Mathematical Models and Methods in Applied Sciences
Vol. 22, No. 12 (2012) 1250042 (45 pages)

DOI: 10.1142/S021820251250042X

DISCONTINUOUS GALERKIN METHODS FOR THE MULTI-DIMENSIONAL VLASOV–POISSON PROBLEM

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Received 10 July 2011
Revised 26 February 2012
Accepted 28 February 2012
Published 26 July 2012
Communicated by F. Brezzi

We introduce and analyze two new semi-discrete numerical methods for the multi-dimensional Vlasov–Poisson system. The schemes are constructed by combining a discontinuous Galerkin approximation to the Vlasov equation together with a mixed finite element method for the Poisson problem. We show optimal error estimates in the case of smooth compactly supported initial data. We propose a scheme that preserves the total energy of the system.

Keywords: Vlasov–Poisson; discontinuous Galerkin; mixed finite elements.

AMS Subject Classification: Primary 65N30; Secondary 65M60, 65M12, 65M15, 82D10

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1. Introduction

The Vlasov–Poisson (VP) system is a classical model in collisionless kinetic theory. It is a mean-field limit description of a large ensemble of interacting particles by electrostatic or gravitational forces. While most of the results in this work are equally valid in both cases under smoothness assumptions of the solutions, we focus our presentation on the plasma physics case.

In kinetic theory, the evolution of the particle number density or mass density $f(\mathbf{x}, \mathbf{v}, t)$ in phase space, i.e. position and velocity $(\mathbf{x}, \mathbf{v})$ at time $t > 0$ is given by the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} \Phi \cdot \nabla_{\mathbf{v}} f = 0, \quad (\mathbf{x}, \mathbf{v}, t) \in \Omega_{\mathbf{x}} \times \mathbb{R}^d \times [0, T], \tag{1.1}$$

considered with periodic boundary conditions in the d-dimensional torus $\Omega_{\mathbf{x}} = [0,1]^d$ with $d = 2, 3$. In order to describe charged particles motion in plasmas, we need to compute the force field from the macroscopic density of particles

$$\rho(\mathbf{x}, t) = \int_{\mathbb{R}^d} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}. \tag{1.2}$$

While in a more accurate model, magnetic effects and Maxwell's equation for the force fields should be considered, we assume that they are negligible and compute the force field from the Poisson equation,

$$-\Delta \Phi = \rho(\mathbf{x}, t) - 1, \quad (\mathbf{x}, t) \in \Omega_x \times [0, T], \tag{1.3}$$

where $\mathbf{E}(\mathbf{x}, t) = \nabla_{\mathbf{x}} \Phi$ is the electrostatic field per unit mass, up to a sign, acting on particles. Here, we set all physical constants appearing in the equations to one for simplicity. Its solution allows us to compute the electric potential $\Phi(\mathbf{x}, t)$ due to both the self-consistent part coming from the macroscopic density $\rho(\mathbf{x}, t)$ and a uniform background ion density normalized to one. In plasma applications the system has to be globally neutral, meaning that the total charge of the system is zero,

$$\int_{\Omega_x} \rho(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega_{\mathbf{x}}} \int_{\mathbb{R}^d} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} d\mathbf{x} = 1. \tag{1.4}$$

This is a compatibility condition imposed by the periodicity of the boundary conditions for the Poisson equation (1.3). We refer to Refs. 49, 16 and 38 for good accounts on the state-of-the-art in the mathematical analysis and properties of the solutions of the Cauchy problem for the VP system. Global classical solutions were constructed in Ref. 9 for the system (1.1)–(1.3) with periodic boundary conditions in space and with compactly supported in velocity $C^2(\Omega_{\mathbf{x}} \times \mathbb{R}^d)$-initial data. Since the solutions are shown to remain compactly supported in velocity if initially so, we will assume without loss of generality that there exists $L > 0$ such that

$v \in \Omega_v = [-L, L]^d$ and that

$$\mathrm{supp}_{(\mathbf{x},\mathbf{v})}(f(\cdot,t)) \subset \Omega_{\mathbf{x}} \times (-L, L)^d$$

for all $0 \leq t \leq T$ for a given fixed $T > 0$. The VP system is an infinite-dimensional Hamiltonian which has infinitely many conserved quantities, in particular all L^p-norms of the distribution function, $1 \leq p \leq \infty$, and the total (kinetic + potential) energy are preserved in time.

The large number of physical applications and technological implications of the behavior of plasmas and the fact that kinetic descriptions are more accurate, have prompted an upsurge in the research of numerical simulations for the VP system, in the last decades. Most of the first attempts were based on particle-like or stochastic methods[14,31,45] due to the computational limitations and the high dimensionality of the system. Nowadays, there is a strong interest in the design and understanding of accurate deterministic solvers. Different Eulerian approaches have already proved their usefulness in the simulation of challenging questions as the Landau damping of Langmuir waves or the two stream instability. We mention splitting schemes[28,43,63,54,39]; finite volume schemes[40]; semi-Lagrangian schemes[42,10,24,25,32]; spectral methods[68] and conforming finite elements,[65,64,15] among others. Extensive comparisons of the performance of different methods are performed in Ref. 41.

Recently, the use of discontinuous Galerkin (DG) techniques has been brought into the design and simulation of Eulerian solvers for kinetic equations, in different contexts. DG techniques are extremely versatile; they share with finite volume method[a] their ability in approximating hyperbolic problems; but they are also extremely flexible for high-order approximations.

For the numerical simulation of one-dimensional VP system, we find a few works using DG techniques: DG methods are used in Ref. 12 for the multi-water-bag approximation of the system; a piecewise constant DG approximation is considered in Refs. 44, 52, and in Ref. 59 results obtained with a semi-Lagrangian method combined with high-order DG interpolation are reported. In this last work the authors also used the positivity preserving limiter introduced in Ref. 66. We also refer to Ref. 51 for a detailed study of the performance and conservation properties of the high-order DG schemes introduced and analyzed in this paper.

All the efforts in the simulation with the different proposed solvers for VP have benefited also from the theoretical studies of the schemes. However, this field is surely less developed, in part due to the not complete understanding of the continuous problem but also to the inherent difficulties of the resulting nonlinear high-dimensional problem. Except from some theoretical works for particle methods[45]; error analysis and convergence results for Eulerian schemes have been given

[a]Since piecewise constant DG methods are completely equivalent to a finite volume scheme.

so far only for the one-dimensional problem: for finite volumes in Ref. 40; for semi-Lagrangian schemes in Refs. 11, 13, 24, 32, and more recently for discontinuous Galerkin methods in Ref. 6.

Here, we take up the issue of convergence and rigorous error analysis of DG approximation for the multi-dimensional ($d = 2, 3$) VP system. This work is the natural continuation of Ref. 6, however new complications arise for the multi-dimensional case. We introduce a family of DG methods for (1.1)–(1.3) based on the coupling of a DG approximation to the Vlasov equation (transport equation) with several mixed finite element methods for the Poisson problem with either $H(\text{div}; \Omega_{\mathbf{x}})$-conforming (and hence classical) or discontinuous finite element spaces. The use of mixed methods is suggested by the structure of the VP system, since the transport in the velocity variable in the Vlasov equation (1.1) is given by the electrostatic field and not the potential. We remark that most of the solvers proposed in the literature, if not all, employ either a primal method or a direct discretization of the closed form of the solution to (1.3), the latter approach being not suitable for the higher-dimensional problem.

By construction (thanks to DG) our family of methods enjoy the charge conservation property. We also introduce a particular DG-LDG method for which we prove that it preserves the total energy of the system, provided finite element spaces contain at least quadratic polynomials. We point out that, recently, in Ref. 51 the validation of such scheme is studied in detail for many benchmark problems in plasma physics and it is also observed that the restriction on the polynomial degree required by our proof is indeed necessary.

We present the error analysis of the proposed family of methods, in the case of smooth compactly supported solutions. Optimal error bounds in L^2 are given for both the distribution function and the electrostatic field. To avoid the loss of half-order, typical of classical error analyses for hyperbolic problems, we introduce some special projections, inspired mainly by Ref. 55, that exploit the structure of the mesh and extend to higher dimension the ones introduced in Ref. 6.

It is worth noticing that, unlike what often happens in the convergence and error analysis of numerical methods for nonlinear problems, in our analysis we do not require any *a priori* assumption on the approximation to the distribution function or the electrostatic field or mesh restriction. As a consequence the error bounds proved are not asymptotic; i.e. they hold for any mesh size $h < 1$, which has special relevance in view of the complexity of the possible computations. We deal with the nonlinearity, by proving L^∞ bounds on the approximate electrostatic field. We wish to mention that the proof of this result, for both the LDG and classical mixed methods, is of independent interest. Although there is a large amount of work in the literature, devoted to the L^∞ and pointwise error analysis for the approximation of a "linear" Poisson problem (see Refs. 46 and 27), the case where the forcing term in the Poisson problem depends itself on the solution, has not been treated before for mixed and DG approximations. Our analysis is partially inspired by Refs. 61 and 60, where the authors deal with the conforming approximation of a "general" Poisson problem taking into account the outside influence of the forcing term. However since Refs. 61 and 60 deal with standard conforming approximation, the results and arguments used in these works cannot be directly applied nor adapted. For the case of classical mixed approximation, the seminal work[46] can be more easily extended to cover the present situation.

The outline of the paper is as follows. In Sec. 2 we present the basic notations needed for the description and analysis of the numerical methods. We also revise some well-known results that will be used in the paper. In Sec. 3 we introduce our numerical methods for approximating the VP system and show stability of the proposed schemes. The error analysis is carried out in Sec. 4, and we discuss the issue of energy conservation in Sec. 4.4. The paper is completed with Appendix A containing the proofs of the error estimates for the electrostatic field.

2. Preliminaries and Basic Notation

In this section we review the basic notation for the discrete setting and the definition of the finite element spaces together with their basic properties. Throughout the paper, we use the standard notation for Sobolev spaces.[2] For a bounded domain $B \subset \mathbb{R}^{2d}$, we denote by $H^m(B)$ the L^2-Sobolev space of order $m \geq 0$ and by $\|\cdot\|_{m,B}$ and $|\cdot|_{m,B}$ the usual Sobolev norm and semi-norm, respectively. For $m = 0$, we write $L^2(B)$ instead of $H^0(B)$. We shall denote by $H^m(B)/\mathbb{R}$ the quotient space consisting of equivalence classes of elements of $H^m(B)$ differing by constants; for $m = 0$ it is denoted by $L^2(B)/\mathbb{R}$. We shall indicate by $L^2_0(B)$ the space of $L^2(B)$ functions having zero average over B. This notation will also be used for periodic Sobolev spaces without any other explicit reference to periodicity to avoid cumbersome notations.

2.1. *Domain partitioning and finite element spaces*

Let $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ and $\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$ be two families of Cartesian partitions of $\Omega_{\mathbf{x}}$ and $\Omega_{\mathbf{v}}$, respectively, formed by rectangles for $d = 2$ and cubes for $d = 3$. Let $\{\mathcal{T}_h\}$ be defined as the Cartesian product of these two partitions: $\mathcal{T}_h := \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}} \times \mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$; i.e.

$$\mathcal{T}_h := \{\mathcal{R} = T^{\mathbf{x}} \times T^{\mathbf{v}} : T^{\mathbf{x}} \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}, T^{\mathbf{v}} \in \mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}\}.$$

The mesh sizes h, $h_{\mathbf{x}}$ and $h_{\mathbf{v}}$ relative to the partitions are defined as usual

$$0 < h_{\mathbf{x}} = \max_{T^{\mathbf{x}} \in \mathcal{T}^{x}_{h_{\mathbf{x}}}} \operatorname{diam}(T^{\mathbf{x}}), \quad 0 < h_{\mathbf{v}} = \max_{T^{\mathbf{v}} \in \mathcal{T}^{v}_{h_{\mathbf{v}}}} \operatorname{diam}(T^{\mathbf{v}}), \quad h = \max(h_{\mathbf{x}}, h_{\mathbf{v}}).$$

We denote by $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{E}_{\mathbf{v}}$ the set of all edges of the partitions $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ and $\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$, respectively and we set $\mathcal{E} = \mathcal{E}_{\mathbf{x}} \times \mathcal{E}_{\mathbf{v}}$. The sets of interior and boundary edges of the partition $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ (respectively, $\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$) are denoted by $\mathcal{E}^0_{\mathbf{x}}$ (respectively, $\mathcal{E}^0_{\mathbf{v}}$) and $\mathcal{E}^{\partial}_{\mathbf{x}}$ (respectively, $\mathcal{E}^{\partial}_{\mathbf{v}}$), so that $\mathcal{E}_{\mathbf{x}} = \mathcal{E}^0_{\mathbf{x}} \cup \mathcal{E}^{\partial}_{\mathbf{x}}$ (respectively, $\mathcal{E}_{\mathbf{v}} = \mathcal{E}^0_{\mathbf{v}} \cup \mathcal{E}^{\partial}_{\mathbf{v}}$).

Trace operators. Observe that due to the structure of the transport equation (1.1), for each $\mathcal{R} = T^{\mathbf{x}} \times T^{\mathbf{v}} \in \mathcal{T}_h$ with $T^{\mathbf{x}} \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ and $T^{\mathbf{v}} \in \mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$ and for each $\varphi \in H^1(T^{\mathbf{x}} \times T^{\mathbf{v}})$ we only need to define the traces of ϕ at $\partial T^{\mathbf{x}} \times T^{\mathbf{v}}$ and $T^{\mathbf{x}} \times \partial T^{\mathbf{v}}$. Hence, for setting the notation, it is enough to consider a general element T in either $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ or $\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$. By $\mathbf{n}^-_{|\partial T}$ we designate the outward normal to the element T and we denote by φ^- the interior trace of $\varphi_{|T}$ on ∂T and φ^+ refers to the outer trace on ∂T of $\varphi_{|T}$. That is,

$$\varphi^{\pm}_T(\mathbf{x}, \cdot) = \lim_{\epsilon \longrightarrow 0} \varphi_T(\mathbf{x} \pm \epsilon \mathbf{n}^-, \cdot) \quad \forall \mathbf{x} \in \partial T. \tag{2.1}$$

We next define the trace operators, but to avoid complications with fixing some privileged direction we follow Ref. 5. Let T_- and T_+ be two neighboring elements in either $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ or $\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$, and let $\mathbf{n}^-$ and $\mathbf{n}^+$ be their outward normal unit vectors, and let $\varphi^{\pm}$ and $\boldsymbol{\tau}^{\pm}$ be the restriction of φ and $\boldsymbol{\tau}$ to $T_{\pm}$. Following Ref. 5 we set

$$\{\varphi\} = \frac{1}{2}(\varphi^- + \varphi^+), \quad [\![\varphi]\!] = \varphi^- \mathbf{n}^- + \varphi^+ \mathbf{n}^+ \qquad \text{on } e \in \mathcal{E}^0_{\mathbf{r}}, \quad \mathbf{r} = \mathbf{x} \text{ or } \mathbf{v}, \tag{2.2}$$

$$\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^- + \boldsymbol{\tau}^+), \quad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^- \cdot \mathbf{n}^- + \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ \quad \text{on } e \in \mathcal{E}^0_{\mathbf{r}}, \quad \mathbf{r} = \mathbf{x} \text{ or } \mathbf{v}. \tag{2.3}$$

We also introduce a weighted average, for both scalar- and vector-valued functions, as follows: with each internal edge $e = T^+ \cap T^-$ and each $0 \leq \delta \leq 1$, we define

$$\{\boldsymbol{\tau}\}_\delta := \delta \boldsymbol{\tau}^+ + (1 - \delta)\boldsymbol{\tau}^- \quad \text{on internal edges.} \tag{2.4}$$

For $e \in \mathcal{E}^{\partial}_{\mathbf{r}}$ (with $\mathbf{r} = \mathbf{x}$ or $\mathbf{v}$), we set $[\![\varphi]\!] = \varphi \mathbf{n}$, $\{\varphi\} = \varphi$ and $\{\boldsymbol{\tau}\} = \boldsymbol{\tau}$. Notice that when referring to elements rather than edges, according to (2.1), φ^- can be seen as the inner trace relative to T^- (i.e. $\varphi^-_{T^-}$) and also as the outer trace relative to T^+ (i.e. $\varphi^+_{T^+}$). Similarly, $\mathbf{n}^-$ denotes the outward normal to T^- and also the inner normal to T^+. Both notations will be used interchangeable. We shall make extensive use of the following identity[4]

$$\sum_{T \in \mathcal{T}^{\mathbf{r}}} \int_{\partial T^{\mathbf{r}}} \boldsymbol{\tau} \cdot \mathbf{n} \varphi ds_{\mathbf{r}} = \int_{\mathcal{E}_{\mathbf{r}}} \{\boldsymbol{\tau}\} \cdot [\![\varphi]\!] ds_{\mathbf{r}} + \int_{\mathcal{E}^0_{\mathbf{r}}} [\![\boldsymbol{\tau}]\!] \{\varphi\} ds_{\mathbf{r}} \quad \mathbf{r} = \mathbf{x}, \mathbf{v}, \tag{2.5}$$

where the shortcut notation $\int_{\mathcal{E}_{\mathbf{r}}} = \sum_{e \in \mathcal{E}_{\mathbf{r}}} \int_e$ is used. Next, for $k \geq 0$, we define the discontinuous finite element spaces V^k_h, $\mathcal{Z}^k_h$ and $\boldsymbol{\Sigma}^k_h$,

$$\begin{aligned}
\mathcal{Z}^k_h &:= \{\varphi \in L^2(\Omega) : \varphi_{|\mathcal{R}} \in \mathbb{Q}^k(T^{\mathbf{x}}) \times \mathbb{Q}^k(T^{\mathbf{v}}),\ \forall \mathcal{R} = T^{\mathbf{x}} \times T^{\mathbf{v}} \in \mathcal{T}_h\},\\
X^k_h &= \{\psi \in L^2(\Omega_{\mathbf{x}}) : \psi_{|T^{\mathbf{x}}} \in \mathbb{Q}^k(T^{\mathbf{x}}),\ \forall T^{\mathbf{x}} \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}\},\\
V^k_h &= \{\psi \in L^2(\Omega_{\mathbf{v}}) : \psi_{|T^{\mathbf{v}}} \in \mathbb{Q}^k(T^{\mathbf{v}}),\ \forall T^{\mathbf{v}} \in \mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}\},\\
\boldsymbol{\Xi}^k_h &= \{\boldsymbol{\tau} \in (L^2(\Omega_{\mathbf{x}}))^d : \boldsymbol{\tau}_{|T^{\mathbf{x}}} \in (\mathbb{Q}^k(T^{\mathbf{x}}))^d,\ \forall T^{\mathbf{x}} \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}\},
\end{aligned}$$

where $\mathbb{Q}^k(T)$ (respectively, $(\mathbb{Q}^k(T)^d)$ is the space of scalar (respectively, vectorial) polynomials of degree at most k in each variable. We also set $Q_h^k = X_h^k \cap L_0^2(\Omega_{\mathbf{x}})$. We finally introduce the Raviart–Thomas finite element space:

$$\mathbf{\Sigma}_h^k = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega_{\mathbf{x}}) : \boldsymbol{\tau}_{|T^{\mathbf{x}}} \in RT^k(T^{\mathbf{x}}),\ \forall T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}\},$$

where

$$H(\mathrm{div};\Omega_{\mathbf{x}}) = \{\boldsymbol{\tau} \in (L^2(\Omega_{\mathbf{x}}))^d \text{ with } \mathrm{div}(\boldsymbol{\tau}) \in L^2(\Omega_{\mathbf{x}}) \text{ and } \boldsymbol{\tau} \cdot \mathbf{n}_{\partial\Omega} \text{ periodic on } \partial\Omega\}$$

and $RT^k(T^{\mathbf{x}}) := \mathbb{Q}^k(T^{\mathbf{x}})^d + \mathbf{x} \cdot \mathbb{Q}^k(T^{\mathbf{x}})$ (see Ref. 21 for further details). We shall denote by $\|\cdot\|_{H(\mathrm{div};\Omega_{\mathbf{x}})}$ the $H(\mathrm{div};\Omega_{\mathbf{x}})$-norm defined by

$$\|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega_{\mathbf{x}})}^2 := \|\boldsymbol{\tau}\|_0^2 + \|\mathrm{div}(\boldsymbol{\tau})\|_0^2 \quad \forall \boldsymbol{\tau} \in H(\mathrm{div};\Omega_{\mathbf{x}}).$$

Remark 2.1. We have chosen to present the classical mixed approximation specifying one $H(\mathrm{div};\Omega_{\mathbf{x}})$-conforming finite element spaces: the Raviart–Thomas finite element spaces. However, we wish to note that one could have chosen the *Brezzi–Douglas–Marini*[19,20] BDM_{k+1} or the *Brezzi–Douglas–Fortin–Marini*[18] BDFM_{k+1} finite element spaces as well and all the results given here hold for them without changes.

2.2. *Technical tools*

We start by defining the following semi-norm and norms that will be used in our analysis:

$$|\varphi|_{1,h}^2 = \sum_{\mathcal{R}\in\mathcal{T}_h} |\varphi|_{1,\mathcal{R}}^2, \quad \|\varphi\|_{m,\mathcal{T}_h}^2 := \sum_{\mathcal{R}\in\mathcal{T}_h} \|\varphi\|_{m,\mathcal{R}}^2 \quad \forall \varphi \in H^m(\mathcal{T}_h),\ m \geq 0,$$

$$\|\varphi\|_{0,\infty,\mathcal{T}_h} = \sup_{\mathcal{R}\in\mathcal{T}_h} \|\varphi\|_{0,\infty,\mathcal{R}}, \quad \|\varphi\|_{L^p(\mathcal{T}_h)}^p := \sum_{\mathcal{R}\in\mathcal{T}_h} \|\varphi\|_{L^p(\mathcal{R})}^p \quad \forall \varphi \in L^p(\mathcal{T}_h),$$

for all $1 \leq p < \infty$. We also introduce the following norms over the skeleton of the finite element partition,

$$\|\varphi\|_{0,\mathcal{E}_{\mathbf{x}}}^2 := \sum_{e\in\mathcal{E}_{\mathbf{x}}} \int_{\Omega_v} \int_e |\varphi|^2 ds_{\mathbf{x}} d\mathbf{v}, \quad \|\varphi\|_{0,\mathcal{E}_{\mathbf{v}}}^2 = \sum_{e\in\mathcal{E}_{\mathbf{v}}} \int_{\Omega_{\mathbf{x}}} \int_e |\varphi|^2 ds_{\mathbf{v}} d\mathbf{x}.$$

Then, we define $\|\varphi\|_{0,\mathcal{E}_h}^2 = \|\varphi\|_{0,\mathcal{E}_{\mathbf{x}}}^2 + \|\varphi\|_{0,\mathcal{E}_{\mathbf{v}}}^2$.

Projection operators. Let $k \geq 0$ and let $\mathcal{P}_h : L^2(\Omega) \to \mathcal{Z}_h^k$ be the standard L^2-projection. We denote by $\mathcal{P}_{\mathbf{x}} : L^2(\Omega) \to X_h^k$ and $\mathcal{P}_{\mathbf{v}} : L^2(\Omega) \to V_h^k$ the standard

d-dimensional L^2-projections onto the spaces X_h^k and V_h^k, respectively, and we note that $\mathcal{P}_h = \mathcal{P}_{\mathbf{x}} \otimes \mathcal{P}_{\mathbf{v}}$ satisfies (see Refs. 29 and 3)

$$\|w - \mathcal{P}_h(w)\|_{0,\mathcal{T}_h} + h^{1/2}\|w - \mathcal{P}_h(w)\|_{0,\mathcal{E}_h} \leq Ch^{k+1}\|w\|_{k+1,\Omega} \quad \forall\, w \in H^{k+1}(\Omega), \tag{2.6}$$

with C depending only on the shape regularity of the triangulation and the polynomial degree. By definition, $\mathcal{P}_h$ is stable in L^2 and it can be further shown to be stable in all L^p-norms (see Ref. 34 for details):

$$\|\mathcal{P}_h(w)\|_{L^p(\mathcal{T}_h)} \leq C\|w\|_{L^p(\Omega)} \quad \forall\, w \in L^p(\Omega),\ \ 1 \leq p \leq \infty. \tag{2.7}$$

We will also need approximation properties in the supremum-norm[29]:

$$\|w - \mathcal{P}_h(w)\|_{0,\infty,\mathcal{T}_h} \leq Ch^{k+1}\|w\|_{k+1,\infty,\Omega} \quad \forall\, w \in W^{k+1,\infty}(\Omega). \tag{2.8}$$

We wish to stress that the projections $\mathcal{P}_{\mathbf{x}}$ and $\mathcal{P}_{\mathbf{v}}$ also satisfy properties (2.7) and (2.8). Furthermore, we will also use

$$\|w - \mathcal{P}_{\mathbf{r}}(w)\|_{0,\mathcal{T}_h} \leq Ch^{k+1}\|w\|_{k+1,\Omega} \quad \forall\, w \in H^{k+1}(\Omega),\ \ \mathbf{r} = \mathbf{x} \text{ or } \mathbf{v}. \tag{2.9}$$

Raviart–Thomas projection. For $k \geq 0$ we denote by $\mathcal{R}_h^k$ the local interpolation operator which satisfies the following commuting diagram:

$$\begin{array}{ccc} H(\mathrm{div};\Omega_{\mathbf{x}}) & \xrightarrow{\ \mathrm{div}\ } & L_0^2(\Omega_{\mathbf{x}}) \\ \mathcal{R}_h^k \Big\downarrow & & \Big\downarrow \widehat{P}_h^k \\ \Sigma_h^k & \xrightarrow{\ \mathrm{div}\ } & Q_h^k, \end{array}$$

where $\widehat{P}_h^k$ refers to the standard L^2-projection operator onto Q_h^k. The above commuting diagram expresses that $\mathrm{div}(\Sigma_h^k) = Q_h^k$ and

$$\mathrm{div}\, \mathcal{R}_h^k(\boldsymbol{\tau}) = \widehat{P}_h^k(\mathrm{div}\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in H(\mathrm{div};\Omega_{\mathbf{x}}). \tag{2.10}$$

In particular (2.10) holds for all $\boldsymbol{\tau} \in H^1(\Omega_{\mathbf{x}})^d$. Optimal L^p-approximation properties, with $2 \leq p \leq \infty$ can be shown for this operator (see Refs. 21 and 47 for details):

$$\|\boldsymbol{\tau} - \mathcal{R}_h^k(\boldsymbol{\tau})\|_{L^p(\Omega_{\mathbf{x}})} + \|\mathrm{div}(\boldsymbol{\tau} - \mathcal{R}_h^k(\boldsymbol{\tau}))\|_{L^p(\Omega_{\mathbf{x}})} \leq Ch^{k+1}\|\boldsymbol{\tau}\|_{W^{k+1,p}(\Omega_{\mathbf{x}})}, \tag{2.11}$$

for all $\boldsymbol{\tau} \in W^{k+1,p}(\Omega_{\mathbf{x}})^d$. We notice that all the approximation and stability results stated here for the standard L^2-projection, hold true also for the L^2-projection onto Q_h^k; i.e. $\widehat{P}_h^k$.

3. Numerical Methods and Stability

In this section we describe the numerical methods we propose for approximating the VP system (1.1)–(1.3) and prove stability for the proposed schemes. Following the work initiated in Ref. 6, the proposed numerical schemes are based on the coupling of a simple DG discretization of the Vlasov equation and some suitable finite element approximation, possibly discontinuous, to the Poisson problem.

Thanks to the special Hamiltonian structure of the Vlasov equation (1.1): $\mathbf{v}$ is independent of $\mathbf{x}$ and $\mathbf{E}$ is independent of $\mathbf{v}$; for all methods the DG approximation for the electron distribution function is done exactly in the same way. Therefore we first present the DG method for the transport equation (1.1), postponing the description of the approximation to the Poisson problem (1.3) to the last part of the section.

While describing the numerical schemes, we will also state a number of approximation results. The proofs of most of them, except for the stability and particle conservation, are postponed to Appendix A.

3.1. *Discontinuous Galerkin approximation for the Vlasov equation*

Throughout this section, we denote by $\mathbf{E}_h \in \widetilde{\mathbf{\Sigma}}$ the FE approximation to the electrostatic field to be specified later. We consider a DG approximation for the Vlasov equation coupled with a finite element approximation to the Poisson problem. The DG approximation to (1.1) reads: Find $(\mathbf{E}_h, f_h) \in \mathcal{C}^1([0,T]; \widetilde{\mathbf{\Sigma}} \times \mathcal{Z}_h^k)$ such that

$$\sum_{R\in\mathcal{T}_h} \mathcal{B}_{h,R}(\mathbf{E}_h; f_h, \varphi_h) = 0 \quad \forall\, \varphi_h \in \mathcal{Z}_h^k, \tag{3.1}$$

and $f_h(0) = \mathcal{P}_h(f_0)$, the L^2-projected initial data, where $\forall\, R = T^{\mathbf{x}} \times T^{\mathbf{v}} \in \mathcal{T}_h$,

$$\begin{aligned}\mathcal{B}_{h,R}(\mathbf{E}_h; f_h, \varphi_h) = {} & \int_R \frac{\partial f_h}{\partial t}\varphi_h d\mathbf{v}d\mathbf{x} - \int_R f_h \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi_h d\mathbf{v}d\mathbf{x} \\ & + \int_R f_h \mathbf{E}_h \cdot \nabla_{\mathbf{v}}\varphi_h d\mathbf{v}d\mathbf{x} + \int_{T^{\mathbf{v}}}\int_{\partial T^{\mathbf{x}}} (\widehat{\mathbf{v}\cdot\mathbf{n}f_h})\varphi_h ds_{\mathbf{x}} d\mathbf{v} \\ & - \int_{T^{\mathbf{x}}}\int_{\partial T^{\mathbf{v}}} (\widehat{\mathbf{E}_h\cdot\mathbf{n}f_h})\varphi_h ds_{\mathbf{v}} d\mathbf{x} \quad \forall\, \varphi_h \in \mathcal{Z}_h^k.\end{aligned}$$

In this paper, we have slightly abused notation by using $\mathbf{n}$ to denote both $\mathbf{n}^-_{|\partial T^{\mathbf{x}}}$ and $\mathbf{n}^-_{|\partial T^{\mathbf{v}}}$ in the boundary integrals in (3.1). On each interior $(2d-1)$-dimensional face $e^x \subset (\partial T^{\mathbf{x}} \cap \mathcal{E}^0_{\mathbf{x}}) \times T^{\mathbf{v}}$ and $e^v \subset T^{\mathbf{x}} \times (\partial T^{\mathbf{v}} \cap \mathcal{E}^0_{\mathbf{v}})$, the numerical fluxes $(\widehat{\mathbf{v}\cdot\mathbf{n}f_h})$

and $(\widehat{\mathbf{E}_h\cdot\mathbf{n}f_h})$ are defined, respectively, by

$$\begin{aligned}\widehat{\mathbf{v}\cdot\mathbf{n}f}_{h|_{e^x}} &= \begin{cases}\mathbf{v}\cdot\mathbf{n}(f_h)^-_{|T^{\mathbf{x}}} & \text{if } \mathbf{v}\cdot\mathbf{n}>0,\\ \mathbf{v}\cdot\mathbf{n}(f_h)^+_{|T^{\mathbf{x}}} & \text{if } \mathbf{v}\cdot\mathbf{n}<0, \quad e^x\subset\partial T^{\mathbf{x}}\times T^{\mathbf{v}},\\ \{\mathbf{v}\cdot\mathbf{n}f_h\} & \text{if } \mathbf{v}\cdot\mathbf{n}=0,\end{cases}\\ \widehat{\mathbf{E}_h\cdot\mathbf{n}f}_{h|_{e^v}} &= \begin{cases}\mathbf{E}_h\cdot\mathbf{n}(f_h)^+_{|T^{\mathbf{v}}} & \text{if } \mathbf{E}_h\cdot\mathbf{n}>0,\\ \mathbf{E}_h\cdot\mathbf{n}(f_h)^-_{|T^{\mathbf{v}}} & \text{if } \mathbf{E}_h\cdot\mathbf{n}<0, \quad e^v\subset T^{\mathbf{x}}\times\partial T^{\mathbf{v}}.\\ \{\mathbf{E}_h\cdot\mathbf{n}f_h\} & \text{if } \mathbf{E}_h\cdot\mathbf{n}=0,\end{cases}\end{aligned} \tag{3.2}$$

On boundary edges, $e^r\in\mathcal{E}^0_{\mathbf{r}}, r=x,v$, we impose the periodicity for $\widehat{\mathbf{v}\cdot\mathbf{n}f_h}$ and compactness for $\widehat{\mathbf{E}_h\cdot\mathbf{n}f_h}$. Notice that the (upwind) fluxes defined in (3.2) are consistent and conservative. Now, taking into account the definition of the weighted average (2.4) and that of the standard trace operators (2.2) and (2.3) and the fact that for each fixed e, $\mathbf{n}^-=-\mathbf{n}^+$, the upwind numerical fluxes (3.2) can be rewritten in terms of the weighted average (see Refs. 22 and 7 for details). More precisely, we have for all $T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$ and for all $T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$

$$\begin{cases}\widehat{\mathbf{v}\cdot\mathbf{n}f_h}=\{\mathbf{v}f_h\}_\alpha\cdot\mathbf{n}:=\left(\{\mathbf{v}f_h\}+\dfrac{|\mathbf{v}\cdot\mathbf{n}|}{2}[\![f_h]\!]\right)\cdot\mathbf{n} & \text{on } \mathcal{E}^0_{\mathbf{x}}\times T^{\mathbf{v}},\\ \widehat{\mathbf{E}_h\cdot\mathbf{n}f_h}=\{\mathbf{E}_hf_h\}_\beta\cdot\mathbf{n}:=\left(\{\mathbf{E}_hf_h\}-\dfrac{|\mathbf{E}_h\cdot\mathbf{n}|}{2}[\![f_h]\!]\right)\cdot\mathbf{n} & \text{on } T^{\mathbf{x}}\times\mathcal{E}^0_{\mathbf{v}},\end{cases} \tag{3.3}$$

with $\alpha=\frac{1}{2}(1\pm\mathrm{sign}(\mathbf{v}\cdot\mathbf{n}^\pm))$ and $\beta=\frac{1}{2}(1\mp\mathrm{sign}(\mathbf{E}_h\cdot\mathbf{n}^\pm))$. Then, using formula (2.5) together with the conservativity property of the numerical fluxes, the DG scheme reads, for all $\varphi_h\in\mathcal{Z}^k_h$,

$$\begin{aligned}0 &= \sum_{\mathcal{R}\in\mathcal{T}_h}\mathcal{B}_{h,R}(\mathbf{E}_h;f_h,\varphi_h)\\ &= \sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}}\frac{\partial f_h}{\partial t}\varphi_h d\mathbf{v}d\mathbf{x}-\int_\Omega f_h\mathbf{v}\cdot\nabla^h_{\mathbf{x}}\varphi_h d\mathbf{v}d\mathbf{x}+\int_\Omega f_h\mathbf{E}_h\cdot\nabla^h_{\mathbf{v}}\varphi_h d\mathbf{v}d\mathbf{x}\\ &\quad+\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}\int_{\mathcal{E}_{\mathbf{x}}}\{\mathbf{v}f_h\}_\alpha\cdot[\![\varphi_h]\!]ds_{\mathbf{x}}d\mathbf{v}\\ &\quad-\sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}\int_{T^{\mathbf{x}}}\int_{\mathcal{E}_{\mathbf{v}}}\{\mathbf{E}_hf_h\}_\beta\cdot[\![\varphi_h]\!]ds_{\mathbf{v}}d\mathbf{x},\end{aligned} \tag{3.4}$$

where $\nabla^h_{\mathbf{x}}\varphi_h$ and $\nabla^h_{\mathbf{v}}\varphi_h$ are the functions whose restriction to each element $\mathcal{R}\in\mathcal{T}_h$ are equal to $\nabla_{\mathbf{x}}\varphi_h$ and $\nabla_{\mathbf{v}}\varphi_h$, respectively. The discrete density, ρ_h is defined by

$$\rho_h=\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}f_h d\mathbf{v}\in X^k_h. \tag{3.5}$$

The following result guarantees the particle conservation and the L^2-stability for the above scheme.

Lemma 3.1. (Particle or Mass conservation) *Let $k \geq 0$ and let $f_h \in \mathcal{C}^1([0,T]; \mathcal{Z}_h^k)$ be the DG approximation to f, satisfying* (3.1). *Then,*

$$\sum_{\mathcal{R}\in\mathcal{T}_h} \int_{\mathcal{R}} f_h(t) d\mathbf{x} d\mathbf{v} = \sum_{\mathcal{R}\in\mathcal{T}_h} \int_{\mathcal{R}} f_h(0) d\mathbf{x} d\mathbf{v} = \sum_{\mathcal{R}\in\mathcal{T}_h} \int_{\mathcal{R}} f_0 = 1 \quad \forall\, t.$$

Proof. The proof follows essentially the same lines as the proof of Lemma 3.1 in Ref. 6, by fixing some arbitrary $\mathcal{R} = \mathcal{R}_1$ and taking in (3.1) a test function φ, such that $\varphi_h = 1$ in $\mathcal{R}_1$ and $\varphi_h = 0$ elsewhere. □

We next show L^2-stability for the numerical method (3.1), which follows from the selection of the numerical fluxes.

Proposition 3.1. (L^2-stability) *Let $f_h \in \mathcal{Z}_h^k$ be the approximation of problem* (1.1)–(1.3), *solution of* (3.1) *with the numerical fluxes defined as in* (3.2). *Then*

$$\|f_h(t)\|_{0,\mathcal{T}_h} \leq \|f_h(0)\|_{0,\mathcal{T}_h} \quad \forall\, t \in [0,T].$$

Proof. The proof follows essentially the same steps as for the case $d = 1$. By setting $\varphi_h = f_h$ in (3.1), integrating the volume terms that result and using (2.5) one easily gets

$$\begin{aligned} 0 &= \sum_{\mathcal{R}\in\mathcal{T}_h} \mathcal{B}_{h,R}(\mathbf{E}_h; f_h, f_h) \\ &= \frac{1}{2} \sum_{\mathcal{R}\in\mathcal{T}_h} \left(\frac{d}{dt} \int_{\mathcal{R}} f_h^2 d\mathbf{v} d\mathbf{x} - \int_{T^{\mathbf{v}}} \int_{\mathcal{E}_{\mathbf{x}}} \mathbf{v} \cdot [\![f_h^2]\!] ds_{\mathbf{x}} d\mathbf{v} + \int_{T^{\mathbf{x}}} \int_{\mathcal{E}_{\mathbf{v}}} \mathbf{E}_h \cdot [\![f_h^2]\!] ds_{\mathbf{v}} d\mathbf{x} \right) \\ &\quad + \sum_{T^{\mathbf{v}}\in\mathcal{T}_{h_{\mathbf{v}}}^{\mathbf{v}}} \int_{T^{\mathbf{v}}} \int_{\mathcal{E}_{\mathbf{x}}} \{\mathbf{v} f_h\}_{\alpha} \cdot [\![f_h]\!] ds_{\mathbf{x}} d\mathbf{v} - \sum_{T^{\mathbf{x}}\in\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \int_{T^{\mathbf{x}}} \int_{\mathcal{E}_{\mathbf{v}}} \{\mathbf{E}_h f_h\}_{\beta} \cdot [\![f_h]\!] ds_{\mathbf{v}} d\mathbf{x}. \end{aligned}$$

Now, from the definition of the trace operators (2.2) it follows that $[\![f_h^2]\!] = 2\{f_h\}[\![f_h]\!]$ on $e \in \mathcal{E}_h^0$. Substituting the above identity together with the definition of the numerical fluxes given in (3.3), and using the periodic boundary conditions in $\mathbf{x}$ and compact support in $\mathbf{v}$, we have that

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{dt} \int_{\Omega} f_h^2 d\mathbf{v} d\mathbf{x} + \sum_{T^{\mathbf{v}}\in\mathcal{T}_{h_{\mathbf{v}}}^{\mathbf{v}}} \int_{T^{\mathbf{v}}} \int_{\mathcal{E}_x^0} \frac{|\mathbf{v}\cdot\mathbf{n}|}{2} [\![f_h]\!]^2 ds_{\mathbf{x}} d\mathbf{v} \\ &\quad + \sum_{T^{\mathbf{x}}\in\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \int_{T^{\mathbf{x}}} \int_{\mathcal{E}_v^0} \frac{|\mathbf{E}_h\cdot\mathbf{n}|}{2} [\![f_h]\!]^2 ds_{\mathbf{v}} d\mathbf{x}. \end{aligned}$$

Integration in time of the above equation, from 0 to t concludes the proof. □

We close this section stating an elementary approximation result that will be required in our analysis. Its proof is given in Appendix A.

Lemma 3.2. *Let $k \geq 0$ and f and f_h be the continuous and approximate solutions to the VP problem. Let ρ and ρ_h be the continuous and discrete densities defined in* (1.2) *and* (3.8). *Then,*

$$\|\rho - \rho_h\|_{0,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \leq C[\mathrm{meas}(\Omega_{\mathbf{v}})]^{\frac{1}{2}} \|f - f_h\|_{0,\mathcal{T}_h} \leq CL^{\frac{d}{2}} \|f - f_h\|_{0,\mathcal{T}_h}. \tag{3.6}$$

Furthermore, if $\rho \in W^{\frac{3}{2},d}(\Omega_{\mathbf{x}})$ and $f \in \mathcal{C}^1([0,T]; H^{k+1}(\Omega))$ we have

$$\|\rho - \rho_h\|_{-1,\infty,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \leq Ch^{\frac{3}{2}} \|\rho\|_{W^{\frac{3}{2},d}(\Omega_{\mathbf{x}})} + CL^{\frac{d}{2}} h^{1-\frac{d}{2}} (Ch^{k+1}\|f\|_{k+1,\Omega} + \|f_h - \mathcal{P}_h(f)\|_{0,\mathcal{T}_h}). \tag{3.7}$$

3.2. *Mixed finite element approximation to the Poisson problem*

We next consider the approximation to the discrete Poisson problem, which can be rewritten as the following first-order system:

$$\mathbf{E} = \nabla_{\mathbf{x}}\Phi \quad \text{in } \Omega_{\mathbf{x}}, \quad -\mathrm{div}_{\mathbf{x}}(\mathbf{E}) = \rho_h - 1 \quad \text{in } \Omega_{\mathbf{x}},$$
$$\rho_h = \sum_{T^{\mathbf{v}} \in \mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}} \int_{T^{\mathbf{v}}} f_h d\mathbf{v}, \tag{3.8}$$

with periodic boundary conditions for $\mathbf{E}$ and Φ. Notice that in view of Lemma 3.1 and by taking $\Phi \in L^2_0(\Omega_{\mathbf{x}})$, we guarantee that the above problem is well-posed. The weak formulation of the above problem reads: Find $(\mathbf{E}, \Phi) \in H(\mathrm{div}; \Omega_{\mathbf{x}}) \times L^2_0(\Omega_{\mathbf{x}})$ such that

$$\int_{\Omega_{\mathbf{x}}} \mathbf{E} \cdot \boldsymbol{\tau} d\mathbf{x} = \int_{\Omega_{\mathbf{x}}} \nabla_{\mathbf{x}}\Phi \cdot \boldsymbol{\tau} d\mathbf{x} = 0 \quad \forall\, \boldsymbol{\tau} \in H(\mathrm{div}; \Omega_{\mathbf{x}}),$$
$$-\int_{\Omega_{\mathbf{x}}} \mathrm{div}_{\mathbf{x}}(\mathbf{E}) q d\mathbf{x} = \int_{\Omega_{\mathbf{x}}} (\rho_h - 1) q d\mathbf{x} \qquad \forall\, q \in L^2_0(\Omega_{\mathbf{x}}).$$

Unlike for the one-dimensional case, where a direct integration of the Poisson equation provides a conforming finite element approximation to the electrostatic potential (see Ref. 6), for higher dimensions, we only consider mixed finite element approximation to the discrete Poisson problem with either $H(\mathrm{div}; \Omega_{\mathbf{x}})$-conforming or discontinuous finite element spaces. Throughout this section, we focus on the detailed description of the methods we consider, stating also the approximation results that will be needed in our subsequent error analysis. The proofs of all these results are postponed to Appendix A.

3.2.1. $H(\mathrm{div};\Omega_{\mathbf{x}})$*-conforming or classical mixed finite element approximation*

The approximation reads: Find $(\mathbf{E}_h,\Phi_h)\in\mathbf{\Sigma}_h^r\times Q_h^r$ satisfying

$$\begin{aligned}\int_{\Omega_{\mathbf{x}}}\mathbf{E}_h\cdot\boldsymbol{\tau}d\mathbf{x}+\int_{\Omega_{\mathbf{x}}}\Phi_h\mathrm{div}_{\mathbf{x}}(\boldsymbol{\tau})d\mathbf{x}&=0 && \forall\,\boldsymbol{\tau}\in\mathbf{\Sigma}_h^k,\\ -\int_{\Omega_{\mathbf{x}}}\mathrm{div}_{\mathbf{x}}(\mathbf{E}_h)q d\mathbf{x}&=\int_{\Omega_{\mathbf{x}}}(\rho_h-1)q d\mathbf{x} && \forall\,q\in Q_h^k.\end{aligned}\tag{3.9}$$

The following lemma provides error estimates in the energy and the uniform norm for the approximate electrostatic field. Let us emphasize that this uniform estimate is essential.

Lemma 3.3. *Let* $k\geq 0$ *and let* $(\mathbf{E}_h,\Phi_h)\in\mathcal{C}^0([0,T];\mathbf{\Sigma}_h^k\times X_h^k)$ *be the* RT_k *approximation to the Poisson problem* (3.8). *Assume* $\Phi\in\mathcal{C}^0([0,T];H^{k+2}(\Omega_{\mathbf{x}}))$. *Then, the following estimates hold for all* $t\in[0,T]$:

$$\begin{aligned}\|\mathbf{E}(t)-\mathbf{E}_h(t)\|_{H(\mathrm{div};\Omega_{\mathbf{x}})}&\leq Ch^{k+1}\|\Phi(t)\|_{k+2,\Omega_{\mathbf{x}}}\\&\quad+CL^{\frac{d}{2}}\|f(t)-f_h(t)\|_{0,\mathcal{T}_h}\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}\|\mathbf{E}(t)-\mathbf{E}_h(t)\|_{0,\infty,\Omega_{\mathbf{x}}}&\leq C\|\mathbf{E}-\mathcal{R}_h^k(\mathbf{E})\|_{0,\infty,\Omega_{\mathbf{x}}}\\&\quad+C|\log(h)|\|\rho-\rho_h\|_{-1,\infty,\Omega_{\mathbf{x}}}.\end{aligned}\tag{3.11}$$

Remark 3.1. We wish to stress that all the results shown in this paper for the Raviart–Thomas–DG method for VP remain valid if the RT_k finite element spaces used in the approximation for the Poisson problem are replaced by either *Brezzi–Douglas–Marini*[19,20] BDM_{k+1} or *Brezzi–Douglas–Fortin–Marini*[18] BDFM_{k+1} finite element spaces.

3.2.2. *Discontinuous Galerkin approximation*

For $r\geq 1$ the method reads: Find $(\mathbf{E}_h,\Phi_h)\in\mathbf{\Xi}_h^r\times Q_h^r$ such that

$$\int_{T^{\mathbf{x}}}\mathbf{E}_h\cdot\boldsymbol{\tau}d\mathbf{x}+\int_{T^{\mathbf{x}}}\Phi_h\,\mathrm{div}_{\mathbf{x}}(\boldsymbol{\tau})d\mathbf{x}-\int_{\partial T^{\mathbf{x}}}\widehat{\Phi_h}\boldsymbol{\tau}\cdot\mathbf{n}ds_{\mathbf{x}}=0\quad\forall\,\boldsymbol{\tau}\in\mathbf{\Xi}_h^r,\tag{3.12}$$

$$\int_{T^{\mathbf{x}}}\mathbf{E}_h\cdot\nabla_{\mathbf{x}}q d\mathbf{x}-\int_{\partial T^{\mathbf{x}}}q\widehat{\mathbf{E}_h}\cdot\mathbf{n}ds_{\mathbf{x}}=\int_{T^{\mathbf{x}}}(\rho_h-1)q d\mathbf{x}\qquad\forall\,q\in Q_h^r.\tag{3.13}$$

On interior edges, the numerical fluxes are defined as

$$\begin{cases}\widehat{\mathbf{E}_h}=\{\mathbf{E}_h\}-\mathbf{C}_{12}[\![\mathbf{E}_h]\!]-C_{11}[\![\Phi_h]\!] & \text{on }\mathcal{E}_{\mathbf{x}}^0,\\ \widehat{\Phi_h}=\{\Phi_h\}+\mathbf{C}_{12}\cdot[\![\Phi_h]\!]-C_{22}[\![\mathbf{E}_h]\!] & \text{on }\mathcal{E}_{\mathbf{x}}^0,\end{cases}\tag{3.14}$$

and on $e \in \mathcal{E}_{\mathbf{x}}^{\partial}$ we impose the periodicity for both $\widehat{\mathbf{E}_h}$ and $\widehat{\Phi_h}$. As for the case $d = 1$, the parameters $C_{11}, \mathbf{C}_{12}$ and C_{22} could be taken in several ways to try to achieve different levels of accuracy. However, all superconvergence results for the Hybridized DG (in $d \geq 2$) are for partitions made of simplexes (and the proof of these results rely strongly on that). As for the *minimal dissipation* MD-DG method (see Ref. 30 for details) one can expect, at most, an improvement of half an order in the error estimate for $\|\mathbf{E} - \mathbf{E}_h\|_{0,\mathcal{T}_{h_\mathbf{x}}^\mathbf{x}}$ for $d = 2$ (for a Poisson problem with Dirichlet boundary conditions). Therefore, throughout this section we will not further distinguish between the possible choices (since no improvement on the final rate of convergence could be achieved) and we set $r = k + 1$. One might stick to the classical LDG method for which $C_{22} = 0$ and $C_{11} = ch^{-1}$ with c a strictly positive constant, see Ref. 5.

Substituting the definition of the numerical fluxes (3.14) into (3.12)–(3.13) and summing over all elements of $\mathcal{T}_{h_\mathbf{x}}^\mathbf{x}$, we arrive at the mixed problem:

$$\begin{cases} a(\mathbf{E}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \Phi_h) = 0 & \forall\, \boldsymbol{\tau} \in \boldsymbol{\Xi}_h^r, \\ -b(\mathbf{E}_h, q) + c(\Phi_h, q) = \displaystyle\int_{\Omega_\mathbf{x}} (\rho_h - 1) q d\mathbf{x} & \forall\, q \in Q_h^r, \end{cases} \tag{3.15}$$

where

$$a(\mathbf{E}_h, \boldsymbol{\tau}) = \int_{\Omega_\mathbf{x}} \mathbf{E}_h \cdot \boldsymbol{\tau} d\mathbf{x}, \quad c(\Phi_h, q) = \int_{\mathcal{E}_\mathbf{x}} C_{11} [\![\Phi_h]\!] \cdot [\![q]\!] ds_\mathbf{x}, \quad \text{and}$$

$$b(\boldsymbol{\tau}, \Phi_h) = \int_{\Omega_\mathbf{x}} \Phi_h \nabla_\mathbf{x}^h \cdot \boldsymbol{\tau} d\mathbf{x} - \int_{\mathcal{E}_x^0} (\{\Phi_h\} + \mathbf{C}_{12} \cdot [\![\Phi_h]\!]) [\![\boldsymbol{\tau}]\!] ds_\mathbf{x} - \int_{\mathcal{E}_x^\partial} \Phi_h \boldsymbol{\tau} \cdot \mathbf{n} ds_\mathbf{x}.$$

Note that integration by parts of the volume term in $b(\boldsymbol{\tau}, \Phi_h)$, together with (2.5), gives

$$\begin{aligned} b(\boldsymbol{\tau}, \Phi_h) = &- \int_{\Omega_\mathbf{x}} \nabla_\mathbf{x}^h \Phi_h \cdot \boldsymbol{\tau} d\mathbf{x} + \int_{\mathcal{E}_\mathbf{x}^0} [\![\Phi_h]\!] \cdot (\{\boldsymbol{\tau}\} - \mathbf{C}_{12} [\![\boldsymbol{\tau}]\!]) ds_\mathbf{x} \\ &+ \int_{\mathcal{E}_\mathbf{x}^\partial} \Phi_h \boldsymbol{\tau} \cdot \mathbf{n} ds_\mathbf{x}. \end{aligned} \tag{3.16}$$

Defining the semi-norm $|(\boldsymbol{\tau}, q)|_\mathcal{A}^2 := \|\boldsymbol{\tau}\|_{0,\mathcal{T}_{h_\mathbf{x}}^\mathbf{x}}^2 + \|C_{11}^{\frac{1}{2}} [\![q]\!]\|_{0,\mathcal{E}_\mathbf{x}}^2$, we state the error estimates for the approximation $(\mathbf{E}_h, \Phi_h)$.

Lemma 3.4. *Let $r \geq 1$ and let $(\mathbf{E}_h, \Phi_h) \in \mathcal{C}^0([0,T]; \boldsymbol{\Xi}_h^r \times Q_h^r)$ be the LDG approximation to the Poisson problem solution of* (3.12)–(3.13). *Assume $(\mathbf{E}, \Phi) \in \mathcal{C}^0([0,T]; H^{r+1}(\Omega_\mathbf{x}) \times H^{r+2}(\Omega_\mathbf{x}))$. Then, the following estimates hold:*

$$|(\mathbf{E} - \mathbf{E}_h, \Phi - \Phi_h)|_\mathcal{A} \leq Ch^s \|\Phi\|_{r+2,\Omega_\mathbf{x}} + CL^{\frac{d}{2}} \|f - f_h\|_{0,\mathcal{T}_h} \tag{3.17}$$

and

$$\|\mathbf{E}-\mathbf{E}_h\|_{0,\infty} \leq C|\log(h)|^{\bar{r}}(\|\mathbf{E}-\mathcal{P}_{\mathbf{x}}(\mathbf{E})\|_{0,\infty,\Omega_{\mathbf{x}}} + h^{-1}\|\Phi-\mathcal{P}_{\mathbf{x}}(\Phi)\|_{0,\infty,\Omega_{\mathbf{x}}}) + C|\log(h)|\|\rho-\rho_h\|_{-1,\infty,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}}, \tag{3.18}$$

for all $t \in [0,T]$, *where* $\bar{r}=1$ *for* $r=1$ *and* $\bar{r}=0$ *for* $r>1$.

In the sequel and whenever there is no ambiguity, the dependence of $\mathbf{E}$, $\mathbf{E}_h$, Φ and Φ_h on t is dropped for the sake of clarity as in (3.17) and (3.18).

4. Main Results and Error Analysis

In this section, we now carry out the error analysis for the proposed DG approximations for the VP system. The main result of this section is the following theorem.

Theorem 4.1. *Let* $\Omega = \Omega_{\mathbf{x}} \times \Omega_{\mathbf{v}} = [0,1]^d \times [-L,L]^d \subset \mathbb{R}^{2d}$, $d=2,3$. *Let* $k \geq 1$ *and let* $f \in \mathcal{C}^1([0,T]; H^{k+2}(\Omega) \cap W^{1,\infty}(\Omega))$ *be the compactly supported solution at time* $t \in [0,T]$ *of the VP problem* (1.1)–(1.3) *and let* $\mathbf{E} \in \mathcal{C}^0([0,T]; H^{k+1}(\Omega_{\mathbf{x}})^d \cap W^{1,\infty}(\Omega_{\mathbf{x}})^d)$ *with* $d=2$ *or* 3 *be the associated electrostatic potential. Then,*

(a) *RT*$_k$*-DG method. If* $((\mathbf{E}_h,\Phi_h),f_h) \in \mathcal{C}^0([0,T]; (\boldsymbol{\Sigma}_h^k \times Q_h^k)) \times \mathcal{C}^1([0,T]; \mathcal{Z}_h^k)$ *is the RT*$_k$*-DG approximation solution of* (3.4)–(3.9), *the following estimates hold:*

$$\|f(t)-f_h(t)\|_{0,\Omega} \leq C_a h^{k+1} \quad \forall t \in [0,T]$$

and

$$\|\mathbf{E}(t)-\mathbf{E}_h(t)\|_{H(\mathrm{div};\Omega_{\mathbf{x}})} \leq C h^{k+1}\|\Phi(t)\|_{k+2,\Omega_{\mathbf{x}}} + C_a h^{k+1} \quad \forall t \in [0,T],$$

where C_a *depends on the final time* T, *the polynomial degree* k, *the shape regularity of the partition and it also depends on* f *through the norms*

$$C_a(t) = C(\|f(t)\|_{k+2,\Omega}, \|f_t(t)\|_{k+1,\Omega}, \|\Phi(t)\|_{k+2,\Omega_{\mathbf{x}}}, \|\mathbf{E}(t)\|_{1,\infty,\Omega_{\mathbf{x}}}),$$

with $C_a = \max\{C_a(t), 0 \leq t \leq T\}$.

(b) *DG-DG method. Let* $r=k+1$ *and let* $((\mathbf{E}_h,\Phi_h),f_h) \in \mathcal{C}^0([0,T]; \boldsymbol{\Xi}_h^r \times Q_h^r) \times \mathcal{C}^1([0,T]; \mathcal{Z}_h^k)$ *be the DG-DG approximation solution of* (3.4), (3.12) *and* (3.13). *If* $\Phi \in \mathcal{C}^0([0,T]; H^{k+3}(\Omega_{\mathbf{x}}))$, *then*

$$\|f(t)-f_h(t)\|_{0,\Omega} \leq C_b h^{k+1} \quad \forall t \in [0,T]$$

and

$$|(\mathbf{E}(t)-\mathbf{E}_h(t), \Phi(t)-\Phi_h(t))|_{\mathcal{A}} \leq C h^{k+1}\|\Phi(t)\|_{k+2,\Omega_{\mathbf{x}}} + C_b h^{k+1} \quad \forall t \in [0,T],$$

where C_b *depends on the final time* T, *the polynomial degree* k, *the shape regularity of the partition and it also depends on* f *(and therefore on* f_0*) through* *the norms*

$$C_b(t) = C(\|f(t)\|_{k+2,\Omega}, \|f_t(t)\|_{k+1,\Omega}, \|\Phi(t)\|_{k+2,\Omega_{\mathbf{x}}},$$
$$\|\mathbf{E}(t)\|_{1,\infty,\Omega_{\mathbf{x}}}, \|\Phi(t)\|_{2,\infty,\Omega_{\mathbf{x}}}),$$

with $C_b = \max\{C_b(t), 0 \leq t \leq T\}$.

Let us make the following remarks:

- The proof of the error estimates for the electric field follows straightforwardly by substituting the error estimates for the distribution function given in Theorem 4.1 into the approximation results of Lemmas 3.3 and 3.4; stated in Sec. 3.
- Unlike what usually happens with the analysis of nonlinear problems, the error estimates given in Theorem 4.1 are not asymptotic; i.e. they can be guaranteed for any $h < 1$. The above theorem is shown without using any *a priori assumption* made on the discrete solution $(\mathbf{E}_h, f_h)$ (as it usually happens in the error analysis of nonlinear problems). We cope with the nonlinearity by proving an L^∞-bound of the approximate electrostatic field and using the assumed regularity of $\mathbf{E}$.
- The optimal rate of convergence for the full DG approximation requires to approximate the Poisson problem using polynomials one degree higher than the ones used for approximating the distribution function. We also note that DG-LDG requires further regularity for the continuous electrostatic field than RT_k-DG.
- The available existence results for the VP system with periodic boundary conditions[9] show the existence of classical solutions, i.e. solutions in $C^m(\Omega)$ spaces for all $t \geq 0$, for initial data in $C^m(\Omega)$. Note that C^m-regularity of solutions together with the compact support in velocity imply the regularity assumptions on f and Φ.

The rest of the section is devoted to prove Theorem 4.1. We start by deriving the error equation and introducing some special projection operators that will be used in our analysis. We then show some auxiliary lemmas and finally, at the very end of the section, we give the proof of the theorem.

4.1. *Error equation and special projection operators*

Notice that the solution $(\mathbf{E}, f)$ to (1.1)–(1.3) satisfies the variational formulation:

$$\begin{aligned}0 = &\sum_{R\in\mathcal{T}_h} \int_R \frac{\partial f}{\partial t}\varphi_h d\mathbf{v}d\mathbf{x} - \int_R f\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi_h d\mathbf{v}d\mathbf{x} + \int_R f\mathbf{E}\cdot\nabla_{\mathbf{v}}\varphi_h d\mathbf{v}d\mathbf{x}\\ &+ \sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}} \int_{T^{\mathbf{v}}}\int_{\mathcal{E}_{\mathbf{x}}} \{\mathbf{v}f\}\cdot[\![\varphi_h]\!] ds_{\mathbf{x}} d\mathbf{v}\\ &- \sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \int_{T^{\mathbf{x}}}\int_{\mathcal{E}_{\mathbf{v}}} \{\mathbf{E}f\}\cdot[\![\varphi_h]\!] ds_{\mathbf{v}} d\mathbf{x} \quad \forall\varphi_h\in\mathcal{Z}^k_h,\end{aligned}$$

where we have allowed for a discontinuous test function. Then subtracting (3.4) from above equation we have

$$a(f - f_h, \varphi_h) + \mathcal{N}(\mathbf{E}; f, \varphi_h) - \mathcal{N}^h(\mathbf{E}_h; f_h, \varphi_h) = 0 \quad \forall \varphi_h \in \mathcal{Z}_h, \tag{4.1}$$

where $a(\cdot,\cdot)$ gathers the linear terms

$$a(f_h, \varphi_h) = \int_\Omega (f_h)_t \varphi_h d\mathbf{x} d\mathbf{v} - \int_\Omega f_h \mathbf{v} \cdot \nabla^h_{\mathbf{x}} \varphi_h d\mathbf{x} d\mathbf{v} + \sum_{T^{\mathbf{v}} \in \mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}} \int_{T^{\mathbf{v}}} \int_{\mathcal{E}_{\mathbf{x}}} \{\mathbf{v} f_h\}_\alpha [\![\varphi]\!] ds_{\mathbf{x}} d\mathbf{v}$$

and $\mathcal{N}^h(\mathbf{E}_h; \cdot, \cdot)$ carries the nonlinear part given by

$$\mathcal{N}^h(\mathbf{E}_h; f_h, \varphi_h) = \int_\Omega f_h \mathbf{E}_h \cdot \nabla^h_{\mathbf{v}} \varphi_h d\mathbf{v} d\mathbf{x} - \sum_{T^{\mathbf{x}} \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \int_{T^{\mathbf{x}}} \int_{\mathcal{E}_{\mathbf{v}}} \{\mathbf{E}_h f_h\}_\beta [\![\varphi_h]\!] ds_{\mathbf{v}} d\mathbf{x}$$

with $\mathcal{N}(\mathbf{E}; f, \varphi_h)) = \mathcal{N}^h(\mathbf{E}; f, \varphi_h)$. We next introduce some special projection operators that will play a crucial role in our error analysis. These projections extend those considered in Ref. 6 to the multi-dimensional case, see Remark 4.1 for further comments on the motivation and origin of the projections. Their definition is based on the use of the one-dimensional projection operators used in Ref. 62, that we recall next. Assume $I_h = \{I_i\}_i$ is a FE partition of the unit interval and let us denote by S^k_h the discontinuous finite element space of degree k associated to that partition. Let $\pi^\pm : H^{\frac{1}{2}+\epsilon}(I) \to S^k_h$ be the projection operators defined by

$$\int_{I_i} (\pi^\pm(w) - w) q_h dx = 0 \quad \forall\, q_h \in \mathbb{P}^{k-1}(I_i) \;\; \forall\, i, \tag{4.2}$$

together with the matching conditions:

$$\pi^+(w(x^+_{i-\frac{1}{2}})) = w(x^+_{i-\frac{1}{2}}); \quad \pi^-(w(x^-_{i+\frac{1}{2}})) = w(x^-_{i+\frac{1}{2}}). \tag{4.3}$$

Notice that more regularity than $L^2(I)$ is required for defining these projections. The following error estimates can be easily shown for all these projections:

$$\|w - \pi^\pm(w)\|_{0,I_i} \le C h^{k+1} |w|_{k+1,I_i} \quad \forall\, w \in H^{k+1}(I_i),$$

where C is a constant depending only on the shape-regularity of the mesh and the polynomial degree.[29,62]

We denote by $\Pi_h : \mathcal{C}^0(\Omega) \to \mathcal{Z}^k_h$ the projection operator defined as follows: Let $R = T^{\mathbf{x}} \times T^{\mathbf{v}}$ be an arbitrary element of $\mathcal{T}_h$ and let $w \in \mathcal{C}^0(\overline{R})$. The restriction of $\Pi_h(w)$ to R is defined by

$$\Pi_h(w) = \begin{cases} (\widetilde{\Pi}_{\mathbf{x}} \otimes \widetilde{\Pi}_{\mathbf{v}})(w) & \text{if } \operatorname{sign}(\mathbf{E}\cdot\mathbf{n}) = \text{constant}, \\ (\widetilde{\Pi}_{\mathbf{x}} \otimes \widetilde{\mathcal{P}}_{\mathbf{v}})(w) & \text{if } \operatorname{sign}(\mathbf{E}\cdot\mathbf{n}) \neq \text{constant}, \end{cases} \tag{4.4}$$

where $\widetilde{\Pi}_{\mathbf{x}} : \mathcal{C}^0(\Omega_{\mathbf{x}}) \to X_h^k$ and $\widetilde{\Pi}_{\mathbf{v}} : \mathcal{C}^0(\Omega_{\mathbf{v}}) \to V_h^k$ are the d-dimensional projection operators which, thanks to the structure of the mesh, can be defined as the tensor product of the one-dimensional projections $\pi^{\pm}$ given in (4.2)–(4.3). For simplicity, we give the detailed definition in the case $d = 2$ (the case $d = 3$ is similar but taking into account more cases). Let $\mathbf{v} = [v_1, v_2]^t$ and $\mathbf{E} = [E_1, E_2]^t$; then

$$\widetilde{\Pi}_{\mathbf{x}}(w) = \begin{cases} \pi^-_{\mathbf{x},1} \otimes \pi^-_{\mathbf{x},2} & \text{if } v_1 > 0,\ \ v_2 > 0, \\ \pi^+_{\mathbf{x},1} \otimes \pi^-_{\mathbf{x},2} & \text{if } v_1 < 0,\ \ v_2 > 0, \\ \pi^+_{\mathbf{x},1} \otimes \pi^+_{\mathbf{x},2} & \text{if } v_1 < 0,\ \ v_2 < 0, \\ \pi^-_{\mathbf{x},1} \otimes \pi^+_{\mathbf{x},2} & \text{if } v_1 > 0,\ \ v_2 < 0, \end{cases}$$
$$\widetilde{\Pi}_{\mathbf{v}}(w) = \begin{cases} \pi^+_{\mathbf{v},1} \otimes \pi^+_{\mathbf{v},2} & \text{if } E_1 > 0,\ \ E_2 > 0, \\ \pi^-_{\mathbf{v},1} \otimes \pi^+_{\mathbf{v},2} & \text{if } E_1 < 0,\ \ E_2 > 0, \\ \pi^-_{\mathbf{v},1} \otimes \pi^-_{\mathbf{v},2} & \text{if } E_1 < 0,\ \ E_2 < 0, \\ \pi^+_{\mathbf{v},1} \otimes \pi^-_{\mathbf{v},2} & \text{if } E_1 > 0,\ \ E_2 < 0, \end{cases} \tag{4.5}$$

where the subscript i in $\pi^{\pm}_{\mathbf{r},i}$ ($\mathbf{r} = \mathbf{x}$ or $\mathbf{v}$) refers to the fact that projection is along the ith component in the $\mathbf{r}$ space.

To complete the definition of the projection Π_h, we need to provide the definition of $\widetilde{\mathcal{P}}_{\mathbf{v}} : L^2(\Omega_{\mathbf{v}}) \to V_h^k$, which accounts for the cases where $\mathbf{E} \cdot \mathbf{n}$ changes sign across any single $(2d-1)$-element $e \subset T^{\mathbf{x}} \times \partial T^{\mathbf{v}}$. From the structure of the partition such condition amounts to have at least one of the components of $\mathbf{E}$ vanishing within the element R (and so in $T^{\mathbf{x}}$). We next give the detailed definition in the case $d = 2$ (the case $d = 3$ is similar but taking into account more cases):

$$\widetilde{\mathcal{P}}_{\mathbf{v}}(w) = \begin{cases} [\mathcal{P}_{\mathbf{v},1} \otimes \tilde{\pi}_{\mathbf{v},2}](w) & \text{if } \operatorname{sign}(E_1) \neq \text{cte and } \operatorname{sign}(E_2) = \text{cte}, \\ [\tilde{\pi}_{\mathbf{v},1} \otimes \mathcal{P}_{\mathbf{v},2}](w) & \text{if } \operatorname{sign}(E_1) = \text{cte and } \operatorname{sign}(E_2) \neq \text{cte}, \\ [\mathcal{P}_{\mathbf{v},1} \otimes \mathcal{P}_{\mathbf{v},2}](w) & \text{if } \operatorname{sign}(E_1) \neq \text{cte and } \operatorname{sign}(E_2) \neq \text{cte}. \end{cases} \tag{4.6}$$

Here, $\mathcal{P}_{\mathbf{v},i}$, $i = 1, 2$, stands for the standard one-dimensional projection along the v_i direction. We have just used $\tilde{\pi}_{\mathbf{v},j}$ to denote $\pi^{\pm}_{\mathbf{v},j}$, $j = 1, 2$, where the $+$ and $-$ signs refer to whether E_j is positive or negative. Note that this is consistent with the definition of $\widetilde{\Pi}_{\mathbf{v}}$ given in (4.5). Observe that conditions (4.4)–(4.6), together with (4.2)–(4.3), define the projection $\Pi_h(w)$ uniquely for any given $w \in \mathcal{C}^0(\Omega)$.

Remark 4.1. The definition of Π_h is inspired in those introduced in the two-dimensional case, for a linear transport equation in Ref. 55 and for a Poisson problem in Ref. 30. In fact, the authors in Ref. 55 display the error analysis by using an "(interpolation) operator" that in each element (a rectangle or square), reproduces the value of the interpolated function at the Gauss–Radau nodes. To the best of our knowledge, this idea was first coded in terms of projection operators in Ref. 30. Notice that the property of collocation at one boundary end of $\pi^{\pm}$ given in (4.3) is just reflecting the fact of using Gauss–Radau nodes for the interpolation operator.

The next lemma, although elementary, provides the basic approximation properties we need in our analysis. Its proof is omitted for the sake of conciseness.

Lemma 4.1. *Let $w \in H^{s+2}(R)$, $s \geq 0$ and let Π_h be the projection operator defined through* (4.4)–(4.5). *Then, for all $e \subset (\partial T^{\mathbf{x}} \times T^{\mathbf{v}}) \cup (T^{\mathbf{x}} \times \partial T^{\mathbf{v}})$, we get*

$$\begin{aligned} \|w - \Pi_h(w)\|_{0,R} &\leq Ch^{\min(s+2,k+1)}\|w\|_{s+1,R}, \\ \|w - \Pi_h(w)\|_{0,e} &\leq Ch^{\min(s+\frac{3}{2},k+\frac{1}{2})}\|w\|_{s+1,R}. \end{aligned} \tag{4.7}$$

Summing estimates (4.7) from Lemma 4.1, over elements of the partition $\mathcal{T}_h$, we have

$$\begin{aligned} &\|w - \Pi_h(w)\|_{0,\mathcal{T}_h} + h^{-\frac{1}{2}}\|w - \Pi_h(w)\|_{0,\mathcal{E}_{\mathbf{x}}\times\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}} \\ &\quad + h^{-\frac{1}{2}}\|w - \Pi_h(w)\|_{0,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}\times\mathcal{E}_{\mathbf{v}}} \\ &\leq Ch^{k+1}\|w\|_{k+1,\Omega}. \end{aligned} \tag{4.8}$$

Next, we write

$$f - f_h = [\Pi_h(f) - f_h] - [\Pi_h(f) - f] = \omega^h - \omega^e. \tag{4.9}$$

Taking now as test function $\varphi_h = \omega^h \in \mathcal{Z}_h^k$, the error equation (4.1) becomes

$$a(\omega^h - \omega^e, \omega^h) + \mathcal{N}(\mathbf{E}; f, \omega^h) - \mathcal{N}^h(\mathbf{E}_h; f_h, \omega^h) = 0. \tag{4.10}$$

Finally, we define

$$\mathcal{K}^1(\mathbf{v}, f, \omega^h) = \sum_{\mathcal{R}\in\mathcal{T}_h} \mathcal{K}^1_{\mathcal{R}}(\mathbf{v}, f, \omega^h), \quad \mathcal{K}^2(\mathbf{E}_h, f, \omega^h) = \sum_{\mathcal{R}\in\mathcal{T}_h} \mathcal{K}^2_{\mathcal{R}}(\mathbf{E}_h, f, \omega^h), \tag{4.11}$$

where

$$\mathcal{K}^1_{\mathcal{R}}(\mathbf{v}, f, \omega^h) = \int_{\mathcal{R}} \omega^e \mathbf{v}\cdot\nabla_{\mathbf{x}}\omega^h d\mathbf{x}d\mathbf{v} - \int_{T^{\mathbf{v}}}\int_{\partial T^{\mathbf{x}}} (\widehat{\mathbf{v}\cdot\mathbf{n}\omega^e})\omega^h d\mathbf{v}ds_{\mathbf{x}}, \tag{4.12}$$

$$\mathcal{K}^2_{\mathcal{R}}(\mathbf{E}_h, f, \omega^h) = \int_{\mathcal{R}} \omega^e \mathbf{E}_h\cdot\nabla_{\mathbf{v}}\omega^h d\mathbf{x}d\mathbf{v} - \int_{T^{\mathbf{x}}}\int_{\partial T^{\mathbf{v}}} (\widehat{\mathbf{E}_h\cdot\mathbf{n}\omega^e})\omega^h d\mathbf{x}ds_{\mathbf{v}}. \tag{4.13}$$

The next two lemmas are the extension to the higher-dimensional case of Lemmas 4.5 and 4.6 in Ref. 6, respectively. They provide estimates for the two expressions defined in (4.11). We refer to Ref. 8 for the proof of both results.

Lemma 4.2. *Let $\mathcal{T}_h = \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}} \times \mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$ be the tensor product of two Cartesian meshes $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ and $\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}$ of $\Omega_{\mathbf{x}}$ and $\Omega_{\mathbf{v}}$, respectively. Let $k \geq 1$ and let $f \in \mathcal{C}^0([0,T]; W^{1,\infty}(\Omega) \times H^{k+2}(\Omega))$ be the distribution function solution of* (1.1)–(1.3). *Let $f_h \in \mathcal{Z}_h^k$ be its approximation satisfying* (3.1) *and let $\mathcal{K}^1$ be defined as in* (4.11)–(4.12). *Assume* *that the partition $\mathcal{T}_h$ is constructed so that none of the components of* $\mathbf{v}$ *vanish inside any element. Then, the following estimate holds*:

$$|\mathcal{K}^1(\mathbf{v}, f, \omega^h)| \le Ch^{k+1}(\|f\|_{k+1,\Omega} + CL\|f\|_{k+2,\Omega})\|\omega^h\|_{0,\mathcal{T}_h}, \tag{4.14}$$

for all $t \in [0, T]$.

Lemma 4.3. *Let $\mathcal{T}_h$ be a Cartesian mesh of Ω, $k \ge 1$ and let $(\mathbf{E}_h, f_h) \in \widetilde{\mathbf{\Sigma}}_h \times \mathcal{Z}_h^k$ be the solution to* (3.4) *with either $\widetilde{\mathbf{\Sigma}}_h = \mathbf{\Sigma}_h^r$ or $\widetilde{\mathbf{\Sigma}}_h = \mathbf{\Xi}_h^r$, $r \ge 1$. Let $(\mathbf{E}, f) \in \mathcal{C}^0([0,T]; W^{1,\infty}(\Omega) \times H^{k+2}(\Omega))$ and let $\mathcal{K}^2$ be defined as in* (4.11)–(4.13). *Then, the following estimate holds*:

$$\begin{aligned}|\mathcal{K}^2(\mathbf{E}_h, f, \omega^h)| &\le Ch^k\|\mathbf{E} - \mathbf{E}_h\|_{0,\infty,\mathcal{T}_{h_\mathbf{x}}^{\mathbf{x}}}\|f\|_{k+1,\Omega}\|\omega^h\|_{0,\mathcal{T}_h} \\ &\quad + Ch^{k+1}(\|f\|_{k+2,\Omega}\|\mathbf{E}\|_{0,\infty,\Omega_\mathbf{x}} \\ &\quad + \|f\|_{k+1,\Omega}|\mathbf{E}|_{1,\infty,\Omega_\mathbf{x}})\|\omega^h\|_{0,\mathcal{T}_h},\end{aligned} \tag{4.15}$$

for all $t \in [0, T]$.

Remark 4.2. We would like to note that, as it happens for $d = 1$ in Ref. 6, the definition (4.4) of Π_h is given in terms of $\mathbf{E}$ (and $\mathbf{v}$), while the definition of the numerical fluxes is given in terms of $\mathbf{E}_h$ (and $\mathbf{v}$). This is due to the nonlinearity of the problem and it is inspired by the ideas used in Ref. 67. By defining Π_h in terms of $\mathbf{E}$ rather than $\mathbf{E}_h$ and using the regularity of the solution, one can estimate optimally the expression $\mathcal{K}^2$ without any further assumption on the mesh partition $\mathcal{T}_h$. See Ref. 8 for more details.

4.2. *Auxiliary results*

We next give two auxiliary results that will be required for our subsequent analysis.

Lemma 4.4. *Let $f \in \mathcal{C}^0(\Omega)$ and let $f_h \in \mathcal{Z}_h^k$ with $k \ge 0$. Then, the following equality holds true,*

$$\begin{aligned}a(f - f_h, \omega^h) &= \sum_{\mathcal{R}\in\mathcal{T}_h} \int_{\mathcal{R}} (\omega_t^h - \omega_t^e)\omega^h d\mathbf{x}d\mathbf{v} \\ &\quad + \sum_{T^\mathbf{v}\in\mathcal{T}_{h_\mathbf{v}}^\mathbf{v}} \int_{T^\mathbf{v}} \int_{\mathcal{E}_\mathbf{x}} \frac{|\mathbf{v}\cdot\mathbf{n}|}{2} [\![\, \omega^h \,]\!]^2 ds_\mathbf{x} d\mathbf{v} + \mathcal{K}^1(\mathbf{v}, f, \omega^h).\end{aligned}$$

Proof. Noting that $a(f - f_h, \omega^h) = a(\omega^h, \omega^h) - a(\omega^e, \omega^h)$, the first term is readily estimated arguing as in the proof of Proposition 3.1,

$$a(\omega^h, \omega^h) = \sum_{\mathcal{R}\in\mathcal{T}_h} \int_{\mathcal{R}} \omega_t^h \omega^h d\mathbf{x}d\mathbf{v} + \sum_{T^\mathbf{v}\in\mathcal{T}_{h_\mathbf{v}}^\mathbf{v}} \int_{T^\mathbf{v}} \int_{\mathcal{E}_\mathbf{x}^0} \frac{|\mathbf{v}\cdot\mathbf{n}|}{2} [\![\, \omega^h \,]\!]^2 ds_\mathbf{x} d\mathbf{v}.$$

For the second term, the continuity of f, the consistency of the numerical fluxes (3.2) and the definition (4.11) give

$$\begin{aligned} a(\omega^e, \omega^h) &= \sum_{\mathcal{R}\in\mathcal{T}_h} \int_{\mathcal{R}} \omega_t^e \omega^h d\mathbf{x}d\mathbf{v} - \int_{\Omega} \omega^e \mathbf{v}\cdot\nabla_{\mathbf{x}}^h \omega^h d\mathbf{x}d\mathbf{v} \\ &\quad + \sum_{T^{\mathbf{v}}\in\mathcal{T}_{h_{\mathbf{v}}}^{\mathbf{v}}} \int_{T^{\mathbf{v}}} \int_{\mathcal{E}_{\mathbf{x}}} \{\mathbf{v}\omega^e\}_\alpha \cdot [\![\omega^h]\!] ds_{\mathbf{x}} d\mathbf{v} \\ &= \sum_{\mathcal{R}\in\mathcal{T}_h} \int_{\mathcal{R}} \omega_t^e \omega^h d\mathbf{x}d\mathbf{v} - \mathcal{K}^1(\mathbf{v}, f, \omega^h), \end{aligned}$$

which concludes the proof. □

The other auxiliary lemma deals with the error coming from the nonlinear term.

Lemma 4.5. *Let* $\mathbf{E} \in \mathcal{C}^0(\Omega_{\mathbf{x}})$, $f \in \mathcal{C}^0(\Omega)$ *and* $f_h \in \mathcal{Z}_h^k$ *with* $k \geq 0$. *Then, the following identity holds true,*

$$\begin{aligned} \mathcal{N}(\mathbf{E}; f; \omega^h) - \mathcal{N}^h(\mathbf{E}_h; f_h, \omega^h) &= \sum_{T^{\mathbf{x}}\in\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \int_{T^{\mathbf{x}}} \int_{\mathcal{E}_v} \frac{|\mathbf{E}_h\cdot\mathbf{n}|}{2} [\![\omega^h]\!]^2 ds_{\mathbf{v}} d\mathbf{x} \\ &\quad - \int_{\Omega} [\mathbf{E}-\mathbf{E}_h]\cdot\nabla_{\mathbf{v}}^h f \omega^h d\mathbf{v}d\mathbf{x} - \mathcal{K}^2(\mathbf{E}_h, f, \omega^h). \end{aligned}$$

Proof. Subtracting the discrete and continuous nonlinear terms, using the continuity of $\mathbf{E}$ and f, the consistency of $\widehat{\mathbf{E}_h f_h}$ together with (2.5), we find

$$\begin{aligned} &\mathcal{N}(\mathbf{E}; f, \omega^h) - \mathcal{N}^h(\mathbf{E}_h; f_h, \omega^h) \\ &\quad = \int_{\Omega} [f(\mathbf{E}-\mathbf{E}_h) + (f-f_h)\mathbf{E}_h]\cdot\nabla_{\mathbf{v}}^h \omega^h d\mathbf{v}d\mathbf{x} \\ &\qquad - \sum_{T^{\mathbf{x}}\in\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \int_{T^{\mathbf{x}}} \int_{\mathcal{E}_{\mathbf{v}}} \{\mathbf{E}f - \mathbf{E}_h f_h\}_\beta \cdot [\![\omega^h]\!] ds_{\mathbf{v}} d\mathbf{x} := T_1 + T_2 + T_3, \end{aligned} \tag{4.16}$$

where $T_1 + T_2$ corresponds to separate the first term into two corresponding to $f(\mathbf{E}-\mathbf{E}_h)$ and $(f-f_h)\mathbf{E}_h$, respectively.

Integrating by parts T_1 and using the continuity of f together with (2.5) and the fact that neither $\mathbf{E}$ nor $\mathbf{E}_h$ depend on v, we have

$$\begin{aligned} T_1 &= -\int_{\Omega} [\mathbf{E}-\mathbf{E}_h]\cdot\nabla_{\mathbf{v}}^h f\omega^h d\mathbf{v}d\mathbf{x} + \sum_{T^{\mathbf{x}}\in\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \int_{T^{\mathbf{x}}} \int_{\mathcal{E}_{\mathbf{v}}} (\mathbf{E}-\mathbf{E}_h)\cdot[\![\omega^h]\!] f ds_{\mathbf{v}} d\mathbf{x} \\ &= T_{1a} + T_{1b}. \end{aligned} \tag{4.17}$$

We next deal with T_2. From the splitting (4.9), direct integration and (2.5), we get

$$\begin{aligned}T_2 &= \frac{1}{2}\int_\Omega \mathbf{E}_h \cdot \nabla^h_{\mathbf{v}}(\omega^h)^2 d\mathbf{v}d\mathbf{x} - \int_\Omega \omega^e \mathbf{E}_h \cdot \nabla^h_{\mathbf{v}}\omega^h d\mathbf{v}d\mathbf{x} \\ &= \frac{1}{2}\sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}\int_{T^{\mathbf{x}}}\int_{\mathcal{E}_{\mathbf{v}}} \mathbf{E}_h \cdot [\![(\omega^h)^2]\!] ds_{\mathbf{v}}d\mathbf{x} \\ &\quad - \int_\Omega \omega^e \mathbf{E}_h \cdot \nabla^h_{\mathbf{v}}\omega^h d\mathbf{v}d\mathbf{x} = T_{2a} + T_{2b}.\end{aligned} \tag{4.18}$$

We finally deal with the boundary terms collected in T_3. Adding and subtracting $f\mathbf{E}_h$ and using the continuity of f and the consistency of $\widehat{\mathbf{E}_h f_h}$, we find

$$\begin{aligned}T_3 &= \sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}\int_{T^{\mathbf{x}}}\int_{\mathcal{E}_{\mathbf{v}}} [f(\mathbf{E}_h - \mathbf{E}) \cdot [\![\omega^h]\!] - \{\mathbf{E}_h\omega^h\}_\beta \cdot [\![\omega^h]\!] \\ &\quad + \{\mathbf{E}_h\omega^e\}_\beta \cdot [\![\omega^h]\!]] ds_{\mathbf{v}}d\mathbf{x} \\ &= T_{3a} + T_{3b} + T_{3c}.\end{aligned}$$

The first term above, T_{3a}, cancels with T_{1b} in (4.17). Arguing as in Proposition 3.1, the sum of the second term above T_{3b} and T_{2a} in (4.18) gives

$$T_{3b} + T_{2a} = \sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}\int_{T^{\mathbf{x}}}\int_{\mathcal{E}_{\mathbf{v}}} \frac{|\mathbf{E}_h \cdot \mathbf{n}|}{2}[\![\omega^h]\!]^2 ds_{\mathbf{v}}d\mathbf{x}.$$

Finally, recalling the definition (4.12) we have

$$T_{2b} + T_{3c} = -\mathcal{K}^2(\mathbf{E}_h, f, \omega^h),$$

and so substituting the above results and T_{1a} into (4.16), the proof is completed.

□

We have now all ingredients to carry out the proof of Theorem 4.1.

4.3. *Proof of Theorem* 4.1

Proof of Theorem 4.1. Substituting in the error equation (4.10) the expressions from Lemmas 4.4 and 4.5 and using standard triangle inequality, we find

$$\begin{aligned}&\frac{d}{dt}\|\omega^h\|^2_{0,\mathcal{T}_h} + \frac{1}{2}\||\mathbf{v}\cdot\mathbf{n}|^{\frac{1}{2}}[\![\omega^h]\!]\|^2_{\mathcal{E}_{\mathbf{x}}\times\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}} + \frac{1}{2}\||\mathbf{E}_h\cdot\mathbf{n}|^{\frac{1}{2}}[\![\omega^h]\!]\|^2_{\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}\times\mathcal{E}_{\mathbf{v}}} \\ &= \sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}} \omega^e_t\omega^h d\mathbf{x}d\mathbf{v} + \int_\Omega [\mathbf{E} - \mathbf{E}_h]\cdot\nabla_{\mathbf{v}} f\omega^h d\mathbf{v}d\mathbf{x} \\ &\quad - \mathcal{K}^1(\mathbf{v}, f, \omega^h) + \mathcal{K}^2(\mathbf{E}_h, f, \omega^h) \\ &= I_1 + I_2 - \mathcal{K}^1 + \mathcal{K}^2 \le |I_1| + |I_2| + |\mathcal{K}^1| + |\mathcal{K}^2|.\end{aligned} \tag{4.19}$$

The first and third terms are independent of the approximation to the electrostatic field $\mathbf{E}_h$ and therefore are estimated in the same way for both cases **(a)** and **(b)**.

For the first term, the Cauchy–Schwarz inequality and the arithmetic–geometric inequality, together with the approximation estimate (4.8), give

$$|I_1| \leq Ch^{2k+2}\|f_t\|_{k+1,\Omega}^2 + C\|\omega^h\|_{0,\mathcal{T}_h}^2. \tag{4.20}$$

The third term is estimated by means of the estimate (4.14) from Lemma 4.2 and the arithmetic–geometric inequality,

$$|\mathcal{K}^1| \leq Ch^{2k+2}(\|f\|_{k+1,\Omega} + CL\|f\|_{k+2,\Omega})^2 + C\|\omega^h\|_{0,\mathcal{T}_h}^2. \tag{4.21}$$

Next we estimate the second and fourth terms in (4.19), which depend on the approximation to the electrostatic field. We first deal with the RT_k-DG method (case **(a)**). For the second term the Hölder inequality, the arithmetic–geometric inequality and estimate (3.10) from Lemma 3.3 together with the approximation estimate (4.8) give

$$\begin{aligned} |I_2| &\leq C\|\mathbf{E} - \mathbf{E}_h\|_{0,\Omega_x}^2 \|\nabla_v f\|_{0,\infty,\Omega} + C\|\nabla_v f\|_{0,\infty,\Omega}\|\omega^h\|_{0,\mathcal{T}_h}^2 \\ &\leq Ch^{2k+2}\|f\|_{1,\infty,\Omega}[(\|\mathbf{E}(t)\|_{k+1,\Omega_\mathbf{x}} + \|\Phi\|_{k+2,\Omega_\mathbf{x}})^2 + C\|f\|_{k+1,\Omega}^2] \\ &\quad + 2C\|f\|_{1,\infty,\Omega}\|\omega^h\|_{0,\mathcal{T}_h}^2. \end{aligned} \tag{4.22}$$

To deal with the last term, we observe that the bound (4.15) in Lemma 4.3

$$\begin{aligned} |\mathcal{K}^2| &\leq Ch^k\|\mathbf{E} - \mathbf{E}_h\|_{0,\infty,\mathcal{T}_{h_\mathbf{x}}^\mathbf{x}}\|f\|_{k+1,\Omega}\|\omega^h\|_{0,\mathcal{T}_h} \\ &\quad + Ch^{k+1}\|f\|_{k+2,\Omega}\|\mathbf{E}\|_{1,\infty,\Omega_\mathbf{x}}\|\omega^h\|_{0,\mathcal{T}_h}, \end{aligned} \tag{4.23}$$

requires an L^∞-bound on the error $\mathbf{E} - \mathbf{E}_h$. This is obtained by combining estimate (3.11) from Lemma 3.3 with the bound (3.7) from Lemma 3.2 and the approximation property (2.11) for $p = \infty$,

$$\begin{aligned} \|\mathbf{E} - \mathbf{E}_h\|_{0,\infty,\Omega_\mathbf{x}} &\leq Ch\|\mathbf{E}\|_{1,\infty,\Omega_\mathbf{x}} + Ch^{\frac{3}{2}}|\log(h)|\|\rho\|_{W^{\frac{3}{2},d}(\Omega_\mathbf{x})} \\ &\quad + CL^{\frac{d}{2}}h^{1-\frac{d}{2}}|\log(h)|(Ch^{k+1}\|f\|_{k+1,\Omega} \\ &\quad + \|f_h - \mathcal{P}_h(f)\|_{0,\mathcal{T}_h}). \end{aligned} \tag{4.24}$$

Notice now that since $\mathcal{P}_h$ is polynomial preserving, $\mathcal{P}_h[\widetilde{\Pi}(f)] = \widetilde{\Pi}(f)$ and so using also that it is stable in L^2, we have

$$\begin{aligned} \|f_h - \mathcal{P}_h(f)\|_{0,\mathcal{T}_h} &\leq \|f_h - \widetilde{\Pi}(f)\|_{0,\mathcal{T}_h} + \|\widetilde{\Pi}(f) - \mathcal{P}_h(f)\|_{0,\mathcal{T}_h} \\ &\leq \|f_h - \widetilde{\Pi}(f)\|_{0,\mathcal{T}_h} + C\|\widetilde{\Pi}(f) - f\|_{0,\mathcal{T}_h}. \end{aligned} \tag{4.25}$$

Substituting the above estimate into (4.24) and using the approximation property (4.8), we find

$$\begin{aligned} \|\mathbf{E} - \mathbf{E}_h\|_{0,\infty,\Omega_\mathbf{x}} &\leq Ch|\mathbf{E}|_{1,\infty,\Omega_\mathbf{x}} + Ch^{\frac{3}{2}}|\log(h)|\|\rho\|_{W^{3/2,d}(\Omega_\mathbf{x})} \\ &\quad + Ch^{k+2-\frac{d}{2}}|\log(h)|\|f\|_{k+1,\Omega} + Ch^{1-\frac{d}{2}}|\log(h)|\|\omega^h\|_{0,\mathcal{T}_h}. \end{aligned}$$

Plugging now the above result in estimate (4.23) and using the arithmetic–geometric inequality, we finally get, for the last term in (4.19),

$$\begin{aligned}|K^2| &\le Ch^{k+1}(\|f\|_{k+2,\Omega}\|\mathbf{E}\|_{1,\infty,\Omega_{\mathbf{x}}}\\ &\quad + C\|f\|_{k+1,\Omega}h^{\frac{1}{2}}|\log(h)|\|\rho\|_{W^{3/2,d}(\Omega_{\mathbf{x}})})\|\omega^h\|_{0,\mathcal{T}_h}\\ &\quad + Ch^{2k+2-\frac{d}{2}}|\log(h)|\|f\|^2_{k+1,\Omega}\|\omega^h\|_{0,\mathcal{T}_h}\\ &\quad + Ch^{k+1-\frac{d}{2}}|\log(h)|\|f\|_{k+1,\Omega}\|\omega^h\|^2_{0,\mathcal{T}_h}\\ &\le Ch^{2k+2}(\|f\|^2_{k+2,\Omega}\|\mathbf{E}\|^2_{1,\infty,\Omega_{\mathbf{x}}} + \|f\|^2_{k+1,\Omega}) + Ch^{4k+4-d}|\log(h)|^2\|f\|^4_{k+1,\Omega}\\ &\quad + C\|\omega^h\|^2_{0,\mathcal{T}_h}(1 + h|\log(h)|^2\|\rho\|^2_{W^{\frac{3}{2},d}(\Omega_{\mathbf{x}})} + h^{k+1-\frac{d}{2}}|\log(h)|\|f\|_{k+1,\Omega}).\end{aligned}$$

Observe that since $k \ge 1$ the coefficient of the term $\|\omega^h\|^2_{0,\mathcal{T}_h}$ is uniformly bounded for all $h < 1$; i.e. there exists a constant $c_1 > 0$ independent of h such that

$$C\|\omega^h\|^2_{0,\mathcal{T}_h}(1 + h|\log(h)|^2\|\rho\|^2_{W^{3/2,d}(\Omega_{\mathbf{x}})} + h^{k+1-\frac{d}{2}}|\log(h)|\|f\|_{k+1,\Omega}) \le c_1\|\omega^h\|^2_{0,\mathcal{T}_h}.$$

Hence,

$$|\mathcal{K}^2| \le Ch^{2k+2}(\|f\|^2_{k+2,\Omega}\|\mathbf{E}\|^2_{1,\infty,\Omega_{\mathbf{x}}} + L^{\frac{d}{2}}\|f\|^2_{k+1,\Omega}) + c_1\|\omega^h\|^2_{0,\mathcal{T}_h},$$

where we have already discarded the higher-order terms. Now, substituting into (4.19) the above estimate together with (4.20), (4.22) and (4.21), we obtain

$$\begin{aligned}&\frac{d}{dt}\|\omega^h\|^2_{0,\mathcal{T}_h} + \frac{1}{2}\||\mathbf{v}\cdot\mathbf{n}^-|^{\frac{1}{2}}[\![\omega^h]\!]\|^2_{\mathcal{E}^0_{\mathbf{x}}\times\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}} + \frac{1}{2}\||\mathbf{E}_h\cdot\mathbf{n}^-|^{\frac{1}{2}}[\![\omega^h]\!]\|^2_{\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}\times\mathcal{E}^0_v}\\ &\quad\le Ch^{2k+2}[\|f\|^2_{k+2,\Omega}(\|\mathbf{E}\|^2_{1,\infty} + CL^{\frac{d}{2}}) + \|f_t\|^2_{k+1,\Omega}\\ &\qquad + \|f\|_{1,\infty,\Omega}(\|\mathbf{E}(t)\|_{k+1,\Omega_{\mathbf{x}}} + \|\Phi\|_{k+2,\Omega_{\mathbf{x}}})^2]\\ &\qquad + (c_1 + 2C\|f\|_{1,\infty,\Omega} + C)\|\omega^h\|^2_{0,\mathcal{T}_h}.\end{aligned}$$

Integrating in time the above inequality, together with a standard application of Gronwall's inequality,[53] gives the error estimate,

$$\|\omega^h(t)\|^2_{0,\mathcal{T}_h} \le C_a^2 h^{2k+2},$$

where C_a is now independent of h and f_h, and depends on t and on the solution (E, f) through its norm. This proves part **(a)** of the theorem.

To prove part **(b)** of the theorem, we only need to modify slightly the estimates for I_2 and $\mathcal{K}^2$ which involve the approximation of the electrostatic field. The term I_2 is estimated similarly but using (3.17) from Lemma 3.4 (with $r = k+1$) to estimate the error $\|\mathbf{E} - \mathbf{E}_h\|_{0,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}$. Thus,

$$|I_2| \le Ch^{2k+2}\|f\|_{1,\infty,\Omega}[\|(\mathbf{E},\Phi)\|^2_{k+2,\Omega_{\mathbf{x}}} + C\|f\|^2_{k+1,\Omega}] + 2C\|f\|_{1,\infty,\Omega}\|\omega^h\|^2_{0,\mathcal{T}_h}.$$

To estimate $\mathcal{K}^2$, we only need to modify the estimate for $\|\mathbf{E} - \mathbf{E}_h\|_{0,\infty,\Omega_\mathbf{x}}$ used to bound $\mathcal{K}^2$ given in (4.23). Using now (3.18) from Lemma 3.4 (with $r = k + 1$) together with the estimate (3.7) from Lemma 3.2 and the approximation properties (2.8), we get

$$\begin{aligned}\|\mathbf{E} - \mathbf{E}_h\|_{0,\infty} \leq{}& C(\|\mathbf{E} - \mathcal{P}_\mathbf{x}(\mathbf{E})\|_{0,\infty,\Omega_\mathbf{x}} + h^{-1}\|\Phi - \mathcal{P}_\mathbf{x}(\Phi)\|_{0,\infty,\Omega_\mathbf{x}}) \\ &+ Ch^{\frac{3}{2}}|\log(h)|\|\rho\|_{W^{\frac{3}{2},d}(\Omega_\mathbf{x})} \\ &+ Ch^{1-\frac{d}{2}}|\log(h)|(\|f - \mathcal{P}_h(f)\|_{0,\mathcal{T}_h} + \|f_h - \mathcal{P}_h(f)\|_{0,\mathcal{T}_h}),\end{aligned}$$

and so making use of (4.25) and the approximation properties (2.8), we get

$$\begin{aligned}\|\mathbf{E} - \mathbf{E}_h\|_{0,\infty,\mathcal{T}_{h_\mathbf{x}}^\mathbf{x}} \leq{}& Ch(\|\mathbf{E}\|_{1,\infty,\Omega_\mathbf{x}} + \|\Phi\|_{2,\infty,\Omega_\mathbf{x}}) + Ch^{\frac{3}{2}}|\log(h)|\|\rho\|_{W^{\frac{3}{2},d}(\Omega_\mathbf{x})} \\ &+ Ch^{1-\frac{d}{2}}|\log(h)|(h^{k+1}\|f\|_{k+1,\Omega} + \|\omega^h\|_{0,\mathcal{T}_h}),\end{aligned}$$

which except for the norm in the first term is the same bound we had for the RT_k approximation in case **(a)**. Hence, the proof of part **(b)** can be completed proceeding exactly as before and therefore the details are omitted. □

4.4. *Energy conservation*

We now discuss how well the proposed schemes for approximating the VP system preserve the total energy. We show, following Ref. 6 that by appropriately tuning the coefficients of the LDG approximation of the Poisson problem, the total discrete energy is indeed conserved for the resulting LDG-DG method for the VP system. As a matter of fact, we can show such result under a technical restriction on the polynomial degree, namely $k \geq 2$.

We wish to observe that the resulting method requires the solution of $2d$ (4 in $d = 2$ and 6 in $d = 3$) Poisson problems in dimension d. Although this might be considered as a drawback of the method, it should be noted that the solution of the Poisson problem is the low-dimensional part (and so the less computational expensive) of the whole computation.

Proposition 4.1. (Energy conservation) *Let* $r = k \geq 2$ *and let* $((\mathbf{E}_h, \Phi_h), f_h) \in \mathcal{C}^1([0,T]; (\mathbf{\Xi}_h^k \times X_h^k) \times \mathcal{Z}_h^k)$ *be the LDG*($\mathbf{v}$)*-DG approximation of the VP problem* (1.1)–(1.3), *given by* (3.4)–(3.15), *with the numerical fluxes* (3.2) *for the approximate density. Let the numerical fluxes for the LDG approximation to* (3.15) *be given by*:

$$\begin{cases}(\widehat{\mathbf{E}_h}) = \{\mathbf{E}_h\} + \dfrac{\mathrm{sign}(\mathbf{v}\cdot\mathbf{n})}{2}[\![\,\mathbf{E}_h\,]\!]\mathbf{n} - C_{11}[\![\,\Phi_h\,]\!] \\ (\widehat{\Phi_h}) = \{\Phi_h\} - \dfrac{\mathrm{sign}(\mathbf{v}\cdot\mathbf{n})}{2}[\![\,\Phi_h\,]\!]\cdot\mathbf{n}\end{cases} \quad \text{on } e \in \mathcal{E}_\mathbf{x}^o,$$

where $C_{11} > 0$ at all edges/faces. Then, the following identity holds true,

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{R\in\mathcal{T}_h}\int_R f_h(t)|\mathbf{v}|^2 d\mathbf{x}d\mathbf{v} + \|\mathbf{E}_h(t)\|^2_{0,\mathcal{T}^{\mathbf{x}}_{h_\mathbf{x}}} + \|C_{11}^{\frac{1}{2}}[\![\Phi_h(t)]\!]\|^2_{0,\mathcal{E}_\mathbf{x}}\right) = 0.$$

The proof follows from Ref. 6 for the one-dimensional case. We report it here for the sake of completeness.

Proof of Proposition 4.1. *First step.* In this step, since $f \in \mathcal{Z}_h^k$ is a scalar function, we set $\boldsymbol{\tau} = \mathbf{v}f \in \boldsymbol{\Xi}_h^k$ in (3.12) and we integrate over all the elements of the partition $\mathcal{T}^{\mathbf{v}}_{h_\mathbf{v}}$

$$\int_{\Omega_\mathbf{v}}\int_{T^\mathbf{x}} \mathbf{E}\cdot\mathbf{v}f d\mathbf{x}d\mathbf{v} + \int_{\Omega_\mathbf{v}}\int_{T^\mathbf{x}} \Phi \mathrm{div}_\mathbf{x}(\mathbf{v}f)d\mathbf{x}d\mathbf{v} - \int_{\Omega_\mathbf{v}}\int_{\partial T^\mathbf{x}} \widehat{\Phi} f\mathbf{v}\cdot\mathbf{n}ds_\mathbf{x}d\mathbf{v} = 0,$$

and integrating by parts again and summing over all elements in $\mathcal{T}^{\mathbf{x}}_{h_\mathbf{x}}$, we get

$$\begin{aligned}\int_\Omega \mathbf{v}\cdot\nabla^h_\mathbf{x}(\Phi)f d\mathbf{x}d\mathbf{v} = &\sum_{R\in\mathcal{T}_h}\int_R \mathbf{E}\cdot\mathbf{v}f d\mathbf{x}d\mathbf{v}\\ &+ \sum_{T^\mathbf{x}\in\mathcal{T}^{\mathbf{x}}_{h_\mathbf{x}}}\int_{\Omega_\mathbf{v}}\int_{\partial T^\mathbf{x}} \Phi f\mathbf{v}\cdot\mathbf{n}ds_\mathbf{x}d\mathbf{v}\\ &- \sum_{T^\mathbf{x}\in\mathcal{T}^{\mathbf{x}}_{h_\mathbf{x}}}\int_{\Omega_\mathbf{v}}\int_{\partial T^\mathbf{x}} \widehat{\Phi} f\mathbf{v}\cdot\mathbf{n}ds_\mathbf{x}d\mathbf{v}. \qquad (4.26)\end{aligned}$$

Next, we set $\varphi_h = \Phi \in X_h^k \subset \mathcal{Z}_h^k$ in (3.4) (Φ as a polynomial in $\mathcal{Z}_h^k$ is constant in $\mathbf{v}$)

$$\begin{aligned}0 = &\sum_{\mathcal{R}\in\mathcal{T}_h}\int_\mathcal{R} \frac{\partial f}{\partial t}\Phi d\mathbf{v}d\mathbf{x} - \int_\Omega f\mathbf{v}\cdot\nabla^h_\mathbf{x}\Phi d\mathbf{v}d\mathbf{x} + \int_\Omega f\mathbf{E}\cdot\nabla^h_\mathbf{v}\Phi d\mathbf{v}d\mathbf{x}\\ &+ \sum_{T^\mathbf{v}\in\mathcal{T}^{\mathbf{v}}_{h_\mathbf{v}}}\int_{T^\mathbf{v}}\int_{\mathcal{E}_\mathbf{x}} \{\mathbf{v}f\}_\alpha\cdot[\![\Phi]\!]ds_\mathbf{x}d\mathbf{v} - \sum_{T^\mathbf{x}\in\mathcal{T}^{\mathbf{x}}_{h_\mathbf{x}}}\int_{T^\mathbf{x}}\int_{\mathcal{E}_\mathbf{v}} \{\mathbf{E}f\}_\beta\cdot[\![\Phi]\!]ds_\mathbf{v}d\mathbf{x}.\end{aligned}$$

Observe that the third and the last terms vanish; since Φ does not depend on $\mathbf{v}$, not only $\nabla^h_\mathbf{v}\Phi = 0$ but also $[\![\Phi]\!] = 0$ (Φ is constant on $\mathbf{v}$), and the terms from the boundary of $\Omega_\mathbf{v}$ cancel due to the compact boundary conditions. Hence,

$$\begin{aligned}0 = &\sum_{\mathcal{R}\in\mathcal{T}_h}\int_\mathcal{R} \frac{\partial f}{\partial t}\Phi d\mathbf{v}d\mathbf{x} - \int_\Omega f\mathbf{v}\cdot\nabla^h_\mathbf{x}\Phi d\mathbf{v}d\mathbf{x}\\ &+ \sum_{T^\mathbf{v}\in\mathcal{T}^{\mathbf{v}}_{h_\mathbf{v}}}\int_{T^\mathbf{v}}\int_{\mathcal{E}_\mathbf{x}} \{\mathbf{v}f\}_\alpha\cdot[\![\Phi]\!]ds_\mathbf{x}d\mathbf{v}.\end{aligned}$$

Then, combing the result with (4.26) and using the periodic boundary conditions in x, we have

$$\sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}}\frac{\partial f}{\partial t}\Phi d\mathbf{v}d\mathbf{x}$$
$$=\sum_{R\in\mathcal{T}_h}\int_R \mathbf{E}\cdot\mathbf{v}f d\mathbf{x}d\mathbf{v}+\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}\int_{\mathcal{E}^0_{\mathbf{x}}}\{\Phi\}[\![\mathbf{v}f]\!]ds_{\mathbf{x}}d\mathbf{v}$$
$$+\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}\int_{\mathcal{E}_{\mathbf{x}}}[[\![\Phi]\!]\{\mathbf{v}f\}-\{\mathbf{v}f\}_\alpha\cdot[\![\Phi]\!]-\widehat{\Phi}[\![\mathbf{v}f]\!]]ds_{\mathbf{x}}d\mathbf{v}. \tag{4.27}$$

Second step. Now, we differentiate with respect to time the first-order system (3.8) and consider its DG approximation. The second equation (3.13) reads,

$$\int_{T^{\mathbf{x}}}\mathbf{E}_t\cdot\nabla_{\mathbf{x}}q d\mathbf{x}-\int_{\partial T^{\mathbf{x}}}\widehat{\mathbf{E}_t}q\cdot\mathbf{n}ds_{\mathbf{x}}=\int_{T^{\mathbf{x}}}\rho_t q d\mathbf{x}\quad\forall\, q\in V_h^r,$$

where the definition for $\widehat{\mathbf{E}_t}$ corresponds to that chosen for $\widehat{\mathbf{E}}$ but with $(\mathbf{E},\Phi)$ replaced by $(\mathbf{E}_t,\Phi_t)$. By setting $q=\Phi$ in the above equation and replacing ρ_t by its definition, we have

$$\int_{T^{\mathbf{x}}}\mathbf{E}_t\cdot\nabla_{\mathbf{x}}\Phi d\mathbf{x}-\int_{\partial T^{\mathbf{x}}}\widehat{\mathbf{E}_t}\Phi\cdot\mathbf{n}ds_{\mathbf{x}}=\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{x}}}\int_{T^{\mathbf{v}}}f_t\Phi d\mathbf{v}d\mathbf{x}. \tag{4.28}$$

Now, taking $\boldsymbol{\tau}=\mathbf{E}_t$ in (3.12) and integrating by parts the volume term on the right-hand side of that equation, we find

$$\int_{T^{\mathbf{x}}}\mathbf{E}\cdot\mathbf{E}_t d\mathbf{x}-\int_{T^{\mathbf{x}}}\mathbf{E}_t\cdot\nabla_{\mathbf{x}}\Phi d\mathbf{x}+\int_{\partial T^{\mathbf{x}}}\Phi\mathbf{E}_t\cdot\mathbf{n}ds_{\mathbf{x}}-\int_{\partial T^{\mathbf{x}}}\widehat{\Phi}\mathbf{E}_t\cdot\mathbf{n}ds_{\mathbf{x}}=0.$$

Then, combining (4.28) with the above equation and summing over all elements of $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ and using (2.5) together with the periodicity of the boundary conditions for the Poisson problem, we get

$$\int_{\Omega_{\mathbf{x}}}\mathbf{E}\cdot\mathbf{E}_t d\mathbf{x}=\int_{\Omega}f_t\Phi d\mathbf{v}d\mathbf{x}+\int_{\mathcal{E}_{\mathbf{x}}}(\widehat{\mathbf{E}_t}[\![\Phi]\!]+\widehat{\Phi}[\![\mathbf{E}_t]\!]-[\![\Phi]\!]\{\mathbf{E}_t\})ds_{\mathbf{x}}$$
$$-\int_{\mathcal{E}^o_{\mathbf{x}}}\{\Phi\}[\![\mathbf{E}_t]\!]ds_{\mathbf{x}}. \tag{4.29}$$

Third step. We now proceed as in the proof for the continuous case and we take $\varphi_h=\frac{|\mathbf{v}|^2}{2}$ in (3.4),

$$0=\sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}}\frac{\partial f}{\partial t}\frac{|\mathbf{v}|^2}{2}d\mathbf{v}d\mathbf{x}-\int_{\Omega}f\mathbf{v}\cdot\nabla^h_{\mathbf{x}}\left(\frac{|\mathbf{v}|^2}{2}\right)d\mathbf{v}d\mathbf{x}$$

$$+ \int_\Omega f\mathbf{E}\cdot\nabla^h_{\mathbf{v}}\left(\frac{|\mathbf{v}|^2}{2}\right)d\mathbf{v}d\mathbf{x} + \frac{1}{2}\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}\int_{\mathcal{E}_{\mathbf{x}}}\{\mathbf{v}f\}_\alpha\cdot[\![\,|\mathbf{v}|^2\,]\!]ds_{\mathbf{x}}d\mathbf{v}$$
$$- \sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{x}}}\int_{\mathcal{E}_{\mathbf{v}}}\{\mathbf{E}f\}_\beta\cdot\left[\!\!\left[\frac{|\mathbf{v}|^2}{2}\right]\!\!\right]ds_{\mathbf{v}}d\mathbf{x}.$$

The second and fourth terms vanish since $\mathbf{v}$ is independent of $\mathbf{x}$, as well as the last term. Then, using the consistency of the numerical fluxes $(\widehat{\mathbf{v}\cdot\mathbf{n}f})$ and $\widehat{\mathbf{E}\cdot\mathbf{n}f}$ (see (3.3)), the boundary terms telescope and no boundary term is left due to the periodic and compact boundary conditions. Hence, we simply get

$$0 = \sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}}\frac{\partial f}{\partial t}\frac{|\mathbf{v}|^2}{2}d\mathbf{v}d\mathbf{x} + \int_\Omega \mathbf{E}\cdot\mathbf{v}fd\mathbf{v}d\mathbf{x}. \tag{4.30}$$

Next, we use Eq. (4.27) to substitute the last term in (4.30),

$$\begin{aligned}0 = &\sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}}\frac{\partial f}{\partial t}\frac{|\mathbf{v}|^2}{2}d\mathbf{v}d\mathbf{x}\\ &+\sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}}\frac{\partial f}{\partial t}\Phi d\mathbf{v}d\mathbf{x} - \sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}\int_{\mathcal{E}^o_{\mathbf{x}}}\{\Phi\}[\![\,\mathbf{v}f\,]\!]ds_{\mathbf{x}}d\mathbf{v}\\ &+\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}\int_{\mathcal{E}_{\mathbf{x}}}[\{\mathbf{v}f\}_\alpha\cdot[\![\,\Phi\,]\!]+\widehat{\Phi}[\![\,\mathbf{v}f\,]\!]-[\![\,\Phi\,]\!]\{\mathbf{v}f\}]ds_{\mathbf{x}}d\mathbf{v}.\end{aligned}$$

Finally, we substitute the second volume term above by means of (4.29),

$$\begin{aligned}0 = &\sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}}\frac{\partial f}{\partial t}\frac{|\mathbf{v}|^2}{2}d\mathbf{v}d\mathbf{x} + \int_{\Omega_{\mathbf{x}}}\mathbf{E}\cdot\mathbf{E}_t d\mathbf{x} - \sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}\int_{\mathcal{E}^o_{\mathbf{x}}}[\![\,\Phi\,]\!]\{\mathbf{v}f\}ds_{\mathbf{x}}d\mathbf{v}\\ &+\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}\int_{\mathcal{E}_{\mathbf{x}}}[\{\mathbf{v}f\}_\alpha\cdot[\![\,\Phi\,]\!]+\widehat{\Phi}[\![\,\mathbf{v}f\,]\!]-\{\Phi\}[\![\,\mathbf{v}f\,]\!]]ds_{\mathbf{x}}d\mathbf{v}\\ &+\int_{\mathcal{E}_{\mathbf{x}}}([\![\,\Phi\,]\!]\{\mathbf{E}_t\}-\widehat{\mathbf{E}_t}[\![\,\Phi\,]\!]-\widehat{\Phi}[\![\,\mathbf{E}_t\,]\!])ds_{\mathbf{x}} + \int_{\mathcal{E}^o_{\mathbf{x}}}\{\Phi\}[\![\,\mathbf{E}_t\,]\!]ds_{\mathbf{x}}.\end{aligned} \tag{4.31}$$

Then, we define

$$\Theta^H_e = \begin{cases}[\![\,\Phi\,]\!]\{\mathbf{E}_t\}-\widehat{\mathbf{E}_t}[\![\,\Phi\,]\!]-\widehat{\Phi}[\![\,\mathbf{E}_t\,]\!]+\{\Phi\}[\![\,\mathbf{E}_t\,]\!] & \text{on } e\in\mathcal{E}^o_{\mathbf{x}},\\ [\![\,\Phi\,]\!]\{\mathbf{E}_t\}-\widehat{\mathbf{E}_t}[\![\,\Phi\,]\!]-\widehat{\Phi}[\![\,\mathbf{E}_t\,]\!] & \text{on } e\in\mathcal{E}^\partial_{\mathbf{x}},\end{cases}$$

$$\Theta^F_e = \begin{cases}\{\mathbf{v}f\}_\alpha\cdot[\![\,\Phi\,]\!]+\widehat{\Phi}[\![\,\mathbf{v}f\,]\!]-\{\Phi\}[\![\,\mathbf{v}f\,]\!]-[\![\,\Phi\,]\!]\{\mathbf{v}f\} & \text{on } e\in\mathcal{E}^o_{\mathbf{x}}\times T^{\mathbf{v}},\\ \{\mathbf{v}f\}_\alpha\cdot[\![\,\Phi\,]\!]+\widehat{\Phi}[\![\,\mathbf{v}f\,]\!]-\{\Phi\}[\![\,\mathbf{v}f\,]\!] & \text{on } e\in\mathcal{E}^\partial_{\mathbf{x}}\times T^{\mathbf{v}},\end{cases}$$

with $T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}$ and $T^{\mathbf{v}} \in \mathcal{T}_{h_{\mathbf{v}}}^{\mathbf{v}}$, so that (4.31) can be rewritten as

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_{\mathcal{R}\in\mathcal{T}_h}\int_{\mathcal{R}} f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int_{\Omega_{\mathbf{x}}} |\mathbf{E}|^2 d\mathbf{x}\right) \\ &\quad + \sum_{T^{\mathbf{v}}\in\mathcal{T}_{h_{\mathbf{v}}}^{\mathbf{v}}}\sum_{e\in\mathcal{E}_{\mathbf{x}}}\int_{T^{\mathbf{v}}}\int_e \Theta_e^F ds_{\mathbf{x}}d\mathbf{v} + \sum_{e\in\mathcal{E}_{\mathbf{x}}}\int_e \Theta_e^H ds_{\mathbf{x}} = 0. \end{aligned} \tag{4.32}$$

Thus, we only need to show that Θ_e^H and Θ_e^F are either zero or the time derivative of a non-negative function for all $e \in \mathcal{E}_{\mathbf{x}}$.

Next, for $e \in \mathcal{E}_{\mathbf{x}}^0$, using the definition of the numerical fluxes (4.1) for the LDG approximation together with the fact that $(\mathbf{E}, \Phi)$ is $\mathcal{C}^1$ in time, we find

$$\begin{aligned} \Theta_e^H &= [\![\Phi]\!]\{\mathbf{E}_t\} - \widehat{\mathbf{E}_t}[\![\Phi]\!] - \widehat{\Phi}[\![\mathbf{E}_t]\!] + \{\Phi\}[\![\mathbf{E}_t]\!] = c_{11}[\![\Phi_t]\!]\cdot[\![\Phi]\!] \\ &= \frac{1}{2}\frac{\partial}{\partial t}(c_{11}|[\![\Phi]\!]|^2) \quad \forall\, e \in \mathcal{E}_{\mathbf{x}}^0. \end{aligned}$$

Similarly, for $e \in \mathcal{E}_{\mathbf{x}}^\partial$ taking into account the definition at boundary edges/faces, we have $\Theta_e^H = c_{11}[\![\Phi_t]\!]\cdot[\![\Phi]\!]$ on $e \in \mathcal{E}_{\mathbf{x}}^\partial$. Hence, arguing as before and putting together the result with the above identity we arrive at

$$\Theta_e^H = c_{11}[\![\Phi_t]\!]\cdot[\![\Phi]\!] = \frac{1}{2}\frac{\partial}{\partial t}(c_{11}|[\![\Phi]\!]|^2) \quad \forall\, e \in \mathcal{E}_{\mathbf{x}}. \tag{4.33}$$

We next deal with Θ_e^F. Notice that for $e \in \mathcal{E}_{\mathbf{x}}^\partial$ it is easy to see, using the definition of the numerical fluxes $\widehat{\mathbf{v}f}$ and $\widehat{\Phi}$ at $\partial\Omega_{\mathbf{x}}$, that $\Theta_e^F \equiv 0$ for all $e \in \mathcal{E}_{\mathbf{x}}^\partial$.

Now, for $e \in \mathcal{E}_{\mathbf{x}}^o \times T^{\mathbf{v}}$, from the definition of the numerical fluxes $\widehat{\mathbf{v}f}$ and $\widehat{\Phi}$ given in (3.3) and (4.1), respectively, we find, for Θ_e^F,

$$\begin{aligned} \Theta_e^F &= \{\mathbf{v}f\}_\alpha \cdot [\![\Phi]\!] + \widehat{\Phi}[\![\mathbf{v}f]\!] - \{\Phi\}[\![\mathbf{v}f]\!] - [\![\Phi]\!]\cdot\{\mathbf{v}f\} \\ &= \frac{|\mathbf{v}\cdot\mathbf{n}|}{2}[\![f]\!]\cdot[\![\Phi]\!] - \mathbf{C_{12}}\cdot[\![\Phi]\!][\![\mathbf{v}f]\!] \\ &= \frac{|\mathbf{v}\cdot\mathbf{n}|}{2}[\![f]\!]\cdot[\![\Phi]\!] - \frac{\mathrm{sign}(\mathbf{v}\cdot\mathbf{n})}{2}\mathbf{n}\cdot[\![\Phi]\!]\mathbf{v}\cdot[\![f]\!] \\ &= \frac{1}{2}[\![f]\!]\cdot[\![\Phi]\!]\,(|\mathbf{v}\cdot\mathbf{n}| - |\mathbf{v}\cdot\mathbf{n}|) = 0 \quad e \in \mathcal{E}_{\mathbf{x}}^0 \times T^{\mathbf{v}}, \end{aligned}$$

and so substituting the above result together with (4.33) into (4.32) we reach (4.1). □

For other DG-DG schemes, energy inequalities similar to those given in Ref. 6 can be proved.

Appendix A. Error Analysis for the Approximation of the Electrostatic Field

This Appendix is devoted to show the results stated in Sec. 3 related to the approximation of the electrostatic field. The results in this Appendix are one of the core differences with respect to the one-dimensional case in which the uniform estimates of the electric field are a trivial consequence of Sobolev embeddings. We start by showing the auxiliary result, Lemma 3.2, which bounds the error in the density in terms of the error in the distribution function. Then we prove the energy norm estimates for the RT_k and LDG approximations, given in Lemmas 3.3 and 3.4, respectively. The L^∞-bounds for both methods are given at the end of the Appendix.

A.1. *Proof of Lemma* 3.2

The proof of the estimate (3.6) follows straightforwardly from the definitions (1.2) and (3.5) of ρ and ρ_h, respectively, and the Hölder inequality.

To show (3.7), we first prove that

$$\|\rho - \rho_h\|_{-1,\infty,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \leq C\|\rho - \rho_h\|_{L^d(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})}. \tag{A.1}$$

Note that from the mass conservation given in (1.4) and Lemma 3.1 for ρ and ρ_h, respectively, it follows that $[\rho - \rho_h]$ is orthogonal to the global constants. Hence, denoting by $\langle q\rangle_{\Omega_{\mathbf{x}}} = \left(\frac{1}{|\Omega_{\mathbf{x}}|}\right)\int_{\Omega_{\mathbf{x}}} q d\mathbf{x}$ the average of a function q, the Hölder inequality, together with the Poincaré–Friedrichs inequality (Ref. 23, Theorem 4.1) (which shows the Sobolev's embedding $W^{1,1}(\Omega_{\mathbf{x}}) \subset L^{q^*}(\Omega_{\mathbf{x}})$ with $q^* = \frac{d}{d-1}$ for the DG functions, see also Lemma 5.10 in Ref. 2 for the continuous counterpart) gives

$$\begin{aligned}
\|\rho - \rho_h\|_{-1,\infty,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} &= \sup_{q\in W_h^{1,1}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} \frac{|\int_{\Omega_{\mathbf{x}}}(\rho - \rho_h)q d\mathbf{x}|}{\|q\|_{W_h^{1,1}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})}} \\
&= \sup_{q\in W_h^{1,1}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} \frac{|\int_{\Omega_{\mathbf{x}}}(\rho - \rho_h)[q - \langle q\rangle] d\mathbf{x}|}{\|q\|_{W_h^{1,1}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})}} \\
&\leq \sup_{q\in W_h^{1,1}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} \frac{\|\rho - \rho_h\|_{L^d(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})}\|q - \langle q\rangle\|_{L^{\frac{d}{d-1}}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})}}{\|q\|_{W_h^{1,1}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})}} \\
&\leq C\|\rho - \rho_h\|_{L^d(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})}.
\end{aligned}$$

To conclude we only need to bound the error in the L^d-norm. Triangle inequality, together with the L^d-stability of the L^2-projection (2.7) and inverse inequality, give

$$\begin{aligned}
\|\rho - \rho_h\|_{L^d(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} &\leq \|\rho - \mathcal{P}_{\mathbf{x}}(\rho)\|_{L^d(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} + \|\mathcal{P}_{\mathbf{x}}(\rho) - \rho_h\|_{L^d(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} \\
&\leq Ch^{\frac{3}{2}}|\rho|_{W^{\frac{3}{2},d}(\Omega_{\mathbf{x}})} + Ch_{\mathbf{x}}^{-d(\frac{1}{2}-\frac{1}{d})}\|\mathcal{P}_{\mathbf{x}}(\rho) - \rho_h\|_{0,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}}.
\end{aligned} \tag{A.2}$$

Next, taking into account the definition of the continuous and discrete density, using that the projection $\mathcal{P}_{\mathbf{x}}$ is independent of $\mathbf{v}$ and Hölder inequality, we find

$$\begin{aligned}\|\mathcal{P}_{\mathbf{x}}(\rho)-\rho_h\|_{0,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} &= \sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}\int_{T^{\mathbf{x}}}\left|\sum_{T^{\mathbf{v}}\in\mathcal{T}^{\mathbf{v}}_{h_{\mathbf{v}}}}\int_{T^{\mathbf{v}}}[\mathcal{P}_{\mathbf{x}}(f)-f_h]d\mathbf{v}\right|d\mathbf{x}\\ &\le C[\mathrm{meas}(\Omega_{\mathbf{v}})]^{\frac{1}{2}}\|\mathcal{P}_{\mathbf{x}}(f)-f_h\|_{0,\mathcal{T}_h}\\ &\le CL^{\frac{d}{2}}(\|\mathcal{P}_{\mathbf{x}}(f)-\mathcal{P}_h(f)\|_{0,\mathcal{T}_h}+\|\mathcal{P}_h(f)-f_h\|_{0,\mathcal{T}_h}), \qquad (A.3)\end{aligned}$$

where in the last step we have added and subtracted $\mathcal{P}_h(f)$ and used triangle inequality. Now, using the L^2-stability of the $\mathcal{P}_{\mathbf{x}}$-projection together with the approximation property (2.9) we have, for the first term above,

$$\begin{aligned}\|\mathcal{P}_{\mathbf{x}}(f)-\mathcal{P}_h(f)\|_{0,\mathcal{T}_h} &= \|[\mathcal{P}_{\mathbf{x}}\otimes\mathbb{I}_{\mathbf{v}}](f)-[\mathcal{P}_{\mathbf{x}}\otimes\mathcal{P}_{\mathbf{v}}](f)\|_{0,\mathcal{T}_h} = \|\mathcal{P}_{\mathbf{x}}[f-\mathcal{P}_{\mathbf{v}}(f)]\|_{0,\mathcal{T}_h}\\ &\le C\|f-\mathcal{P}_{\mathbf{v}}(f)\|_{0,\mathcal{T}_h}\le Ch_{\mathbf{v}}^{k+1}\|f\|_{k+1,\mathcal{T}_h}.\end{aligned}$$

Substituting this estimate in (A.3) and the result in (A.2) we get (A.1), which implies (3.7) and the proof is complete.

A.2. *Proof of Lemma* 3.3

To simplify the notation we drop the dependence on the t-variable. From Proposition II.2.16 in Ref. 21 it follows that

$$\begin{aligned}&\|\mathbf{E}-\mathbf{E}_h\|_{H(\mathrm{div};\Omega_{\mathbf{x}})}+\|\Phi-\Phi_h\|_{0,\Omega_{\mathbf{x}}}\\ &\quad\le C\left(\inf_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k}\|\mathbf{E}-\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega_{\mathbf{x}})}+\inf_{q\in Q_h^k}\|\Phi-q\|_{0,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}+M_{3h}\right), \qquad (A.4)\end{aligned}$$

where M_{3h} is the consistency error:

$$M_{3h}:=\sup_{q\in Q_h^k}\frac{|\int_{\Omega_{\mathbf{x}}}(\rho-\rho_h)q\,d\mathbf{x}|}{\|q\|_{0,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}}.$$

The first two terms in (A.4) are readily estimated from the standard approximation properties of Raviart–Thomas elements; estimate (2.11) and the approximation of the L^2_0-projection (2.6),

$$\begin{aligned}&\inf_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k}\|\mathbf{E}-\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega_{\mathbf{x}})}+\inf_{q\in Q_h^k}\|\Phi-q\|_{0,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}\\ &\quad\le Ch^{k+1}(\|\mathbf{E}(t)\|_{k+1,\Omega_{\mathbf{x}}}+\|\Phi\|_{k+2,\Omega_{\mathbf{x}}}).\end{aligned}$$

Using the Cauchy–Schwarz inequality, together with the estimate (3.6) from Lemma 3.2, we find

$$M_{3h}\le C\|\rho-\rho_h\|_{0,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}}\le CL^{\frac{d}{2}}\|f-f_h\|_{0,\mathcal{T}_h},$$

and the proof of the estimate in the $H(\mathrm{div};\Omega_{\mathbf{x}})$-norm is complete.

A.3. *Proof of Lemma* 3.4

We start by noticing that if we denote by $(\widetilde{\mathbf{E}}, \widetilde{\Phi})$ the solution of the discrete Poisson problem (3.8), the triangle inequality gives

$$|(\mathbf{E} - \mathbf{E}_h, \Phi - \Phi_h)|_{\mathcal{A}} \leq |(\mathbf{E} - \widetilde{\mathbf{E}}, \Phi - \widetilde{\Phi})|_{\mathcal{A}} + |(\widetilde{\mathbf{E}} - \mathbf{E}_h, \widetilde{\Phi} - \Phi_h)|_{\mathcal{A}}. \tag{A.5}$$

The last term above is estimated proceeding exactly as in Ref. 26 (where the Dirichlet problem is treated) and the same error estimate can be shown (for the case of interest, C_{11} of order $O\big(\frac{1}{h}\big)$ and C_{22} either zero or of order $O(1)$):

$$|(\widetilde{\mathbf{E}} - \mathbf{E}_h, \widetilde{\Phi} - \Phi_h)|_{\mathcal{A}} \leq C h^r |||(\widetilde{\mathbf{E}}, \widetilde{\Phi})|||_{r+1}. \tag{A.6}$$

We omit the details for the sake of conciseness. The first term in (A.5) is estimated by using standard regularity theorems for the Poisson problem[48] together with the Poincaré–Friedrichs inequality for discrete functions in Q_h^r (see Ref. 17) and the estimate (3.6) from Lemma 3.2

$$\begin{aligned} |(\mathbf{E} - \widetilde{\mathbf{E}}, \Phi - \widetilde{\Phi})|_{\mathcal{A}} &= \|\mathbf{E} - \widetilde{\mathbf{E}}\|_{0,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \leq C\|\rho - \rho_h\|_{-1,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \\ &= \sup_{q_h \in Q_h^r} \frac{\int_{\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} (\rho - \rho_h) q_h d\mathbf{x}}{\|q_h\|_{1,\mathcal{T}_h}} \\ &\leq C C_p \|\rho - \rho_h\|_{0,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \leq C L^{\frac{d}{2}} \|f - f_h\|_{0,\mathcal{T}_h}. \end{aligned}$$

Hence, substituting this estimate, together with (A.6) into (A.5), concludes the proof.

A.4. *L^∞-error estimates for the approximation to the electrostatic field*

We next show the error estimates in the L^∞-norm for the approximate electrostatic field with RT_k and the LDG methods. For both methods, there are already available in the literature, L^∞ and pointwise error analysis for the approximation of a linear Poisson problem (see Refs. 46 and 27). Here, we will mainly modify the proof of those results in order to account for the nonlinearity of the Poisson problem (1.3). We wish to stress that we are not concerned here with providing pointwise and localized error estimates (although they could be easily derived from the results given here). Our main goal is to prove the uniform estimates in Lemmas 3.3 and 3.4 which in turn give the L^∞-error bounds required by our analysis of VP.

We next recall a result that will be used in the proof of both lemmas.

Let $\boldsymbol{\varphi} \in H(\mathrm{div}, \Omega_{\mathbf{x}})$ be such that $\nabla \cdot \boldsymbol{\varphi} \in L_0^2(\Omega_{\mathbf{x}})$. Let $g \in H^1(\Omega_{\mathbf{x}}) \cap L_0^2(\Omega_{\mathbf{x}})$ be the solution of the problem

$$-\Delta g = \nabla \cdot \varphi \quad \text{in } \Omega_{\mathbf{x}}, \tag{A.7}$$

with g and ∇g subject to periodic boundary conditions on $\partial\Omega_{\mathbf{x}}$. We shall need the following *a priori* estimates in $L^p(\Omega_{\mathbf{x}})$-based norms for problem (A.7),

$$\|g\|_{W^{1,p}(\Omega_{\mathbf{x}})} \leq \frac{C}{p-1}\|\nabla\cdot\boldsymbol{\varphi}\|_{W^{-1,p}(\Omega_{\mathbf{x}})} \leq \frac{C}{p-1}\|\boldsymbol{\varphi}\|_{L^p(\Omega_{\mathbf{x}})}, \quad 1 < p \leq 2. \quad \text{(A.8)}$$

The above estimate can be shown from the *a priori* L^p-estimates for problem (A.7) (see for instance Ref. 48) but tracing the constants through the proof of those results to get a precise dependence on p of the leading constant in the estimate (A.8). We also mention that for general polyhedral domains and Dirichlet or Neumann boundary conditions, the range of p is more restricted (see Refs. 37, 36 and also Ref. 35 for related work).

A.4.1. L^∞-*error estimates for the classical mixed approximation to the electrostatic field*

In Ref. 46 the authors provide a general abstract framework for the $L^\infty(\Omega_{\mathbf{x}})$- error analysis of classical mixed finite element approximation of a linear Poisson problem with Dirichlet boundary conditions. They use Nitsche's method of weighted Sobolev-norms[58] (see also Refs. 56 and 57), in which the key idea is that by using weighted norms one can still work in L^2 rather than in L^∞ and in particular, can still use duality arguments. In fact, the essential ingredient in their analysis is a duality argument combined with an *a priori* estimate in certain weighted norm. Their result is rather general, since it covers Raviart–Thomas–Nédelec mixed methods and also Brezzi–Douglas–Marini and Brezzi–Douglas–Fortin–Marini mixed approximations. Moreover, it is valid for any space dimension $d \geq 2$, and holds for partitions made of simplexes or rectangles.

To prove the estimate (3.11) from Lemma 3.3, we follow Ref. 46, but to account for the nonlinearity on the right-hand side of the Poisson problem, we will need to modify a bit of their proof. We wish to stress that although the authors deal with the Dirichlet problem, all the error estimates proved in Ref. 46 carry over for the periodic Poisson problem.

We recall now some notation that will allow us to use and to refer to the results proved in Ref. 46. The weight function σ is defined by

$$\sigma(x) := (|\mathbf{x}-\mathbf{x}_0|^2 + \theta^2)^{\frac{1}{2}}, \quad \mathbf{x}, \mathbf{x}_0 \in \Omega_{\mathbf{x}}, \quad \text{(A.9)}$$

where $|\cdot|$ denotes here the Euclidean distance in $\mathbb{R}^d$ and $\theta = C^* h$ with a constant $C^* \geq 1$ to be specified later on. The weight satisfies the non-oscillation property[58]:

$$\max_{\mathbf{x}\in T^{\mathbf{x}}} \sigma(\mathbf{x}) \leq C \min_{\mathbf{x}\in T^{\mathbf{x}}} \sigma(\mathbf{x}) \quad \forall T^{\mathbf{x}} \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}.$$

For $\alpha \in \mathbb{R}$, we defined the following weighted Sobolev norm:

$$\|u\|^2_{\sigma^\alpha} := \int_{\Omega_{\mathbf{x}}} \sigma^\alpha |u|^2 d\mathbf{x} \quad \forall u \in L^2(\Omega_{\mathbf{x}}), \ \alpha \in \mathbb{R}.$$

The following relations can be established between the weighted and L^∞-norms:

$$\|u\|_{\sigma^{-\alpha}} \leq C\|u\|_{0,\infty,\Omega_{\mathbf{x}}} \begin{cases} \theta^{\frac{d-\alpha}{2}} & \alpha > d, \\ |\log\theta|^{\frac{1}{2}} & \alpha = d, \end{cases} \quad u \in L^\infty(\Omega_{\mathbf{x}}), \tag{A.10}$$

$$\|\chi\|_{0,\infty,\Omega_{\mathbf{x}}} \leq C\left(\frac{\theta^\alpha}{h^d}\right)^{\frac{1}{2}} \|\chi\|_{\sigma^{-\alpha}}, \quad \alpha \in \mathbb{R}, \ \chi \in \Sigma_h^k \quad \text{or} \quad \chi \in Q_h^k. \tag{A.11}$$

Proof of (3.11) in Lemma 3.3. Let $\mathcal{R}_h^k : H(\mathrm{div};\Omega_{\mathbf{x}}) \to \Sigma_h^k$ be the Raviart–Thomas projection as defined in Sec. 2. Triangle inequality gives

$$\|\mathbf{E} - \mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}} \leq \|\mathbf{E} - \mathcal{R}_h^k(\mathbf{E})\|_{0,\infty,\Omega_{\mathbf{x}}} + \|\mathcal{R}_h^k(\mathbf{E}) - \mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}}.$$

Thus, we only need to estimate the last term above on right-hand side. We shall show

$$\begin{aligned} &\|\mathcal{R}_h^k(\mathbf{E}) - \mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}} \\ &\qquad \leq C\|\mathbf{E} - \mathcal{R}_h^k(\mathbf{E})\|_{0,\infty,\Omega_{\mathbf{x}}} + C|\log(h)|\|\rho - \rho_h\|_{-1,\infty,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}}, \end{aligned} \tag{A.12}$$

and so, substituting this estimate above and using standard approximation properties, the proof of the lemma will be complete. Hence, it is enough to prove (A.12).

Arguing as in Ref. 46, it turns out that we just need to modify one step in the proof of Lemma 4.1 in Ref. 46; the bound for the V-term. In such step the authors were using the Galerkin orthogonality property of $\mathrm{div}(\mathbf{E} - \mathbf{E}_h)$ being orthogonal to Q_h^k, which due to the nonlinearity in Poisson is obviously not true in the present case. Since $\mathcal{R}_h^k(\mathbf{E}) - \mathbf{E}_h \in \mathbf{\Sigma_h^k}$, from (A.11) one has

$$\begin{aligned} &\|\mathcal{R}_h^k(\mathbf{E}) - \mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}} \\ &\qquad \leq C\left(\frac{\theta^{(d+\alpha)}}{h^d}\right)^{\frac{1}{2}} \|\mathcal{R}_h^k(\mathbf{E}) - \mathbf{E}_h\|_{\sigma^{-(d+\alpha)}}, \quad 0 < \alpha < 2. \end{aligned} \tag{A.13}$$

From Ref. 46 one has

$$\begin{aligned} \|\mathcal{R}_h^k(\mathbf{E}) - \mathbf{E}_h\|_{\sigma^{-(d+\alpha)}}^2 &\leq C\left(\frac{h}{\theta}\right)\|\mathcal{R}_h^k(\mathbf{E}) - \mathbf{E}_h\|_{\sigma^{-(d+\alpha)}}^2 \\ &\quad + C\|\mathcal{R}_h^k(\mathbf{E}) - \mathbf{E}\|_{\sigma^{-(d+\alpha)}}^2 + |V|, \end{aligned} \tag{A.14}$$

where the V-term reads (after integration by parts)

$$V = -\int_{\Omega_{\mathbf{x}}} (\mathbf{E} - \mathbf{E}_h)\nabla u d\mathbf{x} = \int_{\Omega_{\mathbf{x}}} \mathrm{div}(\mathbf{E} - \mathbf{E}_h) u d\mathbf{x}, \tag{A.15}$$

with u the solution of the dual problem:

$$\text{Find} \quad u \in H^1(\Omega_{\mathbf{x}}) \cap L_0^2(\Omega_{\mathbf{x}}) : -\Delta u = \mathrm{div}\,\mathcal{R}_h^k(\boldsymbol{\psi}), \tag{A.16}$$

subject to periodic boundary conditions (for u and for $\boldsymbol{\nabla} u$). In the above dual problem, $\boldsymbol{\psi}$ is defined as

$$\boldsymbol{\psi} = \sigma^{-(\alpha+n)}(\mathcal{R}_h^0(\mathbf{E}) - \mathbf{E}_h).$$

Notice that in view of (2.10) the above problem is well-posed. To estimate the term in (A.15), we first observe that $\operatorname{div}\mathbf{E}_h \in Q_h^k$ and $\operatorname{div}(\mathbf{E}-\mathbf{E}_h) = [1-\rho-(1-\rho_h)] = [\rho_h-\rho]$. Hence, we can rewrite the term V as

$$V = \int_{\Omega_{\mathbf{x}}} \operatorname{div}(\mathbf{E}-\mathbf{E}_h)ud\mathbf{x} = \int_{\Omega_{\mathbf{x}}} [\rho_h-\rho]ud\mathbf{x}.$$

Using now the Hölder inequality together with Poincaré–Friedrichs inequality, we find

$$|V| \le \|\rho-\rho_h\|_{-1,\infty,\mathcal{T}_h}\|u\|_{W^{1,1}(\Omega_{\mathbf{x}})/\mathbb{R}}. \tag{A.17}$$

We now estimate the term $\|u\|_{W^{1,1}(\Omega_{\mathbf{x}})}$. Sobolev's embeddings together with the *a priori* estimate (A.8) for problem (A.16) give

$$\begin{aligned}\|u\|_{W^{1,1}(\Omega_{\mathbf{x}})} &\le C\|u\|_{W^{1,p}(\Omega_{\mathbf{x}})} \le \frac{C}{(p-1)}\|\operatorname{div}\mathcal{R}_h^k\boldsymbol{\psi}\|_{W^{-1,p}(\Omega_{\mathbf{x}})} \le \frac{C}{(p-1)}\|\mathcal{R}_h^k\boldsymbol{\psi}\|_{L^p(\Omega_{\mathbf{x}})}\\ &\le \frac{C}{(p-1)}h^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|\mathcal{R}_h^k\boldsymbol{\psi}\|_{0,\Omega_{\mathbf{x}}} \le \frac{C}{(p-1)}h^{\frac{d}{2}}h^{-d\left(1-\frac{1}{p}\right)}\|\boldsymbol{\psi}\|_{0,\Omega_{\mathbf{x}}}\\ &\le \frac{C}{(p-1)}h^{-d\left(1-\frac{1}{p}\right)}h^{\frac{d}{2}}\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{\sigma^{-2(\alpha+d)}},\end{aligned}$$

where we have also used inverse inequality, the L^2-stability of the Raviart–Thomas projection together with the definition of $\boldsymbol{\psi}$. Taking now $p = 1+\frac{1}{\log(1/h)}$ and using the fact that $h^{-d|\log(h)|^{-1}} = O(1)$, we finally have

$$\|u\|_{W^{1,1}(\Omega_{\mathbf{x}})} \le Ch^{\frac{d}{2}}|\log(h)|\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{\sigma^{-2(\alpha+d)}}.$$

Now, from the relations between the weighted norms and the L^∞-norms (A.11) and (A.10) it follows that

$$\begin{aligned}\|u\|_{W^{1,1}(\Omega_{\mathbf{x}})} &\le Ch^{\frac{d}{2}}|\log(h)|\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{\sigma^{-2(\alpha+d)}}\\ &\le C|\log(h)|h^{\frac{d}{2}}\theta^{-\frac{d}{2}-\alpha}\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}}.\end{aligned}$$

Substituting the above estimate into (A.17) we have

$$|V| \le C|\log(h)|\left(\frac{h}{\theta}\right)^{\frac{d}{2}}\theta^{-\alpha}\|\rho-\rho_h\|_{-1,\infty,\mathcal{T}_h}\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}}.$$

Inserting this estimate into (A.14) and choosing $C^* = \frac{\theta}{h}$ large enough to absorb into the left-hand side the terms $\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|^2_{\sigma^{-(d+\alpha)}}$ we get

$$\begin{aligned}&\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|^2_{\sigma^{-(d+\alpha)}}\\ &\quad\le C\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}\|^2_{\sigma^{-(d+\alpha)}} + C|\log(h)|\left(\frac{h}{\theta}\right)^{\frac{d}{2}}\\ &\qquad\times\theta^{-\alpha}\|\rho-\rho_h\|_{-1,\infty,\mathcal{T}_h}\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}}.\end{aligned}$$

Using now (A.10) and (A.13) to transform the above norms into L^∞-norms together with the definition of θ, we finally get:

$$\begin{aligned}
&\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}}^2\\
&\quad\le C\left(\frac{\theta}{h}\right)^d\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}\|_{0,\infty,\Omega_{\mathbf{x}}}^2+C\left(\frac{\theta^{(d+\alpha)}}{h^d}\right)|\log(h)|\left(\frac{h}{\theta}\right)^{\frac{d}{2}}\\
&\qquad\times\theta^{-\alpha}\|\rho-\rho_h\|_{-1,\infty,\mathcal{T}_h}\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}}\\
&\quad\le C\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}\|_{0,\infty,\Omega_{\mathbf{x}}}^2+(C^*)^{\frac{d}{2}}|\log(h)|\|\rho-\rho_h\|_{-1,\infty,\mathcal{T}_h}\|\mathcal{R}_h^k(\mathbf{E})-\mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}}.
\end{aligned}$$

Recalling that $C^*\ge 1$ is a constant, the above estimate readily implies the assertion of the lemma and the proof is concluded.

A.4.2. *L^∞-error estimates for the LDG approximation to the electrostatic field*

In Ref. 27 the author carries out the pointwise error analysis for the LDG method, with a different approach to that used in Ref. 46. He follows the technique introduced in Refs. 61 and 60, in which instead of using global weighted L^2-error estimates, one has to use local L^2-error estimates along with dyadic decompositions of the domain $\Omega_{\mathbf{x}}$. This strategy relies on sharp pointwise bounds for high-order derivatives of the Green's function. These types of the Green's function estimates are well known for smooth domains, but do not hold for general convex polyhedral domains.[b] We wish to note that since we consider periodic boundary conditions, the issue of *a priori* estimates reduces to the classical interior *a priori* estimates (no special treatment of the boundary is required).

To prove (3.18) in Lemma 3.4 we will use the same approach of Ref. 27; but to account for the nonlinearity of the Poisson problem (1.3), we need to prove further results not contained in Ref. 27 and use some completely different arguments.

Prior to show (3.18), we introduce some notations. For each fixed point $\mathbf{z}\in\overline{\Omega_{\mathbf{x}}}$, real number s and arbitrary $\mathbf{x}\in\mathbb{R}^d$ consider the weight function

$$\sigma_{z,h}^s(x):=\left(\frac{h}{|z-x|+h}\right)^s,\quad x,z\in\Omega_{\mathbf{x}},\ \ -\infty<s<\infty. \tag{A.18}$$

We consider the following norm notation introduced in Ref. 27,

$$\begin{aligned}
\|\boldsymbol{\tau}\|_{L^p(D),z,s}&=\|\sigma_{z,h}^s\boldsymbol{\tau}\|_{L^p(D)},\\
\|\boldsymbol{\tau}\|_{a,1,D,z,s}&=\|\boldsymbol{\tau}\|_{L^1(D),z,s}+\sum_{e\in\mathcal{E}_h^0}\int_{e\cap D}h\sigma_{z,h}^s|[\![\boldsymbol{\tau}]\!]|ds_{\mathbf{x}},\\
|q|_{c,1,D,z,s}&=\sum_{e\in\mathcal{E}_h}\int_{e\cap D}\sigma_{z,h}^s|[\![q]\!]|ds_{\mathbf{x}}.
\end{aligned}\tag{A.19}$$

[b]Recently, in Ref. 50, the authors have shown the Hölder-type estimate for the first-order derivatives and the second-order mixed derivatives of the Green's function, which allows to provide pointwise and L^∞-estimates in general polygonal domains.

Following Ref. 60 we note that, if $s > 0$ and $|z - x| = O(h)$ then $\sigma^s_{z,h}(x) = O(1)$ while $\sigma^s_{z,h}(x) = O(h^s)$ when $|z - x| = O(1)$. Obviously for $s = 0$ we recover the norms without weights. Also we note that the denominator in (A.18) could be replaced by $(|z - x|^2 + h^2)^{\frac{1}{2}}$ without affecting the results. Notice, however, that positive powers of this weight correspond to negative powers of the weight function defined in (A.9). Following Ref. 23, we define

$$|q\|_{W_h^{1,1}(D)} = \|q\|_{L^1(D)} + \sum_{T^{\mathbf{x}} \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \|\nabla q\|_{L^1(T^{\mathbf{x}} \cap D)} + \sum_{e \in \mathcal{E}_h} \int_{e \cap D} |[\![q]\!]| ds_{\mathbf{x}}.$$

Proof of (3.18) in Lemma 3.4. Observe that subtracting (3.15) from the mixed formulation of the continuous Poisson problem (1.3), we have the error equations

$$\begin{cases} a(\mathbf{E} - \mathbf{E}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \Phi - \Phi_h) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\Xi}^r_h, \\ -b(\mathbf{E} - \mathbf{E}_h, q) + c(\Phi - \Phi_h, q) = F(q) & \forall q \in Q^r_h, \end{cases} \tag{A.20}$$

where $F(q) = \int_{\Omega_{\mathbf{x}}} (\rho - \rho_h) q d\mathbf{x}\ \forall q \in Q^r_h$.

Let now $T_z \in \mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ be such that $z \in \bar{T}_z$ and let $\boldsymbol{\delta}_{\mathbf{z}} \in \mathcal{C}^\infty(\Omega_{\mathbf{x}})^d$ be a regularization of the Dirac mass satisfying the following properties:

$$\begin{aligned} &\operatorname{supp}(\boldsymbol{\delta}_{\boldsymbol{z}}) \subset \bar{T}_z, \quad \mathbf{E}_h(z) = \int_{\Omega_{\mathbf{x}}} \mathbf{E}_h \cdot \boldsymbol{\delta}_{\boldsymbol{z}} d\mathbf{x}, \\ &\|\boldsymbol{\delta}_{\mathbf{z}}\|_{L^p(\Omega_{\mathbf{x}})} \le C h^{-d(1-\frac{1}{p})}, \quad 1 \le p \le \infty. \end{aligned} \tag{A.21}$$

Using triangle inequality and (A.21), we have

$$|(\mathbf{E} - \mathbf{E}_h)(z)| \le \|\mathbf{E} - \mathcal{P}_{\mathbf{x}}(\mathbf{E})\|_{L^\infty(\Omega_{\mathbf{x}}), z, s} + \left| \int_{\Omega_{\mathbf{x}}} \boldsymbol{\delta}_{\mathbf{z}} (\mathbf{E} - \mathbf{E}_h) d\mathbf{x} \right|.$$

Next, we introduce the regularized Green's function. Let $\tilde{g}_z \in H^1_{\text{per}}(\Omega_{\mathbf{x}}) \cap L^2_0(\Omega_{\mathbf{x}})$ be the solution of

$$-\Delta \tilde{g}_z = \nabla \cdot (\boldsymbol{\delta}_{\mathbf{z}}) - \mathfrak{c}_0, \quad \mathfrak{c}_0 := \int_{\Omega_{\mathbf{x}}} \nabla \cdot (\boldsymbol{\delta}_{\mathbf{z}}) d\mathbf{x}, \tag{A.22}$$

and let $\widetilde{\mathbf{G}}_z := \nabla \tilde{g}_z + \boldsymbol{\delta}_{\mathbf{z}}$ so that $-\nabla \cdot \widetilde{\mathbf{G}}_z = -\mathfrak{c}_0$. The problem is completed by imposing periodic boundary conditions for both $\tilde{g}_z$ and $\widetilde{\mathbf{G}}_z$.

Let now $(\widetilde{\mathbf{G}}_{z,h}, \tilde{g}_{z,h})$ be the DG approximation to $(\widetilde{\mathbf{G}}_z, \tilde{g}_z)$ that satisfies

$$\begin{aligned} a(\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \tilde{g}_z - \tilde{g}_{z,h}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Xi}^r_h, \\ -b(\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}, q) + c(\tilde{g}_z - \tilde{g}_{z,h}, q) = 0 \quad \forall q \in Q^r_h. \end{aligned} \tag{A.23}$$

Arguing as in Ref. 27 one can easily show the estimates:

$$\|\nabla(\tilde{g}_z - \mathcal{P}_{\mathbf{x}}(\tilde{g}_z))\|_{L^1(\Omega),z,-s} + h\|\nabla\cdot(\widetilde{\mathbf{G}}_z - \mathcal{P}_{\mathbf{x}}(\widetilde{\mathbf{G}}_z))\|_{L^1(\Omega_{\mathbf{x}}),z,-s} \le C|\log(h)|^{\bar{r}}, \tag{A.24}$$

$$\|\tilde{g}_z - \tilde{g}_{z,h}\|_{c,1,\Omega,z,-s} + \|\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}\|_{L^1(\Omega),z,-s} \le C|\log(h)|^{\bar{r}}, \tag{A.25}$$

where $\bar{r} = 0$ for $0 \le s < r-1$ and $\bar{r} = 1$ for $s = r-1$.

Next, observe that the solution $(\widetilde{\mathbf{G}}_z, \tilde{g}_z)$ satisfies

$$\begin{aligned} a(\widetilde{\mathbf{G}}_z, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \tilde{g}_z) &= \int_{\Omega_{\mathbf{x}}} \boldsymbol{\delta}_{\mathbf{z}} \boldsymbol{\tau} d\mathbf{x} \quad \forall \boldsymbol{\tau} \in H(\mathrm{div};\Omega_{\mathbf{x}}), \\ -b(\widetilde{\mathbf{G}}_z, q) + c(\tilde{g}_z, q) &= 0 \qquad\qquad \forall q \in L^2_0(\Omega_{\mathbf{x}}). \end{aligned} \tag{A.26}$$

Observe that in the last equation above we have used that since c_0 is constant $(c_0, q) = 0$ for all $q \in L^2_0(\Omega_{\mathbf{x}})$.

By setting now $(\boldsymbol{\tau}, q) = (\mathbf{E} - \mathbf{E}_h, \Phi - \Phi_h)$ in (A.26) and $(\boldsymbol{\tau}, q) = (\widetilde{\mathbf{G}}_{z,h}, \tilde{g}_{z,h})$ in (A.20) and combining both equations we get

$$\begin{aligned} \int_{\Omega_{\mathbf{x}}} (\mathbf{E} - \mathbf{E}_h)\boldsymbol{\delta}_{\mathbf{z}} d\mathbf{x} &= a(\widetilde{\mathbf{G}}_z, \mathbf{E} - \mathbf{E}_h) + b(\mathbf{E} - \mathbf{E}_h, \tilde{g}_z) \\ &= a(\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}, \mathbf{E} - \mathbf{E}_h) + b(\mathbf{E} - \mathbf{E}_h, \tilde{g}_z - \tilde{g}_{z,h}) \\ &\quad - b(\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}, \Phi - \Phi_h) + c(\Phi - \Phi_h, \tilde{g}_z - \tilde{g}_{z,h}) + F(\tilde{g}_{z,h}) \\ &= a(\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}, \mathbf{E} - \mathcal{P}_{\mathbf{x}}(\mathbf{E})) + b(\mathbf{E} - \mathcal{P}_{\mathbf{x}}(\mathbf{E}), \tilde{g}_z - \tilde{g}_{z,h}) \\ &\quad + b(\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}, \mathcal{P}_{\mathbf{x}}(\Phi) - \Phi) \\ &\quad + c(\Phi - \mathcal{P}_{\mathbf{x}}(\Phi), \tilde{g}_z - \tilde{g}_{z,h}) + F(\tilde{g}_{z,h}) \\ &= I_1 + I_2 + I_3 + I_4 + F(\tilde{g}_{z,h}), \end{aligned} \tag{A.27}$$

where in the last step we have used the Galerkin orthogonality given in (A.23). Then, the Hölder inequality gives for the first four terms:

$$\begin{aligned} I_1 &\le \|\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}\|_{L^1(\Omega_{\mathbf{x}}),z,-s}\|\mathbf{E} - \mathcal{P}_{\mathbf{x}}(\mathbf{E})\|_{L^\infty(\Omega_{\mathbf{x}}),z,s}, \\ I_2 &\le h^{-1}\|\Phi - \mathcal{P}_{\mathbf{x}}(\Phi)\|_{L^\infty(\Omega_{\mathbf{x}}),z,s}\|\tilde{g}_z - \tilde{g}_{z,h}\|_{c,1,\Omega_{\mathbf{x}},z,-s}, \\ I_3 &\le \|\mathbf{E} - \mathcal{P}_{\mathbf{x}}(\mathbf{E})\|_{L^\infty(\Omega_{\mathbf{x}}),z,s}(\|\tilde{g}_z - \tilde{g}_{z,h}\|_{c,1,\Omega_{\mathbf{x}},z,-s} + \|\nabla(\tilde{g}_z - \mathcal{P}_{\mathbf{x}}(\tilde{g}_z))\|_{L^1(\Omega_{\mathbf{x}}),z,-s}), \\ I_4 &\le Ch^{-1}\|\Phi - \mathcal{P}_{\mathbf{x}}(\Phi)\|_{L^\infty(\Omega_{\mathbf{x}}),z,s} \\ &\quad \cdot (\|\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}\|_{a,1,s,z} + h\|\nabla\cdot(\widetilde{\mathbf{G}}_z - \mathcal{P}_{\mathbf{x}}(\widetilde{\mathbf{G}}_z))\|_{L^1(\Omega_{\mathbf{x}}),z,-s}), \end{aligned}$$

which in view of (A.24) and (A.25) yield

$$\begin{aligned} &I_1 + I_2 + I_3 + I_4 \\ &\quad \le C|\log(h)|^{\bar{r}}(\|\mathbf{E} - \mathcal{P}_{\mathbf{x}}(\mathbf{E})\|_{L^\infty(\Omega_{\mathbf{x}}),z,s} + h^{-1}\|\Phi - \mathcal{P}_{\mathbf{x}}(\Phi)\|_{L^\infty(\Omega_{\mathbf{x}}),z,s}). \end{aligned}$$

The passage from the localized estimate to an L^∞-estimate can then be achieved by choosing $z \in \Omega_{\mathbf{x}}$ such that $|(\mathbf{E} - \mathbf{E}_h)(z)| = \|\mathbf{E} - \mathbf{E}_h\|_{0,\infty,\Omega_{\mathbf{x}}}$ and setting $s = 0$.

Therefore, we only need to estimate the last term in (A.27). Triangle inequality and Hölder inequality give

$$\begin{aligned} |F(\tilde{g}_{z,h})| &\leq |F(\tilde{g}_z - \tilde{g}_{z,h})| + |F(\tilde{g}_z)| \\ &\leq \|F\|_{W^{-1,\infty}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} \|\tilde{g}_z - \tilde{g}_{z,h}\|_{W_h^{1,1}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} \\ &\quad + \|F\|_{W^{-1,\infty}(\Omega_{\mathbf{x}})} \|\tilde{g}_z\|_{W^{1,1}(\Omega_{\mathbf{x}})}. \end{aligned} \tag{A.28}$$

Hence, to conclude we need to bound the above terms involving the generalized Green function $\tilde{g}_z$. Sobolev's embeddings together with the *a priori* estimate (A.8) for problem (A.22) and the bound (A.21) give for $1 < p \leq 2$

$$\begin{aligned} \|\tilde{g}_z\|_{W^{1,1}(\Omega_{\mathbf{x}})} &\leq C\|\tilde{g}_z\|_{W^{1,p}(\Omega_{\mathbf{x}})} \leq \frac{C}{p-1}\|\boldsymbol{\delta}_{\mathbf{z}}\|_{L^p(T_z)} \\ &\leq \frac{C}{p-1} h^{-d(1-\frac{1}{p})} \leq C|\log(h)|, \end{aligned} \tag{A.29}$$

where in last step we have taken $p = 1 + \frac{1}{\log\left(\frac{1}{h}\right)}$ and used the fact that $h^{-d|\log(h)|^{-1}} = O(1)$.

Now we estimate the first term in (A.28). Let $E_g = \tilde{g}_z - \tilde{g}_{z,h}$ and let $\mathbf{T_g} = \nabla_{\mathbf{x}}^h(\tilde{g}_z - \tilde{g}_{z,h})$. From the definition in (A.19), we have

$$\begin{aligned} \|\tilde{g}_z - \tilde{g}_{z,h}\|_{W_h^{1,1}(\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}})} &= \|E_g\|_{L^1(\Omega_{\mathbf{x}})} + \sum_{T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \|\mathbf{T_g}\|_{L^1(T^{\mathbf{x}})} \\ &\quad + \sum_{e \in \mathcal{E}_{\mathbf{x}}} \int_e |[\![E_g]\!]| ds_{\mathbf{x}}. \end{aligned} \tag{A.30}$$

The last term above is estimated by setting $s = 0$ in the estimate (A.25). We next estimate the second term above. We first recall that for each $T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}$:

$$\begin{aligned} \|\mathbf{T}_{\tilde{g}}\|_{L^1(T^{\mathbf{x}})} &= \sup_{\substack{\boldsymbol{\tau} \in \mathcal{C}_0^\infty(T^{\mathbf{x}}) \\ \|\boldsymbol{\tau}\|_{L^\infty(T^{\mathbf{x}})} = 1}} \left(\int_{T^{\mathbf{x}}} \mathbf{T}_{\tilde{g}} \cdot \boldsymbol{\tau} d\mathbf{x} \right) \\ &= \left(\int_{T^{\mathbf{x}}} \mathbf{T}_{\tilde{g}} \cdot \boldsymbol{\tau}_T^\epsilon d\mathbf{x} \right) - \epsilon, \quad \epsilon > 0, \end{aligned}$$

for some $\boldsymbol{\tau}_T^\epsilon \in \mathcal{C}_0^\infty(T^{\mathbf{x}})$ with $\|\tau^\epsilon\|_{0,\infty,T^{\mathbf{x}}} = 1$. Let $\boldsymbol{\tau}^\epsilon := \sum_T \boldsymbol{\tau}_T^\epsilon \in \mathcal{C}_0^\infty(\Omega)$ be the function such that $\boldsymbol{\tau}^\epsilon|_{T^{\mathbf{x}}} = \boldsymbol{\tau}_T^\epsilon$. Hence, summation over all the elements in $\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}$ gives

$$\begin{aligned} \sum_{T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \|\mathbf{T}_{\tilde{g}}\|_{L^1(T^{\mathbf{x}})} + \sum_{T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \epsilon &= \sum_{T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \left(\int_{T^{\mathbf{x}}} \nabla_{\mathbf{x}}(\tilde{g}_z - \tilde{g}_{z,h}) \cdot \boldsymbol{\tau}_{T^{\mathbf{x}}}^\epsilon d\mathbf{x} \right) \\ &= \int_{\Omega^{\mathbf{x}}} \nabla_{\mathbf{x}}^h(\tilde{g}_z - \tilde{g}_{z,h}) \cdot \boldsymbol{\tau}^\epsilon d\mathbf{x}. \end{aligned}$$

Notice also that summing and subtracting $\mathcal{P}_{\mathbf{x}}(\tau^\epsilon)$ (with $\mathcal{P}_{\mathbf{x}}$ denoting the standard local L^2-projection), we have

$$\sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \|\mathbf{T}_{\tilde{g}}\|_{L^1(T^{\mathbf{x}})} + \sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \epsilon = \int_{\Omega^{\mathbf{x}}} \mathbf{T}_{\tilde{g}} \cdot \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon) d\mathbf{x} + \int_{\Omega^{\mathbf{x}}} \mathbf{T}_{\tilde{g}} \cdot [\boldsymbol{\tau}^\epsilon - \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon)] d\mathbf{x}$$

$$= S_1 + S_2. \tag{A.31}$$

We now estimate each of the above terms. For the first one, using the definition (3.16) of $b(\cdot,\cdot)$ together with the first error equation in (A.23), we have

$$\begin{aligned}
\int_\Omega \mathbf{T}_{\tilde{g}} \cdot \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon) d\mathbf{x} &= b(\mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon), E_g) + \int_{\mathcal{E}^0_{\mathbf{x}}} [\![E_g]\!] \cdot (\{\mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon)\} - \mathbf{C}_{12}[\![\mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon)]\!]) ds_{\mathbf{x}} \\
&\quad + \int_{\mathcal{E}^\partial_{\mathbf{x}}} E_g \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon) \cdot \mathbf{n} ds_{\mathbf{x}} \\
&= -a(\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}, \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon)) \\
&\quad + \int_{\mathcal{E}^0_{\mathbf{x}}} [\![E_g]\!] \cdot (\{\mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon)\} - \mathbf{C}_{12}[\![\mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon)]\!]) ds_{\mathbf{x}} \\
&\quad + \int_{\mathcal{E}^\partial_{\mathbf{x}}} E_g \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon) \cdot \mathbf{n} ds_{\mathbf{x}}.
\end{aligned}$$

Hence, Hölder inequality, the definitions of the norms (A.19), together with the estimate (A.25) with $s = 0$, give

$$\begin{aligned}
|S_1| &\leq \|\mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon)\|_{0,\infty,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} (\|\widetilde{\mathbf{G}}_z - \widetilde{\mathbf{G}}_{z,h}\|_{L^1(\Omega)} + \|\tilde{g}_z - \tilde{g}_{z,h}\|_{c,1,\Omega}) \\
&\leq C|\log(h)|^{\bar{r}} \|\boldsymbol{\tau}^\epsilon\|_{0,\infty,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}},
\end{aligned} \tag{A.32}$$

where in the last step we have also used the L^∞-stability of the L^2-projection. In the above estimate, $\bar{r} = 1$ for $r = 1$ and $\bar{r} = 0$ for $r > 1$. We now estimate the second term in (A.31). From the definition of the standard L^2-projection, we have

$$\begin{aligned}
&\sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \int_{T^{\mathbf{x}}} (\nabla\tilde{g}_z - \nabla\tilde{g}_{z,h}) \cdot [\boldsymbol{\tau}^\epsilon_{T^{\mathbf{x}}} - \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon_{T^{\mathbf{x}}})] d\mathbf{x} \\
&\quad = \sum_{T^{\mathbf{x}}\in\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \int_{T^{\mathbf{x}}} \nabla(\tilde{g}_z - \mathcal{P}_{\mathbf{x}}(\tilde{g}_z)) \cdot [\boldsymbol{\tau}^\epsilon_{T^{\mathbf{x}}} - \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon_{T^{\mathbf{x}}})] d\mathbf{x}.
\end{aligned}$$

Hence, Hölder inequality, estimate (A.24) with $s = 0$ and the L^∞-stability of the L^2-projection yield to

$$\begin{aligned}
|S_2| &\leq \|\nabla(\tilde{g}_z - \mathcal{P}_{\mathbf{x}}(\tilde{g}_z))\|_{L^1(\mathcal{T}_h)} \|\boldsymbol{\tau}^\epsilon - \mathcal{P}_{\mathbf{x}}(\boldsymbol{\tau}^\epsilon)\|_{0,\infty,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}} \\
&\leq C|\log(h)|^{\bar{r}} \|\boldsymbol{\tau}^\epsilon\|_{0,\infty,\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}},
\end{aligned}$$

where as before, $\bar{r} = 1$ for $r = 1$ and $\bar{r} = 0$ for $r > 1$. Thus, substituting the above estimate, together with (A.32) in (A.31), we have

$$\sum_{T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \|\mathbf{T}_{\tilde{g}}\|_{L^1(T^{\mathbf{x}})} + \sum_{T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \epsilon \leq 2C|\log(h)|^{\bar{r}} \|\boldsymbol{\tau}^{\epsilon}\|_{0,\infty,\mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} = 2C|\log(h)|^{\bar{r}},$$

and now letting $\epsilon \downarrow 0$ we finally get

$$\sum_{T^{\mathbf{x}} \in \mathcal{T}_{h_{\mathbf{x}}}^{\mathbf{x}}} \|\nabla(\tilde{g}_z - \tilde{g}_{z,h})\|_{L^1(T^{\mathbf{x}})} \leq 2C|\log(h)|^{\bar{r}}. \tag{A.33}$$

Hence, to conclude we need to provide a bound for $\|\tilde{g}_z - \tilde{g}_{z,h}\|_{L^1(\Omega_{\mathbf{x}})}$. Using the fact that $\Omega_{\mathbf{x}}$ is convex and both $\tilde{g}_z$ and $\tilde{g}_{z,h}$ are functions with zero average over $\Omega_{\mathbf{x}}$, triangle inequality together with the L^1-Poincaré–Friedrichs inequality for $W^{1,p}(\Omega_{\mathbf{x}})$ functions[1] and the L^1-Poincaré–Friedrichs inequality for DG functions,[23] we have

$$\begin{aligned}
\|\tilde{g}_z - \tilde{g}_{z,h}\|_{L^1(\Omega_{\mathbf{x}})} &\leq \|\tilde{g}_z\|_{L^1(\Omega_{\mathbf{x}})} + \|\tilde{g}_{z,h}\|_{L^1(\Omega_{\mathbf{x}})} \\
&\leq \frac{\operatorname{diam}(\Omega_{\mathbf{x}})}{2} \|\nabla \tilde{g}_z\|_{L^1(\Omega_{\mathbf{x}})} + C(\|\nabla \tilde{g}_{z,h}\|_{L^1(\Omega_{\mathbf{x}})} + \|\tilde{g}_{z,h}\|_{c,1,\Omega_{\mathbf{x}}}) \\
&\leq C|\log(h)| + C|\log(h)|^{\bar{r}} \leq C|\log(h)|,
\end{aligned}$$

where in the last step we have also used the bounds (A.29) together with (A.33) and (A.25).

Therefore substituting the above estimate together with the bounds (A.33) and (A.25) into (A.30), we finally get

$$\|\tilde{g}_z - \tilde{g}_{z,h}\|_{W_h^{1,1}(\mathcal{T}_h)} \leq C|\log(h)|,$$

which, together with (A.29), (A.28) and the definition of the functional F, concludes the proof of the lemma.

Acknowledgments

B.A. is grateful to Daniele Boffi from Università degli Studi di Pavia and to Jesús García-Azorero from Universidad Autónoma of Madrid, for helpful and fruitful discussions while carrying out this work. She also thanks Jonnhy Guzmán from Brown University for his comments and careful reading of parts of Appendix A.4. B.A. also acknowledges the kind hospitality from the Division of Applied Mathematics at Brown while she was visiting in Spring 2010. B. A. has been partially supported by projects MTM2011-27739-C04-04 and HI2008-0173. J.A.C. was supported by projects MTM2011-27739-C04-02 and 2009-SGR-345 from Agència de Gestió d'Ajuts Universitaris i de Recerca-Generalitat de Catalunya. C.W.S. is partially supported by National Science Foundation Grant DMS-0809086 and Department of Energy Grant DE-FG02-08ER25863.

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