarks and comments on future work are given in section 5. Finally, in the appendix we provide the proofs for some of the more technical lemmas.

2. DG scheme. In this section, we follow [9] and review the DG scheme for the problem (1.1).

The usual notation of the DG method is adopted here. Let us start by assuming the following mesh to cover the interval $I = (0, 2\pi)$, consisting of cells $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, for $1 \le j \le N$, where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

The cell center and cell length, respectively, are denoted by $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. We use $h = \max_j h_j$ and $\rho = \min_j h_j$ to denote the lengths of the largest cell and the smallest cell, respectively. The mesh is assumed to be regular in the sense that the ratio h/ρ is always bounded during mesh refinements, namely, there exists a positive constant γ such that $\gamma h \le \rho \le h$. We denote by $p^-_{j+\frac{1}{2}}$ and $p^+_{j+\frac{1}{2}}$ the values of p at the discontinuity point $x_{j+\frac{1}{2}}$, from the left cell, I_j, and from the right cell, I_{j+1}, respectively. Moreover, we use $[\![p]\!] = p^+ - p^-$ and $\{\!\{p\}\!\} = \frac{1}{2}(p^+ + p^-)$ to represent the jump and the mean value of p at each element boundary point. The following piecewise polynomials space is chosen as the finite element space:

$$V_h \equiv V_h^k = \left\{ v \in L^2(I) : v|_{I_j} \in P^k(I_j), \quad j = 1, \ldots, N \right\},$$

where $P^k(I_j)$ denotes the set of polynomials of degree up to $k \ge 1$ defined on the cell I_j. Note that functions in V_h are allowed to have discontinuities across element interfaces. Then the approximation of the semidiscrete DG method for solving (1.1) becomes: find, for any time $t \in (0, T]$, the unique function $u_h = u_h(t) \in V_h$ such that

$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} f(u_h)(v_h)_x dx + \hat{f}_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = \int_{I_j} g(x,t) v_h dx \tag{2.1}$$

holds for all $v_h \in V_h$ and all $j = 1, \ldots, N$. Here, in order to achieve the superconvergence property, the numerical flux $\hat{f}_{j+\frac{1}{2}}$ is chosen to be an upwind flux defined on each element boundary point, which includes the well-known Godunov flux, the Engquist–Osher flux, and the Roe flux with an entropy fix.

3. The main results. Before we state the main results, let us begin by introducing the following notation, definitions, and some auxiliary results which will be used later in the proof of the superconvergence property.

3.1. Preliminaries.

3.1.1. Notation for different constants. We denote by C (possibly accompanied by lower or upper indices) a positive constant independent of h that may depend on the exact solution of (1.1) and the polynomial degree k, which could have a different value in each occurrence. To emphasize the nonlinearity of the flux $f(u)$, we employ $C_\star$ to denote a nonnegative constant depending solely on the maximum of $|f''|$ or/and $|f'''|$. We remark that $C_\star = 0$ for a linear flux $f(u) = cu$, where c is a constant.

We assume that the exact solution u is smooth enough, for example, $\|u\|_{k+1}$, $\|u_t\|_{k+1}$ and $\|u_{tt}\|_{k+1}$ are bounded uniformly for any time $t \in [0,\ T]$. Here and from now on, $\|\cdot\|_{k+1}$ represents the standard Sobolev $(k+1)$ norm. Suppose that the initial condition $u_0(x)$ lies in $[m_0, M_0]$; then it follows from the maximum principle that the exact solution is also in this range. Assuming that the flux function satisfies $|f'(u)| \ge \delta$ uniformly on the interval $[m_0, M_0]$ with δ being a positive constant, we then follow [17] to modify the flux f such that $|f'(u)| \geq \frac{1}{2}\delta$ on $[m_0-1, M_0+1]$, and for simplicity, we still denote this lower bound as δ. We also assume that the derivatives of $f(u)$ with respect to u up to third order are bounded on $[m_0-1, M_0+1]$.

3.1.2. Functionals related to the L^2 norm. To get the superconvergence property of the method, two functionals related to the L^2 norm of a function $\mathbb{F}(x)$ on I_j are needed as defined in [6]:

$$\mathcal{B}_j^-(\mathbb{F}) = \int_{I_j} \mathbb{F}(x)\frac{x-x_{j-1/2}}{h_j}\frac{d}{dx}\left(\mathbb{F}(x)\frac{x-x_j}{h_j}\right)dx,$$
$$\mathcal{B}_j^+(\mathbb{F}) = \int_{I_j} \mathbb{F}(x)\frac{x-x_{j+1/2}}{h_j}\frac{d}{dx}\left(\mathbb{F}(x)\frac{x-x_j}{h_j}\right)dx.$$

The functionals defined above have the following properties, which are essential to the proof of the superconvergence.

LEMMA 3.1 (see [6]). *For any function $\mathbb{F}(x) \in C^1$ on I_j, we have*

$$\mathcal{B}_j^-(\mathbb{F}) = \frac{1}{4h_j}\int_{I_j}\mathbb{F}^2(x)dx + \frac{\mathbb{F}^2(x_{j+1/2})}{4}, \tag{3.1}$$

$$\mathcal{B}_j^+(\mathbb{F}) = -\frac{1}{4h_j}\int_{I_j}\mathbb{F}^2(x)dx - \frac{\mathbb{F}^2(x_{j-1/2})}{4}. \tag{3.2}$$

The proof of this lemma is straightforward; see [6].

3.1.3. Projections and interpolation properties. We start by introducing the standard L^2 projection of a function $q \in L^2(I)$ into the finite element space V_h, denoted by $P_h q$, which is the unique function in V_h satisfying for each j,

$$\int_{I_j}(P_h q(x) - q(x))v_h dx = 0 \quad \forall v_h \in V_h.$$

Next, we recall two kinds of Gauss–Radau projections $P_h^{\pm}$ into V_h, which were first introduced by Castillo et al. in [3] in their study of the local DG (LDG) method for time-dependent convection-diffusion problems. For any given function q in the broken Sobolev space $H^{1,h}(I_h) = \{v \in L^2(I) : v|_{I_j} \in H^1(I_j), \ \ j = 1,\dots,N\}$ with I_h being the union of all cells and an arbitrary element $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, the special projection of q, denoted by $P_h^{\pm}q$, is the unique function in V_h satisfying, for each j,

$$\int_{I_j}\left(P_h^+ q(x) - q(x)\right)v_h dx = 0 \quad \forall v_h \in P^{k-1}(I_j), \quad (P_h^+ q)^+_{j-\frac{1}{2}} = q\left(x^+_{j-\frac{1}{2}}\right); \tag{3.3}$$

$$\int_{I_j}\left(P_h^- q(x) - q(x)\right)v_h dx = 0 \quad \forall v_h \in P^{k-1}(I_j), \quad (P_h^- q)^-_{j+\frac{1}{2}} = q\left(x^-_{j+\frac{1}{2}}\right). \tag{3.4}$$

Note that these two special projections have been used to derive optimal error estimates of the DG methods in the literature, for example, in [17, 18]. We would like to mention that the exact collocation at one of the boundary points on each cell plus the orthogonality property for polynomials of degree up to $k-1$ of the Gauss–Radau projections $P_h^{\pm}$ play an important role and are used repeatedly in the proof of the superconvergence property. We denote by $\eta = q(x) - \mathbb{Q}_h q(x)$ ($\mathbb{Q}_h = P_h$, or

$P_h^{\pm}$) the projection error; then by a standard scaling argument [7] together with trace inequality, it is easy to obtain, for smooth enough $q(x)$, that

$$\|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \le Ch^{k+1}, \tag{3.5a}$$

where C is a positive constant depending solely on q and k but independent of h. In particular, in (3.5a), $C = C'\|q\|_{k+1}$, where $\|q\|_{k+1}$ is the standard Sobolev $(k+1)$ norm and C' is a constant independent of q. Here and below, an unmarked norm $\|\cdot\|$ is the usual L^2 norm defined on the interval I, and $\|\cdot\|_{\Gamma_h}$ denotes the L^2 norm defined on the cell interfaces of the mesh. For example, for the one-dimensional case under consideration in this paper, $\|\eta\|_{\Gamma_h}^2 = \sum_{j=1}^{N}((\eta_{j+1/2}^+)^2 + (\eta_{j+1/2}^-)^2)$. Moreover, the Sobolev's inequality implies that

$$\|\eta\|_\infty \le Ch^{k+\frac{1}{2}} \tag{3.5b}$$

with the positive constant C independent of h. This inequality is important for us to make an a priori assumption to be used in our analysis; see section 3.3 below.

3.1.4. Inverse properties. Finally, we list some inverse properties of the finite element space V_h. For any $p_h \in V_h$, there exists a positive constant C independent of p_h and h such that

(i) $\|\partial_x p_h\| \le Ch^{-1}\|p_h\|$; (ii) $\|p_h\|_{\Gamma_h} \le Ch^{-1/2}\|p_h\|$; (iii) $\|p_h\|_\infty \le Ch^{-1/2}\|p_h\|$.

Here and below, $\partial_x(\cdot)$ denotes the partial derivative of a function with respect to the variable x, likewise for $\partial_t(\cdot)$. For more details of these inverse properties, see [7].

3.2. The main results. Let us start by denoting $e = u - u_h$ to be the error between the exact solution and the DG solution. Next, we split it into two parts; one is the projection error, denoted by $\eta = u - \mathbb{Q}_h u$, and the other is the part belonging to the finite element space V_h, denoted by $\xi = \mathbb{Q}_h u - u_h$, which is proved to be $\left(k+\frac{3}{2}\right)$th order superconvergent as shown in the following theorem. Here the projection $\mathbb{Q}_h$ is defined at each time level t corresponding to the sign variation of $f'(u)$; more specifically, for any $t \in [0,T]$ and $x \in I$, if $f'(u(x,t)) > 0$, we choose $\mathbb{Q}_h = P_h^-$, and if $f'(u(x,t)) < 0$, we take $\mathbb{Q}_h = P_h^+$.

We are now ready to state the main theorem.

THEOREM 3.2. *Let u be the exact solution of the problem* (1.1)*, which is assumed to be sufficiently smooth with bounded derivatives, i.e., $\|u\|_{k+1}$, $\|u_t\|_{k+1}$ and $\|u_{tt}\|_{k+1}$ are bounded uniformly for any time $t \in [0,\ T]$. We further assume that $f \in C^3$ and that $|f'(u)|$ is lower bounded uniformly by a positive constant. Let u_h be the numerical solution of scheme* (2.1) *with the initial condition $u_h(\cdot,0) = \mathbb{Q}_h u_0$ when upwind fluxes are used. For regular triangulations of $I = (0,2\pi)$, if the finite element space V_h^k of piecewise polynomials with arbitrary degree $k \ge 1$ is used, then for small enough h there holds the error estimate*

$$\|\xi(\cdot,t)\| \le Ch^{k+3/2} \quad \forall t \in [0,T], \tag{3.6}$$

where the positive constant C depends on the exact solution u, the polynomial degree k, the final time T, and the maximum of $|f^{(m)}|$ $(m = 1,2,3)$ but is independent of h.

3.3. Proof of the main results. Without loss of generality, we will only consider the case $f'(u(x,t)) \ge \delta > 0$ for all $(x,t) \in I \times [0,T]$; the case of $f'(u(x,t)) \le -\delta < 0$ is analogous. Therefore, we take the numerical flux as $\hat{f} = f(u_h^-)$ on each cell interface and choose the projection as $\mathbb{Q}_h = P_h^-$ on each cell element, and the initial condition is chosen as $u_h(\cdot, 0) = P_h^- u_0$. In this case the DG scheme (2.1) can be written as

(3.7)
$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} f(u_h)(v_h)_x dx + f(u_h^-)v_h^-|_{j+\frac{1}{2}} - f(u_h^-)v_h^+|_{j-\frac{1}{2}} = \int_{I_j} g(x,t) v_h dx,$$

which holds for any $v_h \in V_h$. Since the exact solution u also satisfies the weak formulation (3.7), we have the error equation

(3.8)
$$\int_{I_j} e_t v_h dx = \int_{I_j} (f(u) - f(u_h))(v_h)_x dx - (f(u) - f(u_h^-))v_h^-|_{j+\frac{1}{2}} + (f(u) - f(u_h^-))v_h^+|_{j-\frac{1}{2}}$$

for any $v_h \in V_h$. Summing the above error equation over j and using the periodic boundary conditions, we obtain

$$\int_I e_t v_h dx = \sum_{j=1}^N \int_{I_j} (f(u) - f(u_h))(v_h)_x dx + \sum_{j=1}^N \left((f(u) - f(u_h^-)) [\![v_h]\!] \right)_{j+\frac{1}{2}} \tag{3.9}$$

for all $v_h \in V_h$. We now take $v_h = \xi$ to obtain the identity

$$LHS = RHS, \tag{3.10}$$

where

$$LHS = \int_I e_t \xi dx, \tag{3.11a}$$

$$RHS = \sum_{j=1}^N \int_{I_j} (f(u) - f(u_h))\xi_x dx + \sum_{j=1}^N \left((f(u) - f(u_h^-)) [\![\xi]\!] \right)_{j+\frac{1}{2}}. \tag{3.11b}$$

Obviously,

$$LHS = \frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \int_I \eta_t \xi dx. \tag{3.12a}$$

To estimate RHS, let us define $\xi = r_j + \mathbb{S}_j(x)(x - x_j)/h_j$ on each cell I_j with $r_j = \xi(x_j)$ being a constant and $\mathbb{S}_j(x) \in P^{k-1}(I_j)$. We further define the piecewise polynomial $\mathbb{S}(x)$ whose restriction on I_j is $\mathbb{S}_j(x)$. The estimate of RHS is given in the following lemma.

Lemma 3.3. *Suppose that the interpolation property* (3.5a) *is satisfied. Then we have*

$$RHS \le (\mathcal{C}(e) + C_\star h^{-3}\|e\|_\infty^2)\|\xi\|^2 + C_\star h^{k+1}\|\mathbb{S}\| + Ch^{2k+3}, \tag{3.12b}$$

where $\mathcal{C}(e) = C + C_\star h^{-1}\|e\|_\infty$ *and the positive constants* C *and* $C_\star$ *are independent of* h *and the approximate solution* u_h.

Proof. We start by using the second order Taylor expansion with respect to the variable u to write out the nonlinear terms, namely, $f(u) - f(u_h)$ and $f(u) - f(u_h^-)$, as

$$f(u) - f(u_h) = f'(u)\xi + f'(u)\eta - \frac{1}{2}\tilde{f}_u''(\eta + \xi)^2 \triangleq \lambda_1 + \lambda_2 + \lambda_3, \tag{3.13a}$$

$$f(u) - f(u_h^-) = f'(u)\xi^- + f'(u)\eta^- - \frac{1}{2}\tilde{\tilde{f}}''_u(\eta^- + \xi^-)^2 \triangleq \theta_1 + \theta_2 + \theta_3, \tag{3.13b}$$

where $\tilde{f}''_u$ and $\tilde{\tilde{f}}''_u$ are the mean values given by $\tilde{f}''_u = f''(\alpha_1 u + (1-\alpha_1)u_h)$ and $\tilde{\tilde{f}}''_u = f''(\alpha_2 u + (1-\alpha_2)u_h^-)$ with $0 \le \alpha_1,\ \alpha_2 \le 1$. Note that we have dropped the subscript $j+1/2$ in (3.13b) for notational convenience, since all quantities are evaluated at the same points (i.e., the interfaces between the cells). Thus, the identity (3.11b) can be represented by

$$RHS = \sum_{i=1}^{3}(\Lambda_i + \Theta_i)$$

with Λ_i and Θ_i given by

$$\Lambda_i = \sum_{j=1}^{N}\int_{I_j}\lambda_i\xi_x dx \quad \text{and} \quad \Theta_i = \sum_{j=1}^{N}(\theta_i[\![\xi]\!])_{j+\frac{1}{2}} \quad (i=1,2,3),$$

which will be estimated separately later. A simple integration by parts gives us the estimate for $\Lambda_1 + \Theta_1$:

$$\begin{aligned}\Lambda_1 + \Theta_1 &= -\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}\partial_x f'(u)\xi^2 dx + \sum_{j=1}^{N} f'(u_{j+\frac{1}{2}})(\xi^- - \{\!\{\xi\}\!\})_{j+\frac{1}{2}}[\![\xi]\!]_{j+\frac{1}{2}}\\ &\le C_\star\|\xi\|^2 - \frac{1}{2}\sum_{j=1}^{N}\left|f'(u_{j+\frac{1}{2}})\right|[\![\xi]\!]^2_{j+\frac{1}{2}}\\ &\le C_\star\|\xi\|^2. \qquad (3.14a)\end{aligned}$$

Note that $\eta^-_{j+1/2} = 0$, due to the property of the projection P_h^- in (3.4). Thus we have

$$\Theta_2 = \sum_{j=1}^{N}\left(f'(u)\eta^-[\![\xi]\!]\right)_{j+\frac{1}{2}} = 0.$$

To estimate Λ_2, we rewrite the expression for $f'(u)$ as

$$f'(u) = f'(u_j) + (f'(u) - f'(u_j)),$$

where u_j denotes the value of the exact solution u at x_j. Note that η is orthogonal to any polynomials of degree up to $k-1$ by virtue of the property of the projection P_h^- in (3.4); therefore, Λ_2 can be represented by

$$\begin{aligned}\Lambda_2 &= \sum_{j=1}^{N}\int_{I_j}(f'(u) - f'(u_j))\eta\xi_x dx\\ &= \sum_{j=1}^{N}\int_{I_j}(f'(u) - f'(u_j))\eta\left(\mathbb{S}'_j(x)\frac{x-x_j}{h_j} + \frac{\mathbb{S}_j(x)}{h_j}\right)dx.\end{aligned}$$

We now define the piecewise polynomial $\phi(x)$ such that $\phi(x) = (x - x_j)/h_j$ on each I_j. Clearly $\|\phi\|_\infty = \frac{1}{2}$; then the Cauchy–Schwarz inequality together with the inverse property (i) yields that

$$\Lambda_2 \le C_\star\|\eta\|\|\mathbb{S}\| \le C_\star h^{k+1}\|\mathbb{S}\|,$$

where we have also used the interpolation property (3.5a) and the fact that $|f'(u) - f'(u_j)| \le C_\star h$ on each element I_j due to the smoothness of the exact solution u and f. Thus, we arrive at a bound for $\Lambda_2 + \Theta_2$,

$$\Lambda_2 + \Theta_2 \le C_\star h^{k+1} \|\mathbb{S}\|. \tag{3.14b}$$

It is easy to show that

$$\begin{aligned}\Lambda_3 + \Theta_3 &\le C_\star \|e\|_\infty \|e\| \|\xi_x\| + C_\star \|e\|_\infty \|e\|_{\Gamma_h} \|\xi\|_{\Gamma_h} \\ &\le C_\star h^{-1} \|e\|_\infty (\|e\| \|\xi\| + h^{\frac{1}{2}} \|\eta\|_{\Gamma_h} \|\xi\| + \|\xi\|^2) \\ &\le C_\star h^k \|e\|_\infty \|\xi\| + C_\star h^{-1} \|e\|_\infty \|\xi\|^2 \\ &\le (C_\star h^{-1} \|e\|_\infty + C_\star h^{-3} \|e\|_\infty^2) \|\xi\|^2 + C h^{2k+3}, \end{aligned} \tag{3.14c}$$

where for the first step we have used the Cauchy–Schwarz inequality, for the second step we have used the inverse properties (i) and (ii), for the third step we have employed the interpolation properties (3.5a), and a direct application of Young's inequality is used for the last step. To complete the proof of Lemma 3.3, we need only to combine (3.14a)–(3.14c). ∎

We now collect (3.12a) and (3.12b) into (3.10) to get

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 \le \left| \int_I \eta_t \xi dx \right| + (\mathcal{C}(e) + C_\star h^{-3} \|e\|_\infty^2) \|\xi\|^2 + C_\star h^{k+1} \|\mathbb{S}\| + C h^{2k+3}. \tag{3.15}$$

It follows from the orthogonality property of the projection P_h^- in (3.4), the Cauchy–Schwarz inequality, and the interpolation property (3.5a) that the first term on the right-hand side of (3.15), $|\int_I \eta_t \xi dx|$, can be bounded by

$$\left| \int_I \eta_t \xi dx \right| \le \|\eta_t\| \|\mathbb{S}\| \|\phi\|_\infty \le C h^{k+1} \|\mathbb{S}\|, \tag{3.16}$$

where $C = C' \|u_t\|_{k+1}$ is independent of h. A combination of (3.15) and (3.16) yields that

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 \le (\mathcal{C}(e) + C_\star h^{-3} \|e\|_\infty^2) \|\xi\|^2 + C_\star h^{k+1} \|\mathbb{S}\| + C h^{2k+3}. \tag{3.17}$$

In what follows, to deal with the nonlinearity of the flux $f(u)$ we shall make an a priori assumption that for small enough h, there holds

$$\|\mathbb{Q}_h u - u_h\| \le h^2. \tag{3.18}$$

Later we will justify this a priori assumption (3.18) for piecewise polynomials of degree $k \ge 1$.

COROLLARY 3.4. *Suppose that the interpolation property* (3.5b) *is satisfied. Then the a priori assumption* (3.18) *implies that*

$$\|e\|_\infty \le C h^{\frac{3}{2}} \quad \text{and} \quad \|\xi\|_\infty \le C h^{\frac{3}{2}}. \tag{3.19}$$

Proof. This follows from the inverse property (iii), the interpolation property (3.5b), and triangle inequality. ∎

Under this a priori assumption we can first get a crude bound for ξ, which is used to derive a sharp bound for e_t in Lemma 3.7.

COROLLARY 3.5. *Under the same conditions as in Lemma* 3.3*, if the a priori assumption* (3.18) *holds, we have the following error estimates:*

$$\|e\| \le Ch^{k+1} \quad \textit{and} \quad \|\xi\| \le Ch^{k+1}. \tag{3.20}$$

Proof. The proof follows along the same lines as that in Lemma 3.3 except that we can derive a crude bound for $\Lambda_2 + \Theta_2 \le C_\star h^{k+1}\|\xi\|$ instead of $\Lambda_2 + \Theta_2 \le C_\star h^{k+1}\|\mathbb{S}\|$ in (3.14b) and also in (3.16). Then the results in Corollary 3.5 follow by using (3.19) implied by the a priori assumption (3.18) and a simple application of Gronwall's inequality together with the fact that $\xi(\cdot, 0) = 0$ due to the special choice of the initial condition. ∎

We remark that the results in Corollary 3.5 for the semidiscrete case considered in this paper can be viewed as a straightforward consequence of the fully discrete DG method for solving conservation laws when the upwind fluxes are used; see, e.g., [17, 18].

The next lemma gives us a bound for $\mathbb{S}$, which is essential for obtaining the superconvergence property.

LEMMA 3.6. *Under the same conditions as in Theorem* 3.2*, if in addition the a priori assumption* (3.18) *holds, we have*

$$\|\mathbb{S}\| \le Ch\|e_t\| + Ch^{k+2} \tag{3.21}$$

for any $t \in [0, T]$, where the positive constant C is independent of h and the approximate solution u_h.

The proof of this lemma is given in Appendix A.1. We see that we still need to have a bound on e_t, which is given in the following lemma.

LEMMA 3.7. *Under the same conditions as in Theorem* 3.2*, if in addition the a priori assumption* (3.18) *holds, we have*

$$\|e_t\| \le Ch^{k+1} + C_\star h^{-\frac{1}{2}} \sqrt{\int_0^t \|\xi(s)\|^2 ds} \tag{3.22}$$

for any $t \in [0, T]$, where the positive constants C and $C_\star$ are independent of h and the approximate solution u_h.

The proof of this lemma is deferred to Appendix A.2.

We are now ready to get the following important inequality involving ξ by collecting the estimates (3.21) and (3.22) into (3.17), by employing (3.19) implied by the a priori assumption (3.18), and by virtue of Young's inequality

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2 \le C_1\|\xi(t)\|^2 + C_2 \int_0^t \|\xi(s)\|^2 ds + C_3 h^{2k+3}, \tag{3.23}$$

where C_1, C_2, and C_3 are positive constants independent of h. Note that there holds the following identity:

$$\frac{d}{dt}\int_0^t \|\xi(s)\|^2 ds = \|\xi(t)\|^2. \tag{3.24}$$

Adding twice of (3.23) and (3.24), we arrive at

$$\frac{d}{dt}\left(\|\xi(t)\|^2 + \int_0^t \|\xi(s)\|^2 ds\right) \le C_0 \left(\|\xi(t)\|^2 + \int_0^t \|\xi(s)\|^2 ds\right) + Ch^{2k+3}, \tag{3.25}$$

where $C_0 = \max(2C_1 + 1, 2C_2)$ and $C = 2C_3$ are positive constants independent of h.

An application of Gronwall's inequality together with the fact that $\xi(\cdot, 0) = 0$ gives us the desired result,

$$\|\xi(\cdot, t)\| \le Ch^{k+3/2}. \tag{3.26}$$

Finally, let us complete the proof of Theorem 3.2 by verifying the a priori assumption (3.18). First, the a priori assumption is satisfied at $t = 0$ since $\xi(\cdot, 0) = 0$. Next, for piecewise polynomials of degree k ($k \ge 1$), one can choose h small enough such that $Ch^{k+3/2} < \frac{1}{2}h^2$, where C is a constant in (3.6) determined by the final time T. Define $t^\star = \sup\left\{s \le T : \|\mathbb{Q}_h u(t) - u_h(t)\| \le h^2 \text{ for all } t \in [0,\ s]\right\}$; then we have $\|\mathbb{Q}_h u(t^\star) - u_h(t^\star)\| = h^2$ by continuity if $t^\star < T$. However, our main result (3.26) implies that $\|\mathbb{Q}_h u(t^\star) - u_h(t^\star)\| \le Ch^{k+3/2} < \frac{1}{2}h^2$, which is a contradiction. Therefore, there always holds $t^\star = T$, and thus the a priori assumption (3.18) is justified.

4. Numerical examples.

In this section we provide some numerical experiments to verify the superconvergence property of the DG method for hyperbolic conservation laws. To reduce the time errors, we use the five-stage, fourth order strong stability preserving Runge–Kutta discretization (see, e.g., [10]) in time and take $\Delta t = CFL\ h^2$ in the one-dimensional case and $\Delta t = CFL\ h^{\frac{3}{2}}$ in the two-dimensional case. In the computations below, periodic boundary conditions are imposed and our numerical initial condition is taken by the L^2 projection of the initial condition, unless otherwise indicated. In fact, we have observed little difference if we use the $\mathbb{Q}_h$ projection of the initial condition instead, except for the case with a severely nonuniform mesh solving (4.1) in Example 4.1 for short time simulation (for example, $T = 1$); see Table 4.2.

EXAMPLE 4.1. *First we consider the equation*

$$\begin{cases} u_t + (u^3/3 + u)_x = g(x, t), \\ u(x, 0) = \cos(x), \end{cases} \tag{4.1}$$

where $g(x, t)$ is given by

$$g(x, t) = -(2 + \cos^2(x + t)) \sin(x + t).$$

The exact solution is

$$u(x, t) = \cos(x + t).$$

Note that $f'(u) = u^2 + 1 \ge 1 > \delta > 0$; we can use upwind fluxes and choose $\mathbb{Q}_h = P_h^-$. We test this example using P^k polynomials with $1 \le k \le 3$. Table 4.1 lists the numerical errors, ξ and e, and their orders for different final time T using P^1 polynomials on a uniform mesh. We can clearly observe third order accuracy for ξ and second order for e. Also, ξ does not grow much, which ensures that the error e does not grow with respect to time. Table 4.2 lists the results when using P^1 polynomials on a nonuniform mesh which is a 30% random perturbation of the uniform mesh. From the table, we can still observe superconvergence (the order is around 2.5). Let us point out that the usage of P_h^- projection of the initial condition would help to recover the third order accuracy; however, it is not included here to save space. That is, the conclusions also hold true for this nonuniform case. The results for Example 4.1 when using P^2 and P^3 polynomials on a uniform mesh, respectively, are listed

Table 4.1

The errors ξ and e for Example 4.1 when using P^1 polynomials on a uniform mesh of N cells. $CFL = 0.5$.

P^1	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	2.10E-04	–	1.84E-04	–	2.45E-04	–
	40	2.65E-05	2.99	2.73E-05	2.76	3.90E-05	2.65
	80	3.31E-06	3.00	3.65E-06	2.90	5.10E-06	2.93
	160	4.14E-07	3.00	4.61E-07	2.98	6.53E-07	2.97
	320	5.17E-08	3.00	5.77E-08	3.00	8.21E-08	2.99
e	20	4.26E-03	–	4.26E-03	–	4.24E-03	–
	40	1.06E-03	2.00	1.06E-03	2.00	1.06E-03	2.00
	80	2.65E-04	2.00	2.66E-04	2.00	2.65E-04	2.00
	160	6.64E-05	2.00	6.64E-05	2.00	6.64E-05	2.00
	320	1.66E-05	2.00	1.66E-05	2.00	1.66E-05	2.00

Table 4.2

The errors ξ and e for Example 4.1 when using P^1 polynomials on a random mesh of N cells. $CFL = 0.5$.

P^1	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	5.86E-04	–	6.46E-04	–	6.21E-04	–
	40	6.19E-05	3.01	5.86E-05	3.21	5.43E-05	3.26
	80	1.18E-05	2.74	7.71E-06	3.36	8.03E-06	3.16
	160	2.30E-06	2.71	7.81E-07	3.78	9.81E-07	3.47
	320	4.65E-07	2.24	1.14E-07	2.70	1.21E-07	2.94
e	20	5.50E-03	–	4.98E-03	–	5.37E-03	–
	40	1.30E-03	1.93	1.23E-03	1.87	1.28E-03	1.93
	80	3.55E-04	2.15	3.52E-04	2.07	3.52E-04	2.13
	160	8.73E-05	2.32	8.32E-05	2.39	8.36E-05	2.37
	320	2.13E-05	1.98	2.10E-05	1.93	2.15E-05	1.91

in Tables 4.3 and 4.4. We can clearly observe that the orders of convergence of the errors, ξ and e, are $k+2$ and $k+1$, respectively; moreover, neither errors grow with respect to time.

Example 4.2. *In this example, we solve the equation*

$$\begin{cases} u_t + (u^3/3)_x = g(x,t), \\ u(x,0) = \cos(x), \end{cases} \tag{4.2}$$

where $g(x,t)$ is given by

$$g(x,t) = -(1+\cos^2(x+t))\sin(x+t).$$

The exact solution is

$$u(x,t) = \cos(x+t).$$

In this case, $f'(u) = u^2 \geq 0$, and we can still use the upwind flux and choose $\mathbb{Q}_h = P_h^-$. We test this example using P^k polynomials with $1 \leq k \leq 3$ on a nonuniform mesh, which is a 10% random perturbation of the uniform mesh. The results in Tables 4.5 and 4.6 show that the orders of convergence of the errors, ξ and e, are $k+\frac{3}{2}$ and $k+1$, respectively.

Example 4.3. *We consider the Burgers equation*

$$\begin{cases} u_t + (u^2/2)_x = g(x,t), \\ u(x,0) = \cos(x), \end{cases} \tag{4.3}$$

TABLE 4.3
The errors ξ and e for Example 4.1 when using P^2 polynomials on a uniform mesh of N cells. $CFL = 0.5$.

P^2	N	$T = 1$		$T = 50$		$T = 500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	6.35E-06	–	6.70E-06	–	6.69E-06	–
	40	4.12E-07	3.94	4.13E-07	4.02	4.13E-07	4.02
	80	2.57E-08	4.00	2.57E-08	4.00	2.57E-08	4.00
	160	1.61E-09	4.00	1.61E-09	4.00	1.61E-09	4.00
	320	1.00E-10	4.00	1.00E-10	4.00	1.01E-10	3.99
e	20	1.07E-04	–	1.07E-04	–	1.07E-04	–
	40	1.34E-05	3.00	1.34E-05	3.00	1.34E-05	3.00
	80	1.67E-06	3.00	1.67E-06	3.00	1.67E-06	3.00
	160	2.09E-07	3.00	2.09E-07	3.00	2.09E-07	3.00
	320	2.61E-08	3.00	2.61E-08	3.00	2.61E-08	3.00

TABLE 4.4
The errors ξ and e for Example 4.1 when using P^3 polynomials on a uniform mesh of N cells. $CFL = 0.1$.

P^3	N	$T = 10$		$T = 50$		$T = 500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	10	2.82E-06	–	1.81E-06	–	1.98E-06	–
	20	5.47E-08	5.69	5.67E-08	5.00	5.66E-08	5.13
	40	1.74E-09	4.97	1.74E-09	5.02	1.74E-09	5.02
	80	5.42E-11	5.00	5.42E-11	5.00	5.49E-11	4.99
e	10	3.31E-05	–	3.30E-05	–	3.30E-05	–
	20	2.07E-06	4.00	2.07E-06	4.00	2.07E-06	4.00
	40	1.29E-07	4.00	1.29E-07	4.00	1.29E-07	4.00
	80	8.07E-09	4.00	8.07E-09	4.00	8.07E-09	4.00

TABLE 4.5
The errors ξ and e for Example 4.2 when using both P^1 and P^2 polynomials on a random mesh of N cells. $CFL = 0.5$. $T = 1$.

P^k	$k = 1$				$k = 2$			
N	ξ		e		ξ		e	
	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
40	2.28E-04	–	1.08E-03	–	4.29E-06	–	1.40E-05	–
80	4.52E-05	2.41	2.75E-04	2.04	3.25E-07	3.84	1.77E-06	3.08
160	7.95E-06	2.51	6.85E-05	2.01	2.24E-08	3.86	2.18E-07	3.02
320	1.49E-06	2.52	1.72E-05	2.07	1.90E-09	3.70	2.77E-08	3.10
640	2.63E-07	2.50	4.30E-06	2.00	1.66E-10	3.51	3.48E-09	2.99

TABLE 4.6
The errors ξ and e for Example 4.2 when using P^3 polynomials on a random mesh of N cells. $CFL = 0.2$. $T = 1$.

N	ξ		e	
	L^2 error	order	L^2 error	order
10	1.33E-05	–	3.96E-05	–
20	6.03E-07	4.38	2.27E-06	4.05
40	2.84E-08	4.48	1.37E-07	4.12
80	1.53E-09	4.35	9.16E-09	4.03
160	6.96E-11	4.46	5.62E-10	4.03

where $g(x,t)$ *is given by*

$$g(x,t) = -(1+\cos(x+t))\sin(x+t).$$

The exact solution is

$$u(x,t) = \cos(x+t).$$

Since $f'(u)$ changes its sign in the computational domain, we use the Godunov flux, which is an upwind flux. The projection $\mathbb{Q}_h$ is defined element by element as follows. For $t = T$, if $u(x_j,t)$ is positive, we choose $\mathbb{Q}_h = P_h^-$ on the cell I_j; otherwise, we use $\mathbb{Q}_h = P_h^+$. We test this example using P^k polynomials with $1 \le k \le 3$ on a uniform mesh. The results are listed in Tables 4.7–4.9, from which we observe that ξ achieves at least $\left(k+\frac{3}{2}\right)$th order superconvergence and it does not grow with respect to time for most meshes. Meanwhile, the error e achieves the expected $(k+1)$th order of accuracy and it does not grow with respect to time either.

EXAMPLE 4.4. *To better understand that the superconvergence property is valid for conservation laws with general flux functions, let us consider the problem*

$$\begin{cases} u_t + (e^u)_x = g(x,t), \\ u(x,0) = \cos(x), \end{cases} \tag{4.4}$$

where $g(x,t)$ *is given by*

$$g(x,t) = -\left(1+e^{\cos(x+t)}\right)\sin(x+t).$$

TABLE 4.7
The errors ξ *and* e *for Example* 4.3 *when using* P^1 *polynomials on a uniform mesh of* N *cells.* $CFL = 0.5$.

P^1	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	6.31E-04	–	1.61E-03	–	1.64E-03	–
	40	9.03E-05	2.81	2.74E-04	2.56	2.65E-04	2.63
	80	1.25E-05	2.85	3.76E-05	2.86	4.24E-05	2.65
	160	1.82E-06	2.78	8.15E-06	2.21	6.67E-06	2.67
	320	2.59E-07	2.81	1.50E-06	2.44	1.04E-06	2.68
e	20	4.26E-03	–	4.48E-03	–	4.49E-03	–
	40	1.06E-03	2.00	1.09E-03	2.04	1.09E-03	2.04
	80	2.66E-04	2.00	2.68E-04	2.03	2.69E-04	2.02
	160	6.64E-05	2.00	6.68E-05	2.00	6.67E-05	2.01
	320	1.66E-05	2.00	1.67E-05	2.00	1.66E-05	2.00

TABLE 4.8
The errors ξ *and* e *for Example* 4.3 *when using* P^2 *polynomials on a uniform mesh of* N *cells.* $CFL = 0.5$.

P^2	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	20	7.57E-05	–	9.23E-05	–	1.05E-04	–
	40	8.19E-06	3.21	8.76E-06	3.40	9.08E-06	3.53
	80	9.76E-07	3.07	1.01E-06	3.11	9.11E-07	3.32
	160	8.72E-08	3.48	9.03E-08	3.49	8.81E-08	3.37
e	20	1.20E-04	–	1.31E-04	–	1.31E-04	–
	40	1.47E-05	3.03	1.49E-05	3.13	1.49E-05	3.13
	80	1.77E-06	3.05	1.78E-06	3.07	1.78E-06	3.07
	160	2.15E-07	3.04	2.15E-07	3.04	2.15E-07	3.04

TABLE 4.9
The errors ξ and e for Example 4.3 when using P^3 polynomials on a uniform mesh of N cells. $CFL = 0.2$.

P^3	N	$T=1$		$T=50$		$T=500$	
		L^2 error	order	L^2 error	order	L^2 error	order
ξ	10	1.10E-05	–	1.56E-05	–	1.50E-05	–
	20	3.94E-07	4.81	4.16E-07	5.23	4.14E-07	5.18
	40	1.49E-08	4.72	1.29E-08	5.01	1.27E-08	5.02
	80	5.39E-10	4.79	3.92E-10	5.04	3.91E-10	5.02
e	10	3.53E-05	–	3.63E-05	–	3.51E-05	–
	20	2.11E-06	4.06	2.11E-06	4.10	2.11E-06	4.05
	40	1.30E-07	4.02	1.30E-07	4.02	1.30E-07	4.02
	80	8.09E-09	4.01	8.08E-09	4.01	8.08E-09	4.01

TABLE 4.10
The errors ξ and e for Example 4.4 when using both P^1 and P^2 polynomials on a random mesh of N cells. $CFL = 0.1$. $T = 1$.

P^k	$k=1$				$k=2$			
N	ξ		e		ξ		e	
	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
20	3.81E-04	–	4.35E-03	–	1.11E-05	–	1.12E-04	–
40	5.35E-05	2.88	1.08E-03	2.05	6.20E-07	4.23	1.41E-05	3.04
80	6.84E-06	3.07	2.74E-04	2.04	4.09E-08	4.05	1.79E-06	3.08
160	8.25E-07	3.05	6.84E-05	2.00	2.48E-09	4.05	2.20E-07	3.03

The exact solution is

$$u(x,t) = \cos(x+t).$$

We test this example using both P^1 and P^2 polynomials on a nonuniform mesh which is a 10% random perturbation of the uniform mesh. For the numerical initial condition, we use the P_h^- projection. The results in Table 4.10 show that the orders of convergence of the errors, ξ and e, are $k+2$ and $k+1$, respectively. This example demonstrates that the superconvergence property also holds true for conservation laws with a strong nonlinearity that is not a polynomial of u.

EXAMPLE 4.5. *To illustrate that the superconvergence property still holds for the two-dimensional case we solve the problem*

$$\begin{cases} u_t + (u^3/3)_x + (u^3/3)_y = g(x,y,t), \\ u(x,y,0) = \sin(x+y), \end{cases} \tag{4.5}$$

where $g(x,y,t)$ is given by

$$g(x,y,t) = -2\cos^3(x+y-2t).$$

The exact solution is

$$u(x,y,t) = \sin(x+y-2t).$$

We use a rectangular mesh consisting of elements $I_{i,j} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ with $1 \le i \le N_x$ and $1 \le j \le N_y$. The finite element space associated with the mesh is of the form

$$V_h^k = \left\{ v \in L^2(I \times I) : v|_{I_{i,j}} \in Q^k(I_{i,j}), \quad i = 1,\dots,N_x,\ j = 1,\dots,N_y \right\},$$

TABLE 4.11
The errors $\xi = \Pi^- u - u_h$ and e for Example 4.5 when using both Q^1 and Q^2 polynomials on a random mesh of $N \times N$ cells. $CFL = 0.2$. $T = 1$.

Q^k	$k = 1$				$k = 2$			
N	ξ		e		ξ		e	
	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
10	1.56E-02	–	2.44E-02	–	2.12E-04	–	1.23E-03	–
20	2.89E-03	2.57	6.26E-03	2.08	8.89E-06	4.43	1.57E-04	2.87
40	5.29E-04	2.53	1.58E-03	2.05	4.89E-07	4.41	2.03E-05	3.11
80	9.20E-05	2.61	3.90E-04	2.09	2.79E-08	4.37	2.54E-06	3.18

where $Q^k(I_{i,j})$ is the space of tensor product of polynomials of degrees at most k in each variable.

The projection Π^- for a scalar function $q \in C^0(I_{i,j})$ on a rectangle $I_{i,j}$ is defined as

$$\Pi^- q = P^-_{h_x} \otimes P^-_{h_y} q \tag{4.6}$$

with the sub-subscripts x and y indicating the application of the one-dimensional projection P_h^- with respect to the corresponding variable. For more details regarding the approximation error estimates, see [4, 13].

We test this example using both Q^1 and Q^2 polynomials on a nonuniform mesh which is a 10% random perturbation of the uniform mesh. For the numerical initial condition, we use the Π^- projection. The results in Table 4.11 show that the orders of convergence of the errors, ξ and e, are at least $k + \frac{3}{2}$ and $k + 1$, respectively. That is, the conclusions also hold true for the multidimensional cases.

5. Concluding remarks. In this paper, the superconvergence property of the DG method for nonlinear hyperbolic conservation laws is investigated. We prove that the error between the numerical solution and a particular projection of the exact solution achieves $(k + \frac{3}{2})$th order superconvergence when piecewise polynomials of degree k ($k \geq 1$) are used, provided that $|f'(u)|$ is lower bounded uniformly by a positive constant. A series of numerical experiments are given to show that the superconvergence property holds true for nonlinear conservation laws with general flux functions, indicating that the restriction on $f(u)$ is artificial.

Future work includes the study of superconvergence of DG methods for conservation laws in multidimensional cases on structured and unstructured meshes. Analysis of superconvergence of the LDG method for nonlinear diffusion equations also constitutes our future work.

Appendix A. Proof of several lemmas. In this appendix, we give the proofs for some of the technical lemmas.

A.1. The proof of Lemma 3.6. A simple integration by parts of (3.8) yields that

$$\int_{I_j} e_t v_h dx + \int_{I_j} (f(u) - f(u_h))_x v_h dx + [\![f(u) - f(u_h)]\!] v_h^+|_{j-\frac{1}{2}} = 0 \tag{A.1}$$

for all $v_h \in V_h$. We recall that $u - u_h = \eta + \xi$ and $\xi = r_j + \mathbb{S}_j(x)(x - x_j)/h_j$ with r_j being a constant and $\mathbb{S}_j(x) \in P^{k-1}(I_j)$. If we now let $v_h = \mathbb{S}_j(x)(x - x_{j-1/2})/h_j$, then we have the identity

$$\int_{I_j} e_t \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx + \int_{I_j} (f(u) - f(u_h))_x \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx = 0, \tag{A.2}$$

since $v_h(x^+_{j-\frac{1}{2}}) = 0$. To estimate $\|\mathbb{S}\|$, we would like to split the nonlinear term $f(u) - f(u_h)$ in (A.2) into five terms by the following Taylor expansion:

$$\begin{aligned} f(u) - f(u_h) &= f'(u)\xi + f'(u)\eta - \frac{1}{2}f''_u e^2 \\ &= f'(u_j)\xi + (f'(u) - f'(u_j))\xi + f'(u_j)\eta + (f'(u) - f'(u_j))\eta - \frac{1}{2}f''_u e^2 \\ &\triangleq \pi_{1,j} + \pi_{2,j} + \pi_{3,j} + \pi_{4,j} + \pi_{5,j}, \end{aligned}$$

where f''_u is again the mean value. As we have shown before, $|f'(u) - f'(u_j)| \le C_\star h$ on each element I_j. Therefore, (A.2) can be written as

$$\int_{I_j} e_t \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx + \sum_{i=1}^{5} \Pi_{i,j} = 0, \tag{A.3}$$

where

$$\Pi_{i,j} = \int_{I_j} (\pi_{i,j})_x \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx \quad (i = 1, \ldots, 5).$$

In what follows, we will estimate each term above separately.

First, it is easy to show, by the property of $\mathcal{B}^-_j$ in Lemma 3.1, that

$$\Pi_{1,j} = f'(u_j)\mathcal{B}^-_j(\mathbb{S}_j) = f'(u_j)\left[\frac{1}{4h_j}\int_{I_j} \mathbb{S}^2_j(x)dx + \frac{\mathbb{S}^2_j(x_{j+1/2})}{4}\right]. \tag{A.4}$$

Plugging (A.4) into (A.3) and using the assumption that $f'(u(x,t)) \ge \delta > 0$, we get

$$\frac{\delta}{4}\int_{I_j} \mathbb{S}^2_j(x)dx \le h_j\left[-\int_{I_j} e_t \mathbb{S}_j(x)\frac{x - x_{j-1/2}}{h_j}dx - \sum_{i=2}^{5}\Pi_{i,j}\right]. \tag{A.5}$$

We now define the piecewise polynomial $\phi_1(x)$ such that $\phi_1(x) = (x - x_{j-1/2})/h_j$ on each I_j; clearly, $\|\phi_1\|_\infty = 1$. A summation of (A.5) over j yields that

$$\frac{\delta}{4}\|\mathbb{S}\|^2 \le -h\int_I e_t \mathbb{S}(x)\phi_1(x)dx - \sum_{i=2}^{5}\Pi_i \tag{A.6}$$

with $\Pi_i = h\sum_{j=1}^N \Pi_{i,j}$ $(i = 2, \ldots, 5)$. We shall estimate the right-hand side of (A.6) one by one below.

The first integral term can be bounded by using the Cauchy–Schwarz inequality

$$h\left|\int_I e_t \mathbb{S}(x)\phi_1(x)dx\right| \le h\|e_t\|\|\mathbb{S}\|\|\phi_1\|_\infty \le h\|e_t\|\|\mathbb{S}\|. \tag{A.7a}$$

To estimate Π_2, we begin by using the Cauchy–Schwarz inequality and inverse property (i) to get a bound for $\Pi_{2,j}$; it reads

$$\begin{aligned} \Pi_{2,j} &= \int_{I_j} \left[(f'(u) - f'(u_j))_x \xi + (f'(u) - f'(u_j))\xi_x\right] \mathbb{S}_j(x)\frac{x - x_{j-1/2}}{h_j}dx \\ &\le C_\star \|\xi\|_{I_j}\|\mathbb{S}\|_{I_j}. \end{aligned}$$

Here and below, $\|\cdot\|_{I_j}$ denotes the usual L^2 norm defined on the cell I_j. Consequently, again by the Cauchy–Schwarz inequality, we get

$$|\Pi_2| \le C_\star h\|\xi\|\|\mathbb{S}\|. \tag{A.7b}$$

A simple integration by parts together with the property of projection P_h^- in (3.4) gives us

$$\Pi_{3,j} = -f'(u_j)\int_{I_j} \eta \frac{d}{dx}\left(\mathbb{S}_j(x)\frac{x - x_{j-1/2}}{h_j}\right)dx = 0.$$

Thus,

$$|\Pi_3| = 0. \tag{A.7c}$$

To estimate Π_4, we first get a bound for $\Pi_{4,j}$; it reads

$$\begin{aligned}\Pi_{4,j} &= \int_{I_j} \left[(f'(u) - f'(u_j))_x\eta + (f'(u) - f'(u_j))\eta_x\right]\mathbb{S}_j(x)\frac{x - x_{j-1/2}}{h_j}dx\\ &\le C_\star(\|\eta\|_{I_j} + h\|\eta_x\|_{I_j})\|\mathbb{S}\|_{I_j}.\end{aligned}$$

As a consequence, the Cauchy–Schwarz inequality in combination with the interpolation property (3.5a) produces a bound for Π_4,

$$|\Pi_4| \le C_\star h(\|\eta\| + h\|\eta_x\|)\|\mathbb{S}\| \le C_\star h^{k+2}\|\mathbb{S}\|. \tag{A.7d}$$

It is easy to get, for the high order nonlinear term Π_5, that

$$|\Pi_5| \le C_\star\|e\|_\infty(h^{k+1} + \|\xi\|)\|\mathbb{S}\|. \tag{A.7e}$$

Finally, the error estimate (3.21) follows by collecting the estimates (A.7a)–(A.7e) into (A.6) and by using (3.19) implied by the a priori assumption (3.18) and a crude bound for ξ in (3.20), in Corollary 3.4 and Corollary 3.5, respectively. This completes the proof of Lemma 3.6.

A.2. The proof of Lemma 3.7. From the interpolation property (3.5a), we know that $\|\eta_t(\cdot,t)\| \le Ch^{k+1}$. Thus, to get the error estimate (3.22) we need only to prove $\|\xi_t(\cdot,t)\| \le Ch^{k+1} + C_\star h^{-\frac{1}{2}}\sqrt{\int_0^t \|\xi(s)\|^2 ds}$. To this end, we shall first get a bound for the initial error $\|\xi_t(\cdot,0)\|$.

We start by noting that the error equation (3.9) still holds at $t = 0$ for any $v_h \in V_h$. Using the fact that $\xi(\cdot,0) = 0$, we arrive at the following representation of the nonlinear terms in (3.13a) and (3.13b) on the right-hand side of (3.9):

$$f(u) - f(u_h) = f'(u)\eta - \frac{1}{2}\tilde{f}''_u\eta^2, \tag{A.8a}$$

$$f(u) - f(u_h^-) = f'(u)\eta^- - \frac{1}{2}\tilde{\tilde{f}}''_u(\eta^-)^2. \tag{A.8b}$$

By an analysis similar to that in the proof of Lemma 3.3, we can easily get a bound for the right-hand side of (3.9) at $t = 0$, denoted by $\mathcal{RHS}$,

$$\mathcal{RHS} \le C_\star(h^{k+1} + h^k\|\eta(\cdot,0)\|_\infty)\|v_h\|, \tag{A.9}$$

which holds for any $v_h \in V_h$. If we now let $v_h = \xi_t(\cdot, 0)$ in (3.9) as well as in (A.9), we get, after a simple calculation, that

$$\|\xi_t(\cdot,0)\| \le \|\eta_t(\cdot,0)\| + C_\star(h^{k+1} + h^k\|\eta(\cdot,0)\|_\infty) \le Ch^{k+1} \tag{A.10}$$

by the interpolation properties (3.5a) and (3.5b).

We then move on to the estimate of $\|\xi_t(\cdot,t)\|$ for $t > 0$. To do that, we proceed as follows. We take the time derivative of the error equation (3.9) and let $v_h = \xi_t$ to get

$$\int_I e_{tt}\xi_t dx = \sum_{j=1}^N \int_{I_j} (f(u) - f(u_h))_t(\xi_t)_x dx + \sum_{j=1}^N \big((f(u) - f(u_h^-))_t[\![\xi_t]\!]\big)_{j+\frac12}. \tag{A.11}$$

To estimate the right-hand side of (A.11), we would like to use the following Taylor expansion for the nonlinear terms:

$$\begin{aligned}
(f(u) - f(u_h))_t &= (f'(u)\xi)_t + (f'(u)\eta)_t - (R_1e^2)_t \\
&= \partial_t f'(u)\xi + f'(u)\xi_t + \partial_t f'(u)\eta + f'(u)\eta_t - \partial_t R_1 e^2 - 2R_1 e e_t \\
&\triangleq \varphi_1 + \cdots + \varphi_6, && \text{(A.12a)}\\
(f(u) - f(u_h^-))_t &= (f'(u)\xi^-)_t + (f'(u)\eta^-)_t - (R_2(e^-)^2)_t \\
&= \partial_t f'(u)\xi^- + f'(u)\xi_t^- + \partial_t f'(u)\eta^- \\
&\quad + f'(u)\eta_t^- - \partial_t R_2(e^-)^2 - 2R_2 e^- e_t^- \\
&\triangleq \psi_1 + \cdots + \psi_6, && \text{(A.12b)}
\end{aligned}$$

where $R_1 = \int_0^1 (1-\mu) f''(u + \mu(u_h - u))d\mu$ and $R_2 = \int_0^1 (1-\nu) f''(u+\nu(u_h^- - u))d\nu$ are the integral form of the reminders of the second order Taylor expansion. Therefore, the right-hand side of (A.11), denoted by Υ, can be formulated as

$$\Upsilon = \mathcal{K}_1 + \cdots + \mathcal{K}_6, \tag{A.13}$$

where

$$\mathcal{K}_i = \sum_{j=1}^N \int_{I_j} \varphi_i(\xi_t)_x dx + \sum_{j=1}^N (\psi_i[\![\xi_t]\!])_{j+\frac12} \quad (i = 1, \ldots, 6),$$

which will be estimated one by one below. Accordingly, (A.11) can be represented by

$$\frac12 \frac{d}{dt}\|\xi_t\|^2 \le \Upsilon + \|\eta_{tt}\|\|\xi_t\| \le \Upsilon + Ch^{k+1}\|\xi_t\| \tag{A.14}$$

by the interpolation error estimates (3.5a).

We estimate the term $\mathcal{K}_1$ first. It follows from Young's inequality and the inverse property (ii) that

$$\mathcal{K}_1 = \sum_{j=1}^N \int_{I_j} \partial_t f'(u)\xi(\xi_t)_x dx + \varepsilon \sum_{j=1}^N [\![\xi_t]\!]_{j+\frac12}^2 + C_\star h^{-1}\|\xi\|^2, \tag{A.15a}$$

where ε is a small positive constant to be specified later. We would like to point out that the first integral term on the right-hand side of (A.15a) is intractable to get a sharp bound for $\|\xi_t\|$ due to the hybrid of ξ and $(\xi_t)_x$ and is left to be estimated later, together with other terms. A simple integration by parts gives us a bound for $\mathcal{K}_2$,

$$\begin{aligned}\mathcal{K}_2 &= -\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}\partial_x f'(u)(\xi_t)^2dx - \frac{1}{2}\sum_{j=1}^{N} f'(u_{j+\frac{1}{2}})[\![\xi_t]\!]^2_{j+\frac{1}{2}}\\ &\le C_\star\|\xi_t\|^2 - \frac{\delta}{2}\sum_{j=1}^{N}[\![\xi_t]\!]^2_{j+\frac{1}{2}},\end{aligned}$$

where we have used the assumption that $f'(u(x,t)) \ge \delta > 0$. Combining the above estimates for $\mathcal{K}_1$ and $\mathcal{K}_2$, we arrive at

$$\begin{aligned}\mathcal{K}_1+\mathcal{K}_2 &\le C_\star\|\xi_t\|^2 + \sum_{j=1}^{N}\int_{I_j}\partial_t f'(u)\xi(\xi_t)_x dx + C_\star h^{-1}\|\xi\|^2 - \left(\frac{\delta}{2}-\varepsilon\right)\sum_{j=1}^{N}[\![\xi_t]\!]^2_{j+\frac{1}{2}}\\ &\le C_\star\|\xi_t\|^2 + \sum_{j=1}^{N}\int_{I_j}\partial_t f'(u)\xi(\xi_t)_x dx + C_\star h^{-1}\|\xi\|^2, \qquad \text{(A.15b)}\end{aligned}$$

where we have chosen ε to be small enough, for example, $\varepsilon = \delta/4$, to obtain the last inequality. Noting that $\partial_t f'(u) = \partial_t f'(u_j) + (\partial_t f'(u) - \partial_t f'(u_j))$ and $|\partial_t f'(u) - \partial_t f'(u_j)| \le C_\star h$ on each element I_j, we thus have, by the property of the projection P_h^-, that

$$\mathcal{K}_3 = \sum_{j=1}^{N}\int_{I_j}(\partial_t f'(u) - \partial_t f'(u_j))\eta(\xi_t)_x dx \le C_\star\|\eta\|\|\xi_t\| \le C_\star h^{k+1}\|\xi_t\|, \qquad \text{(A.15c)}$$

where we have used the Cauchy–Schwarz inequality, the inverse property (i), and the interpolation error estimates (3.5a). Analogously, it follows from the property of the projection P_h^- and the inverse property (i) that

$$\mathcal{K}_4 \le C_\star\|\eta_t\|\|\xi_t\| \le C_\star h^{k+1}\|\xi_t\|. \qquad \text{(A.15d)}$$

It is easy to show, for the high order term $\mathcal{K}_5$, that

$$\mathcal{K}_5 \le C_\star h^{-1}\|e\|_\infty(\|\xi\| + h^{k+1})\|\xi_t\| \le C_\star h^k\|e\|_\infty\|\xi_t\|, \qquad \text{(A.15e)}$$

where we have employed (3.20) in Corollary 3.5 to obtain the last inequality. For the last term, namely, $\mathcal{K}_6$, we have

$$\mathcal{K}_6 \le C_\star h^{-1}\|e\|_\infty\|\xi_t\|^2 + C_\star h^k\|e\|_\infty\|\xi_t\|. \qquad \text{(A.15f)}$$

Therefore, by collecting the estimates (A.15b)–(A.15f) and (A.13) into (A.14), we get after a straightforward application of Young's inequality that

$$\frac{1}{2}\frac{d}{dt}\|\xi_t\|^2 \le \mathcal{C}^2(e)\|\xi_t\|^2 + \sum_{j=1}^{N}\int_{I_j}\partial_t f'(u)\xi(\xi_t)_x dx + C_\star h^{-1}\|\xi\|^2 + Ch^{2k+2}, \qquad \text{(A.16)}$$

where $\mathcal{C}(e) = C + C_\star h^{-1}\|e\|_\infty$ has been defined in (3.12b). Now we integrate the above inequality (A.16) with respect to time between 0 and t and take into account the initial error estimate (A.10) to obtain

$$\frac{1}{2}\|\xi_t\|^2 \le \mathcal{C}^2(e)\int_0^t\|\xi_t\|^2dt + \mathcal{Q} + C_\star h^{-1}\int_0^t\|\xi\|^2dt + Ch^{2k+2}, \qquad \text{(A.17)}$$

where

$$\mathcal{Q} = \int_0^t \sum_{j=1}^N \int_{I_j} \partial_t f'(u)\xi(\xi_t)_x dxdt.$$

Let us work on the term $\mathcal{Q}$. To do that, we begin by using integration by parts with respect to time to get

$$\begin{aligned}\mathcal{Q} &= \sum_{j=1}^N \int_{I_j} \int_0^t \partial_t f'(u)\xi(\xi_x)_t dtdx \\ &= \sum_{j=1}^N \int_{I_j} \left\{ -\int_0^t \xi_x \partial_t(\partial_t f'(u)\xi)dt + \partial_t f'(u)\xi\xi_x|_0^t \right\} dx \\ &= \sum_{j=1}^N \int_{I_j} \left\{ -\int_0^t \xi_x \partial_t(\partial_t f'(u)\xi)dt + (\partial_t f'(u)\xi\xi_x)(t) \right\} dx,\end{aligned}$$

since $\xi(\cdot, 0) = 0$. Next, by an analysis similar to that in the proof of (3.14b), we have that

$$\begin{aligned}\mathcal{Q} &\le C_\star h^{-1} \int_0^t \|\mathbb{S}\|(\|\xi\| + \|\xi_t\|)dt + C_\star h^{-1}\|\mathbb{S}\|\|\xi\| \\ &\le C_\star \int_0^t \|\xi_t\|^2 dt + C_\star h^{k+1}\|\xi_t\| + Ch^{2k+2} \\ &\le C_\star \int_0^t \|\xi_t\|^2 dt + \frac{1}{4}\|\xi_t\|^2 + Ch^{2k+2}, \qquad \text{(A.18)}\end{aligned}$$

where for the second inequality we have used a crude bound for ξ in (3.20) and a compact bound for $\mathbb{S}$, namely, $\|\mathbb{S}\| \le Ch\|\xi_t\| + Ch^{k+2}$, by taking into account the interpolation properties (3.5a); the last inequality is a direct application of Young's inequality.

Plugging the estimate (A.18) into (A.17) and taking into account (3.19) implied by the a priori assumption (3.18), we get that for small enough h

$$\frac{1}{4}\|\xi_t\|^2 \le \tilde{C} \int_0^t \|\xi_t\|^2 dt + C_\star h^{-1} \int_0^t \|\xi\|^2 dt + Ch^{2k+2}, \qquad \text{(A.19)}$$

where $C, C_\star$, and $\tilde{C}$ are positive constants independent of h. Finally, a direct application of Gronwall's inequality yields that

$$\|\xi_t\| \le Ch^{k+1} + C_\star h^{-\frac{1}{2}} \sqrt{\int_0^t \|\xi(s)\|^2 ds}.$$

This completes the proof of Lemma 3.7.

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