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OPTIMAL ERROR ESTIMATES OF THE SEMIDISCRETE LOCAL DISCONTINUOUS GALERKIN METHODS FOR HIGH ORDER WAVE EQUATIONS*

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Abstract. In this paper, we introduce a general approach for proving optimal L^2 error estimates for the semidiscrete local discontinuous Galerkin (LDG) methods solving linear high order wave equations. The optimal order of error estimates holds not only for the solution itself but also for the auxiliary variables in the LDG method approximating the various order derivatives of the solution. Examples including the one-dimensional third order wave equation, one-dimensional fifth order wave equation, and multidimensional Schrödinger equation are explored to demonstrate this approach. The main idea is to derive energy stability for the various auxiliary variables in the LDG discretization by using the scheme and its time derivatives with different test functions. Special projections are utilized to eliminate the jump terms at the cell boundaries in the error estimate in order to achieve the optimal order of accuracy.

Key words. local discontinuous Galerkin method, high order wave equation, error estimate, energy stability

AMS subject classifications. 65M60, 65M15

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1. Introduction. In this paper, we introduce a general approach for proving optimal L^2 error estimates for the semidiscrete local discontinuous Galerkin (LDG) methods solving high order wave equations. To demonstrate the main idea, we will discuss the following representative examples:

- The one-dimensional third order wave equation

$$u_t + u_{xxx} = 0; \tag{1.1}$$

- The one-dimensional fifth order wave equation

$$u_t + u_{xxxxx} = 0; \tag{1.2}$$

- The multidimensional Schrödinger equation

$$\mathrm{i}u_t + \Delta u = 0. \tag{1.3}$$

We will give the details of the proof for these equations to explain the main ideas on how to obtain optimal L^2 error estimates for the LDG methods solving high order wave equations. These equations are classical model equations for many very important physical applications. KdV-type equations describe the propagation of waves in a variety of nonlinear, dispersive media [2]. The fifth order nonlinear evolution equation

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is known as the critical surface-tension model [13]. The Schrödinger equation describes many phenomena and has important applications in fluid dynamics, nonlinear optics, and plasma physics [3, 4, 10].

The DG method is a class of finite element methods using discontinuous, piecewise polynomials as the solution and the test spaces. It was first designed as a method for solving hyperbolic conservation laws containing only first order spatial derivatives, e.g., Reed and Hill [18], for solving steady state linear equations, and Cockburn and Shu [7, 8], for solving time-dependent nonlinear equations. The LDG method is an extension of the DG method aimed at solving partial differential equations (PDEs) containing higher than first order spatial derivatives. The first LDG method was constructed by Cockburn and Shu in [9] for solving nonlinear convection diffusion equations containing second order spatial derivatives. Their work was motivated by the successful numerical experiments of Bassi and Rebay [1] for the compressible Navier–Stokes equations. The idea of the LDG method is to rewrite the equations with higher order derivatives into a first order system, then apply the DG method on the system. The design of the numerical fluxes is the key ingredient to ensure stability. The LDG techniques have been developed for various high order PDEs, including convection diffusion equations [9] and nonlinear one-dimensional and two-dimensional KdV-type equations [25, 23]. More details about the LDG methods for high order time-dependent PDEs can be found in the review paper [24]. General information about DG methods for elliptic, parabolic, and hyperbolic PDEs can be found in the recent books and lecture notes [16, 12, 19, 20].

The LDG schemes for solving (1.1), (1.2), and (1.3) were developed in [25, 26, 22]. Compared with the status of optimal L^2 error estimates for LDG methods solving time-dependent diffusive PDEs, for example, the convection diffusion equations [9, 20] and the fourth order time-dependent bi-harmonic type equations [11], optimal L^2 error estimates for LDG methods solving high order time-dependent wave equations are much more elusive. The main technical difficulty is the lack of coercivity and hence the control on the auxiliary variables in the LDG method which are approximations to the derivatives of the solution and the lack of control on the interface boundary terms. When these issues are not addressed carefully, optimal L^2 error estimates cannot be obtained. In [25, 22], a priori L^2 error estimates with suboptimal order $O(h^{k+\frac{1}{2}})$ for the LDG method with P^k elements for the linearized KdV equations and the linearized Schrödinger equation in one spatial dimension were obtained. Even for such linear PDEs, there are technical difficulties in deriving the L^2 a priori error estimates from the L^2 stability (cell entropy inequality) and approximation results, because of the possible lack of control on some of the jump terms at cell boundaries, which appear because of the discontinuous nature of the finite element space for the DG method. The remedy in [25, 22] to handle such jump terms was via a special projection on the variable u, which eliminates such troublesome jump terms in the error equation. The lack of control for the auxiliary variables in the LDG method which are approximations to the derivatives of the solution is more troublesome. The special projections cannot be used on the auxiliary variables without causing order reduction (half order or more) in the error estimate.

The main idea in this paper to prove optimal L^2 accuracy order is to obtain the energy stability for the auxiliary variables. We can then use the special projections on the auxiliary variables and eliminate or control the troublesome jump terms in the equations, as in [15, 6]. We will demonstrate the main ideas through the proof of optimal L^2 error estimates for the one-dimensional third order wave equation, one-dimensional fifth order wave equation, and multidimensional Schrödinger equation on

Cartesian meshes. The optimal order of error estimates hold not only for the solution itself but also for the auxiliary variables in the LDG method approximating the various order derivatives of the solution. To our best knowledge, this is the first successful optimal L^2 error estimates of the LDG schemes for such high order wave equations.

The paper is organized as follows. In section 2, we present the details of the optimal error estimates for the one-dimensional third order wave equation. In section 3, the energy stability proof and optimal error estimates for the one-dimensional fifth order wave equation are given. Discussions of the multidimensional Schrödinger equation on Cartesian meshes are given in section 4. Concluding remarks are given in section 5.

2. One-dimensional third order wave equation.

In this section, we use the simple example of the one-dimensional third order wave equation

$$u_t + u_{xxx} = 0 \tag{2.1}$$

to explain the main ideas of our optimal L^2 error estimates for dispersive wave equations. The same technique applies to other third order linear wave equations such as the linearized KdV equation

$$u_t + au_x = u_{xxx}.$$

We will adopt the following convention for different constants. These constants may have a different value in each occurrence. We will denote by C a positive constant independent of h, which may depend on the exact smooth solution of the PDE and its derivatives. For problems considered in this paper, the exact solution is assumed to be smooth with periodic boundary conditions. Notice that the assumption of periodic boundary conditions is for simplicity only and is not essential: the method can be easily designed for nonperiodic boundary conditions. The development of the LDG method for the nonperiodic boundary conditions can be found in, e.g., [17]. Also, we consider $0 \le t \le T$ for a fixed T.

2.1. Notations, definitions, and projections in the one-dimensional case.

In this section, we will first introduce some notations and definitions in the one-dimensional case, which will be used in the error analysis for one-dimensional wave equations.

2.1.1. Tessellation and function spaces.

We tessellate the computational domain Ω with shape-regular elements. The boundary of Ω is denoted as $\Gamma = \partial\Omega$. We denote the mesh by $K_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1, \ldots, N$. The center of the cell is $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $h = \max_j \Delta x_j$. $(\cdot)_{j+\frac{1}{2}}$ denotes the value of function at $x_{j+\frac{1}{2}}$. We denote by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ the value of u at $x_{j+\frac{1}{2}}$, from the right cell, K_{j+1}, and from the left cell, K_j, respectively. We define the real valued piecewise-polynomial space V_h as the space of polynomials of degree at most k in each cell I_j, i.e.,

$$V_h = \{v : \quad v \in P^k(K_j) \quad \text{for } x \in I_j, \ \ j = 1, \ldots, N\}.$$

Note that functions in V_h are allowed to have discontinuities across element interfaces.

For any element $K = K_j$, $j = 1, \ldots, N$, we define the inner product and L^2 norm in K as

$$(w, v)_K = \int_K wv\, dx, \quad \|\eta\|_K^2 = (w, w)_K, \tag{2.2}$$

for the scalar variables w, v.

2.1.2. Projection and interpolation properties. In what follows, we will consider the standard L^2 projection π and two special projections of a function ω with $k+1$ continuous derivatives into the space V_h, $P^{\pm} : H^1(\Omega) \longrightarrow V_h$, which are defined as follows. Given a function $\eta \in H^1(\Omega)$ and an arbitrary subinterval $K_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, the restriction of $P^{\pm}\eta$ to K_j are defined as the elements of $\mathcal{P}^k(K_j)$ that satisfy

$$\int_{K_j} (P^+\eta - \eta) w dx = 0 \quad \forall w \in \mathcal{P}^{k-1}(K_j) \quad \text{and} \quad P^+\eta(x^+_{j-\frac{1}{2}}) = \eta(x_{j-\frac{1}{2}}), \tag{2.3}$$

$$\int_{K_j} (P^-\eta - \eta) w dx = 0 \quad \forall w \in \mathcal{P}^{k-1}(K_j) \quad \text{and} \quad P^-\eta(x^-_{j+\frac{1}{2}}) = \eta(x_{j+\frac{1}{2}}). \tag{2.4}$$

For the projections mentioned above, it is easy to show (c.f. [5])

$$\|\eta^e\|_\Omega \le Ch^{k+1}, \tag{2.5}$$

where $\eta^e = \pi\eta - \eta$ or $\eta^e = P^{\pm}\eta - \eta$. The positive constant C, only depending on η, is independent of h.

2.2. The LDG scheme for the one-dimensional third order wave equation.

2.2.1. The LDG scheme. In order to construct the LDG method, first we rewrite (2.1) as a system containing only first order derivatives:

$$u_t + w_x = 0,\ w - v_x = 0,\ v - u_x = 0. \tag{2.6}$$

The LDG scheme to solve (2.6) is as follows. Find u_h, w_h, $v_h \in V_h$ such that for all test functions η, φ, $\psi \in V_h$

$$((u_h)_t, \eta)_{K_j} - (w_h, \eta_x)_{K_j} + (\widehat{w}_h \eta^-)_{j+\frac{1}{2}} - (\widehat{w}_h \eta^+)_{j-\frac{1}{2}} = 0, \tag{2.7a}$$

$$(w_h, \varphi)_{K_j} + (v_h, \varphi_x)_{K_j} - (\widehat{v}_h \varphi^-)_{j+\frac{1}{2}} + (\widehat{v}_h \varphi^+)_{j-\frac{1}{2}} = 0, \tag{2.7b}$$

$$(v_h, \psi)_{K_j} + (u_h, \psi_x)_{K_j} - (\widehat{u}_h \psi^-)_{j+\frac{1}{2}} + (\widehat{u}_h \psi^+)_{j-\frac{1}{2}} = 0. \tag{2.7c}$$

The "hat" terms in (2.7) in the cell boundary terms from integration by parts are the so-called numerical fluxes, which are functions defined on the edges and should be designed based on guiding principles for different PDEs to ensure stability and local solvability of the intermediate variables w_h, v_h. It turns out that we can take the simple choices with upwinding for $\widehat{v}_h$ and alternating fluxes for $\widehat{w}_h$ and $\widehat{u}_h$, such as

$$\widehat{w}_h = w_h^-, \quad \widehat{v}_h = v_h^+, \quad \widehat{u}_h = u_h^+. \tag{2.8}$$

The scheme here is a special case of the LDG methods in [25] when applied to the simple linear PDE (2.1).

2.2.2. Notation for the DG discretization. To facilitate our proof of the error estimate, we will fix some notation for the DG discretization. We use the usual notations

$$[\eta] = \eta^+ - \eta^-, \quad \bar{\eta} = \frac{1}{2}(\eta^+ + \eta^-)$$

to denote the jump and the average of the function η at each element boundary point, respectively.

For the one-dimensional case, we define the DG discretization operator $\mathcal{D}$, i.e., in an arbitrary subinterval $K_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$,

(2.9)
$$\mathcal{D}_{K_j}(\eta, \phi; \widehat{\eta}) = -(\eta, \phi_x)_{K_j} + (\widehat{\eta}\phi^-)_{j+\frac{1}{2}} - (\widehat{\eta}\phi^+)_{j-\frac{1}{2}},\ \mathcal{D}(\eta, \phi; \widehat{\eta}) = \sum_j \mathcal{D}_{K_j}(\eta, \phi; \widehat{\eta}).$$

Using the definition of the jump and the average of the functions and periodic boundary conditions, we can easily prove the following lemmas.

LEMMA 2.1. *Choosing different numerical fluxes, the DG discretization operator satisfies the following equalities:*

$$\mathcal{D}(\eta, \phi; \eta^-) + \mathcal{D}(\phi, \eta; \phi^+) = 0, \quad \mathcal{D}(\eta, \phi; \eta^+) + \mathcal{D}(\phi, \eta; \phi^-) = 0,$$
$$\mathcal{D}(\eta, \phi; \eta^+) + \mathcal{D}(\phi, \eta; \phi^+) = -\sum_j ([\eta][\phi])_{j-\frac{1}{2}},\ \mathcal{D}(\eta, \phi; \eta^-) + \mathcal{D}(\phi, \eta; \phi^-)$$
$$= \sum_j ([\eta][\phi])_{j-\frac{1}{2}},$$
$$\mathcal{D}(\eta, \eta; \eta^-) = \frac{1}{2}\sum_j [\eta]^2_{j-\frac{1}{2}}, \quad \mathcal{D}(\eta, \eta; \eta^+) = -\frac{1}{2}\sum_j [\eta]^2_{j-\frac{1}{2}}.$$

LEMMA 2.2. *For any $\phi \in V_h$, we have*

$$\mathcal{D}(\eta - P^-\eta, \phi; (\eta - P^-\eta)^-) = 0, \quad \mathcal{D}(\eta - P^+\eta, \phi; (\eta - P^+\eta)^+) = 0, \tag{2.10}$$

where P^-, P^+ are the projections defined in section 2.1.2.

We also define two bilinear forms,

$$\mathcal{A}_{K_j}(\rho, \zeta, \phi; \eta, \varphi, \psi) = ((\rho)_t, \eta)_{K_j} + (\zeta, \varphi)_{K_j} + (\phi, \psi)_{K_j} \tag{2.11}$$

and

$$\mathcal{B}_{K_j}(\rho, \zeta, \phi; \eta, \varphi, \psi) = \mathcal{D}_{K_j}(\zeta, \eta; \zeta^-) - \mathcal{D}_{K_j}(\phi, \varphi; \phi^+) - \mathcal{D}_{K_j}(\rho, \psi; \rho^+), \tag{2.12}$$

which will help to prove the energy stability in the next section. We also use the notation

$$\mathcal{A}(\rho, \zeta, \phi; \eta, \varphi, \psi) = \sum_j \mathcal{A}_{K_j}(\rho, \zeta, \phi; \eta, \varphi, \psi),$$
$$\mathcal{B}(\rho, \zeta, \phi; \eta, \varphi, \psi) = \sum_j \mathcal{B}_{K_j}(\rho, \zeta, \phi; \eta, \varphi, \psi).$$

2.2.3. Energy stability of the LDG scheme. The LDG scheme for the third-order linear wave equation satisfies the following energy stability.

LEMMA 2.3 (energy stability). *The solution to the LDG scheme* (2.7) *and* (2.8) *satisfies the following energy stability:*

$$\|u_h(t)\|^2_\Omega + \|w_h(t)\|^2_\Omega + \|(u_h)_t(t)\|^2_\Omega + \frac{1}{2}\|v_h(t)\|^2_\Omega \tag{2.13}$$
$$\leqslant \|u_h(0)\|^2_\Omega + \left(\frac{1}{2}t + 1\right)\|w_h(0)\|^2_\Omega + \left(\frac{1}{2}t + 1\right)\|(u_h)_t(0)\|^2_\Omega + \frac{1}{2}\|v_h(0)\|^2_\Omega.$$

Proof. We will obtain the energy stability inequality by proving the following four energy equations.

• **The first energy equation.** Adding the three equations in (2.7) and using the notation (2.11)–(2.12), we can get

$$\mathcal{A}_{K_j}(u_h, w_h, v_h; \eta, \varphi, \psi) + \mathcal{B}_{K_j}(u_h, w_h, v_h; \eta, \varphi, \psi) = 0. \tag{2.14}$$

Taking the test functions in (2.14) as $\eta = u_h$, $\varphi = v_h$, $\psi = -w_h$, we obtain

$$\begin{aligned} 0 &= \mathcal{A}_{K_j}(u_h, w_h, v_h; u_h, v_h, -w_h) + \mathcal{B}_{K_j}(u_h, w_h, v_h; u_h, v_h, -w_h) \\ &= ((u_h)_t, u_h)_{K_j} + \mathcal{D}_{K_j}(w_h, u_h; w_h^-) + \mathcal{D}_{K_j}(u_h, w_h; u_h^+) - \mathcal{D}_{K_j}(v_h, v_h; v_h^+). \end{aligned}$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_\Omega^2 + \frac{1}{2}\sum_j [v_h]_{j-\frac{1}{2}}^2 = 0. \tag{2.15}$$

This clearly implies

$$\|u_h(t)\|_\Omega^2 \leqslant \|u_h(0)\|_\Omega^2, \tag{2.16}$$

which is the standard "cell entropy inequality," obtained also in [25].

• **The second energy equation.** We take the time derivative in (2.7b)–(2.7c) and sum with (2.7a) to obtain

$$\mathcal{A}_{K_j}(u_h, (w_h)_t, (v_h)_t; \eta, \varphi, \psi) + \mathcal{B}_{K_j}((u_h)_t, w_h, (v_h)_t; \eta, \varphi, \psi) = 0. \tag{2.17}$$

Taking the test functions in (2.17) as $\eta = -(v_h)_t$, $\varphi = w_h$, $\psi = (u_h)_t$, we obtain

$$\begin{aligned} 0 &= \mathcal{A}_{K_j}(u_h, (w_h)_t, (v_h)_t; -(v_h)_t, w_h, (u_h)_t) + \mathcal{B}_{K_j}((u_h)_t, w_h, (v_h)_t; -(v_h)_t, w_h, (u_h)_t) \\ &= ((w_h)_t, w_h)_{K_j} - \mathcal{D}_{K_j}(w_h, (v_h)_t; w_h^-) \\ &\quad - \mathcal{D}_{K_j}((v_h)_t, w_h; (v_h)_t^+) - \mathcal{D}_{K_j}((u_h)_t, (u_h)_t; (u_h)_t^+). \end{aligned}$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\frac{d}{dt}\|w_h\|_\Omega^2 + \frac{1}{2}\sum_j [(u_h)_t]_{j-\frac{1}{2}}^2 = 0. \tag{2.18}$$

This clearly implies

$$\|w_h(t)\|_\Omega^2 \leqslant \|w_h(0)\|_\Omega^2. \tag{2.19}$$

• **The third energy equation.** We take the time derivative in (2.7) and add the three equations together to obtain

$$\mathcal{A}_{K_j}((u_h)_t, (w_h)_t, (v_h)_t; \eta, \varphi, \psi) + \mathcal{B}_{K_j}((u_h)_t, (w_h)_t, (v_h)_t; \eta, \varphi, \psi) = 0. \tag{2.20}$$

Taking the test functions in (2.20) as $\eta = (u_h)_t$, $\varphi = (v_h)_t$, $\psi = -(w_h)_t$, we obtain

$$\begin{aligned} 0 &= \mathcal{A}_{K_j}((u_h)_t, (w_h)_t, (v_h)_t; (u_h)_t, (v_h)_t, -(w_h)_t) \\ &\quad + \mathcal{B}_{K_j}((u_h)_t, (w_h)_t, (v_h)_t; (u_h)_t, (v_h)_t, -(w_h)_t) \\ &= ((u_h)_{tt}, (u_h)_t)_{K_j} + \mathcal{D}_{K_j}((w_h)_t, (u_h)_t; (w_h)_t^-) \\ &\quad + \mathcal{D}_{K_j}((u_h)_t, (w_h)_t; (u_h)_t^+) - \mathcal{D}_{K_j}((v_h)_t, (v_h)_t; (v_h)_t^+). \end{aligned}$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\frac{d}{dt}\|(u_h)_t\|_\Omega^2 + \frac{1}{2}\sum_j [(v_h)_t]_{j-\frac{1}{2}}^2 = 0. \tag{2.21}$$

This clearly implies

$$\|(u_h)_t(t)\|_\Omega^2 \leqslant \|(u_h)_t(0)\|_\Omega^2. \tag{2.22}$$

• **The fourth energy equation.** We take the time derivative in (2.7c) and sum with (2.7a)–(2.7b) to obtain

$$\mathcal{A}_{K_j}(u_h, w_h, (v_h)_t; \eta, \varphi, \psi) + \mathcal{B}_{K_j}((u_h)_t, w_h, v_h; \eta, \varphi, \psi) = 0. \tag{2.23}$$

Taking the test functions in (2.23) as $\eta = 0$, $\varphi = \frac{1}{2}(u_h)_t$, $\psi = \frac{1}{2}v_h$, we obtain

$$\begin{aligned} 0 &= \mathcal{A}_{K_j}\left(u_h, w_h, (v_h)_t; 0, \frac{1}{2}(u_h)_t, \frac{1}{2}v_h\right) + \mathcal{B}_{K_j}\left((u_h)_t, w_h, v_h; 0, \frac{1}{2}(u_h)_t, \frac{1}{2}v_h\right) \\ &= \frac{1}{2}((v_h)_t, v_h)_{K_j} + \frac{1}{2}(w_h, (u_h)_t)_{K_j} - \frac{1}{2}\mathcal{D}_{K_j}(v_h, (u_h)_t; v_h^+) - \frac{1}{2}\mathcal{D}_{K_j}((u_h)_t, v_h; (u_h)_t^+). \end{aligned}$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{4}\frac{d}{dt}\|v_h\|_\Omega^2 + \frac{1}{2}(w_h, (u_h)_t)_\Omega + \frac{1}{2}\sum_j ([v_h][(u_h)_t])_{j-\frac{1}{2}} = 0. \tag{2.24}$$

Now we sum the four energy equations (2.15), (2.18), (2.21), and (2.24) to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_h\|_\Omega^2 + \|w_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2 + \frac{1}{2}\|v_h\|_\Omega^2\right) + \frac{1}{2}\sum_j ([v_h]^2 + [(u_h)_t]^2 + [(v_h)_t]^2)_{j-\frac{1}{2}} \\ &\quad + \frac{1}{2}(w_h, (u_h)_t)_\Omega + \frac{1}{2}\sum_j ([v_h][(u_h)_t])_{j-\frac{1}{2}} = 0. \end{aligned}$$

After using the Cauchy–Schwarz inequality, we have

$$\frac{d}{dt}\left(\|u_h\|_\Omega^2 + \|w_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2 + \frac{1}{2}\|v_h\|_\Omega^2\right) \leqslant \frac{1}{2}(\|w_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2).$$

Integrating with respect to time between 0 and t, we obtain

$$\begin{aligned} &\|u_h(t)\|_\Omega^2 + \|w_h(t)\|_\Omega^2 + \|(u_h)_t(t)\|_\Omega^2 + \frac{1}{2}\|v_h(t)\|_\Omega^2 \\ &\leqslant \frac{1}{2}\int_0^t (\|w_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2)d\tau + \|u_h(0)\|_\Omega^2 + \|w_h(0)\|_\Omega^2 + \|(u_h)_t(0)\|_\Omega^2 + \frac{1}{2}\|v_h(0)\|_\Omega^2 \\ &\leqslant \|u_h(0)\|_\Omega^2 + \left(\frac{1}{2}t + 1\right)\|w_h(0)\|_\Omega^2 + \left(\frac{1}{2}t + 1\right)\|(u_h)_t(0)\|_\Omega^2 + \frac{1}{2}\|v_h(0)\|_\Omega^2, \end{aligned}$$

where in the second inequality we have used (2.19) and (2.22). This finishes the proof of the lemma. ∎

Remark 2.1. We remark that the main difference between Lemma 2.3 and the results in [25] is the additional result on the L^2 stability for $(u_h)_t$ and the auxiliary variables w_h and v_h. The error estimates will also follow this line, and this additional result will give us control of $(u_h)_t$, w_h, and v_h terms in the error estimates. This is the essential point which helps us to obtain the optimal error estimates.

2.3. A priori error estimates. In this section, we obtain the optimal a priori L^2 error estimates for the approximation u_h, w_h, $v_h \in V_h$, which are given by the LDG methods.

In order to obtain the error estimate to smooth solutions for the considered semi-discrete LDG scheme (2.7), we need to first obtain the error equation.

Notice that the scheme (2.7) is also satisfied when the numerical solutions u_h, w_h, v_h are replaced by the exact solutions u, $w = u_{xx}$, $v = u_x$ (this is the consistency of the LDG scheme). We therefore have the error equations

$$((u-u_h)_t,\eta)_{K_j} - (w-w_h,\eta_x)_{K_j} + ((w-\widehat{w}_h)\eta^-)_{j+\frac{1}{2}} - ((w-\widehat{w}_h)\eta^+)_{j-\frac{1}{2}} = 0, \tag{2.25a}$$

$$(w-w_h,\varphi)_{K_j} + (v-v_h,\varphi_x)_{K_j} - ((v-\widehat{v}_h)\varphi^-)_{j+\frac{1}{2}} + ((v-\widehat{v}_h)\varphi^+)_{j-\frac{1}{2}} = 0, \tag{2.25b}$$

$$(v-v_h,\psi)_{K_j} + (u-u_h,\psi_x)_{K_j} - ((u-\widehat{u}_h)\psi^-)_{j+\frac{1}{2}} + ((u-\widehat{u}_h)\psi^+)_{j-\frac{1}{2}} = 0. \tag{2.25c}$$

Denote

$$\begin{aligned} \boldsymbol{e}_u &= u-u_h = u-P^+u+P^+u-u_h = u-P^+u+P\boldsymbol{e}_u,\\ \boldsymbol{e}_w &= w-w_h = w-P^-w+P^-w-w_h = w-P^-w+P\boldsymbol{e}_w,\\ \boldsymbol{e}_v &= v-v_h = v-P^+v+P^+v-v_h = v-P^+v+P\boldsymbol{e}_v. \end{aligned} \tag{2.26}$$

We choose the initial condition $u_h(x,0)$ such that

$$w_h(x,0) = P^-w(x,0), \quad w(x,0) = u_0''(x). \tag{2.27}$$

It is possible to find a well-defined initial condition $u_h(x,0)$ so that (2.27) holds. We can prove the following error estimates:

$$\|u(x,0)-u_h(x,0)\|_\Omega + \|w(x,0)-w_h(x,0)\|_\Omega + \|v(x,0)-v_h(x,0)\|_\Omega \le Ch^{k+1}. \tag{2.28}$$

The details of the existence of $u_h(x,0)$ to satisfy (2.27) and the error estimates (2.28) can be found in [14, Appendix A.3, Proof of Lemma 5.1]. The proof of the main error estimates results will also need the following error estimates for the initial condition.

LEMMA 2.4. *Assume the initial condition of the LDG scheme* (2.7)–(2.8) *is given by* (2.27)*, which satisfies the error estimates* (2.28)*; then there holds the following error estimate:*

$$\|u_t(x,0)-(u_h)_t(x,0)\|_\Omega \le Ch^{k+1}. \tag{2.29}$$

Proof. We take $t=0$ in the error equation (2.25a) and get

$$\begin{aligned} ((u_t(0)-(u_h)_t(0)),\eta)_{K_j} &- (w(0)-w_h(0),\eta_x)_{K_j}\\ &+ ((w(0)-\widehat{w}_h(0))\eta^-)_{j+\frac{1}{2}} - ((w(0)-\widehat{w}_h(0))\eta^+)_{j-\frac{1}{2}} = 0. \end{aligned} \tag{2.30}$$

Due to the choice of $w_h(0)$, we know that

$$((u_t(0)-(u_h)_t(0)),\eta)_{K_j} = 0. \tag{2.31}$$

Now we choose $\eta = P\boldsymbol{e}_{u_t}(0)$ and the results of the lemma can then be easily obtained by using the standard approximation results. ∎

Remark 2.2. The special choice of the initial condition (2.27) is needed just for the technical proof of the optimal error estimates. In numerical experiments, it does not seem to be necessary; other choices of the initial condition such as the standard L^2 projection for $u_h(x,0)$ also seem to work to obtain the optimal convergence rate.

Next, we will follow the idea of proving the energy stability and get the four important energy equations to obtain the optimal error estimates.

2.3.1. The first energy equation. We first replace u_h, w_h, v_h in (2.14) by $u-u_h$, $w-w_h$, $v-v_h$ and take the test functions $\eta = P\boldsymbol{e}_u, \varphi = P\boldsymbol{e}_v, \psi = -P\boldsymbol{e}_w$. After summing over j we get

$$\begin{aligned}(2.32)\quad &\mathcal{A}(u-u_h, w-w_h, v-v_h; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w)\\ &\qquad + \mathcal{B}(u-u_h, w-w_h, v-v_h; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w) = 0.\end{aligned}$$

Using the notation in (2.26), we have

(2.33)
$$\begin{aligned}&\mathcal{A}(u-P^+u, w-P^-w, v-P^+v; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w)\\ &\quad + \mathcal{B}(u-P^+u, w-P^-w, v-P^+v; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w)\\ &\quad + \mathcal{A}(P\boldsymbol{e}_u, P\boldsymbol{e}_w, P\boldsymbol{e}_v; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w) + \mathcal{B}(P\boldsymbol{e}_u, P\boldsymbol{e}_w, P\boldsymbol{e}_v; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w) = 0.\end{aligned}$$

From the proof of the energy stability, we know that

$$\begin{aligned}(2.34)\quad &\mathcal{A}(P\boldsymbol{e}_u, P\boldsymbol{e}_w, P\boldsymbol{e}_v; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w)\\ &\qquad + \mathcal{B}(P\boldsymbol{e}_u, P\boldsymbol{e}_w, P\boldsymbol{e}_v; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w)\\ &\qquad = \frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_u\|_\Omega^2 + \frac{1}{2}\sum_j [P\boldsymbol{e}_v]_{j-\frac{1}{2}}^2.\end{aligned}$$

Using the definition of $\mathcal{A}$ in (2.11), we have

$$\begin{aligned}(2.35)\quad &\mathcal{A}(u-P^+u, w-P^-w, v-P^+v; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w)\\ &\qquad = ((u-P^+u)_t, P\boldsymbol{e}_u)_\Omega + (w-P^-w, P\boldsymbol{e}_v)_\Omega - (v-P^+v, P\boldsymbol{e}_w)_\Omega.\end{aligned}$$

By the definition of $\mathcal{B}$ in (2.12), we also have

$$\begin{aligned}(2.36)\quad &\mathcal{B}(u-P^+u, w-P^-w, v-P^+v; P\boldsymbol{e}_u, P\boldsymbol{e}_v, -P\boldsymbol{e}_w)\\ &\quad = \mathcal{D}(w-P^-w, P\boldsymbol{e}_u; (w-P^-w)^-) + \mathcal{D}(u-P^+u, P\boldsymbol{e}_w; (u-P^+u)^+)\\ &\qquad - \mathcal{D}(v-P^+v, P\boldsymbol{e}_v; (v-P^+v)^+) = 0,\end{aligned}$$

where the last equality is due to the results in Lemma 2.2. Then combining (2.34), (2.35), and (2.36), we have

$$\begin{aligned}(2.37)\quad &\frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_u\|_\Omega^2 + \frac{1}{2}\sum_j [P\boldsymbol{e}_v]_{j-\frac{1}{2}}^2 + ((u-P^+u)_t, P\boldsymbol{e}_u)_\Omega\\ &\qquad + (w-P^-w, P\boldsymbol{e}_v)_\Omega - (v-P^+v, P\boldsymbol{e}_w)_\Omega = 0.\end{aligned}$$

2.3.2. The second energy equation. Following a process similar to that in section 2.3.1, we replace u_h, w_h, v_h in (2.17) by $u-u_h$, $w-w_h$, $v-v_h$ and take the test functions $\eta = -P\boldsymbol{e}_{v_t}, \varphi = P\boldsymbol{e}_w, \psi = P\boldsymbol{e}_{u_t}$. After summing over j we get

$$\begin{aligned}\mathcal{A}(u-u_h,(w-w_h)_t,(v-v_h)_t;-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})\\ +\mathcal{B}((u-u_h)_t,w-w_h,(v-v_h)_t;-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})=0.\end{aligned} \tag{2.38}$$

Using the notation in (2.26), we have

$$\begin{aligned}&\mathcal{A}(u-P^+u,(w-P^-w)_t,(v-P^+v)_t;-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})\\ &+\mathcal{B}((u-P^+u)_t,w-P^-w,(v-P^+v)_t;-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})\\ &+\mathcal{A}(P\boldsymbol{e}_u,P\boldsymbol{e}_{w_t},P\boldsymbol{e}_{v_t};-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})\\ &+\mathcal{B}(P\boldsymbol{e}_{u_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{v_t};-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})=0.\end{aligned} \tag{2.39}$$

From the proof of the energy stability, we know that

$$\begin{aligned}&\mathcal{A}(P\boldsymbol{e}_u,P\boldsymbol{e}_{w_t},P\boldsymbol{e}_{v_t};-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})\\ &\quad+\mathcal{B}(P\boldsymbol{e}_{u_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{v_t};-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})\\ &\qquad=\frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_w\|_\Omega^2+\frac{1}{2}\sum_j[P\boldsymbol{e}_{u_t}]_{j-\frac{1}{2}}^2.\end{aligned} \tag{2.40}$$

Using the definition of $\mathcal{A}$ in (2.11), we have

$$\begin{aligned}&\mathcal{A}(u-P^+u,(w-P^-w)_t,(v-P^+v)_t;-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})\\ &\quad=-((u-P^+u)_t,P\boldsymbol{e}_{v_t})_\Omega+((w-P^-w)_t,P\boldsymbol{e}_w)_\Omega+((v-P^+v)_t,P\boldsymbol{e}_{u_t})_\Omega.\end{aligned} \tag{2.41}$$

By the definition of $\mathcal{B}$ in (2.12), we also have

$$\begin{aligned}&\mathcal{B}((u-P^+u)_t,w-P^-w,(v-P^+v)_t;-P\boldsymbol{e}_{v_t},P\boldsymbol{e}_w,P\boldsymbol{e}_{u_t})\\ &\quad=-\mathcal{D}(w-P^-w,P\boldsymbol{e}_{v_t};(w-P^-w)^-)-\mathcal{D}((v-P^+v)_t,P\boldsymbol{e}_w;(v-P^+v)_t^+)\\ &\qquad-\mathcal{D}((u-P^+u)_t,P\boldsymbol{e}_{u_t};(u-P^+u)_t^+)=0,\end{aligned} \tag{2.42}$$

where the last equality is due to the results in Lemma 2.2. Then combining (2.40), (2.41), and (2.42), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_w\|_\Omega^2+\frac{1}{2}\sum_j[P\boldsymbol{e}_{u_t}]_{j-\frac{1}{2}}^2-((u-P^+u)_t,P\boldsymbol{e}_{v_t})_\Omega\\ &\qquad+((w-P^-w)_t,P\boldsymbol{e}_w)_\Omega+((v-P^+v)_t,P\boldsymbol{e}_{u_t})_\Omega=0.\end{aligned} \tag{2.43}$$

2.3.3. The third energy equation. We first replace u_h, w_h, v_h in (2.20) by $u-u_h$, $w-w_h$, $v-v_h$ and take the test functions $\eta = P\boldsymbol{e}_{u_t}, \varphi = P\boldsymbol{e}_{v_t}, \psi = -P\boldsymbol{e}_{w_t}$. After summing over j we get

$$\begin{aligned}&\mathcal{A}((u-u_h)_t,(w-w_h)_t,(v-v_h)_t;P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})\\ &\quad+\mathcal{B}((u-u_h)_t,(w-w_h)_t,(v-v_h)_t;P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})=0.\end{aligned} \tag{2.44}$$

Using the notation in (2.26), we have

$$\begin{aligned}(2.45)\qquad &\mathcal{A}((u-P^+u)_t,(w-P^-w)_t,(v-P^+v)_t;P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})\\ &+\mathcal{B}((u-P^+u)_t,(w-P^-w)_t,(v-P^+v)_t;P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})\\ &+\mathcal{A}(P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{w_t},P\boldsymbol{e}_{v_t};P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})\\ &+\mathcal{B}(P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{w_t},P\boldsymbol{e}_{v_t};P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})=0.\end{aligned}$$

From the proof of the energy stability, we know that

$$\begin{aligned}(2.46)\quad &\mathcal{A}(P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{w_t},P\boldsymbol{e}_{v_t};P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})\\ &+\mathcal{B}(P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{w_t},P\boldsymbol{e}_{v_t};P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})\\ &=\frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_{u_t}\|_\Omega^2+\frac{1}{2}\sum_j[P\boldsymbol{e}_{v_t}]^2_{j-\frac{1}{2}}.\end{aligned}$$

Using the definition of $\mathcal{A}$ in (2.11), we have

$$\begin{aligned}(2.47)\quad &\mathcal{A}((u-P^+u)_t,(w-P^-w)_t,(v-P^+v)_t;P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})\\ &=((u-P^+u)_{tt},P\boldsymbol{e}_{u_t})_\Omega+((w-P^-w)_t,P\boldsymbol{e}_{v_t})_\Omega-((v-P^+v)_t,P\boldsymbol{e}_{w_t})_\Omega.\end{aligned}$$

By the definition of $\mathcal{B}$ in (2.12), we also have

$$\begin{aligned}(2.48)\quad &\mathcal{B}((u-P^+u)_t,(w-P^-w)_t,(v-P^+v)_t;P\boldsymbol{e}_{u_t},P\boldsymbol{e}_{v_t},-P\boldsymbol{e}_{w_t})\\ &=\mathcal{D}((w-P^-w)_t,P\boldsymbol{e}_{u_t};(w-P^-w)_t^-)+\mathcal{D}((u-P^+u)_t,P\boldsymbol{e}_{w_t};(u-P^+u)_t^+)\\ &\quad-\mathcal{D}((v-P^+v)_t,P\boldsymbol{e}_{v_t};(v-P^+v)_t^+)=0,\end{aligned}$$

where the last equality is due to the results in Lemma 2.2. Then combining (2.46), (2.47), and (2.48), we have

$$\begin{aligned}(2.49)\qquad &\frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_{u_t}\|_\Omega^2+\frac{1}{2}\sum_j[P\boldsymbol{e}_{v_t}]^2_{j-\frac{1}{2}}+((u-P^+u)_{tt},P\boldsymbol{e}_{u_t})_\Omega\\ &+((w-P^-w)_t,P\boldsymbol{e}_{v_t})_\Omega-((v-P^+v)_t,P\boldsymbol{e}_{w_t})_\Omega=0.\end{aligned}$$

2.3.4. The fourth energy equation. Following a process similar to that in section 2.3.1, we replace u_h, w_h, v_h in (2.23) by $u-u_h$, $w-w_h$, $v-v_h$ and take the test functions $\eta=0,\varphi=\frac{1}{2}P\boldsymbol{e}_{u_t},\psi=\frac{1}{2}P\boldsymbol{e}_v$. After summing over j we get

$$\begin{aligned}(2.50)\quad &\mathcal{A}\left(u-u_h,w-w_h,(v-v_h)_t;0,\frac{1}{2}P\boldsymbol{e}_{u_t},\frac{1}{2}P\boldsymbol{e}_v\right)\\ &+\mathcal{B}\left((u-u_h)_t,w-w_h,v-v_h;0,\frac{1}{2}P\boldsymbol{e}_{u_t},\frac{1}{2}P\boldsymbol{e}_v\right)=0.\end{aligned}$$

Using the notation in (2.26), we have

$$
\begin{aligned}
(2.51)\qquad &\mathcal{A}\left(u - P^+u, w - P^-w, (v - P^+v)_t; 0, \frac{1}{2}P\boldsymbol{e}_{u_t}, \frac{1}{2}P\boldsymbol{e}_v\right)\\
&+ \mathcal{B}\left((u - P^+u)_t, w - P^-w, v - P^+v; 0, \frac{1}{2}P\boldsymbol{e}_{u_t}, \frac{1}{2}P\boldsymbol{e}_v\right)\\
&+ \mathcal{A}\left(P\boldsymbol{e}_u, P\boldsymbol{e}_w, P\boldsymbol{e}_{v_t}; 0, \frac{1}{2}P\boldsymbol{e}_{u_t}, \frac{1}{2}P\boldsymbol{e}_v\right)\\
&+ \mathcal{B}\left(P\boldsymbol{e}_{u_t}, P\boldsymbol{e}_w, P\boldsymbol{e}_v; 0, \frac{1}{2}P\boldsymbol{e}_{u_t}, \frac{1}{2}P\boldsymbol{e}_v\right) = 0.
\end{aligned}
$$

From the proof of the energy stability, we know that

$$
\begin{aligned}
(2.52)\qquad &\mathcal{A}\left(P\boldsymbol{e}_u, P\boldsymbol{e}_w, P\boldsymbol{e}_{v_t}; 0, \frac{1}{2}P\boldsymbol{e}_{u_t}, \frac{1}{2}P\boldsymbol{e}_v\right) + \mathcal{B}\left(P\boldsymbol{e}_{u_t}, P\boldsymbol{e}_w, P\boldsymbol{e}_v; 0, \frac{1}{2}P\boldsymbol{e}_{u_t}, \frac{1}{2}P\boldsymbol{e}_v\right)\\
&= \frac{1}{4}\frac{d}{dt}\|P\boldsymbol{e}_v\|_\Omega^2 + \frac{1}{2}(P\boldsymbol{e}_w, P\boldsymbol{e}_{u_t})_\Omega + \frac{1}{2}\sum_j([P\boldsymbol{e}_v][P\boldsymbol{e}_{u_t}])_{j-\frac{1}{2}}.
\end{aligned}
$$

Using the definition of $\mathcal{A}$ in (2.11), we have

$$
\begin{aligned}
(2.53)\qquad &\mathcal{A}\left(u - P^+u, w - P^-w, (v - P^+v)_t; 0, \frac{1}{2}P\boldsymbol{e}_{u_t}, \frac{1}{2}P\boldsymbol{e}_v\right)\\
&= \frac{1}{2}(w - P^-w, P\boldsymbol{e}_{u_t})_\Omega + \frac{1}{2}((v - P^+v)_t, P\boldsymbol{e}_v)_\Omega.
\end{aligned}
$$

By the definition of $\mathcal{B}$ in (2.12), we also have

$$
\begin{aligned}
(2.54)\qquad &\mathcal{B}\left((u - P^+u)_t, w - P^-w, v - P^+v; 0, \frac{1}{2}P\boldsymbol{e}_{u_t}, \frac{1}{2}P\boldsymbol{e}_v\right)\\
&= -\frac{1}{2}\mathcal{D}((u - P^+u)_t, P\boldsymbol{e}_v; (u - P^+u)_t^+) - \frac{1}{2}\mathcal{D}(v - P^+v, P\boldsymbol{e}_{u_t}; (v - P^+v)^+) = 0,
\end{aligned}
$$

where the last equality is due to the results in Lemma 2.2. Then combining (2.52), (2.53), and (2.54), we have

$$
\begin{aligned}
(2.55)\qquad &\frac{1}{4}\frac{d}{dt}\|P\boldsymbol{e}_v\|_\Omega^2 + \frac{1}{2}(P\boldsymbol{e}_w, P\boldsymbol{e}_{u_t})_\Omega + \frac{1}{2}\sum_j([P\boldsymbol{e}_v][P\boldsymbol{e}_{u_t}])_{j-\frac{1}{2}}\\
&+ \frac{1}{2}(w - P^-w, P\boldsymbol{e}_{u_t})_\Omega + \frac{1}{2}((v - P^+v)_t, P\boldsymbol{e}_v)_\Omega = 0.
\end{aligned}
$$

2.3.5. Proof of the error estimates. Now we are ready to combine the four energy equations (2.37), (2.43), (2.49), and (2.55) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|P\boldsymbol{e}_u\|_\Omega^2 + \|P\boldsymbol{e}_w\|_\Omega^2 + \|P\boldsymbol{e}_{u_t}\|_\Omega^2 + \frac{1}{2}\|P\boldsymbol{e}_v\|_\Omega^2\right)\\
&+ \frac{1}{2}\sum_j([P\boldsymbol{e}_{v_t}]^2 + [P\boldsymbol{e}_v]^2 + [P\boldsymbol{e}_{u_t}]^2)_{j-\frac{1}{2}}\\
(2.56)\qquad &+ \frac{1}{2}\sum_j([P\boldsymbol{e}_v][P\boldsymbol{e}_{u_t}])_{j-\frac{1}{2}} + \Upsilon + \Theta = 0,
\end{aligned}
$$

where

$$\begin{aligned}\Theta =&((u-P^+u)_t, P\boldsymbol{e}_u)_\Omega + (w-P^-w, P\boldsymbol{e}_v)_\Omega - (v-P^+v, P\boldsymbol{e}_w)_\Omega\\ &+((w-P^-w)_t, P\boldsymbol{e}_w)_\Omega + ((v-P^+v)_t, P\boldsymbol{e}_{u_t})_\Omega + ((u-P^+u)_{tt}, P\boldsymbol{e}_{u_t})_\Omega\\ &+\frac{1}{2}(w-P^-w, P\boldsymbol{e}_{u_t})_\Omega + \frac{1}{2}((v-P^+v)_t, P\boldsymbol{e}_v)_\Omega\end{aligned}$$

and

$$\Upsilon = ((w-P^-w)_t, P\boldsymbol{e}_{v_t})_\Omega - ((u-P^+u)_t, P\boldsymbol{e}_{v_t})_\Omega - ((v-P^+v)_t, P\boldsymbol{e}_{w_t})_\Omega. \tag{2.57}$$

Using the projection property, we can easily get

$$|\Theta| \leqslant Ch^{2k+2} + \frac{1}{8}\left(\|P\boldsymbol{e}_u\|_\Omega^2 + \|P\boldsymbol{e}_w\|_\Omega^2 + \|P\boldsymbol{e}_{u_t}\|_\Omega^2 + \frac{1}{2}\|P\boldsymbol{e}_v\|_\Omega^2\right). \tag{2.58}$$

Integrating Υ with respect to time between 0 and t, we can get the following equation after integration by parts:

$$\begin{aligned}\int_0^t \Upsilon dt =&((w-P^-w)_t, P\boldsymbol{e}_v)_\Omega|_0^t - \int_0^t ((w-P^-w)_{tt}, P\boldsymbol{e}_v)_\Omega dt - ((u-P^+u)_t, P\boldsymbol{e}_v)_\Omega|_0^t\\ &+\int_0^t ((u-P^+u)_{tt}, P\boldsymbol{e}_v)_\Omega dt - ((v-P^+v)_t, P\boldsymbol{e}_w)_\Omega|_0^t\\ &+\int_0^t ((v-P^+v)_{tt}, P\boldsymbol{e}_w)_\Omega dt.\end{aligned}$$

We can easily get the following estimates after using the approximation property of the projections and the estimates for the initial condition

$$\left|\int_0^t \Upsilon dt\right| \leqslant Ch^{2k+2} + \frac{1}{4}\left(\|P\boldsymbol{e}_w\|_\Omega^2 + \frac{1}{2}\|P\boldsymbol{e}_v\|_\Omega^2\right) + \frac{1}{8}\int_0^t \left(\|P\boldsymbol{e}_w\|_\Omega^2 + \frac{1}{2}\|P\boldsymbol{e}_v\|_\Omega^2\right) dt. \tag{2.59}$$

Now we integrate (2.56) with respect to time between 0 and t, combine with (2.58) and (2.59), and use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}&\frac{1}{2}\left(\|P\boldsymbol{e}_u\|_\Omega^2 + \|P\boldsymbol{e}_w\|_\Omega^2 + \|P\boldsymbol{e}_{u_t}\|_\Omega^2 + \frac{1}{2}\|P\boldsymbol{e}_v\|_\Omega^2\right)\\ &\quad+\frac{1}{4}\int_0^t \sum_j (2[P\boldsymbol{e}_{v_t}]^2 + [P\boldsymbol{e}_v]^2 + [P\boldsymbol{e}_{u_t}]^2)_{j-\frac{1}{2}} dt\\ &\leqslant Ch^{2k+2} + \frac{1}{8}\int_0^t \left(\|P\boldsymbol{e}_u\|_\Omega^2 + 2\|P\boldsymbol{e}_w\|_\Omega^2 + \|P\boldsymbol{e}_{u_t}\|_\Omega^2 + \|P\boldsymbol{e}_v\|_\Omega^2\right) dt\\ &\quad+\frac{1}{4}\left(\|P\boldsymbol{e}_w\|_\Omega^2 + \frac{1}{2}\|P\boldsymbol{e}_v\|_\Omega^2\right).\end{aligned}$$

Finally we obtain

$$\begin{aligned}&\|P\boldsymbol{e}_u\|_\Omega^2 + \frac{1}{2}\|P\boldsymbol{e}_w\|_\Omega^2 + \|P\boldsymbol{e}_{u_t}\|_\Omega^2 + \frac{1}{4}\|P\boldsymbol{e}_v\|_\Omega^2\\ &\leqslant Ch^{2k+2} + \frac{1}{4}\int_0^t \left(\|P\boldsymbol{e}_u\|_\Omega^2 + 2\|P\boldsymbol{e}_w\|_\Omega^2 + \|P\boldsymbol{e}_{u_t}\|_\Omega^2 + \|P\boldsymbol{e}_v\|_\Omega^2\right) dt.\end{aligned} \tag{2.60}$$

After employing Gronwall's inequality, we get

$$\max_t \|P\boldsymbol{e}_u\|_\Omega^2 + \max_t \|P\boldsymbol{e}_w\|_\Omega^2 + \max_t \|P\boldsymbol{e}_{u_t}\|_\Omega^2 + \max_t \|P\boldsymbol{e}_v\|_\Omega^2 \leqslant Ch^{2k+2}. \tag{2.61}$$

After using the standard approximation result, we can get the following error estimates.

THEOREM 2.5. *Assume that* (2.1) *with periodic boundary conditions has a smooth exact solution u. Let u_h be the numerical solution of the semidiscrete LDG scheme* (2.7) *and* (2.8)*. For any triangulation of Ω, if the finite element space is the piecewise polynomials of degree $k \geq 0$, then there holds the following error estimate:*

$$\max_t \|\boldsymbol{e}_u\|_\Omega^2 + \max_t \|\boldsymbol{e}_w\|_\Omega^2 + \max_t \|\boldsymbol{e}_{u_t}\|_\Omega^2 + \max_t \|\boldsymbol{e}_v\|_\Omega^2 \leqslant Ch^{2k+2}, \tag{2.62}$$

where C depends on the final time T, $\|u\|_{L^\infty((0,T);H^{k+3}(\Omega))}$, $\|u_t\|_{L^\infty((0,T);H^{k+1}(\Omega))}$ and $\|u_{tt}\|_{L^\infty((0,T);H^{k+3}(\Omega))}$.

3. One-dimensional fifth order wave equation.

In this section, we will discuss the error estimates for the one-dimensional fifth order wave equation

$$u_t + u_{xxxxx} = 0. \tag{3.1}$$

The idea to prove the error estimates is similar to the proof for the third order wave equation. The key point is to get the energy stability for all the auxiliary variables, and this will help to control these auxiliary variables in the error equation. The special projections which are defined in section 2.1.2 are also important to get rid of the jump terms at the cell boundary. The technique to prove the energy stability is slightly different from the proof in section 2.2.3; therefore we will give details of the LDG schemes and the proof of the energy stability for (3.1). The proof technique for the error estimates follows the same lines as those in section 2.3. We will therefore omit the details and only state the result to save space. We will also use the same notation as that in section 2.

3.1. The LDG scheme for the one-dimensional fifth order wave equation.

In order to construct the LDG method, first we rewrite (3.1) as a system containing only first order derivatives:

$$u_t + s_x = 0,\ s - r_x = 0,\ r - p_x = 0,\ p - q_x = 0,\ q - u_x = 0. \tag{3.2}$$

The LDG scheme to solve (3.2) is as follows. Find u_h, s_h, r_h, p_h, $q_h \in V_h$ such that for all test functions η, φ, ψ, ϕ, $\zeta \in V_h$

$$((u_h)_t, \eta)_{K_j} - (s_h, \eta_x)_{K_j} + (\widehat{s}_h \eta^-)_{j+\frac{1}{2}} - (\widehat{s}_h \eta^+)_{j-\frac{1}{2}} = 0, \tag{3.3a}$$

$$(s_h, \varphi)_{K_j} + (r_h, \varphi_x)_{K_j} - (\widehat{r}_h \varphi^-)_{j+\frac{1}{2}} + (\widehat{r}_h \varphi^+)_{j-\frac{1}{2}} = 0, \tag{3.3b}$$

$$(r_h, \psi)_{K_j} + (p_h, \psi_x)_{K_j} - (\widehat{p}_h \psi^-)_{j+\frac{1}{2}} + (\widehat{p}_h \psi^+)_{j-\frac{1}{2}} = 0, \tag{3.3c}$$

$$(p_h, \phi)_{K_j} + (q_h, \phi_x)_{K_j} - (\widehat{q}_h \phi^-)_{j+\frac{1}{2}} + (\widehat{q}_h \phi^+)_{j-\frac{1}{2}} = 0, \tag{3.3d}$$

$$(q_h, \zeta)_{K_j} + (u_h, \zeta_x)_{K_j} - (\widehat{u}_h \zeta^-)_{j+\frac{1}{2}} + (\widehat{u}_h \zeta^+)_{j-\frac{1}{2}} = 0. \tag{3.3e}$$

The numerical fluxes can be chosen with upwinding for $\widehat{p}_h$ and alternating for the pair $\widehat{s}_h$ and $\widehat{u}_h$ and for the pair $\widehat{r}_h$ and $\widehat{q}_h$, such as

$$\widehat{s}_h = s_h^-, \quad \widehat{r}_h = r_h^+, \quad \widehat{p}_h = p_h^-, \quad \widehat{q}_h = q_h^-, \quad \widehat{u}_h = u_h^+. \tag{3.4}$$

The scheme here is a special case of the LDG methods in [21] when applied to the simple linear PDE (3.1).

3.2. Energy stability of the LDG scheme. The LDG scheme for the fifth order linear equation satisfies the following energy stability.

LEMMA 3.1 (energy stability). *The solution to the LDG scheme* (3.3) *and* (3.4) *satisfies the energy stability*

$$\|u_h(t)\|_\Omega^2 + \|s_h(t)\|_\Omega^2 + \|(u_h)_t(t)\|_\Omega^2 + \|q_h(t)\|_\Omega^2 + \int_0^t \left(\frac{1}{4}\|p_h\|_\Omega^2 + \frac{3}{2}\|r_h\|_\Omega^2\right) dt \\ \leqslant C(\|u_h(0)\|_\Omega^2 + \|s_h(0)\|_\Omega^2 + \|(u_h)_t(0)\|_\Omega^2 + \|q_h(0)\|_\Omega^2). \tag{3.5}$$

Proof. We will get the energy stability by proving the following energy equations.

• **Energy equation I.** Taking the test functions in (3.3) as $\eta = u_h$, $\varphi = q_h$, $\psi = -p_h$, $\phi = r_h$, $\zeta = -s_h$, we obtain

$$((u_h)_t, u_h)_{K_j} + \mathcal{D}_{K_j}(s_h, u_h; s_h^-) + \mathcal{D}_{K_j}(u_h, s_h; u_h^+) \\ + \mathcal{D}_{K_j}(p_h, p_h; p_h^-) - \mathcal{D}_{K_j}(q_h, r_h; q_h^-) - \mathcal{D}_{K_j}(r_h, q_h; r_h^+) = 0.$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_\Omega^2 + \frac{1}{2}\sum_j [p_h]_{j-\frac{1}{2}}^2 = 0. \tag{3.6}$$

This clearly implies

$$\|u_h(t)\|_\Omega^2 \leqslant \|u_h(0)\|_\Omega^2. \tag{3.7}$$

• **Energy equation II.** We take the time derivative in (3.3b)–(3.3e) and then choose the test functions $\eta = -(r_h)_t$, $\varphi = s_h$, $\psi = (u_h)$, $\phi = -(q_h)_t$, $\zeta = (p_h)_t$ to obtain

$$((s_h)_t, s_h)_{K_j} - \mathcal{D}_{K_j}(s_h, (r_h)_t; s_h^-) - \mathcal{D}_{K_j}((r_h)_t, s_h; (r_h)_t^+) + \mathcal{D}_{K_j}((q_h)_t, (q_h)_t; (q_h)_t^-) \\ - \mathcal{D}_{K_j}((p_h)_t, (u_h)_t; (p_h)_t^-) - \mathcal{D}_{K_j}((u_h)_t, (p_h)_t; (u_h)_t^+) = 0.$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\frac{d}{dt}\|s_h\|_\Omega^2 + \frac{1}{2}\sum_j [(q_h)_t]_{j-\frac{1}{2}}^2 = 0. \tag{3.8}$$

This clearly implies

$$\|s_h(t)\|_\Omega^2 \leqslant \|s_h(0)\|_\Omega^2. \tag{3.9}$$

• **Energy equation III.** We take the time derivative in (3.3) and then choose the test functions $\eta = (u_h)_t$, $\varphi = (q_h)_t$, $\psi = -(p_h)_t$, $\phi = (r_h)_t$, $\zeta = -(s_h)_t$ to obtain

$$((u_h)_{tt}, (u_h)_t)_{K_j} + \mathcal{D}_{K_j}((s_h)_t, (u_h)_t; (s_h)_t^-) + \mathcal{D}_{K_j}((u_h)_t, (s_h)_t; (u_h)_t^+) \\ + \mathcal{D}_{K_j}((p_h)_t, (p_h)_t; (p_h)_t^-) - \mathcal{D}_{K_j}((q_h)_t, (r_h)_t; (q_h)_t^-) - \mathcal{D}_{K_j}((r_h)_t, (q_h)_t; (r_h)_t^+) = 0.$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\frac{d}{dt}\|(u_h)_t\|_\Omega^2 + \frac{1}{2}\sum_j [(p_h)_t]_{j-\frac{1}{2}}^2 = 0. \tag{3.10}$$

This clearly implies

$$\|(u_h)_t(t)\|_\Omega^2 \leqslant \|(u_h)_t(0)\|_\Omega^2. \tag{3.11}$$

• **Energy equation IV.** We take the time derivative in (3.3e) and choose the test function $\zeta = q_h$. We also choose the test function $\phi = (u_h)_t$ in (3.3d) and sum the two equalities together to get

$$((q_h)_t, q_h)_{K_j} + (p_h, (u_h)_t)_{K_j} - \mathcal{D}_{K_j}(q_h, (u_h)_t; q_h^-) - \mathcal{D}_{K_j}((u_h)_t, q_h; (u_h)_t^+) = 0.$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\frac{d}{dt}\|q_h\|_\Omega^2 + (p_h, (u_h)_t)_\Omega = 0. \tag{3.12}$$

This clearly implies

$$\frac{1}{2}\frac{d}{dt}\|q_h\|_\Omega^2 \leqslant \frac{1}{8}\|p_h\|_\Omega^2 + C\|(u_h)_t\|_\Omega^2. \tag{3.13}$$

• **Energy equation V.** Taking the test functions in (3.3c)–(3.3d) as $\psi = \frac{1}{2}q_h$, $\phi = \frac{1}{2}p_h$ and summing them, we obtain

$$\frac{1}{2}\|p_h\|_{K_j}^2 + \frac{1}{2}(r_h, q_h)_{K_j} - \frac{1}{2}\mathcal{D}_{K_j}(q_h, p_h; q_h^-) - \frac{1}{2}\mathcal{D}_{K_j}(p_h, q_h; p_h^-) = 0.$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\|p_h\|_\Omega^2 + \frac{1}{2}(r_h, q_h)_\Omega - \frac{1}{2}\sum_j([p_h][q_h])_{j-\frac{1}{2}} = 0. \tag{3.14}$$

This clearly implies

$$\frac{1}{2}\|p_h\|_\Omega^2 \leqslant \frac{1}{4}\|r_h\|_\Omega^2 + \frac{1}{4}\|q_h\|_\Omega^2 + \frac{1}{4}\sum_j([p_h]^2 + [q_h]^2)_{j-\frac{1}{2}}. \tag{3.15}$$

• **Energy equation VI.** Taking the test functions in (3.3b)–(3.3c) as $\varphi = p_h$, $\psi = r_h$ and summing them, we obtain

$$\|r_h\|_{K_j}^2 + (s_h, p_h)_{K_j} - \frac{1}{2}\mathcal{D}_{K_j}(r_h, p_h; r_h^+) - \frac{1}{2}\mathcal{D}_{K_j}(p_h, r_h; p_h^-) = 0.$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\|r_h\|_\Omega^2 + (s_h, p_h)_\Omega = 0. \tag{3.16}$$

This clearly implies

$$\|r_h\|_\Omega^2 \leqslant \frac{1}{8}\|p_h\|_\Omega^2 + C\|s_h\|_\Omega^2. \tag{3.17}$$

Energy equation VII. Taking the test functions in (3.3d) as $\phi = -q_h$, we obtain

$$-(p_h, q_h)_{K_j} + \mathcal{D}_{K_j}(q_h, q_h; q_h^-) = 0.$$

Summing the above equation over j and using the property of the operator $\mathcal{D}$ in Lemma 2.1, we can get

$$\frac{1}{2}\sum_j [q_h]^2_{j-\frac{1}{2}} = (p_h, q_h)_\Omega. \tag{3.18}$$

This clearly implies

$$\frac{1}{2}\sum_j [q_h]^2_{j-\frac{1}{2}} \leqslant \frac{1}{8}\|p_h\|^2_\Omega + C\|q_h\|^2_\Omega. \tag{3.19}$$

Now we are ready to obtain the energy stability. We sum the seven energy equations (3.6), (3.8), (3.10), (3.12), (3.14), (3.16), and (3.18) to obtain

(3.20)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_h\|^2_\Omega + \|s_h\|^2_\Omega + \|(u_h)_t\|^2_\Omega + \|q_h\|^2_\Omega\right) + \frac{1}{2}\|p_h\|^2_\Omega + \|r_h\|^2_\Omega \\
&\quad + \frac{1}{2}\sum_j ([(p_h)_t]^2 + [p_h]^2 + [(q_h)_t]^2 + [q_h]^2)_{j-\frac{1}{2}} \\
&\leqslant \frac{3}{8}\|p_h\|^2_\Omega + \frac{1}{4}\|r_h\|^2_\Omega + C(\|s_h\|^2_\Omega + \|(u_h)_t\|^2_\Omega + \|q_h\|^2_\Omega) + \frac{1}{4}\sum_j ([p_h]^2 + [q_h]^2)_{j-\frac{1}{2}}.
\end{aligned}$$

After using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_h\|^2_\Omega + \|s_h\|^2_\Omega + \|(u_h)_t\|^2_\Omega + \|q_h\|^2_\Omega\right) + \frac{1}{8}\|p_h\|^2_\Omega + \frac{3}{4}\|r_h\|^2_\Omega \\
&\quad \leqslant C(\|s_h\|^2_\Omega + \|(u_h)_t\|^2_\Omega + \|q_h\|^2_\Omega).
\end{aligned}$$

Integrating with respect to time between 0 and t and using (3.7), (3.9), (3.11), and (3.13) and the Gronwall inequality, we obtain

$$\begin{aligned}
&\|u_h(t)\|^2_\Omega + \|s_h(t)\|^2_\Omega + \|(u_h)_t(t)\|^2_\Omega + \|q_h(t)\|^2_\Omega + \int_0^t \left(\frac{1}{4}\|p_h\|^2_\Omega + \frac{3}{2}\|r_h\|^2_\Omega\right) dt \\
&\quad \leqslant C(\|u_h(0)\|^2_\Omega + \|s_h(0)\|^2_\Omega + \|(u_h)_t(0)\|^2_\Omega + \|q_h(0)\|^2_\Omega),
\end{aligned}$$

where the constant C depends on t. This finishes the proof of the lemma. ∎

3.3. The main result for the error estimates. In this section, we will only state the main error estimate result for the one-dimensional fifth order wave equation. The proof technique for this error estimate follows the same lines as that in section 2.3. We will therefore omit the details.

THEOREM 3.2. *Assume that* (3.1) *with periodic boundary conditions has a smooth exact solution u. Let u_h be the numerical solution of the semidiscrete LDG scheme* (3.3) *and* (3.4). *For any triangulation of Ω, if the finite element space is the piecewise polynomial of degree $k \geq 0$, then there holds the following error estimate:*

$$\begin{aligned}
&\max_t \|\boldsymbol{e}_u\|^2_\Omega + \max_t \|\boldsymbol{e}_s\|^2_\Omega + \max_t \|\boldsymbol{e}_{u_t}\|^2_\Omega + \max_t \|\boldsymbol{e}_q\|^2_\Omega \\
&\quad + \int_0^T \left(\frac{1}{4}\|\boldsymbol{e}_p\|^2_\Omega + \frac{3}{2}\|\boldsymbol{e}_r\|^2_\Omega\right) dt \leqslant Ch^{2k+2},
\end{aligned} \tag{3.21}$$

where C depends on the final time T, $\|u\|_{L^\infty((0,T);H^{k+5}(\Omega))}$, $\|u_t\|_{L^\infty((0,T);H^{k+1}(\Omega))}$, and $\|u_{tt}\|_{L^\infty((0,T);H^{k+5}(\Omega))}$.

Remark 3.1. We remark that this general framework for energy stability and optimal L^2 error estimates can be applied for general odd-order linear one-dimensional wave PDEs. We have carefully checked that the proof can go through for such odd-order wave PDEs up to ninth order; however, we will not present the details here to save space.

4. Multidimensional linear Schrödinger equation. To demonstrate that our general approach also works for certain multidimensional problems, in this section we will give the optimal L^2 error analysis for the linear Schrödinger equation

$$\mathrm{i}u_t + \Delta u = 0, \quad \boldsymbol{x} \in \Omega, \tag{4.1}$$

where u is a complex function and $\mathrm{i}^2 = -1$. The initial condition is a smooth function

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}). \tag{4.2}$$

For simplicity we always consider $\Omega = \Pi_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$ to be a bounded rectangular domain with dimension $d \leq 3$.

Remark 4.1. Although the Schrödinger equation is a second order equation, the coefficient i makes it a typical wave equation. There is no coercivity, which is the main difference from the parabolic equation. The essential key idea is to get control for auxiliary variable approximating the first derivatives, which can be easily obtained for elliptic or parabolic equations but is significantly more difficult for wave equations. The treatment for the interface terms uses the idea from [6].

4.1. Notations, definitions, and projections in the general case. We will introduce some notations, definitions, and projections to be used later which are slightly different from the real valued one-dimensional case in section 2.1.

4.1.1. Tessellation and function spaces. Let $\mathcal{T}_h$ denote a tessellation of Ω with shape-regular rectangular elements K. The boundary of Ω is denoted as $\Gamma = \partial\Omega$.

Let $\mathcal{Q}^k(K)$ be the space of tensor product of polynomials of degree at most $k \geq 0$ on $K \in \mathcal{T}_h$ in each variable. The finite element spaces are denoted by

$$\begin{aligned} {}_cV_h &= \Big\{ \eta \in L^2(\Omega) : \quad \eta|_K \in \mathcal{Q}^k(K) \quad \forall K \in \mathcal{T}_h \Big\}, \\ {}_c\Sigma_h &= \Big\{ \boldsymbol{\phi} = (\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_d)^T \in (L^2(\Omega))^d : \quad \boldsymbol{\phi}_l|_K \in \mathcal{Q}^k(K), \ \ l = 1 \dots d, \quad \forall K \in \mathcal{T}_h \Big\}. \end{aligned}$$

For the one-dimensional case, we have $\mathcal{Q}^k(K) = \mathcal{P}^k(K)$, which is the space of polynomials of degree at most $k \geq 0$ defined on K. Note that functions in ${}_cV_h$ and ${}_c\Sigma_h$ are complex valued functions and are allowed to have discontinuities across element interfaces. We note that the polynomial functions can be real valued or complex valued depending on the solutions of different PDEs.

We denote w^* as the conjugate of w and define the inner product as

$$(w, v)_K = \int_K w v^* \, dK, \quad (w, v)_{\partial K} = \int_{\partial K} w v^* \, ds \tag{4.3}$$

for the scalar variables w, v. The definition we use for the L^2 norm in K and on the boundary ∂K is given by

$$\|\eta\|_K^2 = \int_K |\eta|^2 \, dK, \quad \|\boldsymbol{q}\|_K^2 = \int_K |\boldsymbol{q}|^2 \, dK.$$

4.2. Projection and interpolation properties. The projections in the one-dimensional case are the same as those in section 2.1.2.

To prove the error estimates for two-dimensional problems in Cartesian meshes, we need a suitable projection $P^{\pm}$ similar to the one-dimensional case; see [6]. The projections P^- for scalar functions are defined as

$$P^- = P_x^- \otimes P_y^-, \tag{4.4}$$

where the subscripts x and y indicate the one-dimensional projections defined by (2.4) on a two-dimensional rectangle element $I \otimes J = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$.

The projection Π^+ for vector-valued function $\boldsymbol{\rho} = (\rho_1(x,y), \rho_2(x,y))$ is defined as

$$\Pi^+ \boldsymbol{\rho} = (P_x^+ \otimes \pi_y \rho_1, \pi_x \otimes P_y^+ \rho_2). \tag{4.5}$$

Here π_x, π_y are the standard L^2 projections in the x and y directions, respectively. It is easy to see that for any $\boldsymbol{\rho} \in [H^1(\Omega)]^2$, the restriction of $\Pi^+\boldsymbol{\rho}$ to $I \otimes J$ are elements of $[\mathcal{Q}^k(I \otimes J)]^2$ that satisfy

$$(\Pi^+ \boldsymbol{\rho} - \boldsymbol{\rho}, \nabla w)_{I \otimes J} = 0 \tag{4.6}$$

for any $w \in \mathcal{Q}^k(I \otimes J)$, and

$$\Big((\Pi^+ \boldsymbol{\rho}(x_{i-\frac{1}{2}}, \cdot) - \boldsymbol{\rho}(x_{i-\frac{1}{2}}, \cdot)) \cdot \boldsymbol{\nu}, w(x_{i-\frac{1}{2}}^+, \cdot)\Big)_J = 0 \quad \forall w \in \mathcal{Q}^k(I \otimes J), \tag{4.7}$$

$$\Big((\Pi^+ \boldsymbol{\rho}(\cdot, y_{j-\frac{1}{2}}) - \boldsymbol{\rho}(\cdot, y_{j-\frac{1}{2}})) \cdot \boldsymbol{\nu}, w(\cdot, y_{j-\frac{1}{2}}^+)\Big)_I = 0 \quad \forall w \in \mathcal{Q}^k(I \otimes J), \tag{4.8}$$

where $\boldsymbol{\nu}$ is the normal vector of the domain integrated. For a definition of similar projections for the three-dimensional case, see [6].

Similar to the one-dimensional case, there are some approximation results for the projections (4.4) and (4.5) $\forall \eta \in H^{k+1}(\Omega)$ and $\forall \boldsymbol{\rho} \in [H^{k+1}(\Omega)]^d$,

$$\|\eta^e\|_\Omega \le Ch^{k+1}\|\eta\|_{H^{k+1}(\Omega)}, \quad \|\boldsymbol{\rho}^e\|_\Omega \le Ch^{k+1}\|\boldsymbol{\rho}\|_{H^{k+1}(\Omega)},$$

where $\eta^e = \pi\eta - \eta$, $\boldsymbol{\rho}^e = \pi\boldsymbol{\rho} - \boldsymbol{\rho}$ or $\eta^e = P^{\pm}\eta - \eta$, $\boldsymbol{\rho}^e = \Pi^{\pm}\boldsymbol{\rho} - \boldsymbol{\rho}$ and C is independent of h.

The projection P^- on the Cartesian meshes has the following super-convergence property (see [11, Lemma 3.7]).

LEMMA 4.1. *Suppose $\eta \in H^{k+2}(\Omega)$, $\boldsymbol{\rho} \in \Sigma_h$, and the projection. Then we have*

$$|(\eta - P^-\eta, \nabla \cdot \boldsymbol{\rho})_\Omega - (\eta - \widehat{P^-\eta}, \boldsymbol{\rho} \cdot \boldsymbol{\nu})_\Gamma| \le Ch^{k+1}\|\eta\|_{H^{k+2}(\Omega)}\|\boldsymbol{\rho}\|_\Omega, \tag{4.9}$$

where the "hat" term is the numerical flux.

4.3. The LDG scheme for the Schrödinger equation.

4.3.1. The LDG scheme. In order to construct the LDG method, first we rewrite the Schrödinger equation (4.1) as a system containing only first order derivatives:

$$\mathrm{i}u_t + \nabla \cdot \boldsymbol{q} = 0, \tag{4.10a}$$

$$\boldsymbol{q} - \nabla u = 0. \tag{4.10b}$$

The LDG scheme to solve (4.10) is as follows. Find $u_h \in {}_cV_h$ and $\boldsymbol{q}_h \in {}_c\Sigma_h$ such that for all test functions $\eta \in {}_cV_h$ and $\boldsymbol{\phi} \in {}_c\Sigma_h$

$$\mathrm{i}((u_h)_t, \eta)_K - (\boldsymbol{q}_h, \nabla\eta)_K + (\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{\nu}, \eta)_{\partial K} = 0, \tag{4.11a}$$

$$(\boldsymbol{q}_h, \boldsymbol{\phi})_K + (u_h, \nabla \cdot \boldsymbol{\phi})_K - (\widehat{u}_h, \boldsymbol{\phi} \cdot \boldsymbol{\nu})_{\partial K} = 0. \tag{4.11b}$$

The "hat" terms in (4.11) in the cell boundary terms from integration by parts are the "numerical fluxes." The LDG scheme here is a special case of the LDG methods in [22] when applied to the simple linear PDE (4.1).

To give the definition of the numerical fluxes in the multidimensional case, we follow a convention similar to that in [11]. Let e be a face shared by elements K_1 and K_2, and define the unit normal vectors $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ on e pointing exterior to K_1 and K_2, respectively. The jump and the average of a scalar-valued function η on e are given by

$$[\eta\boldsymbol{\nu}] = \eta_1\boldsymbol{\nu}_1 + \eta_2\boldsymbol{\nu}_2, \quad \bar{\eta} = \frac{1}{2}(\eta_1 + \eta_2),$$

where $\eta_i = \eta|_{\partial K_i}$. For a vector value function $\boldsymbol{\rho}$, we define $\boldsymbol{\rho}_i = \boldsymbol{\rho}|_{\partial K_i}$ and set

$$[\boldsymbol{\rho} \cdot \boldsymbol{\nu}] = \boldsymbol{\rho}_1 \cdot \boldsymbol{\nu}_1 + \boldsymbol{\rho}_2 \cdot \boldsymbol{\nu}_2, \quad \bar{\boldsymbol{\rho}} = \frac{1}{2}(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)$$

on e. Let $\boldsymbol{v}_0$ be an arbitrarily chosen but fixed nonzero vector, and we define $u^\pm = \bar{u} \pm \boldsymbol{\beta} \cdot [u\boldsymbol{\nu}]$, where $\boldsymbol{\beta}$ is a function defined on the faces $\boldsymbol{\beta} \cdot \boldsymbol{\nu}_K = \frac{1}{2}\mathrm{sign}(\boldsymbol{v}_0 \cdot \boldsymbol{\nu}_K)$.

Now we are ready to give the numerical fluxes. It turns out that we can take the simple choices such as

$$\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^+, \quad \widehat{u}_h = u_h^-. \tag{4.12}$$

4.3.2. Notations for the DG discretization. We define the numerical entropy flux as

$$H_{\partial K}(\eta, \boldsymbol{\phi}; \widehat{\eta}, \widehat{\boldsymbol{\phi}}) = (\eta, \widehat{\boldsymbol{\phi}} \cdot \boldsymbol{\nu})_{\partial K} + (\widehat{\eta}, \boldsymbol{\phi} \cdot \boldsymbol{\nu})_{\partial K} - (\eta, \boldsymbol{\phi} \cdot \boldsymbol{\nu})_{\partial K}, \tag{4.13}$$

where the "hat" term is the numerical flux and $\eta \in {}_cV_h$, $\boldsymbol{\phi} \in {}_c\Sigma_h$.

Using the numerical flux defined above we have the following property (see [11, Lemma 2.2]).

LEMMA 4.2. *Suppose e is an interelement face shared by the elements K_1 and K_2; then*

$$H_{\partial K_1 \cap e}(\eta, \boldsymbol{\phi}; \widehat{\eta}, \widehat{\boldsymbol{\phi}}) + H_{\partial K_2 \cap e}(\eta, \boldsymbol{\phi}; \widehat{\eta}, \widehat{\boldsymbol{\phi}}) = 0$$

for any $\eta \in {}_cV_h$ and $\boldsymbol{\phi} \in {}_c\Sigma_h$. Here $\widehat{\eta}_e = \eta_R$ and $\widehat{\boldsymbol{\phi}}_e = \boldsymbol{\phi}_L$. Moreover, the periodic boundary conditions give

$$\sum_{K \in \mathcal{T}_h} H_{\partial K}(\eta, \boldsymbol{\phi}; \widehat{\eta}, \widehat{\boldsymbol{\phi}}) = 0.$$

We define the bilinear form

$$\widetilde{\mathcal{A}}_K(\rho, \boldsymbol{\psi}; \eta, \boldsymbol{\phi}) = \mathrm{i}(\rho_t, \eta)_K + (\boldsymbol{\psi}, \boldsymbol{\phi})_K \tag{4.14}$$

and

$$\widetilde{\mathcal{B}}_K(\rho,\boldsymbol{\psi};\eta,\boldsymbol{\phi}) = (\rho,\nabla\cdot\boldsymbol{\phi})_K - (\widehat{\rho},\boldsymbol{\phi}\cdot\boldsymbol{\nu})_{\partial K} - (\boldsymbol{\psi},\nabla\eta)_K + (\widehat{\boldsymbol{\psi}}\cdot\boldsymbol{\nu},\eta)_{\partial K}, \tag{4.15}$$

which will help to prove the L^2 stability in the next section. We also use the notation

$$\widetilde{\mathcal{A}}(\rho,\boldsymbol{\psi};\eta,\boldsymbol{\phi}) = \sum_K \widetilde{\mathcal{A}}_K(\rho,\boldsymbol{\psi};\eta,\boldsymbol{\phi}), \quad \widetilde{\mathcal{B}}(\rho,\boldsymbol{\psi};\eta,\boldsymbol{\phi}) = \sum_K \widetilde{\mathcal{B}}_K(\rho,\boldsymbol{\psi};\eta,\boldsymbol{\phi}).$$

4.3.3. L^2 stability. It can be proved that the LDG scheme satisfies the L^2 stability.

LEMMA 4.3 (L^2 stability). *The solution to the LDG scheme* (4.11) *and* (4.12) *satisfies the L^2 stability*

$$\frac{d}{dt}\|u_h\|_\Omega^2 = 0, \quad \frac{d}{dt}\|\boldsymbol{q}_h\|_\Omega^2 = 0. \tag{4.16}$$

Proof. We will get the energy stability by proving the following two energy equations.

• **The first energy equation.** Adding the two equations in (4.11), we can get the following equality:

$$\begin{aligned} 0 &= \widetilde{\mathcal{A}}_K(u_h,\boldsymbol{q}_h;\eta,\boldsymbol{\phi}) + \widetilde{\mathcal{B}}_K(u_h,\boldsymbol{q}_h;\eta,\boldsymbol{\phi}) \\ &= \mathrm{i}((u_h)_t,\eta)_K - (\boldsymbol{q}_h,\nabla\eta)_K + (\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{\nu},\eta)_{\partial K} \\ &\quad + (\boldsymbol{q}_h,\boldsymbol{\phi})_K + (u_h,\nabla\cdot\boldsymbol{\phi})_K - (\widehat{u}_h,\boldsymbol{\phi}\cdot\boldsymbol{\nu})_{\partial K}. \end{aligned} \tag{4.17}$$

We choose the test functions $\eta = u_h$, $\boldsymbol{\phi} = \boldsymbol{q}_h$ and get

$$\begin{aligned} 0 &= \widetilde{\mathcal{A}}_K(u_h,\boldsymbol{q}_h;u_h,\boldsymbol{q}_h) + \widetilde{\mathcal{B}}_K(u_h,\boldsymbol{q}_h;u_h,\boldsymbol{q}_h) \\ &\quad - \big(\widetilde{\mathcal{A}}_K(u_h,\boldsymbol{q}_h;u_h,\boldsymbol{q}_h) + \widetilde{\mathcal{B}}_K(u_h,\boldsymbol{q}_h;u_h,\boldsymbol{q}_h)\big)^* \end{aligned}$$

$$= \mathrm{i}\frac{d}{dt}\|u_h\|_K^2 + H_{\partial K}(\boldsymbol{q}_h,u_h,;\widehat{\boldsymbol{q}_h},\widehat{u_h}) - H_{\partial K}(u_h,\boldsymbol{q}_h;\widehat{u_h},\widehat{\boldsymbol{q}_h}). \tag{4.18}$$

Using the results in Lemma 4.2 and the numerical flux (4.12), we can get the first equality in (4.16) after summing over all elements K.

• **The second energy equation.** We first take the time derivative in (4.11b) and then sum with (4.11a) to get the equality

$$\begin{aligned} 0 &= \widetilde{\mathcal{A}}_K(u_h,(\boldsymbol{q}_h)_t;\eta,\boldsymbol{\phi}) + \widetilde{\mathcal{B}}_K((u_h)_t,\boldsymbol{q}_h;\eta,\boldsymbol{\phi}) \\ &= \mathrm{i}((u_h)_t,\eta)_K - (\boldsymbol{q}_h,\nabla\eta)_K + (\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{\nu},\eta)_{\partial K} \\ &\quad + ((\boldsymbol{q}_h)_t,\boldsymbol{\phi})_K + ((u_h)_t,\nabla\cdot\boldsymbol{\phi})_K - (\widehat{(u_h)_t},\boldsymbol{\phi}\cdot\boldsymbol{\nu})_{\partial K}. \end{aligned} \tag{4.19}$$

We choose the test functions $\eta = -(u_h)_t$, $\boldsymbol{q}_h = \boldsymbol{q}_h$ and get

$$\begin{aligned} 0 &= \widetilde{\mathcal{A}}_K(u_h,(\boldsymbol{q}_h)_t;-(u_h)_t,\boldsymbol{q}_h) + \widetilde{\mathcal{B}}_K((u_h)_t,\boldsymbol{q}_h;-(u_h)_t,\boldsymbol{q}_h) \\ &\quad + (\widetilde{\mathcal{A}}_K(u_h,(\boldsymbol{q}_h)_t;-(u_h)_t,\boldsymbol{q}_h) + \widetilde{\mathcal{B}}_K((u_h)_t,\boldsymbol{q}_h;-(u_h)_t,\boldsymbol{q}_h))^* \end{aligned} \tag{4.20}$$

$$= \frac{d}{dt}\|\boldsymbol{q}_h\|_K^2 - H_{\partial K}((u_h)_t,\boldsymbol{q}_h;\widehat{(u_h)}_t,\widehat{\boldsymbol{q}}_h) - H_{\partial K}(\boldsymbol{q}_h,(u_h)_t;\widehat{\boldsymbol{q}_h},\widehat{(u_h)}_t). \tag{4.21}$$

Again using the results in Lemma 4.2 and the numerical flux (4.12), we can get the second equality in (4.16) after summing over all elements K. ∎

Remark 4.2. We remark that the main difference between Lemma 4.3 and the results in [22] is the additional L^2 stability for $\boldsymbol{q}_h$. The error estimates will also follow this line and this result will give us the control of $\boldsymbol{q}_h$ terms in the error estimates. This is the essential point which helps us to obtain optimal error estimates.

4.4. A priori error estimates. In order to obtain the error estimate to smooth solutions for the considered semidiscrete LDG scheme (4.11), we need to first obtain the error equation.

Notice that the scheme (4.11) is also satisfied when the numerical solutions u_h, $\boldsymbol{q}_h$ are replaced by the exact solution u, $\boldsymbol{q} = \nabla u$ (the consistency of the LDG scheme). We then have the error equations

$$\mathrm{i}((u-u_h)_t,\eta)_K - (\boldsymbol{q}-\boldsymbol{q}_h,\nabla\eta)_K + ((\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{\nu},\eta)_{\partial K} = 0, \tag{4.22a}$$

$$(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\phi})_K + (u-u_h,\nabla\cdot\boldsymbol{\phi})_K - (u-\widehat{u}_h,\boldsymbol{\phi}\cdot\boldsymbol{\nu})_{\partial K} = 0. \tag{4.22b}$$

Denote

$$\boldsymbol{e}_u = u - u_h = u - Pu + Pu - u_h = u - Pu + P\boldsymbol{e}_u, \tag{4.23}$$

$$\boldsymbol{e}_{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}_h = \boldsymbol{q} - \Pi\boldsymbol{q} + \Pi\boldsymbol{q} - \boldsymbol{q}_h = \boldsymbol{q} - \Pi\boldsymbol{q} + \Pi\boldsymbol{e}_{\boldsymbol{q}}. \tag{4.24}$$

Let P and Π be the projections onto the finite element spaces ${}_cV_h$ and ${}_c\sum_h$, respectively, which have been defined in section 4.2. In this section we choose the projections as follows:

$$(P,\Pi) = (P^-,P^+) \quad \text{in one dimension,} \quad (P,\Pi) = (P^-,\Pi^+) \quad \text{in multidimensions.} \tag{4.25}$$

We choose the initial condition $u_h(x,0) = P^+u(x,0)$. Taking $\boldsymbol{\phi} = \Pi\boldsymbol{e}_{\boldsymbol{q}}$ in the error equation of (4.11b), we have

$$(q-\boldsymbol{q}_h,\Pi\boldsymbol{e}_{\boldsymbol{q}})_K + (u-u_h,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K - (u-\widehat{u}_h,\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K} = 0.$$

With the help of (2.5) and Lemma 4.1 we obtain the error estimates for the numerical initial conditions

$$\|u(x,0)-u_h(x,0)\|_\Omega \le Ch^{k+1}, \quad \|\boldsymbol{q}(x,0)-\boldsymbol{q}_h(x,0)\|_\Omega \le Ch^{k+1}. \tag{4.26}$$

Next we will follow the idea of proving the L^2 stability and get the two important energy equations to prove the error estimates.

4.4.1. The first energy equation. We first replace u_h, $\boldsymbol{q}_h$ in (4.17) by $u-u_h$, $\boldsymbol{q}-\boldsymbol{q}_h$ and take the test functions $\eta = P\boldsymbol{e}_u$, $\boldsymbol{\phi} = \Pi\boldsymbol{e}_{\boldsymbol{q}}$. After summing over all elements K, we obtain

$$\widetilde{\mathcal{A}}(u-u_h,\boldsymbol{q}-\boldsymbol{q}_h;P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}}) + \widetilde{\mathcal{B}}(u-u_h,\boldsymbol{q}-\boldsymbol{q}_h;P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}}) = 0. \tag{4.27}$$

Using the notations in (4.23), we have

$$\begin{aligned}\widetilde{\mathcal{A}}(u-Pu,\boldsymbol{q}-\Pi\boldsymbol{q};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}}) + \widetilde{\mathcal{B}}(u-Pu,\boldsymbol{q}-\Pi\boldsymbol{q};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}})\\ + \widetilde{\mathcal{A}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}}) + \widetilde{\mathcal{B}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}}) = 0.\end{aligned} \tag{4.28}$$

From the proof of the L^2 stability, we have

$$\begin{aligned}\mathrm{i}\frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_u\|_\Omega^2 &= \widetilde{\mathcal{A}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}}) + \widetilde{\mathcal{B}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}})\\ &\quad - \left(\widetilde{\mathcal{A}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}}) + \widetilde{\mathcal{B}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}})\right)^*.\end{aligned} \tag{4.29}$$

From the definition of $\widetilde{\mathcal{A}}$, we have

$$\begin{aligned}(4.30)\quad &\widetilde{\mathcal{A}}(u-Pu,\boldsymbol{q}-\Pi\boldsymbol{q};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}})-(\widetilde{\mathcal{A}}(u-Pu,\boldsymbol{q}-\Pi\boldsymbol{q};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}}))^*\\ &\quad= \mathrm{i}(((u-Pu)_t,P\boldsymbol{e}_u)_\Omega+((u-Pu)_t,P\boldsymbol{e}_u)^*_\Omega)\\ &\qquad+((\boldsymbol{q}-\Pi\boldsymbol{q},\Pi\boldsymbol{e}_{\boldsymbol{q}})_\Omega-(\boldsymbol{q}-\Pi\boldsymbol{q},\Pi\boldsymbol{e}_{\boldsymbol{q}})^*_\Omega).\end{aligned}$$

From the definition of $\widetilde{\mathcal{B}}$ and the properties of the projection Π^+, we can obtain

$$\begin{aligned}(4.31)\quad &\widetilde{\mathcal{B}}(u-Pu,\boldsymbol{q}-\Pi\boldsymbol{q};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}})-\widetilde{\mathcal{B}}(u-Pu,\boldsymbol{q}-\Pi\boldsymbol{q};P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}})^*\\ &=\sum_K\Big((\widehat{\boldsymbol{q}-\Pi\boldsymbol{q}}\cdot\boldsymbol{\nu},P\boldsymbol{e}_u)_{\partial K}-(\boldsymbol{q}-\Pi\boldsymbol{q},\nabla P\boldsymbol{e}_u)_K\\ &\qquad-((\widehat{\boldsymbol{q}-\Pi\boldsymbol{q}}\cdot\boldsymbol{\nu},P\boldsymbol{e}_u)_{\partial K}-(\boldsymbol{q}-\Pi\boldsymbol{q},\nabla P\boldsymbol{e}_u)_K)^*\Big)\\ &\quad+\sum_K\Big((u-Pu,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u-Pu},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K}\\ &\qquad-((u-Pu,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u-Pu},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K})^*\Big)\\ &=\sum_K\Big((u-Pu,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u-Pu},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K}\\ &\qquad-((u-Pu,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u-Pu},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K})^*\Big).\end{aligned}$$

Now we combine (4.29), (4.30), and (4.31) and use the results in Lemma 4.1 to obtain

$$(4.32)\qquad \frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_u\|^2_\Omega\leqslant Ch^{2k+2}+\frac{1}{4}(\|P\boldsymbol{e}_u\|^2_\Omega+\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega).$$

We integrate (4.32) with respect to time between 0 and t and obtain

$$(4.33)\qquad \frac{1}{2}\|P\boldsymbol{e}_u\|^2_\Omega\leqslant Ch^{2k+2}+\frac{1}{4}\int_0^t(\|P\boldsymbol{e}_u\|^2_\Omega+\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega)dt.$$

4.4.2. The second energy equation. We first replace u_h, $\boldsymbol{q}_h$ in (4.19) by $u-u_h$, $\boldsymbol{q}-\boldsymbol{q}_h$ and take the test functions $\eta=-P\boldsymbol{e}_{u_t}$, $\boldsymbol{\phi}=\Pi\boldsymbol{e}_{\boldsymbol{q}}$. After summing over all elements K, we obtain

$$(4.34)\quad \widetilde{\mathcal{A}}(u-u_h,(\boldsymbol{q}-\boldsymbol{q}_h)_t;-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})+\widetilde{\mathcal{B}}((u-u_h)_t,\boldsymbol{q}-\boldsymbol{q}_h;-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})=0.$$

Using the notations in (4.23), we have

$$\begin{aligned}(4.35)\quad &\widetilde{\mathcal{A}}(u-Pu,(\boldsymbol{q}-\Pi\boldsymbol{q})_t;-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})+\widetilde{\mathcal{B}}((u-Pu)_t,\boldsymbol{q}-\Pi\boldsymbol{q};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})\\ &\quad+\widetilde{\mathcal{A}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}_t};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})+\widetilde{\mathcal{B}}(P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})=0.\end{aligned}$$

From the proof of the L^2 stability, we have

(4.36)
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega&=\widetilde{\mathcal{A}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}_t};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})+\widetilde{\mathcal{B}}(P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})\\ &\quad+(\widetilde{\mathcal{A}}(P\boldsymbol{e}_u,\Pi\boldsymbol{e}_{\boldsymbol{q}_t};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})+\widetilde{\mathcal{B}}(P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}}))^*.\end{aligned}$$

From the definition of $\widetilde{\mathcal{A}}$, we have

$$\widetilde{\mathcal{A}}(u-Pu,(\boldsymbol{q}-\Pi\boldsymbol{q})_t;-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})+\widetilde{\mathcal{A}}(u-Pu,(\boldsymbol{q}-\Pi\boldsymbol{q})_t;-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})^* \\ =-i(((u-Pu)_t,P\boldsymbol{e}_{u_t})_\Omega-((u-Pu)_t,P\boldsymbol{e}_{u_t})^*_\Omega) \\ +(((\boldsymbol{q}-\Pi\boldsymbol{q})_t,\Pi\boldsymbol{e}_{\boldsymbol{q}})_\Omega+((\boldsymbol{q}-\Pi\boldsymbol{q})_t,\Pi\boldsymbol{e}_{\boldsymbol{q}})^*_\Omega). \tag{4.37}$$

From the definition of $\widetilde{\mathcal{B}}$ and the properties of the projection Π^+, we can obtain

$$\widetilde{\mathcal{B}}((u-Pu)_t,\boldsymbol{q}-\Pi\boldsymbol{q};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})+\widetilde{\mathcal{B}}((u-Pu)_t,\boldsymbol{q}-\Pi\boldsymbol{q};-P\boldsymbol{e}_{u_t},\Pi\boldsymbol{e}_{\boldsymbol{q}})^* \\ =\sum_K\Big((\boldsymbol{q}-\Pi\boldsymbol{q},\nabla P\boldsymbol{e}_{u_t})_K-(\widehat{\boldsymbol{q}-\Pi\boldsymbol{q}}\cdot\boldsymbol{\nu},P\boldsymbol{e}_{u_t})_{\partial K} \\ +((\boldsymbol{q}-\Pi\boldsymbol{q},\nabla P\boldsymbol{e}_{u_t})_K-(\widehat{\boldsymbol{q}-\Pi\boldsymbol{q}}\cdot\boldsymbol{\nu},P\boldsymbol{e}_{u_t})_{\partial K})^*\Big) \\ +\sum_K\Big(((u-Pu)_t,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u_t-Pu_t},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K} \\ +(((u-Pu)_t,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u_t-Pu_t},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K})^*\Big) \\ =\sum_K\Big(((u-Pu)_t,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u_t-Pu_t},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K} \\ +(((u-Pu)_t,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u_t-Pu_t},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K})^*\Big). \tag{4.38}$$

Now we combine (4.36), (4.37), and (4.38) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega=i(((u-Pu)_t,P\boldsymbol{e}_{u_t})_\Omega-((u-Pu)_t,P\boldsymbol{e}_{u_t})^*_\Omega) \\ -(((\boldsymbol{q}-\Pi\boldsymbol{q})_t,\Pi\boldsymbol{e}_{\boldsymbol{q}})_\Omega+((\boldsymbol{q}-\Pi\boldsymbol{q})_t,\Pi\boldsymbol{e}_{\boldsymbol{q}})^*_\Omega) \\ -\sum_K\Big(((u-Pu)_t,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u_t-Pu_t},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K} \\ +(((u-Pu)_t,\nabla\cdot\Pi\boldsymbol{e}_{\boldsymbol{q}})_K-(\widehat{u_t-Pu_t},\Pi\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{\nu})_{\partial K})^*\Big). \tag{4.39}$$

We integrate (4.39) with respect to time between 0 and t. For the first term we take integration by parts over t and get

$$\int_0^t((u-Pu)_t,P\boldsymbol{e}_{u_t})_\Omega dt=((u-Pu)_t,P\boldsymbol{e}_u)_\Omega|_0^t-\int_0^t(u-Pu)_{tt},P\boldsymbol{e}_u)_\Omega dt. \tag{4.40}$$

Combining (4.39) and (4.40) and using the results in Lemma 4.1, we obtain

$$\frac{1}{2}\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega\leqslant Ch^{2k+2}+\frac{1}{4}\|P\boldsymbol{e}_u\|^2_\Omega+\frac{1}{4}\int_0^t(\|P\boldsymbol{e}_u\|^2_\Omega+\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega)dt. \tag{4.41}$$

4.4.3. Proof of the error estimates. We are now ready to combine the two energy inequalities (4.33) and (4.41) and get

$$\frac{1}{2}(\|P\boldsymbol{e}_u\|^2_\Omega+\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega)\leqslant Ch^{2k+2}+\frac{1}{4}\|P\boldsymbol{e}_u\|^2_\Omega+\frac{1}{2}\int_0^t(\|P\boldsymbol{e}_u\|^2_\Omega+\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega)dt. \tag{4.42}$$

After employing the Gronwall inequality, we obtain

$$\max_t\|P\boldsymbol{e}_u\|^2_\Omega+\max_t\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|^2_\Omega\leqslant Ch^{2k+2}. \tag{4.43}$$

After using the approximation error estimates, we have therefore proved the following error estimates.

THEOREM 4.4. *Assume that* (4.1) *with periodic boundary conditions has a smooth exact solution u. Let u_h be the numerical solution of the semidiscrete LDG scheme* (4.11) *and* (4.12). *For any rectangular triangulation of Ω, if the finite element space is the tensor-product piecewise polynomial of degree $k \geq 0$, then there holds the following error estimate:*

$$\max_t \|\boldsymbol{e}_u\|_\Omega^2 + \max_t \|\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 \leqslant Ch^{2k+2}, \tag{4.44}$$

where C depends on the final time T, $\|u\|_{L^\infty((0,T);H^{k+2}(\Omega))}$ and $\|u_t\|_{L^\infty((0,T);H^{k+1}(\Omega))}$.

5. Concluding remarks. In this paper, we have presented a general approach to prove optimal L^2 error estimates for the LDG methods for solving linear high order wave equations. The optimal order of error estimates holds not only for the solution itself but also for the auxiliary variables in the LDG method approximating the various order derivatives of the solution. Examples including the one-dimensional third order wave equation, one-dimensional fifth order wave equation, and multidimensional Schrödinger equation are given to demonstrate this approach. The key idea is to derive energy stability results for the auxiliary variables of the LDG discretizations. Special projections help to eliminate the jump terms at the cell boundaries.

Future work will include the study for more multidimensional wave PDEs, in which the treatment for the auxiliary variables for mixed derivatives should be carried out carefully in order not to lose half an order or even one order in accuracy. Error estimates for nonlinear high order wave equations with smooth solutions also constitute future work.

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