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MAXIMUM-PRINCIPLE-SATISFYING HIGH ORDER FINITE VOLUME WEIGHTED ESSENTIALLY NONOSCILLATORY SCHEMES FOR CONVECTION-DIFFUSION EQUATIONS*

XIANGXIONG ZHANG†, YUANYUAN LIU‡, AND CHI-WANG SHU§

Abstract. To easily generalize the maximum-principle-satisfying schemes for scalar conservation laws in [X. Zhang and C.-W. Shu, *J. Comput. Phys.*, 229 (2010), pp. 3091–3120] to convection diffusion equations, we propose a nonconventional high order finite volume weighted essentially nonoscillatory (WENO) scheme which can be proved maximum-principle-satisfying. Two-dimensional extensions are straightforward. We also show that the same idea can be used to construct high order schemes preserving the maximum principle for two-dimensional incompressible Navier–Stokes equations in the vorticity stream-function formulation. Numerical tests for the fifth order WENO schemes are reported.

Key words. convection diffusion equations, finite volume scheme, weighted essentially nonoscillatory scheme, maximum principle, high order accuracy, strong stability preserving time discretization, incompressible flow, Navier–Stokes equations

AMS subject classifications. 65M06, 65M60, 65M12

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1. Introduction.

Consider the initial value problem for the convection diffusion equation

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}f(u(x,t))=\frac{\partial^2}{\partial x^2}A(u(x,t)),\quad u(x,0)=u_0(x), \tag{1.1}$$

where $A'(u)\geq 0$. An important property of the exact solutions of (1.1) is that it satisfies a strict maximum principle, i.e., if

$$M=\max_x u_0(x),\quad m=\min_x u_0(x),$$

then $u(x,t)\in[m,M]$ for any x and $t\geq 0$. This property is also desired for numerical solutions because values outside of $[m,M]$ are meaningless physically in many applications, for example, radionuclide transport calculations [1].

To construct maximum-principle-satisfying numerical schemes solving (1.1), the first step is to construct maximum-principle-satisfying schemes for scalar conservation laws

$$u_t+f(u)_x=0,\quad u(x,0)=u_0(x). \tag{1.2}$$

A practical method was proposed recently in [14] to obtain arbitrarily high order maximum-principle-satisfying finite volume or discontinuous Galerkin schemes. It

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†Department of Mathematics, Brown University, Providence, RI 02912. Current address: Department of Mathematics, MIT, Cambridge, MA 02139 (zhangxx@dam.brown.edu).

‡Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China (xiaoliu@mail.ustc.edu.cn).

§Division of Applied Mathematics, Brown University, Providence, RI 02912 (shu@dam.brown.edu).

was the first time that a genuinely high order maximum-principle-satisfying scheme for multidimensional nonlinear scalar conservation laws was available. Suitable generalizations result in high order schemes satisfying positivity preserving property for certain quantities for systems [13, 15, 16, 18]. For a survey of the development of such schemes, see [17].

We first review the main idea in [14]. Assume that we have a uniform mesh $x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}}$ and that $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ is a constant. Define the cell centers as $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$. A conservative finite volume scheme with Euler forward time discretization solving (1.2) has the following form:

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left[\widehat{f}(u^-_{i+\frac{1}{2}}, u^+_{i+\frac{1}{2}}) - \widehat{f}(u^-_{i-\frac{1}{2}}, u^+_{i-\frac{1}{2}})\right], \tag{1.3}$$

where n refers to the time step and i to the spatial cell, $\bar{u}_i^n$ is the approximation to the cell averages of $u(x,t)$ in the cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ at time level n, and $u^-_{i+\frac{1}{2}}$, $u^+_{i+\frac{1}{2}}$ are the approximations of the nodal values $u(x_{i+\frac{1}{2}}, t^n)$ within the cells I_i and I_{i+1}, respectively. The numerical flux $\widehat{f}$ is a monotone flux, for example, the Lax–Friedrichs flux. Assume that there is a polynomial $p_i(x)$ (for example, the reconstruction polynomial in a finite volume scheme) defined on I_i such that $\bar{u}_i^n$ is the cell average of $p_i(x)$ on I_i, $u^+_{i-\frac{1}{2}} = p_i(x_{i-\frac{1}{2}})$ and $u^-_{i+\frac{1}{2}} = p_i(x_{i+\frac{1}{2}})$. The main idea in [14] can be summarized as follows:

- Use strong stability preserving (SSP) high order time discretizations. For more detail, see [10, 11, 3, 2]. Then it suffices to find a way to preserve the maximum principle for (1.3) since SSP high order discretizations are convex combinations of Euler forward.
- Monotonicity: the right-hand side of (1.3) is equivalent to a monotonically increasing function with respect to some point values of $p_i(x)$ (the Gauss–Lobatto quadrature points). Therefore, if $p_i(x) \in [m, M]$ $\forall x \in I_i$, or if this holds at suitable Gauss–Lobatto quadrature points, then $\bar{u}_i^{n+1} \in [m, M]$.
- A linear scaling limiter: the following modified polynomial $\widetilde{p}_i(x)$ is still an accurate approximation and satisfies $\widetilde{p}_i(x) \in [m, M]$ $\forall x \in I_i$:

  $$\widetilde{p}_i(x) = \theta(p_i(x) - \bar{u}_i^n) + \bar{u}_i^n, \qquad \theta = \min\left\{\left|\frac{M - \bar{u}_i^n}{M_i - \bar{u}_i^n}\right|, \left|\frac{m - \bar{u}_i^n}{m_i - \bar{u}_i^n}\right|, 1\right\}, \tag{1.4}$$

  $$M_i = \max_{x\in I_i} p_i(x), \qquad m_i = \min_{x\in I_i} p_i(x). \tag{1.5}$$

  These maximum and minimum values can be replaced by those evaluated at the Gauss–Lobatto quadrature points, hence greatly reducing the computational complexity.

To generalize the idea above to the convection diffusion equations (1.1), it suffices to seek a direct generalization to the diffusion equations. Unfortunately, it seems extremely difficult to do so, if not impossible. Consider the simplest heat equation $u_t = u_{xx}$. Integrate the equation on I_i; we have

$$\frac{d}{dt}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)dx = u_x(x_{i+\frac{1}{2}}, t) - u_x(x_{i-\frac{1}{2}}, t), \tag{1.6}$$

so a conservative finite volume scheme with Euler forward time discretization has the form

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}), \tag{1.7}$$

where $h_{i+\frac{1}{2}}$ is an approximation to u_x at $x = x_{i+\frac{1}{2}}$. The monotonicity (with respect to selected point values), the most crucial step in [14], seems to be achievable only for first and second order approximations.

For an arbitrarily high order approximation, it seems very difficult to establish the monotonicity of (1.7) with respect to selected point values, mainly because h approximates u_x, not u. One way to overcome this difficulty is to remove the spatial derivatives in (1.6) by integrating the equation one more time,

$$\frac{d}{dt}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x} u(\xi,t)d\xi dx = u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t). \tag{1.8}$$

Define the double cell averages of a function $u(x)$ over the intervals $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ as

$$\bar{\bar{u}}_i = \frac{1}{\Delta x^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u(\xi)d\xi\right)dx;$$

then a conservative spatial approximation of (1.8) with Euler forward time discretization has the form

$$\bar{\bar{u}}_i^{n+1} = \bar{\bar{u}}_i^{n} + \frac{\Delta t}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1}). \tag{1.9}$$

For a scheme of the type (1.9), we will show in section 3 that monotonicity (with respect to a selected set of point values) is true for high order schemes. Thus it is straightforward to obtain high order maximum-principle-satisfying schemes approximating the following twice-integrated version of (1.1):

$$\begin{aligned}\frac{d\bar{\bar{u}}_i(t)}{dt} + \frac{1}{\Delta x^2}\left[\int_{x_i}^{x_{i+1}} f(u(x))dx - \int_{x_{i-1}}^{x_i} f(u(x))dx\right]& \\ = \frac{1}{\Delta x^2}\left[A(u(x_{i+1})) - 2A(u(x_i)) + A(u(x_{i-1}))\right].&\end{aligned} \tag{1.10}$$

To this end, in this paper we will construct a high order nonconventional finite volume scheme approximating (1.10) and we will show a straightforward application of the methodology in [14] to enforce the maximum principle of this scheme.

For conventional finite volume schemes whose numerical solutions are cell averages, one can use the essentially nonoscillatory (ENO) and the weighted ENO (WENO) reconstruction procedures [4, 6, 5, 12] to construct point values needed in (1.3). For a nonconventional finite volume scheme like (1.9), we can also use WENO reconstruction based on the double cell averages to construct point values u_i. Such reconstructions were used in finite difference WENO schemes for parabolic equations discussed in [8].

The paper is organized as follows. We first describe the fifth order accurate WENO reconstruction based on the double cell averages in section 2. Then we prove the maximum principle for the fifth order scheme in one space dimension in section 3. In section 4, we provide a straightforward extension to two space dimensions on rectangular meshes. Section 5 is the application to two dimensional incompressible Navier–Stokes equations in the vorticity stream-function formulation. Concluding remarks are given in section 6.

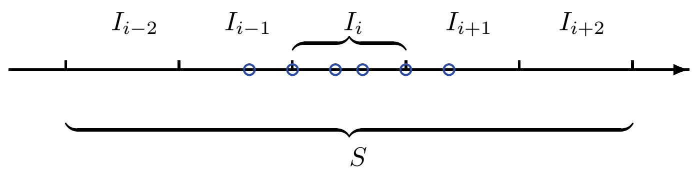


FIG. 2.1. *An illustration of the six points* $x^{\alpha}_{i\pm\frac{1}{2}}$ $(\alpha = 1, 2, 3)$.

2. A fifth order WENO reconstruction based on double cell averages.

2.1. Preliminaries. Given the double cell averages $\bar{\bar{u}}_i$ of a smooth function $u(x)$, we would like to find a fifth order accurate approximation to $u(x)$ at any given point.

We first list the points needed in this paper, which are quadrature points for exact integrations of polynomials of degree four because we would like to construct a fifth order scheme.

I. Three-point Gauss quadrature points on $[\mathbf{x_{i-1}}, \mathbf{x_i}]$. To obtain fifth order accuracy, it suffices to use the three-point Legendre–Gauss quadrature rule to approximate the integrals in (1.10). The three Legendre–Gauss quadrature points and the weights on $[-\frac{1}{2}, \frac{1}{2}]$ are

$$x_\alpha = \left\{-\frac{\sqrt{15}}{10}, 0, \frac{\sqrt{15}}{10}\right\}, \qquad w_\alpha = \left\{\frac{5}{18}, \frac{4}{9}, \frac{5}{18}\right\}.$$

Denote the Gauss quadrature points on the interval $[x_i, x_{i+1}]$ as $x^{\alpha}_{i+\frac{1}{2}} = x_{i+\frac{1}{2}} + x_\alpha \Delta x$ for $\alpha = 1, 2, 3$. See Figure 2.1 for an illustration of $x^{\alpha}_{i\pm\frac{1}{2}}$ $(\alpha = 1, 2, 3)$.

II. Quadrature points for the double integral. It is convenient to introduce quadrature rules for the double integral $\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u(\xi) d\xi dx$. To this end, consider a polynomial of degree four $p(x)$. Replacing the integrals by the three-point Gauss quadrature,

$$\begin{aligned}\frac{1}{\Delta x^2} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p(\xi) d\xi dx &= \frac{1}{\Delta x} \sum_{\alpha=1}^{3} w_\alpha \int_{x_i + x_\alpha \Delta x - \frac{\Delta x}{2}}^{x_i + x_\alpha \Delta x + \frac{\Delta x}{2}} p(\xi) d\xi \\ &= \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} w_\alpha w_\beta p(x_i + x_\alpha \Delta x + x_\beta \Delta x).\end{aligned}$$

Notice that some of the points $x_i + x_\alpha \Delta x + x_\beta \Delta x$ $(\alpha, \beta = 1, 2, 3)$ are overlapped. Thus there are actually only five different points. For simplicity, we use $\bar{x}^{\alpha}_i$ $(\alpha = 1, \dots, 5)$ to denote these five points $\{x_i - \frac{\sqrt{10}}{5}\Delta x, x_i - \frac{\sqrt{10}}{10}\Delta x, x_i, x_i + \frac{\sqrt{10}}{10}\Delta x, x_i + \frac{\sqrt{10}}{5}\Delta x\}$ and $\bar{w}_\alpha$ $(\alpha = 1, \dots, 5)$ to denote the corresponding weights satisfying $\sum_{\alpha=1}^{5} \bar{w}_\alpha = 1$. See Figure 2.2. Then

$$\frac{1}{\Delta x^2} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p(\xi) d\xi dx = \sum_{\alpha=1}^{5} \bar{w}_\alpha p(\bar{x}^{\alpha}_i). \tag{2.1}$$

III. Quadrature points for the double integral of a piecewise polynomial. Consider a piecewise polynomial function $p(x)$ defined by $p(x) = p_j(x), x \in$

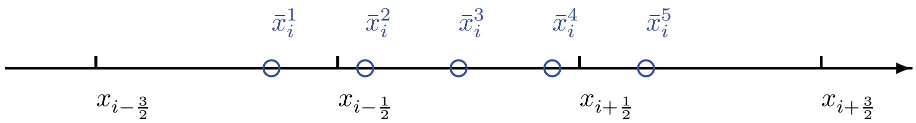


FIG. 2.2. *The five-point quadrature for the double integral.*

$[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, where $p_j(x)$ are polynomials of degree four. Notice that

$$\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p(\xi) d\xi dx &= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{x-\frac{\Delta x}{2}}^{x_i} p(\xi) d\xi dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{x_i}^{x+\frac{\Delta x}{2}} p(\xi) d\xi dx \\
&= \int_{x_{i-1}}^{x_i} \int_{x_{i-\frac{1}{2}}}^{\xi+\frac{\Delta x}{2}} p(\xi) dx d\xi + \int_{x_i}^{x_{i+1}} \int_{\xi-\frac{\Delta x}{2}}^{x_{i+\frac{1}{2}}} p(\xi) dx d\xi \\
&= \int_{x_{i-1}}^{x_i} p(\xi)(\xi - x_{i-1}) d\xi + \int_{x_i}^{x_{i+1}} p(\xi)(x_{i+1} - \xi) d\xi \\
&= \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} p_{i-1}(\xi)(\xi - x_{i-1}) d\xi + \int_{x_{i-\frac{1}{2}}}^{x_i} p_i(\xi)(\xi - x_{i-1}) d\xi \\
&\quad + \int_{x_i}^{x_{i+\frac{1}{2}}} p_i(\xi)(x_{i+1} - \xi) d\xi + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} p_{i+1}(\xi)(x_{i+1} - \xi) d\xi.
\end{aligned}$$

For each of the four intervals $[x_{i-1}, x_{i-\frac{1}{2}}]$, $[x_{i-\frac{1}{2}}, x_i]$, $[x_i, x_{i+\frac{1}{2}}]$, and $[x_{i+\frac{1}{2}}, x_{i+1}]$, we can use the three-point Gauss quadrature. Then

$$\begin{aligned}
&\frac{1}{\Delta x^2} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p(\xi) d\xi dx \\
&= \frac{1}{2\Delta x} \sum_{\alpha=1}^{3} w^\alpha p_{i-1}\left(x_{i-\frac{3}{4}} + \frac{x_\alpha}{2}\Delta x\right)\left(x_{i-\frac{3}{4}} + \frac{x_\alpha}{2}\Delta x - x_{i-1}\right) \\
&\quad + \frac{1}{2\Delta x} \sum_{\alpha=1}^{3} w^\alpha p_{i}\left(x_{i-\frac{1}{4}} + \frac{x_\alpha}{2}\Delta x\right)\left(x_{i-\frac{1}{4}} + \frac{x_\alpha}{2}\Delta x - x_{i-1}\right) \\
&\quad + \frac{1}{2\Delta x} \sum_{\alpha=1}^{3} w^\alpha p_{i}\left(x_{i+\frac{1}{4}} + \frac{x_\alpha}{2}\Delta x\right)\left(x_{i+1} - \left(x_{i+\frac{1}{4}} + \frac{x_\alpha}{2}\Delta x\right)\right) \\
&\quad + \frac{1}{2\Delta x} \sum_{\alpha=1}^{3} w^\alpha p_{i+1}\left(x_{i+\frac{3}{4}} + \frac{x_\alpha}{2}\Delta x\right)\left(x_{i+1} - \left(x_{i+\frac{3}{4}} + \frac{x_\alpha}{2}\Delta x\right)\right).
\end{aligned} \tag{2.2}$$

Let $\tilde{x}_i^\alpha$ ($\alpha = 1, \ldots, 12$) denote $\{x_{i-\frac{3}{4}} - \frac{\sqrt{15}}{20}\Delta x, x_{i-\frac{3}{4}}, x_{i-\frac{3}{4}} + \frac{\sqrt{15}}{20}\Delta x, x_{i-\frac{1}{4}} - \frac{\sqrt{15}}{20}\Delta x, x_{i-\frac{1}{4}}, x_{i-\frac{1}{4}} + \frac{\sqrt{15}}{20}\Delta x, x_{i+\frac{1}{4}} - \frac{\sqrt{15}}{20}\Delta x, x_{i+\frac{1}{4}}, x_{i+\frac{1}{4}} + \frac{\sqrt{15}}{20}\Delta x, x_{i+\frac{3}{4}} - \frac{\sqrt{15}}{20}\Delta x, x_{i+\frac{3}{4}}, x_{i+\frac{3}{4}} + \frac{\sqrt{15}}{20}\Delta x\}$ and let $\tilde{w}_\alpha$ ($\alpha = 1, \ldots, 12$) denote the corresponding weights $\{\frac{1}{2}(\frac{1}{4} + \frac{1}{2}x_1)w_1, \frac{1}{2}(\frac{1}{4} + \frac{1}{2}x_2)w_2, \frac{1}{2}(\frac{1}{4} + \frac{1}{2}x_3)w_3, \frac{1}{2}(\frac{3}{4} + \frac{1}{2}x_1)w_1, \frac{1}{2}(\frac{3}{4} + \frac{1}{2}x_2)w_2, \frac{1}{2}(\frac{3}{4} + \frac{1}{2}x_3)w_3, \frac{1}{2}(\frac{3}{4} - \frac{1}{2}x_1)w_1, \frac{1}{2}(\frac{3}{4} - \frac{1}{2}x_2)w_2, \frac{1}{2}(\frac{3}{4} - \frac{1}{2}x_3)w_3, \frac{1}{2}(\frac{1}{4} - \frac{1}{2}x_1)w_1, \frac{1}{2}(\frac{1}{4} - \frac{1}{2}x_2)w_2, \frac{1}{2}(\frac{1}{4} - \frac{1}{2}x_3)w_3\}$. The weights satisfy $\sum_{\alpha=1}^{12} \tilde{w}_\alpha = 1$. Then (2.2) becomes

$$\frac{1}{\Delta x^2} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p(\xi) d\xi dx = \sum_{\alpha=1}^{12} \tilde{w}_\alpha p(\tilde{x}_i^\alpha).$$

2.2. Linear weights. For each big stencil $S = \{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\}$, we seek a polynomial of degree four

$$p(x) = \sum_{l=0}^{4} a_l \left(\frac{x - x_i}{\Delta x}\right)^l$$

with undetermined coefficients a_l such that

$$\frac{1}{\Delta x^2} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p(\xi) d\xi \right) dx = \bar{\bar{u}}_j, \qquad j = i-2, \ldots, i+2.$$

The coefficients a_l are obtained explicitly by solving the linear system

$$\begin{aligned}
a_0 &= \frac{1}{180}(2\bar{\bar{u}}_{i-2} - 23\bar{\bar{u}}_{i-1} + 222\bar{\bar{u}}_i - 23\bar{\bar{u}}_{i+1} + 2\bar{\bar{u}}_{i+2}), \\
a_1 &= \frac{1}{8}(\bar{\bar{u}}_{i-2} - 6\bar{\bar{u}}_{i-1} + 6\bar{\bar{u}}_{i+1} - \bar{\bar{u}}_{i+2}), \\
a_2 &= \frac{1}{12}(-\bar{\bar{u}}_{i-2} + 10\bar{\bar{u}}_{i-1} - 18\bar{\bar{u}}_i + 10\bar{\bar{u}}_{i+1} - \bar{\bar{u}}_{i+2}), \\
a_3 &= \frac{1}{12}(-\bar{\bar{u}}_{i-2} + 2\bar{\bar{u}}_{i-1} - 2\bar{\bar{u}}_{i+1} + \bar{\bar{u}}_{i+2}), \\
a_4 &= \frac{1}{24}(\bar{\bar{u}}_{i-2} - 4\bar{\bar{u}}_{i-1} + 6\bar{\bar{u}}_i - 4\bar{\bar{u}}_{i+1} + \bar{\bar{u}}_{i+2}).
\end{aligned}$$

The approximation polynomial $p(x)$ is fifth order accurate

$$p(x) = u(x) + O(\Delta x^5) \qquad \forall x \in I_j, \quad j = i-2, \ldots, i+2,$$

provided that the function $u(x)$ is smooth on the big stencil S.

Following a similar argument, we obtain three polynomials of degree two, $p_m(x)$ on each small stencil $S_m = \{I_{i-2+m}, I_{i-1+m}, I_{i+m}\}$ with $m = 0, 1, 2$,

$$\begin{aligned}
p_0(x) &= \frac{1}{12}(-\bar{\bar{u}}_{i-2} + 2\bar{\bar{u}}_{i-1} + 11\bar{\bar{u}}_i) + \frac{1}{2}(\bar{\bar{u}}_{i-2} - 4\bar{\bar{u}}_{i-1} + 3\bar{\bar{u}}_i)\tilde{x} \\
&\quad + \frac{1}{2}(\bar{\bar{u}}_{i-2} - 2\bar{\bar{u}}_{i-1} + \bar{\bar{u}}_i)\tilde{x}^2, \\
p_1(x) &= \frac{1}{12}(-\bar{\bar{u}}_{i-1} + 14\bar{\bar{u}}_i - \bar{\bar{u}}_{i+1}) + \frac{1}{2}(-\bar{\bar{u}}_{i-1} + \bar{\bar{u}}_{i+1})\tilde{x} + \frac{1}{2}(\bar{\bar{u}}_{i-1} - 2\bar{\bar{u}}_i + \bar{\bar{u}}_{i+1})\tilde{x}^2, \\
p_2(x) &= \frac{1}{12}(11\bar{\bar{u}}_i + 2\bar{\bar{u}}_{i+1} - \bar{\bar{u}}_{i+2}) + \frac{1}{2}(-3\bar{\bar{u}}_i + 4\bar{\bar{u}}_{i+1} - \bar{\bar{u}}_{i+2})\tilde{x} \\
&\quad + \frac{1}{2}(\bar{\bar{u}}_i - 2\bar{\bar{u}}_{i+1} + \bar{\bar{u}}_{i+2})\tilde{x}^2,
\end{aligned} \tag{2.3}$$

where $\tilde{x} = \frac{x - x_i}{\Delta x}$ and they are third order accurate,

$$p_m(x) = u(x) + O(\Delta x^3) \qquad \forall x \in I_j, \quad j = i-2, \ldots, i+2. \tag{2.4}$$

For any fixed $x \in I_j, j = i-2, \ldots, i+2$, define the *linear weights*, $d_m(x)$ as the combination coefficients satisfying

$$p(x) = \sum_{m=0}^{2} d_m(x) p_m(x) \tag{2.5}$$

Table 2.1
Linear weights for $x^{\alpha}_{i\pm\frac{1}{2}}$.

x	$d_1(x)$	$d_2(x)$	$d_3(x)$
$x_{i-\frac{1}{2}} - \frac{\sqrt{15}}{10}\Delta x$	$(307+72\sqrt{15})/960$	$(8377-1542\sqrt{15})/6720$	$173(-11+3\sqrt{15})/3360$
$x_{i-\frac{1}{2}}$	$341/1200$	$337/600$	$37/240$
$x_{i-\frac{1}{2}} + \frac{\sqrt{15}}{10}\Delta x$	$(307-72\sqrt{15})/960$	$(8377+1542\sqrt{15})/6720$	$-173(11+3\sqrt{15})/3360$
$x_{i+\frac{1}{2}} - \frac{\sqrt{15}}{10}\Delta x$	$-173(11+3\sqrt{15})/3360$	$(8377+1542\sqrt{15})/6720$	$(307-72\sqrt{15})/960$
$x_{i+\frac{1}{2}}$	$37/240$	$337/600$	$341/1200$
$x_{i+\frac{1}{2}} + \frac{\sqrt{15}}{10}\Delta x$	$173(-11+3\sqrt{15})/3360$	$(8377-1542\sqrt{15})/6720$	$(307+72\sqrt{15})/960$

Table 2.2
Linear weights for $\bar{x}^{\alpha}_{i}$.

x	$d_1(x)$	$d_2(x)$	$d_3(x)$
$x_i - \frac{\sqrt{15}}{5}\Delta x$	$(427+87\sqrt{15})/1590$	$368/795$	$(427-87\sqrt{15})/1590$
$x_i - \frac{\sqrt{15}}{10}\Delta x$	$(29147-246\sqrt{15})/129360$	$35533/64680$	$(29147+246\sqrt{15})/129360$
x_i	$-2/15$	$19/15$	$-2/15$
$x_i + \frac{\sqrt{15}}{10}\Delta x$	$(29147+246\sqrt{15})/129360$	$35533/64680$	$(29147-246\sqrt{15})/129360$
$x_i + \frac{\sqrt{15}}{5}\Delta x$	$(427-87\sqrt{15})/1590$	$368/795$	$(427+87\sqrt{15})/1590$

Table 2.3
Linear weights for $\tilde{x}^{\alpha}_{i}$.

x	$d_1(x)$	$d_2(x)$	$d_3(x)$
$x_{i-\frac{3}{4}} - \frac{\sqrt{15}}{20}\Delta x$	$\frac{323006+31701\sqrt{15}}{628320}$	$\frac{2170712596-194035923\sqrt{15}}{5274432240}$	$\frac{599306-110127\sqrt{15}}{8058720}$
$x_{i-\frac{3}{4}}$	$7481/16320$	$214861/448800$	$3319/52800$
$x_{i-\frac{3}{4}} + \frac{\sqrt{15}}{20}\Delta x$	$\frac{323006-31701\sqrt{15}}{628320}$	$\frac{2170712596+194035923\sqrt{15}}{5274432240}$	$\frac{599306+110127\sqrt{15}}{8058720}$
$x_{i-\frac{1}{4}} - \frac{\sqrt{15}}{20}\Delta x$	$\frac{9698+3309\sqrt{15}}{89760}$	$\frac{1862056-201081\sqrt{15}}{1929840}$	$\frac{-25586+23625\sqrt{15}}{350880}$
$x_{i-\frac{1}{4}}\Delta x$	$2201/16320$	$4241/11424$	$3319/6720$
$x_{i-\frac{1}{4}} + \frac{\sqrt{15}}{20}\Delta x$	$\frac{9698-3309\sqrt{15}}{89760}$	$\frac{1862056+201081\sqrt{15}}{1929840}$	$\frac{-25586-23625\sqrt{15}}{350880}$

and $\sum_{m=0}^{2} d_m(x) = 1$ by consistency. In [7], the positivity of linear weights for various WENO procedures and the existence and nonexistence of linear weights for various situations were studied. Except for special situations, the linear weights $d_m(x)$ are rational functions.

For a fixed point x, the corresponding linear weights are obtained by solving the linear system (2.5). See Tables 2.1, 2.2, and 2.3 for the linear weights needed in this paper. We only listed the linear weights of $\tilde{x}^{\alpha}_{i}$ for $\alpha = 1, \ldots, 6$ since $d_m(\tilde{x}^{\alpha}_{i}) = d_{4-m}(\tilde{x}^{13-\alpha}_{i})$.

2.3. Treatment for negative linear weights. Notice that the linear weights for $x_{i-\frac{1}{2}} + \frac{\sqrt{15}}{10}\Delta x$ and $x_{i+\frac{1}{2}} - \frac{\sqrt{15}}{10}\Delta x$ in Table 2.1, for x_i in Table 2.2, and for

$x_{i-\frac{1}{4}}+\frac{\sqrt{15}}{20}\Delta x$ in Table 2.3 are negative. Numerical test cases for both scalar equations and systems were shown in [9], indicating that the presence of negative weights without special treatment may lead to instability (blow-up of the numerical solution) of WENO schemes. We use the technique in [9] to treat the negative weights. Split the linear weights into two parts, positive and negative, by defining

$$\tilde{\gamma}_m^+ = \frac{1}{2}(d_m + \theta|d_m|), \qquad \tilde{\gamma}_m^- = \tilde{\gamma}_m^+ - d_m,$$

with $m = 0, 1, 2$ and $\theta = 3$. Then scale them by

$$\gamma_m^\pm = \tilde{\gamma}_m^\pm/\sigma^\pm, \qquad \sigma^\pm = \sum_{m=0}^{2} \tilde{\gamma}_m^\pm. \tag{2.6}$$

It is easy to check that

$$\sum_{m=0}^{2} \gamma_m^\pm = 1, \qquad d_m = \sigma^+\gamma_m^+ - \sigma^-\gamma_m^-. \tag{2.7}$$

2.4. The smoothness indicators and nonlinear weights. The *smoothness indicator* β_m is a measure of the relative smoothness of the function $u(x)$ based on the small stencils S_m with $m = 0, 1, 2$. The larger the smoothness indicator β_m, the less smooth the function $u(x)$ in the stencil S_m. Following [5, 12], the smoothness indicator can be defined by

$$\beta_m = \sum_{l=1}^{2} \Delta x^{2l-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^l}{dx^l} p_m(x)\right)^2 dx, \tag{2.8}$$

where $p_m(x)$ is given in (2.3). The right-hand side of (2.8) is just a sum of the squares of scaled L^2 norms for all derivatives of the reconstruction polynomial $p_m(x)$ over the interval $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$. The factor Δx^{2l-1} is introduced to remove any Δx dependency in the smoothness indicators, in order to preserve self-similarity.

The smoothness indicators in terms of the double averages are given by

$$\begin{aligned}
\beta_0 &= \frac{13}{12}(\bar{\bar{u}}_{i-2} - 2\bar{\bar{u}}_{i-1} + \bar{\bar{u}}_i)^2 + \frac{1}{4}(\bar{\bar{u}}_{i-2} - 4\bar{\bar{u}}_{i-1} + 3\bar{\bar{u}}_i)^2,\\
\beta_1 &= \frac{13}{12}(\bar{\bar{u}}_{i-1} - 2\bar{\bar{u}}_i + \bar{\bar{u}}_{i+1})^2 + \frac{1}{4}(\bar{\bar{u}}_{i-1} - \bar{\bar{u}}_{i+1})^2,\\
\beta_2 &= \frac{13}{12}(\bar{\bar{u}}_i - 2\bar{\bar{u}}_{i+1} + \bar{\bar{u}}_{i+2})^2 + \frac{1}{4}(3\bar{\bar{u}}_i - 4\bar{\bar{u}}_{i+1} + \bar{\bar{u}}_{i+2})^2.
\end{aligned} \tag{2.9}$$

For each fixed x, if the linear weights $d_m(x)$ in (2.5) are positive for all m, define the nonlinear weights based on the smoothness indicators in (2.9),

$$\omega_m = \frac{\tilde{\omega}_m}{\sum_{l=0}^{2}\tilde{\omega}_l}, \qquad \tilde{\omega}_m = \frac{d_m}{(\epsilon+\beta_m)^2}. \tag{2.10}$$

If negative linear weights are present, define the nonlinear weights for the positive and negative groups $\omega_m^\pm$, respectively, based on the same smoothness indicators,

$$\omega_m^\pm = \frac{\tilde{\omega}_m^\pm}{\sum_{l=0}^{2}\tilde{\omega}_l^\pm}, \qquad \tilde{\omega}_m^\pm = \frac{\gamma_m^\pm}{(\epsilon+\beta_m)^2}, \tag{2.11}$$

and form the nonlinear weights by rewriting the positive and negative nonlinear weights together:

$$\omega_m = \sigma^+\omega_m^+ - \sigma^-\omega_m^-. \tag{2.12}$$

Here ϵ in (2.10) and (2.11) is introduced to prevent the denominator from becoming zero (we take, as usual, $\epsilon = 10^{-6}$ in this paper), and $\gamma_m^\pm$ and $\sigma^\pm$ are given in (2.6).

2.5. Analysis of the accuracy. First, through a Taylor expansion analysis, we have

$$\beta_m = D(1 + O(\Delta x^2)), \qquad m = 0, 1, 2,$$

where D is a nonzero quantity independent of m but may depend on the derivatives of $\bar{\bar{u}}(x) = \frac{1}{\Delta x^2}\int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x}\left(\int_{\eta-\frac{\Delta x}{2}}^{\eta+\frac{\Delta x}{2}} u(\xi)d\xi\right)d\eta$ and Δx.

Then we show that $\beta_m = D(1 + O(\Delta x^2))$ is the sufficient condition so that $\omega_m = d_m + O(\Delta x^2)$ holds. By the Taylor expansion $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 \ldots$ near $x = 0$ and neglecting ϵ, we obtain

$$\frac{\gamma_m^\pm}{(\epsilon + \beta_m)^2} = \frac{\gamma_m^\pm}{(D(1 + O(\Delta x^2)))^2} = \frac{\gamma_m^\pm}{D^2}(1 + O(\Delta x^2)).$$

Equations (2.11) and (2.7) imply

$$\begin{aligned}\gamma_m^\pm &= \omega_m^\pm\left(\sum_{l=0}^{2}\frac{\gamma_l^\pm}{(\epsilon + \beta_l)^2}\right)(\epsilon + \beta_m)^2\\ &= \omega_m^\pm\left(\frac{1}{D^2}(1 + O(\Delta x^2))\right)(D(1 + O(\Delta x^2)))^2\\ &= \omega_m^\pm + O(\Delta x^2), \qquad m = 0, 1, 2.\end{aligned}$$

Thus the linear weights in (2.7) and the nonlinear weights defined in (2.12) satisfy

$$\omega_m = d_m + O(\Delta x^2), \qquad m = 0, 1, 2. \tag{2.13}$$

Finally, the WENO reconstruction is fifth order accurate,

$$\sum_{m=0}^{2}\omega_m p_m(x) = u(x) + O(\Delta x^5),$$

if the function $u(x)$ is smooth in the big stencil S, because

$$\begin{aligned}\sum_{m=0}^{2}\omega_m p_m(x) - \sum_{m=0}^{2} d_m p_m(x) &= \sum_{m=0}^{2}(\omega_m - d_m)(p_m(x) - u(x))\\ &= \sum_{m=0}^{2} O(\Delta x^2)O(\Delta x^3) = O(\Delta x^5),\end{aligned}$$

where in the first equality we use the conditions $\sum_{m=0}^{2} d_m = 1$ and $\sum_{m=0}^{2}\omega_m = 1$, and in the second equality we use (2.4) and (2.13).

2.6. WENO reconstruction. Now we can formulate the WENO reconstruction as follows.

Procedure 1. WENO reconstruction on the stencil $\mathbf{S} = \{\mathbf{I_{i-2}}, \dots, \mathbf{I_{i+2}}\}$. Given $\bar{\bar{u}}_j$ $(j = i-2, \dots, i+2)$, a fixed point x, and the linear weights $d_m(x)$,

1. Find the smoothness indicators β_m with $m = 0, 1, 2$ in (2.9) and the nonlinear weights ω_m using (2.10) if all the $d_m(x)$ are nonnegative. Otherwise use (2.12) to evaluate the nonlinear weights.
2. Find the fifth order approximation of $u(x)$ by $\sum_{m=0}^{2} \omega_m p_m(x)$.

For convenience, we denote the one-dimensional WENO reconstruction procedures at different points used in this paper by the following:

- Procedure 2. Apply Procedure 1 at 6 points $x^\alpha_{i\pm\frac{1}{2}}$ $(\alpha = 1, 2, 3)$.
- Procedure 3. Apply Procedure 1 at 1 point x_i.
- Procedure 4. Apply Procedure 1 at 5 points $\bar{x}^\alpha_i$ $(\alpha = 1, \dots, 5)$.
- Procedure 5. Apply Procedure 1 at 12 points $\tilde{x}^\alpha_i$ $(\alpha = 1, \dots, 12)$.

3. Maximum-principle-satisfying high order finite volume WENO schemes in one dimension.

3.1. Spatial discretization. In this section, we will construct a fifth order finite volume WENO scheme for (1.1). Replace the integral by the quadrature. Then (1.10) becomes

$$\frac{d\bar{\bar{u}}_i(t)}{dt} + \frac{1}{\Delta x}\sum_{\alpha=1}^{3} w_\alpha \left[f(u(x^\alpha_{i+\frac{1}{2}})) - f(u(x^\alpha_{i-\frac{1}{2}})) \right] = \frac{1}{\Delta x^2}\left(A(u(x_{i+1})) - 2A(u(x_i)) + A(u(x_{i-1}))\right). \tag{3.1}$$

Given the double cell averages $\bar{\bar{u}}_i$, we can use the WENO procedure to reconstruct a fifth order approximation to the function $u(x)$ at any fixed point. Let $u^{\alpha,+}_{i-\frac{1}{2}}$ and $u^{\alpha,-}_{i+\frac{1}{2}}$ denote the values reconstructed by Procedure 2 on the stencil $S = \{I_{i-2}, \dots, I_{i+2}\}$ approximating $u(x^\alpha_{i-\frac{1}{2}})$ and $u(x^\alpha_{i+\frac{1}{2}})$, respectively. Let u_i denote the value reconstructed by Procedure 3 on the stencil $S = \{I_{i-2}, \dots, I_{i+2}\}$ approximating $u(x_i)$.

An additional complication is that the solution to the conservation laws follows characteristics, hence a stable numerical scheme should also propagate its information in the same characteristic direction, which is referred to as *upwinding*. This is achieved by replacing $f(u(x^\alpha_{i+\frac{1}{2}}))$ by

$$\widehat{f}(u^{\alpha,-}_{i+\frac{1}{2}}, u^{\alpha,+}_{i+\frac{1}{2}}),$$

where $\widehat{f}(\cdot\,,\cdot)$ is a monotone numerical flux satisfying the following:

- The flux $\widehat{f}(\cdot\,,\cdot)$ is Lipschitz continuous with respect to both arguments.
- The flux $\widehat{f}(\cdot\,,\cdot)$ is nondecreasing in its fist argument and nonincreasing in its second argument symbolically $\widehat{f}(\uparrow, \downarrow)$.
- The flux $\widehat{f}(\cdot\,,\cdot)$ is consistent with the physical flux, that is, $\widehat{f}(u, u) = f(u)$.

In this paper we use the Lax–Friedrichs flux

$$\widehat{f}(u, v) = \frac{1}{2}(f(u) + f(v) - a(v - u)), \tag{3.2}$$

where $a = \max_u |f'(u)|$.

Then the semidiscrete finite volume WENO scheme can be written as

$$\frac{d\bar{\bar{u}}_i(t)}{dt} + \frac{1}{\Delta x}\sum_{\alpha=1}^{3} w_\alpha \left[\widehat{f}(u^{\alpha,-}_{i+\frac{1}{2}}, u^{\alpha,+}_{i+\frac{1}{2}}) - \widehat{f}(u^{\alpha,-}_{i-\frac{1}{2}}, u^{\alpha,+}_{i-\frac{1}{2}})\right] = \frac{1}{\Delta x^2}\left(A(u_{i+1}) - 2A(u_i) + A(u_{i-1})\right). \tag{3.3}$$

By Taylor expansion, it is straightforward to check the scheme (3.3) is fifth order accurate if $f(u)$ and $A(u)$ are smooth functions.

3.2. High order time discretization. We use SSP high order time discretizations. For more detail, see [10, 11, 3, 2]. For example, the third order SSP Runge–Kutta method [10] (with the CFL coefficient $c = 1$) is

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}), \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}), \end{aligned} \tag{3.4}$$

where $L(u)$ is the spatial operator, and the third order SSP multistep method [11] (with the CFL coefficient $c = \frac{1}{3}$) is

$$u^{n+1} = \frac{16}{27}(u^n + 3\Delta t L(u^n)) + \frac{11}{27}\left(u^{n-3} + \frac{12}{11}\Delta t L(u^{n-3})\right). \tag{3.5}$$

Here the CFL coefficient c for a SSP time discretization refers to the fact that if we assume the Euler forward time discretization for solving the equation $u_t = L(u)$ is stable in a norm or a seminorm under a time step restriction $\Delta t \le \Delta t_0$, then the high order SSP time discretization is also stable in the same norm or seminorm under the time step restriction $\Delta t \le c\Delta t_0$.

3.3. Maximum principle. We consider only the Euler forward time discretization in this subsection since the high order SSP time discretizations are convex combinations of Euler forward and thus will keep the maximum principle if Euler forward does. The WENO scheme (3.3) with first order Euler forward time discretization can be written as

$$\begin{aligned} \bar{\bar{u}}_i^{n+1} = \bar{\bar{u}}_i^n - \lambda \sum_{\alpha=1}^{3} w_\alpha &\left[\widehat{f}(u^{\alpha,-}_{i+\frac{1}{2}}, u^{\alpha,+}_{i+\frac{1}{2}}) - \widehat{f}(u^{\alpha,-}_{i-\frac{1}{2}}, u^{\alpha,+}_{i-\frac{1}{2}})\right] \\ &+ \mu\left(A(u_{i+1}) - 2A(u_i) + A(u_{i-1})\right), \end{aligned} \tag{3.6}$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $\mu = \frac{\Delta t}{\Delta x^2}$. $\bar{\bar{u}}_i^n$ is the approximation to the double cell average of $u(x,t)$ on the cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ at time level n, and $u^{\alpha,+}_{i-\frac{1}{2}}$, $u^{\alpha,-}_{i+\frac{1}{2}}$, u_i are the fifth order WENO reconstructions of the point values $u(x^\alpha_{i-\frac{1}{2}})$, $u(x^\alpha_{i+\frac{1}{2}})$, $u(x_i)$, respectively, based on the stencil $S = \{I_{i-2}, \dots, I_{i+2}\}$.

Given the scheme (3.6), assuming $\bar{\bar{u}}_i^n \in [m, M]$, we will derive some additional conditions such that $\bar{\bar{u}}_i^{n+1} \in [m, M]$ under certain CFL constraints.

We can rewrite (3.6) as $\bar{\bar{u}}_i^{n+1} = \frac{1}{2}\sum_{\alpha=1}^{3} w_\alpha \mathbf{C}_\alpha + \frac{1}{2}\mathbf{D}$, where

$$\mathbf{C}_\alpha = \bar{\bar{u}}_i^n - 2\lambda \left[\widehat{f}(u^{\alpha,-}_{i+\frac{1}{2}}, u^{\alpha,+}_{i+\frac{1}{2}}) - \widehat{f}(u^{\alpha,-}_{i-\frac{1}{2}}, u^{\alpha,+}_{i-\frac{1}{2}})\right] \tag{3.7}$$

and

$$\mathbf{D} = \bar{\bar{u}}_i^n + 2\mu \left(A(u_{i+1}) - 2A(u_i) + A(u_{i-1})\right). \tag{3.8}$$

It suffices to derive the sufficient conditions for $\mathbf{C}_\alpha, \mathbf{D} \in [m, M]$ since $\bar{\bar{u}}_i^{n+1}$ is a convex combination of them.

Part I. The convection part $\mathbf{C}_\alpha$. It is well known that a first order monotone scheme solving $u_t + f(u)_x = 0$ satisfies the strict maximum principle. A first order monotone scheme has the form

$$u_j^{n+1} = u_j^n - \lambda[\widehat{f}(u_j^n, u_{j+1}^n) - \widehat{f}(u_{j-1}^n, u_j^n)] \equiv H_\lambda(u_{j-1}^n, u_j^n, u_{j+1}^n). \tag{3.9}$$

Under suitable CFL conditions, typically of the form $a\lambda \leq 1, \quad a = \max|f'(u)|$, e.g., the Lax–Friedrichs scheme and the Godunov scheme, one can prove that the function $H_\lambda(a, b, c)$ is increasing in all three arguments, and consistency implies $H_\lambda(a, a, a) = a$. We therefore immediately have the strict maximum principle

$$m = H_\lambda(m, m, m) \leq u_j^{n+1} = H_\lambda(u_{j-1}^n, u_j^n, u_{j+1}^n) \leq H_\lambda(M, M, M) = M$$

provided $m \leq u_{j-1}^n, u_j^n, u_{j+1}^n \leq M$.

Assume there exists a polynomial of degree four $p_i^\alpha(x)$ satisfying

- Fifth order accurate: $p_i^\alpha(x) = u(x) + O(\Delta x^5)\ \forall x \in [x_{i-1}, x_{i+1}]$.
- $\frac{1}{\Delta x^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p_i^\alpha(\xi)d\xi dx = \bar{\bar{u}}_i^n$.
- $p_i^\alpha(x^\alpha_{i-\frac{1}{2}}) = u^{\alpha,+}_{i-\frac{1}{2}}$ and $p_i^\alpha(x^\alpha_{i+\frac{1}{2}}) = u^{\alpha,-}_{i+\frac{1}{2}}$.

The existence of such polynomials can be established by interpolation. For example, there exists a unique polynomial of degree four satisfying $p_i^\alpha(x^\alpha_{i-\frac{1}{2}}) = u^{\alpha,+}_{i-\frac{1}{2}}$, $p_i^\alpha(x^\alpha_{i+\frac{1}{2}}) = u^{\alpha,-}_{i+\frac{1}{2}}$, and

$$\frac{1}{\Delta x^2}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p_i^\alpha(\xi)d\xi dx = \bar{\bar{u}}_j^n, \qquad j = i-1, i, i+1.$$

Then $\bar{\bar{u}}_i^n$ can be written as a convex combinations of some point values of $p_i^\alpha(x)$ including $u^{\alpha,\pm}_{i\mp\frac{1}{2}}$. To find the explicit decomposition of $\bar{\bar{u}}_i^n$, we need the N-point Legendre–Gauss–Lobatto quadrature rule on the interval $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, which is exact for the integral of polynomials of degree up to $2N-3$. The four-point Legendre–Gauss–Lobatto quadrature points and weights on $[-\frac{1}{2}, \frac{1}{2}]$ are denoted by

$$\{\widehat{x}_\alpha, \alpha = 1, \ldots, 4\} = \left\{-\frac{1}{2}, -\frac{1}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \frac{1}{2}\right\},$$
$$\{\widehat{w}_\alpha, \alpha = 1, \ldots, 4\} = \left\{\frac{1}{12}, \frac{5}{12}, \frac{5}{12}, \frac{1}{12}\right\}.$$

Replacing the outer integral by the four-point Gauss–Lobatto quadrature and the inner integral by the three-point Gauss quadrature, for each $p_i^\alpha(x)$, we have

$$
\begin{aligned}
\bar{\bar{u}}_i^n &= \frac{1}{\Delta x^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p_i^\alpha(\xi)d\xi\right)dx \\
&= \frac{1}{\Delta x}\sum_{\beta=1}^{4}\widehat{w}_\beta\int_{x_i+\widehat{x}_\beta\Delta x-\frac{\Delta x}{2}}^{x_i+\widehat{x}_\beta\Delta x+\frac{\Delta x}{2}} p_i^\alpha(\xi)d\xi \\
&= \sum_{\beta=1}^{4}\sum_{\gamma=1}^{3}\widehat{w}_\beta w_\gamma p_i^\alpha(x_i+\widehat{x}_\beta\Delta x+x_\gamma\Delta x) \\
&= \sum_{\beta=2}^{3}\sum_{\gamma=1}^{3}\widehat{w}_\beta w_\gamma p_i^\alpha(x_i+(\widehat{x}_\beta+x_\gamma)\Delta x) \\
&\quad+\sum_{\gamma=1}^{3}\widehat{w}_1 w_\gamma p_i^\alpha(x_{i-\frac{1}{2}}+x_\gamma\Delta x)+\sum_{\gamma=1}^{3}\widehat{w}_4 w_\gamma p_i^\alpha(x_{i+\frac{1}{2}}+x_\gamma\Delta x) \\
&= \sum_{\beta=2}^{3}\sum_{\gamma=1}^{3}\widehat{w}_\beta w_\gamma p_i^\alpha(x_i+(\widehat{x}_\beta+x_\gamma)\Delta x)+\widehat{w}_1 w_\alpha p_i^\alpha(x_{i-\frac{1}{2}}^\alpha)+\widehat{w}_4 w_\alpha p_i^\alpha(x_{i+\frac{1}{2}}^\alpha) \\
&\quad+\sum_{\gamma=1,\gamma\neq\alpha}^{3}\widehat{w}_1 w_\gamma p_i^\alpha(x_{i-\frac{1}{2}}+x_\gamma\Delta x)+\sum_{\gamma=1,\gamma\neq\alpha}^{3}\widehat{w}_4 w_\gamma p_i^\alpha(x_{i+\frac{1}{2}}+x_\gamma\Delta x).
\end{aligned}
\tag{3.10}
$$

By the mean value theorem, there exists a point $x_i^{\alpha,*}\in[x_{i-1},x_{i+1}]$ such that

$$
\begin{aligned}
p_i^\alpha(x_i^{\alpha,*}) = \frac{1}{1-\widehat{w}_1 w_\alpha-\widehat{w}_4 w_\alpha}\Bigg[&\sum_{\beta=2}^{3}\sum_{\gamma=1}^{3}\widehat{w}_\beta w_\gamma p_i^\alpha(x_i+(\widehat{x}_\beta+x_\gamma)\Delta x) \\
&+\sum_{\gamma=1,\gamma\neq\alpha}^{3}\widehat{w}_1 w_\gamma p_i^\alpha(x_{i-\frac{1}{2}}+x_\gamma\Delta x) \\
&+\sum_{\gamma=1,\gamma\neq\alpha}^{3}\widehat{w}_4 w_\gamma p_i^\alpha(x_{i+\frac{1}{2}}+x_\gamma\Delta x)\Bigg].
\end{aligned}
$$

Notice that $\widehat{w}_1=\widehat{w}_4$; thus we can rewrite (3.10) as

$$
\bar{\bar{u}}_i^n = (1-2\widehat{w}_1 w_\alpha)p_i^\alpha(x_i^{\alpha,*})+\widehat{w}_1 w_\alpha\left[p_i^\alpha(x_{i-\frac{1}{2}}^\alpha)+p_i^\alpha(x_{i+\frac{1}{2}}^\alpha)\right]. \tag{3.11}
$$

With (3.11), by adding and subtracting $\widehat{f}(u_{i-\frac{1}{2}}^{\alpha,+},u_{i+\frac{1}{2}}^{\alpha,-})$, (3.7) can be written as

$$
\begin{aligned}
\mathbf{C}_\alpha &= (1-2\widehat{w}_1 w_\alpha)p_i^\alpha(x_i^{\alpha,*})+\left(u_{i+\frac{1}{2}}^{\alpha,-}-\frac{2\lambda}{\widehat{w}_1 w_\alpha}\left[\widehat{f}(u_{i+\frac{1}{2}}^{\alpha,-},u_{i+\frac{1}{2}}^{\alpha,+})-\widehat{f}(u_{i-\frac{1}{2}}^{\alpha,+},u_{i+\frac{1}{2}}^{\alpha,-})\right]\right) \\
&\quad+\left(u_{i-\frac{1}{2}}^{\alpha,+}-\frac{2\lambda}{\widehat{w}_1 w_\alpha}\left[\widehat{f}(u_{i-\frac{1}{2}}^{\alpha,+},u_{i+\frac{1}{2}}^{\alpha,-})-\widehat{f}(u_{i-\frac{1}{2}}^{\alpha,-},u_{i-\frac{1}{2}}^{\alpha,+})\right]\right) \\
&= (1-2\widehat{w}_1 w_\alpha)p_i^\alpha(x_i^{\alpha,*})+H_{\frac{2\lambda}{\widehat{w}_1 w_\alpha}}(u_{i-\frac{1}{2}}^{\alpha,+},u_{i+\frac{1}{2}}^{\alpha,-},u_{i+\frac{1}{2}}^{\alpha,+})+H_{\frac{2\lambda}{\widehat{w}_1 w_\alpha}}(u_{i-\frac{1}{2}}^{\alpha,-},u_{i-\frac{1}{2}}^{\alpha,+},u_{i+\frac{1}{2}}^{\alpha,-}),
\end{aligned}
$$

where H is the same function as in (3.9). Therefore, $\mathbf{C}_\alpha$ is a monotonically increasing function of $u^{\alpha,\pm}_{i\mp\frac{1}{2}}$, $u^{\alpha,\pm}_{i\pm\frac{1}{2}}$, and $p^{\alpha}_i(x^{\alpha,*}_i)$ if $\frac{2\lambda}{\widehat{w}_1 w_\alpha} a \leq 1$. The same proof for the first order monotone scheme now applies to imply $\mathbf{C}_\alpha \in [m, M]$ if $u^{\alpha,\pm}_{i\mp\frac{1}{2}}$, $u^{\alpha,\pm}_{i\pm\frac{1}{2}}$, and $p^{\alpha}_i(x^{\alpha,*}_i)$ are in the range $[m, M]$.

Part II. The diffusion part D. Assume there exists a polynomial of degree four $p_i(x)$ satisfying

- Fifth order accurate: $p_i(x) = u(x) + O(\Delta x^5)\ \forall x \in [x_{i-1}, x_{i+1}]$.
- Conservation: $\frac{1}{\Delta x^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p_i(\xi) d\xi dx = \bar{\bar{u}}^n_i, \quad p_i(x_i) = u_i.$

The existence of such polynomials can be established by interpolation. We can construct $p_i(x)$ by setting $p_i(x_i) = u_i$ and

$$\frac{1}{\Delta x^2}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p^{\alpha}_i(\xi) d\xi dx = \bar{\bar{u}}^n_j, \qquad j = i-1, i, i+1, i+2.$$

Then $\bar{\bar{u}}^n_i$ can be written as a convex combinations of some point values of $p_i(x)$ including u_i. By the mean value theorem, there exists a point $x^*_i \in [x_{i-1}, x_{i+1}]$ such that

$$\begin{aligned}\bar{\bar{u}}^n_i &= \frac{1}{\Delta x^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p_i(\xi) d\xi\right) dx \\ &= \sum_{\alpha=1}^{5} \bar{w}_\alpha p_i(\bar{x}^\alpha_i) = \sum_{\alpha=1,\alpha\neq 3}^{5} \bar{w}_\alpha p_i(\bar{x}^\alpha_i) + \bar{w}_3 u_i = (1-\bar{w}_3) p_i(x^*_i) + \bar{w}_3 u_i,\end{aligned} \tag{3.12}$$

where $\bar{w}_3 = 2w_1 w_3 + w_2 w_2 = \frac{19}{54}$.

With (3.12), (3.8) can be written as

$$\mathbf{D} = (1-\bar{w}_3) p_i(x^*_i) + \bar{w}_3\left(u_i - \frac{4\mu}{\bar{w}_3} A(u_i)\right) + 2\mu(A(u_{i+1}) + A(u_{i-1})).$$

Therefore, $\mathbf{D}$ is a monotonically increasing function of u_{i-1}, u_i, u_{i+1} and $p_i(x^*_i)$ if $\frac{4\mu}{\bar{w}_3} A'(u) \leq 1$. Thus $\mathbf{D} \in [m, M]$ if u_{i-1}, u_i, u_{i+1} and $p_i(x^*_i)$ are in the range $[m, M]$.

We have obtained the monotonicity for (3.6).

THEOREM 3.1. *The scheme* (3.6) *satisfies the maximum principle, namely,* $\bar{\bar{u}}^{n+1}_i \in [m, M]$ *if* $u^{\alpha,\pm}_{i\mp\frac{1}{2}}$, $u^{\alpha,\pm}_{i\pm\frac{1}{2}}, p^{\alpha}_i(x^{\alpha,*}_i), u_{i-1}, u_i, u_{i+1}, p_i(x^*_i) \in [m, M]$ *under the CFL conditions*

$$\lambda \max_u |f'(u)| \leq \frac{1}{2}\widehat{w}_1 \min_\alpha w_\alpha = \frac{5}{432}, \qquad \mu \max_u A'(u) \leq \frac{1}{4}\bar{w}_3 = \frac{19}{216}. \tag{3.13}$$

Remark 3.2. The CFL condition (3.13) is much more restrictive than the commonly used ones. However, it is a sufficient condition rather than a necessary one to keep the maximum principle. Therefore, in practice, (3.13) can be strictly enforced only when a precalculation with a normal time stepping to the next time step or stage violates the maximum principle. In general, the percentage of small time steps required by (3.13) depends on the problem. For instance, in Example 4.2, around 90 percent of the time steps are small ones, whereas zero percent of time steps are small in Example 5.2.

3.4. A linear scaling limiter. Given a smooth function $u(x) \in [m, M]\ \forall x$ and a polynomial $p_i(x)$ of degree k defined on $[x_{i-1}, x_{i+1}]$ satisfying

- accuracy: $p_i(x) = u(x) + O(\Delta x^{k+1})\ \forall x \in [x_{i-1}, x_{i+1}]$,
- conservation: $\bar{\bar{u}}_i = \frac{1}{\Delta x^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p_i(\xi)d\xi\right)dx$,

we define the scaled polynomial by

$$\widetilde{p}_i(x) = \theta(p_i(x) - \bar{\bar{u}}_i) + \bar{\bar{u}}_i, \quad \theta = \min\left\{\left|\frac{M - \bar{\bar{u}}_i}{M_i - \bar{\bar{u}}_i}\right|, \left|\frac{m - \bar{\bar{u}}_i}{m_i - \bar{\bar{u}}_i}\right|, 1\right\}, \tag{3.14}$$

where

$$M_i = \max_{x\in[x_{i-1},x_{i+1}]} p_i(x), \qquad m_i = \min_{x\in[x_{i-1},x_{i+1}]} p_i(x).$$

It is easy to check that the double cell average of $\widetilde{p}_i(x)$ is still $\bar{\bar{u}}_i$ and $\widetilde{p}_i(x) \in [m, M]\ \forall x \in [x_{i-1}, x_{i+1}]$. Following [14], we have the next lemma.

LEMMA 3.3. *The modified polynomial is still accurate:* $\widetilde{p}_i(x) = p_i(x) + O(\Delta x^{k+1})$ $\forall x \in [x_{i-1}, x_{i+1}]$.

Proof. We only prove the case that $p_i(x)$ is not a constant and $\theta = |\frac{M-\bar{\bar{u}}_i}{M_i-\bar{\bar{u}}_i}|$, the other cases being similar. $u(x) \in [m, M]$ implies $\bar{\bar{u}}_i \le M$ and $p_i(x) \in [m_i, M_i]$ implies $\bar{\bar{u}}_i \le M_i$. Thus, $\theta = \frac{M-\bar{\bar{u}}_i}{M_i-\bar{\bar{u}}_i}$. Therefore,

$$\begin{aligned}
\widetilde{p}_i(x) - p_i(x) &= \theta(p_i(x) - \bar{\bar{u}}_i) + \bar{\bar{u}}_i - p_i(x)\\
&= (\theta - 1)(p_i(x) - \bar{\bar{u}}_i)\\
&= \frac{M - M_i}{M_i - \bar{\bar{u}}_i}(p_i(x) - \bar{\bar{u}}_i)\\
&= (M - M_i)\frac{p_i(x) - \bar{\bar{u}}_i}{M_i - \bar{\bar{u}}_i}.
\end{aligned}$$

By the definition of θ, $\theta = |\frac{M-\bar{\bar{u}}_i}{M_i-\bar{\bar{u}}_i}|$ implies that $\theta = |\frac{M-\bar{\bar{u}}_i}{M_i-\bar{\bar{u}}_i}| < 1$, i.e., there is an overshoot $M_i > M$, and the overshoot $M_i - M = O(\Delta x^{k+1})$ since $p_i(x)$ is an approximation to $u(x)$ with error $O(\Delta x^{k+1})$. Thus we only need to prove that $|\frac{p_i(x)-\bar{\bar{u}}_i}{M_i-\bar{\bar{u}}_i}| \le C_k$, where C_k is a constant depending only on the polynomial degree k. Assume $p_i(x) = a_0 + a_1(\frac{x-x_i}{\Delta x}) + \cdots + a_4(\frac{x-x_i}{\Delta x})^4$ and $p(x) = a_0 + a_1 x + \cdots + a_4 x^4$; then the double cell average of $p(x)$ on $I = [-\frac{1}{2}, \frac{1}{2}]$ is $\bar{\bar{p}} = \bar{\bar{u}}_i$ and $\max_{x\in[-1,1]} p(x) = M_i$. So we have

$$\max_{x\in[x_{i-1},x_{i+1}]}\left|\frac{p_i(x) - \bar{\bar{u}}_i}{M_i - \bar{\bar{u}}_i}\right| = \max_{x\in[-1,1]}\left|\frac{p(x) - \bar{\bar{p}}}{\max_{y\in[-1,1]} p(y) - \bar{\bar{p}}}\right|.$$

Let $q(x) = p(x) - \bar{\bar{p}}$. Then it suffices to prove the existence of C_k such that

$$\left|\frac{\min_{x\in[-1,1]} p(x) - \bar{\bar{p}}}{\max_{x\in[-1,1]} p(x) - \bar{\bar{p}}}\right| = \left|\frac{\min_{x\in[-1,1]} q(x)}{\max_{x\in[-1,1]} q(x)}\right| \le C_k.$$

It is easy to check that $|\min_{x\in[-1,1]} q(x)|$ and $|\max_{x\in[-1,1]} q(x)|$ are both norms on the finite dimensional linear space consisting of all polynomials of degree k whose double cell averages on the interval I are zero. Any two norms on this finite dimensional space are equivalent, and hence their ratio is bounded by a constant C_k. ∎

3.5. Implementation details. Following [17], we formulate the algorithm of the fifth order maximum-principle-satisfying WENO scheme with Euler forward time discretization as follows:

- At time level n, given $\bar{\bar{u}}_j^n$ with $j = i-2, \dots, i+2$, use WENO reconstruction to obtain point values $u_{i-\frac{1}{2}}^{\alpha,+}, u_{i+\frac{1}{2}}^{\alpha,-}$ and u_i.
- Get the revised point values:
  1. For each i and each α,

(3.15)
$$\widetilde{u}_{i\pm\frac{1}{2}}^{\alpha,\mp} = \theta(u_{i\pm\frac{1}{2}}^{\alpha,\mp} - \bar{\bar{u}}_i^n) + \bar{\bar{u}}_i^n, \quad \theta = \min\left\{ \left| \frac{M - \bar{\bar{u}}_i}{M_i - \bar{\bar{u}}_i} \right|, \left| \frac{m - \bar{\bar{u}}_i}{m_i - \bar{\bar{u}}_i} \right|, 1 \right\},$$

(3.16)
$$M_i = \max\{u_{i+\frac{1}{2}}^{\alpha,-}, u_{i-\frac{1}{2}}^{\alpha,+}, p_i^\alpha(x_i^{\alpha,*})\}, \quad m_i = \min\{u_{i+\frac{1}{2}}^{\alpha,-}, u_{i-\frac{1}{2}}^{\alpha,+}, p_i^\alpha(x_i^{\alpha,*})\},$$

where we need neither the explicit formula of $p_i^\alpha(x)$ nor the exact location $x_i^{\alpha,*}$ since $p_i^\alpha(x_i^{\alpha,*}) = [\bar{\bar{u}}_i^n - \frac{1}{12}w_\alpha(u_{i+\frac{1}{2}}^{\alpha,-} + u_{i-\frac{1}{2}}^{\alpha,+})]/(1 - \frac{1}{6}w_\alpha)$ by (3.11).

  2. For each i, abusing the notation by denoting

$$\widetilde{u}_i = \theta(u_i - \bar{\bar{u}}_i^n) + \bar{\bar{u}}_i^n, \quad \theta = \min\left\{ \left| \frac{M - \bar{\bar{u}}_i}{M_i - \bar{\bar{u}}_i} \right|, \left| \frac{m - \bar{\bar{u}}_i}{m_i - \bar{\bar{u}}_i} \right|, 1 \right\}, \tag{3.17}$$

$$M_i = \max\{u_i, p_i(x_i^*)\}, \quad m_i = \min\{u_i, p_i(x_i^*)\}, \tag{3.18}$$

where $p_i(x_i^*) = (\bar{\bar{u}}_i^n - \bar{w}_3 u_i)/(1 - \bar{w}_3)$ by (3.12).

- Get the revised scheme

$$\begin{aligned} \bar{\bar{u}}_i^{n+1} = \bar{\bar{u}}_i^n - \lambda \sum_{\alpha=1}^{3} w_\alpha \left[\widehat{f}(\widetilde{u}_{i+\frac{1}{2}}^{\alpha,-}, \widetilde{u}_{i+\frac{1}{2}}^{\alpha,+}) - \widehat{f}(\widetilde{u}_{i-\frac{1}{2}}^{\alpha,-}, \widetilde{u}_{i-\frac{1}{2}}^{\alpha,+}) \right] \\ + \mu \left(A(\widetilde{u}_{i+1}) - 2A(\widetilde{u}_i) + A(\widetilde{u}_{i-1}) \right). \end{aligned} \tag{3.19}$$

The limiters (3.15), (3.16), (3.17), and (3.18) are weaker than (3.14), so the accuracy will not be destroyed. By Theorem 3.1 the scheme (3.19) satisfies the maximum principle; thus we have the following stability result.

THEOREM 3.4. *Assuming periodic or zero boundary conditions, the numerical solution of* (3.19) *satisfies*

$$\sum_i |\bar{\bar{u}}_i^{n+1} - m| = \sum_i |\bar{\bar{u}}_i^n - m|, \qquad \sum_i |\bar{\bar{u}}_i^{n+1} - M| = \sum_i |\bar{\bar{u}}_i^n - M|.$$

Proof. Taking the sum of (3.19) over i, we obtain $\sum_i \bar{\bar{u}}_i^{n+1} = \sum_i \bar{\bar{u}}_i^n$. Since the numerical solutions are maximum-principle-satisfying, namely, $\bar{\bar{u}}_i^{n+1}, \bar{\bar{u}}_i^n \in [m, M]$, we have

$$\sum_i |\bar{\bar{u}}_i^{n+1} - m| = \sum_i (\bar{\bar{u}}_i^{n+1} - m) = \sum_i (\bar{\bar{u}}_i^n - m) = \sum_i |\bar{\bar{u}}_i^n - m|.$$

The other equality follows similarly. ∎

Remark 3.5. As an easy corollary, if the solution is nonnegative, namely, if $m \geq 0$, then we have the L^1 stability $\sum_i |\bar{\bar{u}}_i^{n+1}| = \sum_i |\bar{\bar{u}}_i^n|$.

Since an SSP high order time discretization is a convex combination of Euler forward, the full scheme with a high order SSP time discretization will still satisfy the maximum principle. The limiters (3.15), (3.16), (3.17), and (3.18) should be used for each stage in a Runge–Kutta method or each step in a multistep method.

TABLE 3.1
One-dimensional accuracy test. $\Delta x = \frac{2\pi}{N}$. *The third order SSP Runge–Kutta and fifth order finite volume WENO scheme.*

N	L^1 error	Order	L^∞ error	Order
10	4.74E-3	–	9.84E-3	–
20	1.99E-4	4.57	3.78E-4	4.70
40	6.03E-6	5.04	1.29E-5	4.87
80	1.82E-7	5.04	3.94E-7	5.03
160	5.61E-9	5.02	1.15E-8	5.10
320	1.75E-10	4.99	3.22E-10	5.16
640	5.64E-12	4.96	9.67E-12	5.05

TABLE 3.2
Maximum and minimum of the numerical solutions for the Burgers equation at $T = 0.05$.

	With limiter		Without limiter	
Mesh	Min	Max	Min	Max
10	6.3026091802586501E-004	1.999859730036674	-3.2226586770600023E-5	2.003544004561570
20	1.3324291965545294E-010	1.999999999775453	-1.3448900728524493E-5	2.000067631685169
40	1.3802843390151276E-013	2.000000000000000	-2.6839623398535806E-5	2.000034090070941
80	4.4921377592873801E-022	2.000000000000000	-3.2578105791273408E-5	2.000066674567972
160	6.7955494010063555E-041	2.000000000000000	-6.5709076166245168E-5	2.000072578186707
320	2.4563860333258520E-072	2.000000000000000	-8.7174689124698581E-5	2.000092063419499
640	3.4653838656373672E-141	2.000000000000000	-7.5173275378005444E-5	2.000090009279301

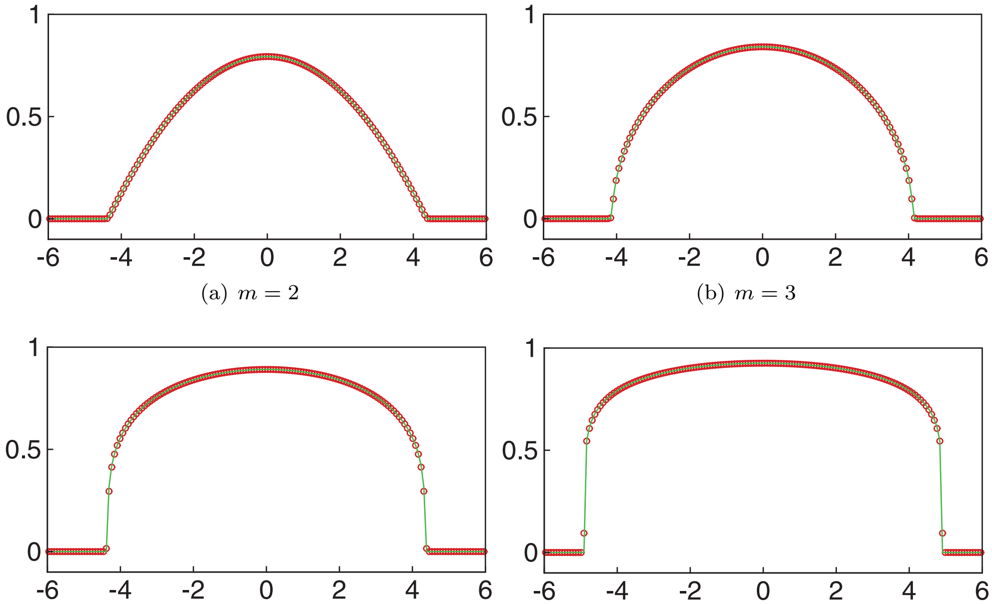
3.6. Numerical tests. Since we use explicit time stepping, the technique in this paper is more relevant for convection dominated convection-diffusion equations. Therefore, our numerical examples below have small diffusion coefficients.

Example 3.1 (accuracy test). We test the accuracy of the scheme for the linear equation $u_t + u_x = \varepsilon u_{xx}$ with initial condition $\sin(x)$ on $[0, 2\pi]$ and periodic boundary conditions. Here ε is set as $\varepsilon = 0.00001$. The exact solution is $e^{-\varepsilon t}\sin(x - t)$. The time step is taken as (3.13). See Table 3.1 for the errors at $T = 1$. We can observe the designed fifth order accuracy. However, we remark that for Runge–Kutta time discretizations, the order of accuracy may degenerate because Lemma 3.3 will not hold due to the lower order accuracy in the intermediate stages of the Runge–Kutta method. Even though the order reduction was not observed in any of the accuracy test problems in this paper, this phenomenon may be observed when the mesh is fine enough. For multistep time discretizations, there are no such problems. For more discussion on this issue of the order reduction, see [14].

Example 3.2 (Burgers equation). Consider the equation $u_t + (u^2)_x = \varepsilon u_{xx}$ with initial condition $u_0(x) = 2$ if $|x| < \frac{1}{2}$ and $u_0(x) = 0$ otherwise on $[-1, 1]$ and periodic boundary conditions. Here $\varepsilon = 0.0001$. The maximum and minimum of numerical solutions are listed in Table 3.2 for the fifth order WENO scheme with or without the limiter. We can observe that the WENO scheme with the limiter satisfies the strict maximum principle. So we refer to it as maximum-principle-satisfying WENO scheme.

Example 3.3 (porous medium equation). Consider the equation $u_t = (u^m)_{xx}$, where $m > 1$. We test our finite volume scheme for the Barenblatt solution

$$B_m(x,t) = t^{-k}\left[\left(1 - \frac{k(m-1)}{2m}\frac{|x|^2}{t^{2k}}\right)_+\right]^{1/(m-1)},$$



(a) $m = 2$ (b) $m = 3$ (c) $m = 5$ (d) $m = 8$

FIG. 3.1. *The Barenblatt solution at $t = 2$. The solid line denotes the exact solution and the symbols are numerical solutions of the maximum-principle-satisfying finite volume WENO scheme.*

where $u_+ = \max(u, 0)$ and $k = (m + 1)^{-1}$. The initial condition is the Barenblatt solution at $t = 1$ and the boundary condition is zero for both ends. In [8], it was reported that high order finite difference WENO schemes may produce undershoots. We do the same tests for the maximum-principle-satisfying WENO scheme as in [8]. The numerical solutions approximating $B_m(x, t)$ at $t = 2$ for $m = 2, 3, 5$, and 8 are shown in Figure 3.1, where the numerical solutions are always nonnegative.

4. High order schemes satisfying the maximum principle in two dimensions.

4.1. Preliminaries.

We consider the two-dimensional convection diffusion equations in the form

$$u_t + f(u)_x + g(u)_y = A(u)_{xx} + B(u)_{yy}, \tag{4.1}$$

where $A'(u) \geq 0$ and $B'(u) \geq 0$. Assume that we have a uniform rectangular mesh with $x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}}$ and $y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N-\frac{1}{2}} < y_{N+\frac{1}{2}}$. Let $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, $\Delta y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$, $x_i = (x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}})/2$, and $y_j = (y_{j+\frac{1}{2}} + y_{j-\frac{1}{2}})/2$. We define the double cell average of a function $u(x, y)$ on the (i, j) cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ as

$$\bar{\bar{u}}_{ij} = \frac{1}{\Delta x^2}\frac{1}{\Delta y^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u(\xi, \eta)\,d\xi dx d\eta dy. \tag{4.2}$$

We integrate (4.1) over the (i,j) cell to obtain

$$\begin{aligned}
&\frac{d\bar{\bar{u}}_{ij}(t)}{dt} + \frac{1}{\Delta x^2 \Delta y^2}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\left[\int_{x_i}^{x_{i+1}} f(u(x,\eta))dx - \int_{x_{i-1}}^{x_i} f(u(x,\eta))dx\right] d\eta dy \\
&\quad + \frac{1}{\Delta x^2 \Delta y^2}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\left[\int_{y_j}^{y_{j+1}} g(u(\xi,y))dy - \int_{y_{j-1}}^{y_j} g(u(\xi,y))dy\right] d\xi dx \\
&= \frac{1}{\Delta x^2 \Delta y^2}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} \left[A(u(x_{i+1},\eta)) - 2A(u(x_i,\eta)) + A(u(x_{i-1},\eta))\right] d\eta dy \\
&\quad + \frac{1}{\Delta x^2 \Delta y^2}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} \left[B(u(\xi,y_{j+1})) - 2B(u(\xi,y_j)) + B(u(\xi,y_{j-1}))\right] d\xi dx.
\end{aligned}$$

Replacing the integrals by proper quadratures, we get

$$\begin{aligned}
\frac{d\bar{\bar{u}}_{ij}(t)}{dt} &= -\frac{1}{\Delta x}\sum_{\alpha=1}^{5}\sum_{\beta=1}^{3}\bar{w}_\alpha w_\beta \left[f(u(x^\beta_{i+\frac12},\bar{y}^\alpha_j)) - f(u(x^\beta_{i-\frac12},\bar{y}^\alpha_j))\right] \\
&\quad - \frac{1}{\Delta y}\sum_{\alpha=1}^{5}\sum_{\beta=1}^{3}\bar{w}_\alpha w_\beta \left[g(u(\bar{x}^\alpha_i,y^\beta_{j+\frac12})) - g(u(\bar{x}^\alpha_i,y^\beta_{j-\frac12}))\right] \\
&\quad + \frac{1}{\Delta x^2}\sum_{\alpha=1}^{5}\bar{w}_\alpha \left[A(u(x_{i+1},\bar{y}^\alpha_j)) - 2A(u(x_i,\bar{y}^\alpha_j)) + A(u(x_{i-1},\bar{y}^\alpha_j))\right] \\
&\quad + \frac{1}{\Delta y^2}\sum_{\alpha=1}^{5}\bar{w}_\alpha \left[B(u(\bar{x}^\alpha_i,y_{j+1})) - 2B(u(\bar{x}^\alpha_i,y_j)) + B(u(\bar{x}^\alpha_i,y_{j-1}))\right].
\end{aligned}$$

For convenience, we will abuse the notation by using $u(x,y,t)$ to denote the exact solution and $u(x,y)$ to denote the approximation reconstructed by WENO.

4.2. A fifth order finite volume WENO scheme. We can denote the numerical scheme as

$$\begin{aligned}
\frac{d\bar{\bar{u}}_{ij}}{dt} &= -\frac{1}{\Delta x}(\widehat{f}_{i+\frac12,j} - \widehat{f}_{i-\frac12,j}) - \frac{1}{\Delta y}(\widehat{g}_{i,j+\frac12} - \widehat{g}_{i,j-\frac12}) \\
&\quad + \frac{1}{\Delta x^2}(\widehat{A}_{i+1,j} - 2\widehat{A}_{i,j} + \widehat{A}_{i-1,j}) + \frac{1}{\Delta y^2}(\widehat{B}_{i,j+1} - 2\widehat{B}_{i,j} + \widehat{B}_{i,j-1}),
\end{aligned}$$

where

$$\begin{aligned}
\widehat{f}_{i+\frac12,j} &= \sum_{\alpha=1}^{5}\sum_{\beta=1}^{3}\bar{w}_\alpha w_\beta \widehat{f}\left(u(x^\beta_{i+\frac12},\bar{y}^\alpha_j)^-, u(x^\beta_{i+\frac12},\bar{y}^\alpha_j)^+\right), \\
\widehat{g}_{i,j+\frac12} &= \sum_{\alpha=1}^{5}\sum_{\beta=1}^{3}\bar{w}_\alpha w_\beta \widehat{g}\left(u(\bar{x}^\alpha_i,y^\beta_{j+\frac12})^-, u(\bar{x}^\alpha_i,y^\beta_{j+\frac12})^+\right), \\
\widehat{A}_{i,j} &= \sum_{\alpha=1}^{5}\bar{w}_\alpha A(u(x_i,\bar{y}^\alpha_j)), \quad \widehat{B}_{i,j} = \sum_{\alpha=1}^{5}\bar{w}_\alpha B(u(\bar{x}^\alpha_i,y_j)).
\end{aligned}$$

The point values $u(x^\beta_{i+\frac12}, \bar y^\alpha_j)^-, u(x^\beta_{i+\frac12}, \bar y^\alpha_j)^+$ are reconstructed in a dimension by dimension fashion as follows:

- Apply Procedure 4 to $\{\bar{\bar u}_{i,j-2}, \bar{\bar u}_{i,j-1}, \bar{\bar u}_{i,j}, \bar{\bar u}_{i,j+1}, \bar{\bar u}_{i,j+2}\}$ at the points $\bar y^\alpha_j$ ($\alpha = 1,\dots,5$) to reconstruct the one-dimensional double average in x-direction, and denote it as $\bar{\bar u}_i(\bar y^\alpha_j)$ (approximating $\frac{1}{\Delta x^2}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u(\xi, \bar y^\alpha_j, t)d\xi dx$).
- Apply Procedure 2 to $\{\bar{\bar u}_{i-2}(\bar y^\alpha_j), \bar{\bar u}_{i-1}(\bar y^\alpha_j), \bar{\bar u}_{i}(\bar y^\alpha_j), \bar{\bar u}_{i+1}(\bar y^\alpha_j), \bar{\bar u}_{i+2}(\bar y^\alpha_j)\}$ at the points $x^\beta_{i\pm\frac12}$ ($\beta = 1,2,3$) to reconstruct $u(x^\beta_{i+\frac12}, \bar y^\alpha_j)^-$, $u(x^\beta_{i-\frac12}, \bar y^\alpha_j)^+$.

Similarly, $u(\bar x^\alpha_i, y^\beta_{j+\frac12})^-, u(\bar x^\alpha_i, y^\beta_{j+\frac12})^+$ are reconstructed as follows:

- Apply Procedure 4 to $\{\bar{\bar u}_{i-2,j}, \bar{\bar u}_{i-1,j}, \bar{\bar u}_{i,j}, \bar{\bar u}_{i+1,j}, \bar{\bar u}_{i+2,j}\}$ at the points $\bar x^\alpha_i$ ($\alpha = 1,\dots,5$) to reconstruct the one-dimensional double average in y-direction, and denote it as $\bar{\bar u}_j(\bar x^\alpha_i)$ (approximating $\frac{1}{\Delta y^2}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} u(\bar x^\alpha_i, \eta, t)d\eta dy$).
- Apply Procedure 2 to $\{\bar{\bar u}_{j-2}(\bar x^\alpha_i), \bar{\bar u}_{j-1}(\bar x^\alpha_i), \bar{\bar u}_{j}(\bar x^\alpha_i), \bar{\bar u}_{j+1}(\bar x^\alpha_i), \bar{\bar u}_{j+2}(\bar x^\alpha_i)\}$ at the points $y^\beta_{j\pm\frac12}$ ($\beta = 1,2,3$) to reconstruct $u(\bar x^\alpha_i, y^\beta_{j+\frac12})^-$, $u(\bar x^\alpha_i, y^\beta_{j-\frac12})^+$.

The point values $u(x_i, \bar y^\alpha_j)$ are reconstructed as follows:

- Apply Procedure 3 to $\{\bar{\bar u}_{i-2,j}, \bar{\bar u}_{i-1,j}, \bar{\bar u}_{i,j}, \bar{\bar u}_{i+1,j}, \bar{\bar u}_{i+2,j}\}$ at the point x_i to reconstruct the one-dimensional double average in y-direction, and denote it as $\bar{\bar u}_j(x_i)$ (approximating $\frac{1}{\Delta y^2}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} u(x_i, \eta, t)d\eta dy$).
- Apply Procedure 4 to $\{\bar{\bar u}_{j-2}(x_i), \bar{\bar u}_{j-1}(x_i), \bar{\bar u}_{j}(x_i), \bar{\bar u}_{j+1}(x_i), \bar{\bar u}_{j+2}(x_i)\}$ at the points $\bar y^\alpha_j$ ($\alpha = 1,\dots,5$) to reconstruct $u(x_i, \bar y^\alpha_j)$.

Similarly, $u(\bar x^\alpha_i, y_j)$ are reconstructed as follows:

- Apply Procedure 3 to $\{\bar{\bar u}_{i,j-2}, \bar{\bar u}_{i,j-1}, \bar{\bar u}_{i,j}, \bar{\bar u}_{i,j+1}, \bar{\bar u}_{i,j+2}\}$ at the point y_j to reconstruct the one-dimensional double average in x-direction, and denote it as $\bar{\bar u}_i(y_j)$ (approximating $\frac{1}{\Delta x^2}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u(\xi, y_j, t)d\xi dx$).
- Apply Procedure 4 to $\{\bar{\bar u}_{i-2}(y_j), \bar{\bar u}_{i-1}(y_j), \bar{\bar u}_{i}(y_j), \bar{\bar u}_{i+1}(y_j), \bar{\bar u}_{i+2}(y_j)\}$ at the points $\bar x^\alpha_i$ ($\alpha = 1,\dots,5$) to reconstruct $u(\bar x^\alpha_i, y_j)$.

4.3. Maximum principle. Consider the Euler forward

$$\begin{aligned}\bar{\bar u}^{n+1}_{ij} = \bar{\bar u}^{n}_{ij} &- \lambda_1(\widehat f_{i+\frac12,j} - \widehat f_{i-\frac12,j}) - \lambda_2(\widehat g_{i,j+\frac12} - \widehat g_{i,j-\frac12})\\ &+ \mu_1(\widehat A_{i+1,j} - 2\widehat A_{i,j} + \widehat A_{i-1,j}) + \mu_2(\widehat B_{i,j+1} - 2\widehat B_{i,j} + \widehat B_{i,j-1}),\end{aligned} \tag{4.3}$$

where $\lambda_1 = \frac{\Delta t}{\Delta x}$, $\lambda_2 = \frac{\Delta t}{\Delta y}$, $\mu_1 = \frac{\Delta t}{\Delta x^2}$, and $\mu_2 = \frac{\Delta t}{\Delta y^2}$. It can be rewritten as

$$\bar{\bar u}^{n+1}_{ij} = \frac12\frac{\lambda_1}{\lambda_1+\lambda_2}\mathbf{F} + \frac12\frac{\lambda_2}{\lambda_1+\lambda_2}\mathbf{G} + \frac12\frac{\mu_1}{\mu_1+\mu_2}\mathbf{A} + \frac12\frac{\mu_2}{\mu_1+\mu_2}\mathbf{B},$$

where

$$\begin{aligned}\mathbf{F} &= \bar{\bar u}^{n}_{ij} - 2(\lambda_1+\lambda_2)(\widehat f_{i+\frac12,j} - \widehat f_{i-\frac12,j}),\\ \mathbf{G} &= \bar{\bar u}^{n}_{ij} - 2(\lambda_1+\lambda_2)(\widehat g_{i,j+\frac12} - \widehat g_{i,j-\frac12}),\\ \mathbf{A} &= \bar{\bar u}^{n}_{ij} + 2(\mu_1+\mu_2)(\widehat A_{i+1,j} - 2\widehat A_{i,j} + \widehat A_{i-1,j}),\\ \mathbf{B} &= \bar{\bar u}^{n}_{ij} + 2(\mu_1+\mu_2)(\widehat B_{i,j+1} - 2\widehat B_{i,j} + \widehat B_{i,j-1}).\end{aligned}$$

To have maximum principle, it suffices to check the monotonicity and consistency of $\mathbf{F}, \mathbf{G}, \mathbf{A}$, and $\mathbf{B}$.

Part I. The convection terms F and G. We have $\mathbf{F} = \sum_{\beta=1}^{3} w_\beta \mathbf{F}_\beta$ with

$$\mathbf{F}_\beta = \bar{\bar{u}}_{ij}^n - 2(\lambda_1+\lambda_2)\sum_{\alpha=1}^{5}\bar{w}_\alpha\Bigg[\widehat{f}\left(u(x^\beta_{i+\frac{1}{2}},\bar{y}^\alpha_j)^-, u(x^\beta_{i+\frac{1}{2}},\bar{y}^\alpha_j)^+\right) - \widehat{f}\left(u(x^\beta_{i-\frac{1}{2}},\bar{y}^\alpha_j)^-, u(x^\beta_{i-\frac{1}{2}},\bar{y}^\alpha_j)^+\right)\Bigg].$$

Let Q^k denote the space of tensor products of one-dimensional polynomials of degree up to k. By interpolation, there exists $p_{ij}^{1,\beta}(x,y) \in Q^4$ such that

- it is a fifth order accurate approximation to $u(x,y,t^n)$ in $[x_{i-1},x_{i+1}]\times[y_{j-1}, y_{j+1}]$;
- $\bar{\bar{u}}_{ij}^n = \frac{1}{\Delta x^2}\frac{1}{\Delta y^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p_{ij}^{1,\beta}(\xi,\eta)d\xi dx d\eta dy$;
- $p_{ij}^{1,\beta}(x^\beta_{i-\frac{1}{2}},\bar{y}^\alpha_j) = u(x^\beta_{i-\frac{1}{2}},\bar{y}^\alpha_j)^+$ and $p_{ij}^{1,\beta}(x^\beta_{i+\frac{1}{2}},\bar{y}^\alpha_j) = u(x^\beta_{i+\frac{1}{2}},\bar{y}^\alpha_j)^-$ for $\alpha = 1,\ldots,5$.

Replacing the integrals by proper quadrature, we have

$$\begin{aligned}\bar{\bar{u}}_{ij}^n &= \frac{1}{\Delta x^2}\frac{1}{\Delta y^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} p_{ij}^{1,\beta}(\xi,\eta)d\xi dx d\eta dy\\ &= \sum_{\alpha=1}^{5}\sum_{\beta=1}^{3}\sum_{\gamma=1}^{4}\bar{w}_\alpha w_\beta \widehat{w}_\gamma p_{ij}^{1,\beta}(x_i + x_\beta\Delta x + \widehat{x}_\gamma\Delta x, \bar{y}^\alpha_j).\end{aligned}$$

By the mean value theorem, there exists a point $(x_i^{1,\beta}, y_j^{1,\beta}) \in [x_{i-1},x_{i+1}]\times[y_{j-1},y_{j+1}]$ such that

$$\bar{\bar{u}}_{ij}^n = \sum_{\alpha=1}^{5}\bar{w}_\alpha w_\beta \widehat{w}_1\left[u(x^\beta_{i-\frac{1}{2}},\bar{y}^\alpha_j)^+ + u(x^\beta_{i+\frac{1}{2}},\bar{y}^\alpha_j)^-\right] + (1-2w_\beta\widehat{w}_1)p_{ij}^{1,\beta}(x_i^{1,\beta}, y_j^{1,\beta}).$$

Therefore, following the previous section, we have the next lemma.

LEMMA 4.1. *If* $u(x^\beta_{i\pm\frac{1}{2}},\bar{y}^\alpha_j)^\mp, u(x^\beta_{i\mp\frac{1}{2}},\bar{y}^\alpha_j)^\mp, p_{ij}^{1,\beta}(x_i^{1,\beta}, y_j^{1,\beta}) \in [m,M]$, *then* $\mathbf{F}_\beta \in [m,M]$ *under the CFL constraint* $(\lambda_1+\lambda_2)\max_u|f'(u)| \le \frac{1}{2}w_\beta\widehat{w}_1$.

Similarly, we have $\mathbf{G} = \sum_{\beta=1}^{3} w_\beta \mathbf{G}_\beta$ with

$$\mathbf{G}_\beta = \bar{\bar{u}}_{ij}^n - 2(\lambda_1+\lambda_2)\sum_{\alpha=1}^{5}\bar{w}_\alpha\Bigg[\widehat{g}\left(u(\bar{x}^\alpha_i, y^\beta_{j+\frac{1}{2}})^-, u(\bar{x}^\alpha_i, y^\beta_{j+\frac{1}{2}})^+\right) - \widehat{g}\left(u(\bar{x}^\alpha_i, y^\beta_{j-\frac{1}{2}})^-, u(\bar{x}^\alpha_i, y^\beta_{j-\frac{1}{2}})^+\right)\Bigg].$$

There exists a Q^4 polynomial $p_{ij}^{2,\beta}(x,y)$ and a point $(x_i^{2,\beta}, y_j^{2,\beta})$ defined similarly and the following conclusion holds.

LEMMA 4.2. *If* $u(\bar{x}^\alpha_i, y^\beta_{j\pm\frac{1}{2}})^\mp, u(\bar{x}^\alpha_i, y^\beta_{j\mp\frac{1}{2}})^\mp, p_{ij}^{2,\beta}(x_i^{2,\beta}, y_j^{2,\beta}) \in [m,M]$, *then* $\mathbf{G}_\beta \in [m,M]$ *under the CFL constraint* $(\lambda_1+\lambda_2)\max_u|g'(u)| \le \frac{1}{2}w_\beta\widehat{w}_1$.

Part II. The diffusion terms A and B. We have $\mathbf{A} = \bar{\bar{u}}_{ij}^n + 2(\mu_1+\mu_2)\sum_{\alpha=1}^{5}\bar{w}_\alpha [A(u(x_{i+1},\bar{y}^\alpha_j)) - 2A(u(x_i,\bar{y}^\alpha_j)) + A(u(x_{i-1},\bar{y}^\alpha_j))]$.

Following the arguments above, there exists a polynomial of degree four $q^1_{ij}(x,y)$ satisfying the following:

- It is a fifth order accurate approximation to $u(x,y,t^n)$ in $[x_{i-1},x_{i+1}]\times[y_{j-1}, y_{j+1}]$.
- $\bar{\bar{u}}^n_{ij}=\frac{1}{\Delta x^2}\frac{1}{\Delta y^2}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} q^1_{ij}(\xi,\eta)d\xi dx d\eta dy.$
- $q^1_{ij}(x_i,\bar{y}^\alpha_j)=u(x_i,\bar{y}^\alpha_j)$ for $\alpha=1,\dots,5$.

Replacing the integrals by proper quadrature, we have

$$\begin{aligned}\bar{\bar{u}}^n_{ij}&=\frac{1}{\Delta x^2}\frac{1}{\Delta y^2}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} q^1_{ij}(\xi,\eta)d\xi dx d\eta dy\\&=\sum_{\alpha=1}^{5}\sum_{\beta=1}^{5}\bar{w}_\alpha\bar{w}_\beta q^1_{ij}(\bar{x}^\beta_i,\bar{y}^\alpha_j).\end{aligned}$$

Notice that $\bar{x}^3_i=x_i$. By the mean value theorem, there exists a point $(x^{1,*}_i,y^{1,*}_j)\in[x_{i-1},x_{i+1}]\times[y_{j-1},y_{j+1}]$ such that

$$\bar{\bar{u}}^n_{ij}=\sum_{\alpha=1}^{5}\bar{w}_\alpha\bar{w}_3 u(x_i,\bar{y}^\alpha_j)+(1-\bar{w}_3)q^{1,*}_{ij}(x^{1,*}_i,y^{1,*}_j).$$

With $A'(u)\geq 0$, we have the next lemma.

Lemma 4.3. *If $u(x_i,\bar{y}^\alpha_j), q^{1,*}_{ij}(x^{1,*}_i,y^{1,*}_j)\in[m,M]$ $\forall i$ and j, then $\mathbf{A}\in[m,M]$ under the CFL constraint $(\mu_1+\mu_2)\max_u A'(u)\leq\frac14\bar{w}_3$.*

Similarly, we have

$$\mathbf{B}=\bar{\bar{u}}^n_{ij}+2(\mu_1+\mu_2)\sum_{\alpha=1}^{5}\bar{w}_\alpha\left[B(u(\bar{x}^\alpha_i,y_{j+1}))-2B(u(\bar{x}^\alpha_i,y_j))+B(u(\bar{x}^\alpha_i,y_{j-1}))\right].$$

There exists a Q^4 polynomial $q^2_{ij}(x,y)$ and a point $(x^{2,*}_i,y^{2,*}_j)$ defined similarly and the following conclusion holds.

Lemma 4.4. *If $u(\bar{x}^\alpha_i,y_j), q^{2,*}_{ij}(x^{2,*}_i,y^{2,*}_j)\in[m,M]$ $\forall i$ and j, then $\mathbf{B}\in[m,M]$ under the CFL constraint $(\mu_1+\mu_2)\max_u B'(u)\leq\frac14\bar{w}_3$.*

So we get the next theorem.

Theorem 4.5. *If $u(x^\beta_{i\pm\frac12},\bar{y}^\alpha_j)^\mp$, $p^{1,\beta}_{ij}(x^{1,\beta}_i,y^{1,\beta}_j), u(\bar{x}^\alpha_i,\ y^\beta_{j\pm\frac12})^\mp, p^{2,\beta}_{ij}(x^{2,\beta}_i,y^{2,\beta}_j)$, $u(x_i,\bar{y}^\alpha_j)$, $q^{1,*}_{ij}(x^{1,*}_i,y^{1,*}_j)$, $u(\bar{x}^\alpha_i,y_j), q^{2,*}_{ij}(x^{2,*}_i,y^{2,*}_j)\in[m,M]$ $\forall i$ and j, then* (3.19) *satisfies the maximum principle, namely, $\bar{\bar{u}}^{n+1}_{ij}\in[m,M]$ under the CFL conditions*

$$\begin{aligned}&(\lambda_1+\lambda_2)\max\{\max_u|f'(u)|,\max_u|g'(u)|\}\leq\frac12\widehat{w}_1\min_\beta w_\beta,\\&(\mu_1+\mu_2)\max\{\max_u A'(u),\max_u B'(u)\}\leq\frac14\bar{w}_3.\end{aligned}\tag{4.4}$$

4.4. The limiter. At time level n, given $\bar{\bar{u}}^n_{ij}\in[m,M]$ and the point values $u(x^\beta_{i\pm\frac12},\bar{y}^\alpha_j)^\mp, u(\bar{x}^\alpha_i,\ y^\beta_{j\pm\frac12})^\mp, u(x_i,\bar{y}^\alpha_j),\ u(\bar{x}^\alpha_i,y_j)$ from the WENO reconstruction, we get the revised point values by the following:

- For each i, j, and each β,

$$\widetilde{u}(x^{\beta}_{i\pm\frac{1}{2}}, \bar{y}^{\alpha}_j)^{\mp} = \theta(u(x^{\beta}_{i\pm\frac{1}{2}}, \bar{y}^{\alpha}_j)^{\mp} - \bar{\bar{u}}^n_{ij}) + \bar{\bar{u}}^n_{ij}, \tag{4.5}$$

$$\theta = \min\left\{\left|\frac{M-\bar{\bar{u}}^n_{ij}}{M_{ij}-\bar{\bar{u}}^n_{ij}}\right|, \left|\frac{m-\bar{\bar{u}}^n_{ij}}{m_{ij}-\bar{\bar{u}}^n_{ij}}\right|, 1\right\},$$

$$M_{ij} = \max_{\alpha}\{u(x^{\beta}_{i\pm\frac{1}{2}}, \bar{y}^{\alpha}_j)^{\mp}, p^{1,\beta}_{ij}(x^{1,\beta}_i, y^{1,\beta}_j)\}, \tag{4.6}$$

$$m_{ij} = \min_{\alpha}\{u(x^{\beta}_{i\pm\frac{1}{2}}, \bar{y}^{\alpha}_j)^{\mp}, p^{1,\beta}_{ij}(x^{1,\beta}_i, y^{1,\beta}_j)\},$$

where

$$p^{1,\beta}_{ij}(x^{1,\beta}_i, y^{1,\beta}_j) = \frac{1}{1-2w_\beta \widehat{w}_1}\Bigg[\bar{\bar{u}}^n_{ij} - \sum_{\alpha=1}^{5} \bar{w}_\alpha w_\beta \widehat{w}_1 \Big(u(x^{\beta}_{i-\frac{1}{2}}, \bar{y}^{\alpha}_j)^{+} + u(x^{\beta}_{i+\frac{1}{2}}, \bar{y}^{\alpha}_j)^{-}\Big)\Bigg].$$

- For each i, j, and each β,

$$\widetilde{u}(\bar{x}^{\alpha}_i, y^{\beta}_{j\pm\frac{1}{2}})^{\mp} = \theta(u(\bar{x}^{\alpha}_i, y^{\beta}_{j\pm\frac{1}{2}})^{\mp} - \bar{\bar{u}}^n_{ij}) + \bar{\bar{u}}^n_{ij}, \tag{4.7}$$

$$\theta = \min\left\{\left|\frac{M-\bar{\bar{u}}^n_{ij}}{M_{ij}-\bar{\bar{u}}^n_{ij}}\right|, \left|\frac{m-\bar{\bar{u}}^n_{ij}}{m_{ij}-\bar{\bar{u}}^n_{ij}}\right|, 1\right\},$$

$$M_{ij} = \max_{\alpha}\{u(\bar{x}^{\alpha}_i, y^{\beta}_{j\pm\frac{1}{2}})^{\mp}, p^{2,\beta}_{ij}(x^{2,\beta}_i, y^{2,\beta}_j)\}, \tag{4.8}$$

$$m_{ij} = \min_{\alpha}\{u(\bar{x}^{\alpha}_i, y^{\beta}_{j\pm\frac{1}{2}})^{\mp}, p^{2,\beta}_{ij}(x^{2,\beta}_i, y^{2,\beta}_j)\},$$

where

$$p^{2,\beta}_{ij}(x^{2,\beta}_i, y^{2,\beta}_j) = \frac{1}{1-2w_\beta \widehat{w}_1}\Bigg[\bar{\bar{u}}^n_{ij} - \sum_{\alpha=1}^{5} \bar{w}_\alpha w_\beta \widehat{w}_1 \Big(u(\bar{x}^{\alpha}_i, y^{\beta}_{j-\frac{1}{2}})^{+} + u(\bar{x}^{\alpha}_i, y^{\beta}_{j+\frac{1}{2}})^{-}\Big)\Bigg].$$

- For each i, j,

$$\widetilde{u}(x_i, \bar{y}^{\alpha}_j) = \theta(u(x_i, \bar{y}^{\alpha}_j) - \bar{\bar{u}}^n_{ij}) + \bar{\bar{u}}^n_{ij}, \quad \theta = \min\left\{\left|\frac{M-\bar{\bar{u}}^n_{ij}}{M_{ij}-\bar{\bar{u}}^n_{ij}}\right|, \left|\frac{m-\bar{\bar{u}}^n_{ij}}{m_{ij}-\bar{\bar{u}}^n_{ij}}\right|, 1\right\}, \tag{4.9}$$

$$M_{ij} = \max_{\alpha}\{u(x_i, \bar{y}^{\alpha}_j), q^{1,\beta}_{ij}(x^{1,*}_i, y^{1,*}_j)\}, \quad m_{ij} = \min_{\alpha}\{u(x_i, \bar{y}^{\alpha}_j), q^{1,\beta}_{ij}(x^{1,*}_i, y^{1,*}_j)\}, \tag{4.10}$$

where

$$q^{1,\beta}_{ij}(x^{1,*}_i, y^{1,*}_j) = \frac{1}{1-\bar{w}_3}\left[\bar{\bar{u}}^n_{ij} - \sum_{\alpha=1}^{5} \bar{w}_\alpha \bar{w}_3 u(x_i, \bar{y}^{\alpha}_j)\right].$$

- For each i, j,

(4.11)
$$\widetilde{u}(\bar{x}_i^\alpha, y_j) = \theta(u(\bar{x}_i^\alpha, y_j) - \bar{\bar{u}}_{ij}^n) + \bar{\bar{u}}_{ij}^n, \quad \theta = \min\left\{\left|\frac{M - \bar{\bar{u}}_{ij}^n}{M_{ij} - \bar{\bar{u}}_{ij}^n}\right|, \left|\frac{m - \bar{\bar{u}}_{ij}^n}{m_{ij} - \bar{\bar{u}}_{ij}^n}\right|, 1\right\},$$

(4.12)
$$M_{ij} = \max_\alpha\{u(\bar{x}_i^\alpha, y_j), q_{ij}^{2,\beta}(x_i^{2,*}, y_j^{2,*})\}, \quad m_{ij} = \min_\alpha\{u(\bar{x}_i^\alpha, y_j), q_{ij}^{2,\beta}(x_i^{2,*}, y_j^{2,*})\},$$

where

$$q_{ij}^{2,\beta}(x_i^{2,*}, y_j^{2,*}) = \frac{1}{1 - \bar{w}_3}\left[\bar{\bar{u}}_{ij}^n - \sum_{\alpha=1}^{5} \bar{w}_\alpha \bar{w}_3 u(\bar{x}_i^\alpha, y_j)\right].$$

Finally, replacing the point values by the revised ones in the limiter (4.5)–(4.12), we get the revised scheme.

4.5. Numerical tests.

Example 4.1 (accuracy test). Consider the linear equation $u_t + u_x + u_y = \varepsilon(u_{xx} + u_{yy})$ with initial data $u_0(x, y) = \sin(x + y)$ on $[0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions. The exact solution is $u(x, y, t) = \exp(-2t\varepsilon)\sin(x + y - 2t)$. Here $\varepsilon = 0.001$ and the final time is $T = 0.1$. The time step is taken as (4.4). See Table 4.1. We observe the fifth order accuracy of the maximum-principle-satisfying WENO scheme.

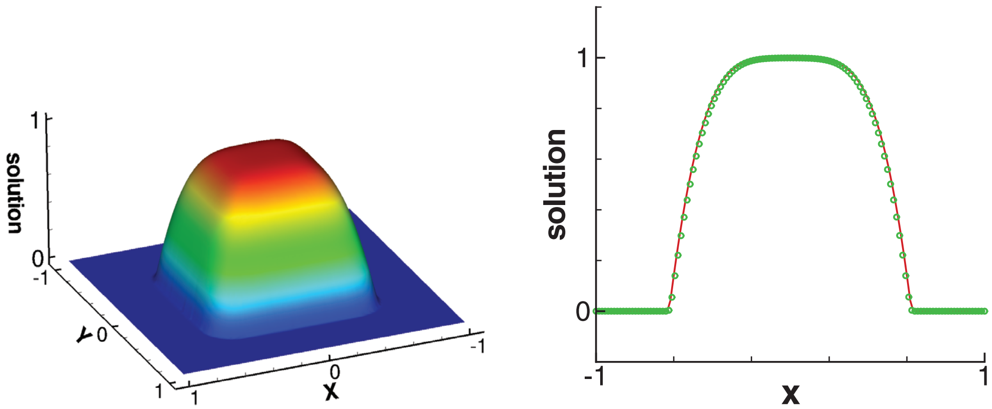
Example 4.2 (porous medium equation). Consider $u_t = (u^2)_{xx} + (u^2)_{yy}$ with the initial condition is $u_0(x, y) = 1$ if $(x, y) \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$; $u_0(x, y) = 0$ otherwise on $[-1, 1] \times [-1, 1]$. The boundary conditions are periodic. We compare the numerical solutions of the scheme in [8] and the maximum-principle-satisfying WENO scheme. See Table 4.2 and Figure 4.1. Even though it seems that the two solutions match very well in the figure, only the maximum-principle-satisfying WENO scheme can maintain strict nonnegativity.

TABLE 4.1
Accuracy test.

Mesh	L^1 error	Order	L^∞ error	Order
8×8	3.42E-3	–	3.96E-3	–
16×16	1.34E-4	4.67	2.64E-4	3.90
32×32	3.60E-6	5.22	7.07E-6	5.22
64×64	1.13E-7	4.98	2.29E-7	4.95
128×128	3.45E-9	5.03	6.96E-9	5.04
256×256	1.06E-10	5.01	1.99E-10	5.12

TABLE 4.2
The minimum of the numerical solutions of two schemes at $T = 0.005$.

Mesh	8^2	16^2	32^2	64^2	128^2
Scheme in [8]	−8.07E-7	−9.54E-7	−1.11E-5	−1.73E-3	−3.72e-4
WENO with limiter	0	0	0	0	0



(a) Surface (b) Cut along $y = 0$

FIG. 4.1. *The surface on the left and the solid curve on the right are the numerical solutions of the maximum-principle-satisfying WENO scheme on a* 128^2 *mesh. The symbol on the right is the numerical solution of finite difference WENO scheme in* [8].

5. Applications to two-dimensional incompressible Navier–Stokes equations.

5.1. Preliminaries. We are interested in solving the two-dimensional incompressible Navier–Stokes equations in the vorticity stream-function formulation with high Reynolds number $\mathrm{Re} \gg 1$:

$$\omega_t + (u\omega)_x + (v\omega)_y = \frac{1}{\mathrm{Re}}\Delta\omega, \tag{5.1}$$

$$\Delta\psi = \omega, \qquad \langle u, v\rangle = \langle -\psi_y, \psi_x\rangle, \tag{5.2}$$

$$\omega(x, y, 0) = \omega_0(x, y), \qquad \langle u, v\rangle \cdot \mathbf{n} = \text{given on } \partial\Omega.$$

The exact solution satisfies the maximum principle $\omega(x, y, t) \in [m, M]\ \forall(x, y, t)$, where $m = \min_{x,y} \omega_0(x, y)$ and $M = \max_{x,y} \omega_0(x, y)$. In [14], the maximum-principle-satisfying high order schemes for scalar conservation laws were applied to constructing maximum-principle-satisfying arbitrarily high order schemes solving the conservative incompressible Euler equation $\omega_t + (u\omega)_x + (v\omega)_y = 0$.

The key step in [14] is to achieve the consistency of the scheme, which requires the numerical velocity field $\langle u, v\rangle$ to satisfy the following:

1. They should be divergence free everywhere.
2. They are continuous in the normal direction across cell boundaries.
3. The quadrature used in the numerical scheme should be exact for any integral involved for the velocity field.

The first two properties can be achieved by using the continuous finite element method to solve the Poisson equation. Given the double cell averages $\bar{\bar{\omega}}^n_{ij}$, we can use them to reconstruct a unique Q^4 polynomial $\Omega_{ij}(x, y)$ satisfying

$$\frac{1}{\Delta x^2}\frac{1}{\Delta y^2}\int_{y_{l-\frac{1}{2}}}^{y_{l+\frac{1}{2}}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x_{m-\frac{1}{2}}}^{x_{m+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} \Omega_{ij}(\xi, \eta)\,d\xi dx d\eta dy = \bar{\bar{\omega}}^n_{ml}$$

for $m = i-2, \ldots, i+2$ and $l = j-2, \ldots, j+2$. Let $\psi(x, y)$ be the solution of Q^4

continuous finite element method solving

$$\Delta\psi(x,y) = \Omega_{ij}(x,y), \quad (x,y) \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}].$$

Then we have the piecewise polynomial velocity field by

$$u(x,y) = -\frac{\partial\psi(x,y)}{\partial y}, \qquad v(x,y) = \frac{\partial\psi(x,y)}{\partial x}. \tag{5.3}$$

Since $u(x,y)$ and $v(x,y)$ are piecewise polynomials, the last property cannot be maintained if we use the same scheme for the convection part in section 4. To this end, we need to approximate the double integral by the quadrature for piecewise polynomials. The diffusion part can be discretized by the same quadrature as in section 4, which will not affect the consistency.

5.2. A consistent double average finite volume scheme. Since the discretization for the diffusion term is the same as in section 4, we first discuss how to construct a consistent double average finite volume scheme for the Euler equation $\omega_t + (u\omega)_x + (v\omega)_y = 0$.

Recall that $\tilde{x}_i^\alpha$ ($\alpha = 1, \ldots, 12$) are the quadrature points for the double integral of piecewise polynomials. $\tilde{x}_i^\alpha$ ($\alpha = 7, 8, 9$) and $\tilde{x}_i^\alpha$ ($\alpha = 10, 11, 12$) are the three-point Gauss quadratures for integrals on the intervals $[x_i, x_{i+\frac{1}{2}}]$ and $[x_{i+\frac{1}{2}}, x_{i+1}]$, respectively. Recall w_α ($\alpha = 1, 2, 3$) are three-point Gauss quadrature weights for $[-\frac{1}{2}, \frac{1}{2}]$. For convenience, we define w_α ($\alpha = 7, \ldots, 12$) by $w_\alpha = \frac{1}{2}w_{\alpha-6}$ for $\alpha = 7, 8, 9$ and $w_\alpha = \frac{1}{2}w_{\alpha-9}$ for $\alpha = 10, 11, 12$ to denote the corresponding normalized three-point Gauss quadrature weights for the intervals $[x_i, x_{i+\frac{1}{2}}]$ and $[x_{i+\frac{1}{2}}, x_{i+1}]$.

The scheme with Euler forward time discretization can be written as

$$\bar{\bar{\omega}}_{ij}^{n+1} = \bar{\bar{\omega}}_{ij}^{n} - \lambda_1(\widehat{f}_{i+\frac{1}{2},j} - \widehat{f}_{i-\frac{1}{2},j}) - \lambda_2(\widehat{g}_{i,j+\frac{1}{2}} - \widehat{g}_{i,j-\frac{1}{2}}) \tag{5.4}$$

with

$$\widehat{f}_{i+\frac{1}{2},j} = \sum_{\alpha=1}^{12}\sum_{\beta=7}^{12} \tilde{w}_\alpha w_\beta \widehat{f}\left(\omega(\tilde{x}_i^\beta, \tilde{y}_j^\alpha)^-, \omega(\tilde{x}_i^\beta, \tilde{y}_j^\alpha)^+, u(\tilde{x}_i^\beta, \tilde{y}_j^\alpha)\right), \tag{5.5}$$

$$\widehat{g}_{i,j+\frac{1}{2}} = \sum_{\alpha=1}^{12}\sum_{\beta=7}^{12} \tilde{w}_\alpha w_\beta \widehat{g}\left(\omega(\tilde{x}_i^\alpha, \tilde{y}_j^\beta)^-, \omega(\tilde{x}_i^\alpha, \tilde{y}_j^\beta)^+, v(\tilde{x}_i^\alpha, \tilde{y}_j^\beta)\right), \tag{5.6}$$

where the Lax–Friedrichs numerical fluxes are defined by

$$\widehat{f}(\omega^-, \omega^+, u) = \frac{1}{2}[u(\omega^+ + \omega^-) - a_1(\omega^+ - \omega^-)],$$
$$\widehat{g}(\omega^-, \omega^+, v) = \frac{1}{2}[v(\omega^+ + \omega^-) - a_2(\omega^+ - \omega^-)],$$

$a_1 = \max|u(x,y)|$ and $a_2 = \max|v(x,y)|$ with the maximum taken either locally or globally.

Notice that $\tilde{x}_{i-1}^{\beta+6}$ are the same points as $\tilde{x}_i^\beta$ for $\beta = 1, \ldots, 6$. The point values $\omega(\tilde{x}_i^\beta, \tilde{y}_j^\alpha)^-$, $\omega(\tilde{x}_{i-1}^\beta, \tilde{y}_j^\alpha)^+$ ($\alpha = 1, \ldots, 12; \beta = 7, \ldots, 12$) are constructed as follows, for each rectangle i, j:

- Apply Procedure 5 to $\{\bar{\bar{\omega}}^n_{i,j-2}, \bar{\bar{\omega}}^n_{i,j-1}, \bar{\bar{\omega}}^n_{i,j}, \bar{\bar{\omega}}^n_{i,j+1}, \bar{\bar{\omega}}^n_{i,j+2}\}$ at the points $\tilde{y}^\alpha_j$ ($\alpha = 1, \dots, 12$) to reconstruct the one-dimensional double average in x-direction, and denote it as $\bar{\bar{\omega}}_i(\tilde{y}^\alpha_j)$ (approximating $\frac{1}{\Delta x^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} \omega(\xi, \tilde{y}^\alpha_j, t^n)d\xi\, dx$).
- Apply Procedure 5 to $\{\bar{\bar{\omega}}_{i-2}(\bar{y}^\alpha_j), \bar{\bar{\omega}}_{i-1}(\bar{y}^\alpha_j), \bar{\bar{\omega}}_i(\bar{y}^\alpha_j), \bar{\bar{\omega}}_{i+1}(\bar{y}^\alpha_j), \bar{\bar{\omega}}_{i+2}(\bar{y}^\alpha_j)\}$ at the points $\tilde{x}^\beta_i$ ($\beta = 1, \dots, 12$) to reconstruct $\omega(\tilde{x}^{\beta+6}_{i-1}, \tilde{y}^\alpha_j)^+$ (for $\beta = 1, \dots, 6$) and $\omega(\tilde{x}^\beta_i, \tilde{y}^\alpha_j)^-$ (for $\beta = 7, \dots, 12$).

The points values $\omega(\tilde{x}^\alpha_i, \tilde{y}^\beta_j)^\pm$ are constructed similarly.

LEMMA 5.1. *The scheme* (5.4)–(5.6) *is consistent, namely, if all the point values and double cell averages of* ω *at time level* n *are replaced by a constant* M*, then* $\bar{\bar{\omega}}^{n+1}_{ij} = M$*.*

Proof. Since all the quadratures are exact for the integration of the velocity field, we get

$$\widehat{f}_{i+\frac{1}{2},j} = \sum_{\alpha=1}^{12}\sum_{\beta=7}^{12} \tilde{w}_\alpha w_\beta \widehat{f}\left(M, M, u(\tilde{x}^\beta_i, \tilde{y}^\alpha_j)\right) = \sum_{\alpha=1}^{12}\sum_{\beta=7}^{12} \tilde{w}_\alpha w_\beta u(\tilde{x}^\beta_i, \tilde{y}^\alpha_j)$$
$$= \frac{1}{\Delta x \Delta y^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x_i}^{x_{i+1}} u(x,\eta)dx d\eta dy.$$

Similarly, we have

$$\widehat{g}_{i,j+\frac{1}{2}} = \frac{1}{\Delta x^2 \Delta y}\int_{y_j}^{y_{j+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u(\xi, y)d\xi dx dy.$$

The divergence theorem implies

$$\bar{\bar{\omega}}^{n+1}_{ij} = \bar{\bar{\omega}}^n_{ij} - \lambda_1(\widehat{f}_{i+\frac{1}{2},j} - \widehat{f}_{i-\frac{1}{2},j}) - \lambda_2(\widehat{g}_{i,j+\frac{1}{2}} - \widehat{g}_{i,j-\frac{1}{2}})$$
$$= \bar{\bar{\omega}}^n_{ij} - \frac{\Delta t}{\Delta x^2 \Delta y^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} \nabla\cdot\langle u, v\rangle(\xi,\eta)d\xi dx d\eta dy.$$

Equation (5.3) implies $\nabla \cdot \langle u, v\rangle \equiv 0$, and thus $\bar{\bar{\omega}}^{n+1}_{ij} = \bar{\bar{\omega}}^n_{ij} = M$. ∎

5.3. Maximum principle. Now consider the high order spatial discretization with Euler forward for the Navier–Stokes equations

$$\begin{aligned}\bar{\bar{\omega}}^{n+1}_{ij} &= \bar{\bar{\omega}}^n_{ij} - \lambda_1(\widehat{f}_{i+\frac{1}{2},j} - \widehat{f}_{i-\frac{1}{2},j}) - \lambda_2(\widehat{g}_{i,j+\frac{1}{2}} - \widehat{g}_{i,j-\frac{1}{2}}) \\ &\quad + \mu_1(\widehat{A}_{i+1,j} - 2\widehat{A}_{i,j} + \widehat{A}_{i-1,j}) + \mu_2(\widehat{B}_{i,j+1} - 2\widehat{B}_{i,j} + \widehat{B}_{i,j-1}),\end{aligned} \tag{5.7}$$

with (5.5), (5.6), and the same $\widehat{A}$, $\widehat{B}$ as in section 4.

It can be written as

$$\bar{\bar{\omega}}^{n+1}_{ij} = \frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}\mathbf{F} + \frac{1}{2}\frac{\lambda_2}{\lambda_1+\lambda_2}\mathbf{G} + \frac{1}{2}\frac{\mu_1}{\mu_1+\mu_2}\mathbf{A} + \frac{1}{2}\frac{\mu_2}{\mu_1+\mu_2}\mathbf{B},$$

where the diffusion terms are the same as in section 4.

It suffices to derive the monotonicity of the convection terms $\mathbf{F}$ and $\mathbf{G}$ since the full scheme is consistent in the sense that $\bar{\bar{\omega}}_{ij}^{n+1} = M$ if all the point values and double cell averages of ω at time level n are replaced by M.

We have $\mathbf{F} = \sum_{\alpha=1}^{12}\sum_{\beta=7}^{12}\tilde{w}_\alpha w_\beta \mathbf{F}_{\beta,\alpha}$ with

$$\begin{aligned}\mathbf{F}_{\beta,\alpha} = \bar{\bar{\omega}}_{ij}^n - 2(\lambda_1+\lambda_2)\Big[&\widehat{f}\left(\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^-,\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^+,u(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)\right)\\ &-\widehat{f}\left(\omega(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^-,\omega(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^+,u(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)\right)\Big].\end{aligned}$$

By interpolation, for each i,j, each $\alpha\in\{1,\dots,12\}$, and each $\beta\in\{7,\dots,12\}$, there exists a polynomial of degree four $p_{i,j}^{\beta,\alpha}(x,y)$ satisfying

- $p_{i,j}^{\beta,\alpha}(x,y)$ is a fifth order approximation to $\omega(x,y,t^n)$ on $[x_{i-1},x_{i+1}]\times[y_{j-1},y_{j+1}]$,
- the double cell average of $p_{i,j}^{\beta,\alpha}(x,y)$ on $[x_{i-\frac12},x_{i+\frac12}]\times[y_{j-\frac12},y_{j+\frac12}]$ is $\bar{\bar{\omega}}_{ij}^n$,
- $p_{i,j}^{\beta,\alpha}(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)=\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^-$ and $p_{i,j}^{\beta,\alpha}(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)=\omega(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^+$.

By quadrature and the mean value theorem, for each i,j, each $\alpha\in\{1,\dots,12\}$, and each $\beta\in\{7,\dots,12\}$, there exists a point $(x_i^{\beta,*},y_j^{\alpha,*})\in[x_{i-1},x_{i+1}]\times[y_{j-1},y_{j+1}]$ such that

$$\begin{aligned}\bar{\bar{\omega}}_{ij}^n &= \sum_{\alpha'=1}^{12}\sum_{\beta'=1}^{12}\tilde{w}_{\alpha'}\tilde{w}_{\beta'}p_{i,j}^{\beta,\alpha}(\tilde{x}_i^{\beta'},\tilde{y}_j^{\alpha'})\\ &=\tilde{w}_\alpha\tilde{w}_\beta\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^- + \tilde{w}_\alpha\tilde{w}_{\beta-6}\omega(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^+\\ &\quad+(1-\tilde{w}_\alpha\tilde{w}_\beta-\tilde{w}_\alpha\tilde{w}_{\beta-6})p_{i,j}^{\beta,\alpha}(x_i^{\beta,*},y_j^{\alpha,*}).\end{aligned}$$

Therefore, following arguments in section 4, we derive that $\mathbf{F}$ is monotonically increasing with respect to $\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^\pm,\omega(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^\pm$, and $p_{i,j}^{\beta,\alpha}(x_i^{\beta,*},y_j^{\alpha,*})$. We can get similar monotonicity for $\mathbf{G}$. The monotonicity results for the diffusion terms are the same as in section 4. Moreover, if all these point values are in the range $[m,M]$, by consistency, we have $\bar{\bar{\omega}}_{ij}^{n+1}\in[m,M]$ under the CFL constraints

$$(\lambda_1+\lambda_2)\max|\langle u,v\rangle|\le\frac12(\min_\alpha\tilde{w}_\alpha)^2,\quad(\mu_1+\mu_2)\le\frac14\bar{w}_3\mathrm{Re}.$$

5.4. The limiter. We only discuss how to limit the point values for the convection terms.

At time level n, given $\bar{\bar{\omega}}_{ij}^n\in[m,M]$ and the point values $\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^\mp,\omega(\tilde{x}_i^\alpha,\tilde{y}_j^\beta)^\mp$ for $\alpha=1,\dots,12$ and $\beta=7,\dots,12$ from the WENO reconstruction, get the revised point values by the following:

- For each i,j,α,β, limit the point values for $\mathbf{F}$,

$$\begin{aligned}\widetilde{\omega}(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^- &= \theta(\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^- - \bar{\bar{\omega}}_{ij}^n)+\bar{\bar{\omega}}_{ij}^n,\\ \widetilde{\omega}(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^+ &= \theta(\omega(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^+ - \bar{\bar{\omega}}_{ij}^n)+\bar{\bar{\omega}}_{ij}^n,\\ \theta &= \min\left\{\left|\frac{M-\bar{\bar{\omega}}_{ij}^n}{M_{ij}-\bar{\bar{\omega}}_{ij}^n}\right|,\left|\frac{m-\bar{\bar{\omega}}_{ij}^n}{m_{ij}-\bar{\bar{\omega}}_{ij}^n}\right|,1\right\},\\ M_{ij} &= \max\{\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^-,\omega(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^+,p_{i,j}^{\beta,\alpha}(x_i^{\beta,*},y_j^{\alpha,*})\},\\ m_{ij} &= \min\{\omega(\tilde{x}_i^\beta,\tilde{y}_j^\alpha)^-,\omega(\tilde{x}_{i-1}^\beta,\tilde{y}_j^\alpha)^+,p_{i,j}^{\beta,\alpha}(x_i^{\beta,*},y_j^{\alpha,*})\},\end{aligned}$$

where

$$p_{i,j}^{\beta,\alpha}(x_i^{\beta,*}, y_j^{\alpha,*}) = \frac{1}{1 - \tilde{w}_\alpha \tilde{w}_\beta - \tilde{w}_\alpha \tilde{w}_{\beta-6}} \Bigg[\bar{\bar{\omega}}_{ij}^n - \tilde{w}_\alpha \tilde{w}_\beta \omega(\tilde{x}_i^\beta, \tilde{y}_j^\alpha)^- - \tilde{w}_\alpha \tilde{w}_{\beta-6} \omega(\tilde{x}_{i-1}^\alpha, \tilde{y}_j^\alpha)^+ \Bigg].$$

- For each i, j, α, β, limit the point values for $\mathbf{G}$,

$$\widetilde{\omega}(\tilde{x}_i^\alpha, \tilde{y}_j^\beta)^- = \theta(\omega(\tilde{x}_i^\alpha, \tilde{y}_j^\beta)^- - \bar{\bar{\omega}}_{ij}^n) + \bar{\bar{\omega}}_{ij}^n,$$
$$\widetilde{\omega}(\tilde{x}_i^\alpha, \tilde{y}_{j-1}^\beta)^+ = \theta(\omega(\tilde{x}_i^\alpha, \tilde{y}_{j-1}^\beta)^+ - \bar{\bar{\omega}}_{ij}^n) + \bar{\bar{\omega}}_{ij}^n,$$
$$\theta = \min\left\{ \left| \frac{M - \bar{\bar{\omega}}_{ij}^n}{M_{ij} - \bar{\bar{\omega}}_{ij}^n} \right|, \left| \frac{m - \bar{\bar{\omega}}_{ij}^n}{m_{ij} - \bar{\bar{\omega}}_{ij}^n} \right|, 1 \right\},$$
$$M_{ij} = \max\{\omega(\tilde{x}_i^\alpha, \tilde{y}_j^\beta)^-, \omega(\tilde{x}_i^\alpha, \tilde{y}_{j-1}^\beta)^+, q_{i,j}^{\alpha,\beta}(x_i^{\alpha,*}, y_j^{\beta,*})\},$$
$$m_{ij} = \min\{\omega(\tilde{x}_i^\alpha, \tilde{y}_j^\beta)^-, \omega(\tilde{x}_i^\alpha, \tilde{y}_{j-1}^\beta)^+, q_{i,j}^{\alpha,\beta}(x_i^{\alpha,*}, y_j^{\beta,*})\},$$

where

$$q_{i,j}^{\alpha,\beta}(x_i^{\alpha,*}, y_j^{\beta,*}) = \frac{1}{1 - \tilde{w}_\alpha \tilde{w}_\beta - \tilde{w}_\alpha \tilde{w}_{\beta-6}} \Bigg[\bar{\bar{\omega}}_{ij}^n - \tilde{w}_\alpha \tilde{w}_\beta \omega(\tilde{x}_i^\alpha, \tilde{y}_j^\beta)^- - \tilde{w}_\alpha \tilde{w}_{\beta-6} \omega(\tilde{x}_i^\alpha, \tilde{y}_{j-1}^\beta)^+ \Bigg].$$

5.5. Numerical tests.

Example 5.1 (accuracy test). We test the accuracy of the scheme constructed in this section for the Navier–Stokes equations with Re = 100 and periodic boundary conditions. The exact solution is $\omega(x, y, t) = -2\sin(x)\sin(y)\exp(-2t/\mathrm{Re})$. The final time is $T = 0.1$. See Table 5.1.

Example 5.2 (the vortex patch problem). We solve the Navier–Stokes equations with Re = 100 in $[0, 2\pi] \times [0, 2\pi]$ with the initial condition

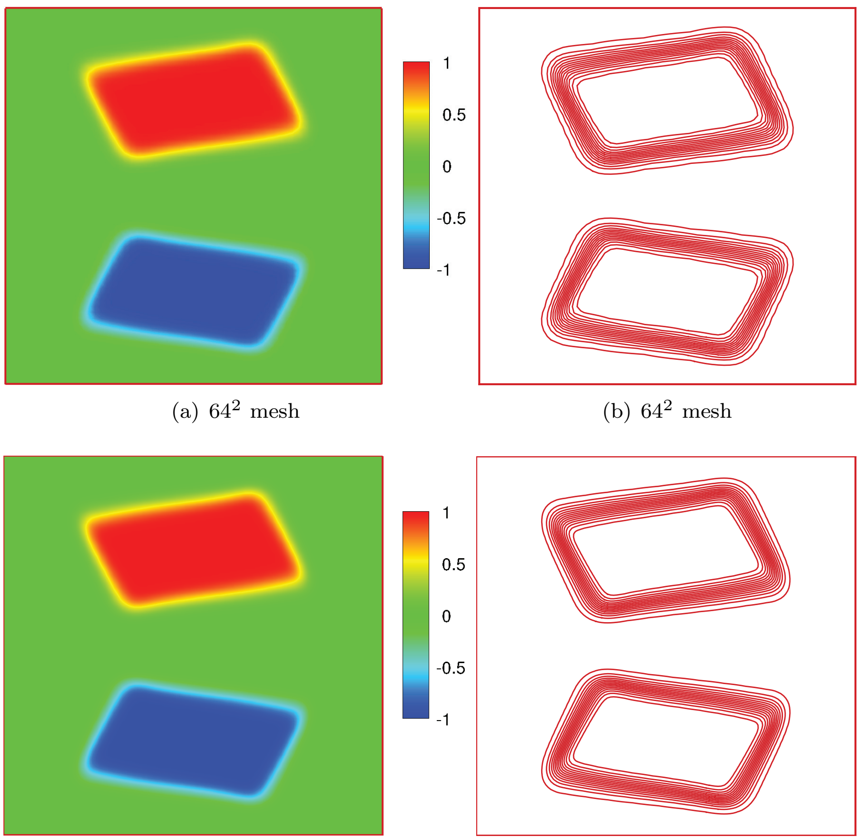
$$\omega(x, y, 0) = \begin{cases} -1, & \frac{\pi}{2} \le x \le \frac{3\pi}{2}, \frac{\pi}{4} \le y \le \frac{3\pi}{4}, \\ 1, & \frac{\pi}{2} \le x \le \frac{3\pi}{2}, \frac{5\pi}{4} \le y \le \frac{7\pi}{4}, \\ 0, & \text{otherwise}, \end{cases}$$

TABLE 5.1
Accuracy test for the Navier–Stokes equations.

$N \times N$	L^1 error	Order	L^∞ error	Order
8×8	1.40E-3	–	2.53E-3	–
16×16	5.59E-5	4.65	1.24E-4	4.35
32×32	1.78E-6	4.98	4.52E-6	4.79
64×64	5.38E-8	5.04	1.15E-7	5.29
128×128	1.61E-9	5.07	3.37E-9	5.10

Table 5.2
Maximum and minimum of the numerical solutions at $T = 0.1$.

	Scheme in section 5		Scheme in section 4	
Mesh	Min	Max	Min	Max
8×8	−0.9201050287386018	0.9201050287386021	−0.923002427257675	0.923002427257675
16×16	−0.9995382842829470	0.9995382842829469	−1.000880764352643	1.000880764352645
32×32	−0.9999999967438283	0.9999999967438288	−1.000074544710187	1.000074544710187
64×64	−0.9999999999996423	0.9999999999996408	−1.000051852836616	1.000051852836616

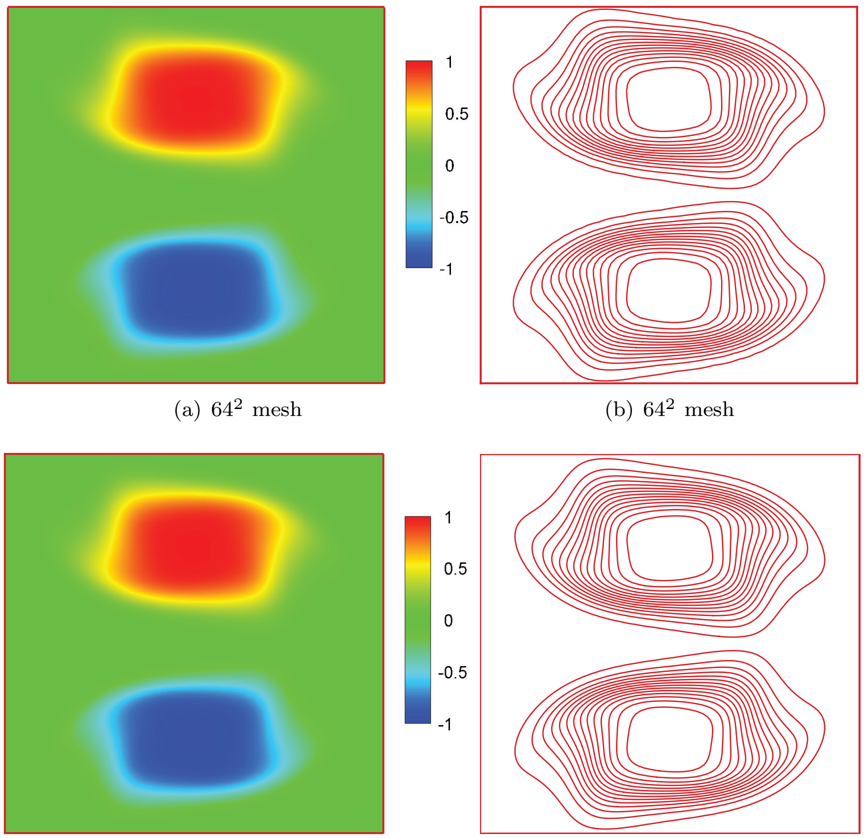


(a) 64^2 mesh (b) 64^2 mesh

(c) 128^2 mesh (d) 128^2 mesh

Fig. 5.1. *The vortex patch problem at $t = 1$. Contours of vorticity. The right are* 30 *equally spaced contour lines from -1 to* 1.

and periodic boundary conditions. We test the scheme in this section and the scheme in section 4. The maximum and minimum of numerical solutions are listed in Table 5.2. We can see that only the scheme in section 5 can keep the strict maximum principle, which confirms the necessity of achieving consistency by using the 12-point quadrature rule for the convection terms. See Figures 5.1 and 5.2 for the numerical solutions of the fifth order maximum-principle-satisfying WENO scheme in section 5 at $t = 1$ and $t = 5$.



(a) 64^2 mesh (b) 64^2 mesh

(c) 128^2 mesh (d) 128^2 mesh

FIG. 5.2. *The vortex patch problem at* $t = 5$. *Contours of vorticity. The right are* 30 *equally spaced contour lines from* -1 *to* 1.

6. Concluding remarks. In this paper, we have proposed a nonconventional fifth order finite volume WENO scheme which can be proved maximum-principle-satisfying for convection diffusion equations. We also show an extension to two dimensions. Moreover, the same idea applies to the two-dimensional incompressible Navier–Stokes equations in the vorticity stream-function formulation. We have tested the fifth order finite volume WENO scheme and clearly observed the strict maximum principle in all these tests.

Since we use explicit time stepping, the technique in this paper is more relevant for convection dominated convection-diffusion equations. Even though the CFL condition derived to preserve maximum principle is very small compared to the ones for conventional finite volume schemes, we emphasize that it is not a necessary condition. To save computational costs, one can strictly enforce the CFL conditions only when a precalculation with a usual time step produces overshoot or undershoot.

The scheme in this paper cannot be extended to nonuniform meshes in a straightforward way. High order maximum-principle-satisfying schemes on unstructured meshes and generalizations to compressible Navier–Stokes equations in gas dynamics in the context of positivity preserving of density and pressure will be explored in the future.

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