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Superconvergence of the local discontinuous Galerkin method for linear fourth-order time-dependent problems in one space dimension

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In this paper we investigate the superconvergence of local discontinuous Galerkin (LDG) methods for solving one-dimensional linear time-dependent fourth-order problems. We prove that the error between the LDG solution and a particular projection of the exact solution, $\bar{e}_u$, achieves $\left(k+\frac{3}{2}\right)$th-order superconvergence when polynomials of degree k ($k \geqslant 1$) are used. Numerical experiments with P^k polynomials, with $1 \leqslant k \leqslant 3$, are displayed to demonstrate the theoretical results, which show that the error $\bar{e}_u$ actually achieves $(k+2)$th-order superconvergence, indicating that the error bound for $\bar{e}_u$ obtained in this paper is suboptimal. Initial boundary value problems, nonlinear equations and solutions having singularities, are numerically investigated to verify that the conclusions hold true for very general cases.

Keywords: local discontinuous Galerkin method; superconvergence; fourth-order problems; error estimates.

1. Introduction

In this paper we are interested in the superconvergence of local discontinuous Galerkin (LDG) methods for a class of one-dimensional linear fourth-order problems formulated as

$$u_t + \alpha u_x + \beta u_{xx} + u_{xxxx} = 0, \tag{1.1}$$

where α and β are arbitrary constants. Note that, for $\beta > 0$, there is an antidiffusion term βu_{xx} in the equation, which is, however, dominated by the higher-order diffusion term u_{xxxx}. The general problem (1.1) includes the following linear time-dependent biharmonic equation

$$u_t + u_{xxxx} = 0, \tag{1.2}$$

and the linearized Cahn–Hilliard equation

$$u_t + u_{xx} + u_{xxxx} = 0, \tag{1.3}$$

as its special cases.

The discontinuous Galerkin (DG) method is a class of finite element methods using discontinuous piecewise polynomials as the solution and the test spaces. It was first introduced by Reed & Hill (1973) for solving a first-order steady-state linear conservation law and later developed by Cockburn *et al.* (Cockburn & Shu, 1989, 1998b; Cockburn *et al.*, 1989, 1990) for solving time-dependent nonlinear equations. Motivated by the successful numerical experiments of Bassi & Rebay (1997) for the compressible Navier–Stokes equations, the LDG methods were developed for solving nonlinear convection–diffusion equations (Cockburn & Shu, 1998a) containing second-order spatial derivatives in which L^2 stability and a suboptimal L^2 error estimates were obtained for linear equations with smooth solutions. Later, the LDG methods were generalized to solve various partial differential equations (PDEs) involving higher-order derivatives. For Korteweg-de Vries-type equations containing third-order derivatives, an LDG method was developed in Yan & Shu (2002a), where a suboptimal error estimate was proved for the linear case, and more recently, an optimal L^2 error estimate was obtained in Xu & Shu (in press). In Yan & Shu (2002b), Xu & Shu (2004); Xu & Shu (2006) and Xia *et al.* (2007), LDG techniques were developed for solving other types of high-order PDEs including the time-dependent biharmonic equations, the fully nonlinear $K(n, n, n)$ equations, the Kuramoto–Sivashinsky-type equations, the Cahn–Hilliard-type equations and so on. In Dong & Shu (2009), optimal error estimates for the LDG method applied to the linear biharmonic equation and linearized Cahn–Hilliard-type equations were obtained in one dimension and in multidimensions for Cartesian and triangular meshes. For more details of the DG and LDG methods we refer to the lecture notes, Cockburn (1999), and review papers, Cockburn & Shu (2001); Xu & Shu (2010).

Apart from the LDG methods mentioned above, there are also other finite element methods in the literature for solving fourth-order time-dependent problems. For example, Elliott & Zheng (1986) applied a conforming finite element method to the Cahn–Hilliard equation and obtained optimal error estimates in L^2 and L^∞ norms provided the approximate solution is bounded in L^∞ and the polynomial degree $k \geqslant 3$. Feng & Prohl (2004) applied a mixed finite element method for solving Cahn–Hillard equations on quasiuniform triangular meshes and obtained an optimal error estimate under minimum regularity assumptions on the initial data and the domain.

Adjerid & Issaev (2005) and Adjerid & Klauser (2005) showed that the LDG solution is superconvergent at Radau points for solving convection- or diffusion-dominant time-dependent equations. Based on Fourier analysis, Cheng & Shu (2008, 2009) proved superconvergence of the DG and LDG solutions towards a particular projection of the exact solution in the case of piecewise linear polynomials on uniform meshes for the linear conservation law and heat equation, respectively. The results were later improved, using a different technique, in Cheng & Shu (2010) for arbitrary nonuniform regular meshes and schemes of any order. In this paper we follow the approach in Cheng & Shu (2010) to obtain the superconvergence property of the LDG method for a class of fourth-order problems. An important motivation for studying such superconvergence is to set a firm theoretical foundation for the excellent behaviour of DG and LDG methods for long-time simulations, which have been repeatedly observed by practitioners. Indeed, if superconvergence for the error between the DG or LDG solution and a particular projection of the exact solution of the order $\left(k + \frac{3}{2}\right)$, with linear growth in time, can be shown for polynomials of degree k, then the error between the numerical solution and the exact solution does not grow for a long time $t = \mathcal{O}\left(\frac{1}{\sqrt{h}}\right)$, where h is the mesh size (Cheng & Shu, 2010). The generalization from first- and second-order equations in Cheng & Shu (2010) to the fourth-order equation in this paper involves several technical difficulties, including the estimate of different combinations of the LDG solution and auxiliary variables that approximate derivatives of different orders, and the design and analysis of a special operator to guarantee the superconvergence property of the initial condition.

This paper is organized as follows. In Section 2 we define the LDG scheme for fourth-order time-dependent problems, state the main results and present the details of the proof of the superconvergence property. In Section 3 various numerical experiments, including linear equations, nonlinear equations, initial boundary value problems and solutions having singularities, are shown to demonstrate that the conclusions hold true for very general cases. Concluding remarks and comments on future work are given in Section 4. The proofs for some of the technical lemmas are collected in Appendix.

2. LDG scheme for fourth-order problems

We consider the following linear fourth-order equation

$$u_t + \alpha u_x + \beta u_{xx} + u_{xxxx} = 0 \tag{2.1a}$$

with initial condition

$$u(x,0) = u_0(x) \tag{2.1b}$$

and periodic boundary conditions

$$u(0,t) = u(2\pi,t). \tag{2.1c}$$

We would like to remark that the assumption of periodic boundary conditions is for simplicity only and not essential: see Cheng & Shu (2010) for discussion related to initial boundary value problems for conservation laws.

2.1 *The LDG scheme*

We assume the following mesh to cover the computational domain $I = [0, 2\pi]$, consisting of cells $I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right)$, for $1 \leqslant j \leqslant N$, where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

The cell centre is denoted by $x_j = \left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right)\big/2$. We also set $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ and $h = \max_j h_j$. We denote by $(v_h)^-_{j+\frac{1}{2}}$ and $(v_h)^+_{j+\frac{1}{2}}$ the values of v_h at the discontinuity point $x_{j+\frac{1}{2}}$ from the left cell, I_j, and from the right cell, I_{j+1}, respectively. The following piecewise polynomial space is chosen as the finite element space:

$$V_h^k = \{v \colon v|_{I_j} \in P^k(I_j),\ j = 1, \ldots, N\},$$

where $P^k(I_j)$ denotes the set of polynomials of degree up to k defined on the cell I_j. Note that functions in V_h^k are allowed to have discontinuities across element interfaces.

In order to construct the LDG scheme, firstly, we introduce some auxiliary variables approximating various order derivatives of the solution and rewrite equation (2.1a) as a first-order system,

$$u_t + (\alpha u + \beta q + r)_x = 0, \quad r - p_x = 0, \quad p - q_x = 0, \quad q - u_x = 0.$$

Then, the semidiscrete LDG scheme is defined as follows: find $u_h, q_h, p_h, r_h \in V_h^k$, such that

$$\int_{I_j} (u_h)_t \rho \, dx - \int_{I_j} \alpha u_h \rho_x \, dx + \alpha \tilde{u}_h \rho^- |_{j+\frac{1}{2}} - \alpha \tilde{u}_h \rho^+ |_{j-\frac{1}{2}} - \int_{I_j} \beta q_h \rho_x \, dx$$

$$+ \beta \hat{q}_h \rho^- |_{j+\frac{1}{2}} - \beta \hat{q}_h \rho^+ |_{j-\frac{1}{2}} - \int_{I_j} r_h \rho_x \, dx + \hat{r}_h \rho^- |_{j+\frac{1}{2}} - \hat{r}_h \rho^+ |_{j-\frac{1}{2}} = 0, \tag{2.2a}$$

$$\int_{I_j} r_h \eta \, dx + \int_{I_j} p_h \eta_x \, dx - \hat{p}_h \eta^- |_{j+\frac{1}{2}} + \hat{p}_h \eta^+ |_{j-\frac{1}{2}} = 0, \tag{2.2b}$$

$$\int_{I_j} p_h \xi \, dx + \int_{I_j} q_h \xi_x \, dx - \hat{q}_h \xi^- |_{j+\frac{1}{2}} + \hat{q}_h \xi^+ |_{j-\frac{1}{2}} = 0, \tag{2.2c}$$

$$\int_{I_j} q_h \psi \, dx + \int_{I_j} u_h \psi_x \, dx - \hat{u}_h \psi^- |_{j+\frac{1}{2}} + \hat{u}_h \psi^+ |_{j-\frac{1}{2}} = 0 \tag{2.2d}$$

hold for any $\rho, \psi, \xi, \eta \in V_h^k$, where $\tilde{u}_h$ is the upwind flux depending on the sign of α. Without loss of generality we assume that $\alpha \geqslant 0$ and take $\tilde{u}_h = u_h^-$ and then choose alternating fluxes for the diffusion terms as follows:

$$\hat{u}_h = u_h^-, \quad \hat{q}_h = q_h^+, \quad \hat{p}_h = p_h^-, \quad \hat{r}_h = r_h^+. \tag{2.3}$$

2.2 *Notation and auxiliary results*

To prove superconvergence of the LDG method, we would like to introduce the following notation, definitions and useful lemmas.

2.2.1 *Notation for the DG discretization.*

First, we use $[\xi] = \xi^+ - \xi^-$ to denote the jump in the function ξ at each cell boundary point. For the linear problems discussed in this paper, we introduce the DG discretization operator $\mathcal{D}$ as in Xu & Shu (in press): for each cell $I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right)$,

$$\mathcal{D}_{I_j}(\xi, \eta; \hat{\xi}) = -\int_{I_j} \xi \eta_x \, dx + \hat{\xi} \eta^- |_{j+\frac{1}{2}} - \hat{\xi} \eta^+ |_{j-\frac{1}{2}}.$$

We also use the notation

$$\mathcal{D}(\xi, \eta; \hat{\xi}) = \sum_j \mathcal{D}_{I_j}(\xi, \eta; \hat{\xi}).$$

Using the definition of this operator, we have the following lemmas, whose proof is straightforward (see Xu & Shu, in press and also Zhang & Shu, 2010).

LEMMA 2.1 (Xu & Shu, in press) Choosing different numerical fluxes the DG discretization operator satisfies the equalities

$$\mathcal{D}(\xi, \eta; \xi^-) + \mathcal{D}(\eta, \xi; \eta^+) = 0, \tag{2.4a}$$

$$\mathcal{D}(\xi,\eta;\xi^{+}) + \mathcal{D}(\eta,\xi;\eta^{-}) = 0, \tag{2.4b}$$

$$\mathcal{D}(\xi,\eta;\xi^{+}) + \mathcal{D}(\eta,\xi;\eta^{+}) = -\sum_{j}[\xi]_{j+\frac{1}{2}}[\eta]_{j+\frac{1}{2}}, \tag{2.4c}$$

$$\mathcal{D}(\xi,\eta;\xi^{-}) + \mathcal{D}(\eta,\xi;\eta^{-}) = \sum_{j}[\xi]_{j+\frac{1}{2}}[\eta]_{j+\frac{1}{2}}, \tag{2.4d}$$

$$\mathcal{D}(\xi,\xi;\xi^{-}) = \frac{1}{2}\sum_{j}[\xi]^2_{j+\frac{1}{2}}, \tag{2.4e}$$

$$\mathcal{D}(\xi,\xi;\xi^{+}) = -\frac{1}{2}\sum_{j}[\xi]^2_{j+\frac{1}{2}}. \tag{2.4f}$$

LEMMA 2.2 By integration by parts we also have

$$\mathcal{D}_{I_j}(\xi,\eta;\xi^{-}) = \int_{I_j}\xi_x\eta\,\mathrm{d}x + [\xi]\eta^{+}|_{j-\frac{1}{2}}, \tag{2.5}$$

$$\mathcal{D}_{I_j}(\xi,\eta;\xi^{+}) = \int_{I_j}\xi_x\eta\,\mathrm{d}x + [\xi]\eta^{-}|_{j+\frac{1}{2}}. \tag{2.6}$$

2.2.2 *Projections and interpolation properties.* In what follows we define two special projections, $P_h^{\pm}$ into V_h^k, which are commonly used in the analysis of DG methods. For any given function $u \in H^1(I)$ and arbitrary subinterval $I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right)$, the special projections of u, denoted by $P_h^{+}u$ and $P_h^{-}u$, are the unique functions in the finite element space V_h^k satisfying, for each j,

$$\int_{I_j}(P_h^{+}u(x) - u(x))\rho(x)\,\mathrm{d}x = 0 \quad \forall\rho\in P^{k-1}(I_j), \qquad (P_h^{+}u)^{+}_{j-\frac{1}{2}} = u\left(x_{j-\frac{1}{2}}\right), \tag{2.7}$$

$$\int_{I_j}(P_h^{-}u(x) - u(x))\rho(x)\,\mathrm{d}x = 0 \quad \forall\rho\in P^{k-1}(I_j), \qquad (P_h^{-}u)^{-}_{j+\frac{1}{2}} = u\left(x_{j+\frac{1}{2}}\right). \tag{2.8}$$

For the special projections mentioned above, we have, by the standard approximation theory (Ciarlet, 1978), that

$$\|P_h^{\pm}u(\cdot) - u(\cdot)\|_{L^2} \leqslant Ch^{k+1}, \tag{2.9}$$

where both here and below C is a positive constant (which may have a different value at each occurrence) depending solely on u and its derivatives but independent of h. In particular, in equation (2.9), $C = C'\|u\|_{k+1}$, where $\|u\|_{k+1}$ is the standard Sobolev $(k+1)$ norm and C' is a positive constant independent of u.

In the proof of the error estimates the following inverse properties are needed: for any $v_h \in V_h^k$ there exists a positive constant C independent of h, such that

$$\|v_h\|_\Gamma \leqslant Ch^{-\frac{1}{2}}\|v_h\|_{L^2}, \tag{2.10}$$

$$\|\partial_x v_h\|_{L^2} \leqslant Ch^{-1}\|v_h\|_{L^2}, \tag{2.11}$$

where $\|v_h\|_\Gamma$ is the usual L^2 norm on the cell interfaces of the mesh.

2.2.3 *Functionals related to the L^2 norm.* To get the superconvergence property of the method two functionals related to the L^2 norm of a function on I_j are needed as defined in Cheng & Shu (2010):

$$\mathcal{B}_j^-(f) = \int_{I_j} f(x)\frac{x - x_{j-\frac{1}{2}}}{h_j}\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\frac{x - x_j}{h_j}\right)\mathrm{d}x,$$

$$\mathcal{B}_j^+(f) = \int_{I_j} f(x)\frac{x - x_{j+\frac{1}{2}}}{h_j}\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\frac{x - x_j}{h_j}\right)\mathrm{d}x.$$

The functionals defined above have the following properties, which are essential to the proof of superconvergence.

LEMMA 2.3 (Cheng & Shu, 2010) For any function $f(x) \in C^1$ on I_j we have

$$\mathcal{B}_j^-(f) = \frac{1}{4h_j}\int_{I_j} f^2(x)\,\mathrm{d}x + \frac{f^2(x_{j+\frac{1}{2}})}{4}, \tag{2.12}$$

$$\mathcal{B}_j^+(f) = -\frac{1}{4h_j}\int_{I_j} f^2(x)\,\mathrm{d}x - \frac{f^2(x_{j-\frac{1}{2}})}{4}. \tag{2.13}$$

The proof of this lemma is straightforward; see Cheng & Shu (2010).

2.2.4 *Initial condition.* To obtain the superconvergence property of the method, the initial condition of the numerical scheme should be chosen carefully to be compatible with the superconvergence error estimate. To this end we define an operator P_h^* as follows: for any function u, then $P_h^* u \in V_h^k$, and suppose $q_h, p_h, r_h \in V_h^k$ are the unique solutions (with given $P_h^* u$) to

$$\int_{I_j} r_h \eta\,\mathrm{d}x + \int_{I_j} p_h \eta_x\,\mathrm{d}x - p_h^- \eta^-|_{j+\frac{1}{2}} + p_h^- \eta^+|_{j-\frac{1}{2}} = 0, \tag{2.14a}$$

$$\int_{I_j} p_h \xi\,\mathrm{d}x + \int_{I_j} q_h \xi_x\,\mathrm{d}x - q_h^+ \xi^-|_{j+\frac{1}{2}} + q_h^+ \xi^+|_{j-\frac{1}{2}} = 0, \tag{2.14b}$$

$$\int_{I_j} q_h \psi\,\mathrm{d}x + \int_{I_j} P_h^* u \psi_x\,\mathrm{d}x - (P_h^* u)^- \psi^-|_{j+\frac{1}{2}} + (P_h^* u)^- \psi^+|_{j-\frac{1}{2}} = 0, \tag{2.14c}$$

for any $\psi, \xi, \eta \in V_h^k$, then we require

$$\int_{I_j} ((P_h^- u - P_h^* u) - (P_h^+ q - q_h) + (P_h^+ r - r_h))\rho \, dx = 0 \tag{2.15}$$

for any $\rho \in P^{k-1}$ on I_j and

$$(P_h^- u - P_h^* u)^- = (P_h^+ q - q_h)^+ - (P_h^+ r - r_h)^+ \quad \text{at } x_{j-\frac{1}{2}}. \tag{2.16}$$

For the regular mesh considered in this paper we denote $\lambda = \max_j h_j / \min_j h_j$, which is a constant during mesh refinements. As to the operator defined above we have the following lemma.

LEMMA 2.4 $P_h^* u$ exists and is unique. Moreover, there holds the error estimate

$$\| P_h^- u - P_h^* u \|_{L^2} \leqslant C(\lambda, \|u\|_{k+4}) h^{k+3/2}. \tag{2.17}$$

The proof of this lemma is given in Appendix.

We would like to remark that the purpose for introducing the operator P_h^* is only theoretical: it is needed for the technical proof of superconvergence. In actual numerical computation we have observed that we can use the usual L^2 projection of u as the initial condition and still observe superconvergence; see the numerical experiments in Section 3. Of course, if the standard L^2 projection is used for the initial condition, then the superconvergence result does not hold at $t = 0$ nor for small t. For later time the dissipativity in the PDE and the numerical scheme seems to help to recover the superconvergent performance.

2.3 *Main results*

Before we state the main results we would like to introduce the following notation:

$$e_u = u - u_h = (u - P_h^- u) + (P_h^- u - u_h) = \varepsilon_u + \bar{e}_u,$$

$$e_q = q - q_h = (q - P_h^+ q) + (P_h^+ q - q_h) = \varepsilon_q + \bar{e}_q,$$

$$e_p = p - p_h = (p - P_h^- p) + (P_h^- p - p_h) = \varepsilon_p + \bar{e}_p,$$

$$e_r = r - r_h = (r - P_h^+ r) + (P_h^+ r - r_h) = \varepsilon_r + \bar{e}_r.$$

For the case $\alpha \geqslant 0$ we have the following error estimates.

THEOREM 2.5 Let u, $p = u_{xx}$ be the exact solution of the fourth-order problem (2.1), which is assumed to be sufficiently smooth, i.e. $\|u\|_{k+4}$, $\|u_t\|_{k+4}$, $\|u_{tt}\|_{k+4}$ and $\|u_{ttt}\|_{k+1}$ are bounded uniformly for any time $t \in [0, T]$. Let u_h, p_h be the LDG solution of equation (2.2) when the diffusion alternating fluxes (2.3) are used. We choose the initial condition as $u_h(\cdot, 0) = P_h^* u_0$. For regular triangulations of $I = [0, 2\pi]$, if the finite element space V_h^k with $k \geqslant 1$ is used, then there holds the following error estimate:

$$\|\bar{e}_u(\cdot, t)\|_{L^2}^2 + \int_0^t \|\bar{e}_p(\cdot, t)\|_{L^2}^2 \, dt \leqslant C e^{2C_1 t} h^{2k+3}, \tag{2.18}$$

and, in particular,

$$\|\bar{e}_u(\cdot,t)\|_{L^2} \leqslant C\mathrm{e}^{C_1 t} h^{k+3/2},$$

where $C = C(\alpha, \beta, \lambda, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1})$ and both here and below $C_1 = C_1(\alpha, \beta) \geqslant 0$.

REMARK 2.6 For the case $\alpha \leqslant 0$ we can choose $\tilde{u}_h = u_h^+$ and take the diffusion alternating fluxes as

$$\hat{u}_h = u_h^+, \quad \hat{q}_h = q_h^-, \quad \hat{p}_h = p_h^+, \quad \hat{r}_h = r_h^-. \tag{2.19}$$

Theorem 2.5 still holds in this case with the obvious change of the projections.

For the case $\alpha = \beta = 0$ equation (2.1a) reduces to the biharmonic equation (1.2), and we have the following result.

THEOREM 2.7 Let u, $p = u_{xx}$ be the exact solution of the fourth-order problem (1.2), which is assumed to be sufficiently smooth, i.e. $\|u\|_{k+4}$, $\|u_t\|_{k+4}$, $\|u_{tt}\|_{k+4}$ and $\|u_{ttt}\|_{k+1}$ are bounded uniformly for any time $t \in [0,\ T]$. Let u_h, p_h be the LDG solution of equation (2.2) when $\alpha = \beta = 0$ and diffusion alternating fluxes (2.3) are used. We choose the initial condition as $u_h(\cdot, 0) = P_h^* u_0$. For regular triangulations of $I = [0, 2\pi]$, if the finite element space V_h^k with $k \geqslant 1$ is used, then there holds the following error estimate:

$$\|\bar{e}_u(\cdot,t)\|_{L^2}^2 + \int_0^t \|\bar{e}_p(\cdot,t)\|_{L^2}^2 \,\mathrm{e}t \leqslant C(1+t)^2 h^{2k+3},$$

and, in particular,

$$\|\bar{e}_u(\cdot,t)\|_{L^2} \leqslant C(1+t) h^{k+3/2},$$

where $C = C(\lambda, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1})$.

The proof of this theorem is similar to that for the previous theorem, except that we need to carefully evaluate and estimate several terms to obtain a linear growth bound without employing Gronwall's inequality. The detailed proof is given in Appendix.

The case with the fluxes (2.19) is the same with the obvious change of the projections.

Note that, for the general cases including the antidiffusive case $\beta > 0$, the exponential growth of the constant with respect to time in Theorem 2.5 is expected, as the exact solution may have such growth in time for small wave numbers.

2.4 *Proof of Theorem* 2.5

By using the DG discretization operator, the LDG scheme (2.2) with the fluxes (2.3) can be written as

$$\int_{I_j} (u_h)_t \rho \,\mathrm{d}x + \alpha \mathcal{D}_{I_j}(u_h, \rho; u_h^-) + \beta \mathcal{D}_{I_j}(q_h, \rho; q_h^+) + \mathcal{D}_{I_j}(r_h, \rho; r_h^+) = 0, \tag{2.20a}$$

$$\int_{I_j} r_h \eta \,\mathrm{d}x - \mathcal{D}_{I_j}(p_h, \eta; p_h^-) = 0, \tag{2.20b}$$

$$\int_{I_j} p_h \xi \, \mathrm{d}x - \mathcal{D}_{I_j}(q_h, \xi; q_h^+) = 0, \tag{2.20c}$$

$$\int_{I_j} q_h \psi \, \mathrm{d}x - \mathcal{D}_{I_j}(u_h, \psi; u_h^-) = 0, \tag{2.20d}$$

for any $\rho, \psi, \xi, \eta \in V_h^k$. Since the exact solutions $u, q = u_x, p = u_{xx}, r = u_{xxx}$ also satisfy the scheme (2.2), we have therefore the error equations

$$\int_{I_j} (e_u)_t \rho \, \mathrm{d}x + \alpha \mathcal{D}_{I_j}(e_u, \rho; e_u^-) + \beta \mathcal{D}_{I_j}(e_q, \rho; e_q^+) + \mathcal{D}_{I_j}(e_r, \rho; e_r^+) = 0,$$

$$\int_{I_j} e_r \eta \, \mathrm{d}x - \mathcal{D}_{I_j}(e_p, \eta; e_p^-) = 0,$$

$$\int_{I_j} e_p \xi \, \mathrm{d}x - \mathcal{D}_{I_j}(e_q, \xi; e_q^+) = 0,$$

$$\int_{I_j} e_q \psi \, \mathrm{d}x - \mathcal{D}_{I_j}(e_u, \psi; e_u^-) = 0,$$

which, by the properties of the projections P_h^- and P_h^+, given in equations (2.7) and (2.8), is

$$\int_{I_j} (e_u)_t \rho \, \mathrm{d}x + \alpha \mathcal{D}_{I_j}(\bar{e}_u, \rho; \bar{e}_u^-) + \beta \mathcal{D}_{I_j}(\bar{e}_q, \rho; \bar{e}_q^+) + \mathcal{D}_{I_j}(\bar{e}_r, \rho; \bar{e}_r^+) = 0, \tag{2.21a}$$

$$\int_{I_j} e_r \eta \, \mathrm{d}x - \mathcal{D}_{I_j}(\bar{e}_p, \eta; \bar{e}_p^-) = 0, \tag{2.21b}$$

$$\int_{I_j} e_p \xi \, \mathrm{d}x - \mathcal{D}_{I_j}(\bar{e}_q, \xi; \bar{e}_q^+) = 0, \tag{2.21c}$$

$$\int_{I_j} e_q \psi \, \mathrm{d}x - \mathcal{D}_{I_j}(\bar{e}_u, \psi; \bar{e}_u^-) = 0, \tag{2.21d}$$

for any $\rho, \psi, \xi, \eta \in V_h^k$. Taking $(\rho, \psi, \xi, \eta) = (\bar{e}_u, -\bar{e}_r, \bar{e}_p, \bar{e}_q)$ in equation (2.21), adding them up and summing over all j, we obtain

$$\int_I (\bar{e}_u)_t \bar{e}_u \, \mathrm{d}x + \int_I \bar{e}_p^2 \, \mathrm{d}x + \int_I (\varepsilon_u)_t \bar{e}_u \, \mathrm{d}x + \int_I \varepsilon_p \bar{e}_p \, \mathrm{d}x + \int_I \varepsilon_r \bar{e}_q \, \mathrm{d}x - \int_I \varepsilon_q \bar{e}_r \, \mathrm{d}x + \alpha \mathcal{D}(\bar{e}_u, \bar{e}_u; \bar{e}_u^-)$$

$$+ \beta \mathcal{D}(\bar{e}_q, \bar{e}_u; \bar{e}_q^+) + \mathcal{D}(\bar{e}_r, \bar{e}_u; \bar{e}_r^+) + \mathcal{D}(\bar{e}_u, \bar{e}_r; \bar{e}_u^-) - \mathcal{D}(\bar{e}_q, \bar{e}_p; \bar{e}_q^+) - \mathcal{D}(\bar{e}_p, \bar{e}_q; \bar{e}_p^-) = 0.$$

Using the property of the operator $\mathcal{D}$ in Lemma 2.1 we thus have

$$\int_I (\bar{e}_u)_t \bar{e}_u \,\mathrm{d}x + \int_I \bar{e}_p^2 \,\mathrm{d}x + \int_I (\varepsilon_u)_t \bar{e}_u \,\mathrm{d}x + \int_I \varepsilon_p \bar{e}_p \,\mathrm{d}x$$

$$+ \int_I \varepsilon_r \bar{e}_q \,\mathrm{d}x - \int_I \varepsilon_q \bar{e}_r \,\mathrm{d}x + \frac{\alpha}{2} \sum_j [\bar{e}_u]^2_{j+\frac{1}{2}} + \beta \mathcal{D}(\bar{e}_q, \bar{e}_u; \bar{e}_q^+) = 0. \tag{2.22}$$

By taking $\xi = \bar{e}_u$ in equation (2.21c) and summing over all j we get

$$\mathcal{D}(\bar{e}_q, \bar{e}_u; \bar{e}_q^+) = \int_I e_p \bar{e}_u \,\mathrm{d}x. \tag{2.23}$$

Combining equations (2.22) and (2.23) we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|^2_{L^2} + \|\bar{e}_p\|^2_{L^2} \leqslant -\int_I (\varepsilon_u)_t \bar{e}_u \,\mathrm{d}x - \int_I \varepsilon_p \bar{e}_p \,\mathrm{d}x - \int_I \varepsilon_r \bar{e}_q \,\mathrm{d}x + \int_I \varepsilon_q \bar{e}_r \,\mathrm{d}x - \beta \int_I e_p \bar{e}_u \,\mathrm{d}x. \tag{2.24}$$

It follows from the Cauchy–Schwarz inequality that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|^2_{L^2} + \frac{1}{2}\|\bar{e}_p\|^2_{L^2} \leqslant \left|\int_I (\varepsilon_u)_t \bar{e}_u \,\mathrm{d}x\right| + \left|\int_I \varepsilon_p \bar{e}_p \,\mathrm{d}x\right| + \left|\int_I \varepsilon_r \bar{e}_q \,\mathrm{d}x\right|$$

$$+ \left|\int_I \varepsilon_q \bar{e}_r \,\mathrm{d}x\right| + |\beta| \left|\int_I \varepsilon_p \bar{e}_u \,\mathrm{d}x\right| + \frac{\beta^2}{2}\|\bar{e}_u\|^2_{L^2}. \tag{2.25}$$

On the other hand, using Lemma 2.2, equation (2.21) can be rewritten as

$$\int_{I_j} (e_u)_t \rho \,\mathrm{d}x + \alpha \mathcal{D}_{I_j}(\bar{e}_u, \rho; \bar{e}_u^-) + \int_{I_j} (\bar{e}_r + \beta \bar{e}_q)_x \rho \,\mathrm{d}x + [\bar{e}_r + \beta \bar{e}_q]\rho^-|_{j+\frac{1}{2}} = 0, \tag{2.26a}$$

$$\int_{I_j} e_r \eta \,\mathrm{d}x - \int_{I_j} (\bar{e}_p)_x \eta \,\mathrm{d}x - [\bar{e}_p]\eta^+|_{j-\frac{1}{2}} = 0, \tag{2.26b}$$

$$\int_{I_j} e_p \xi \,\mathrm{d}x - \int_{I_j} (\bar{e}_q)_x \xi \,\mathrm{d}x - [\bar{e}_q]\xi^-|_{j+\frac{1}{2}} = 0, \tag{2.26c}$$

$$\int_{I_j} e_q \psi \,\mathrm{d}x - \int_{I_j} (\bar{e}_u)_x \psi \,\mathrm{d}x - [\bar{e}_u]\psi^+|_{j-\frac{1}{2}} = 0. \tag{2.26d}$$

Denote

$$\bar{e}_u = r_j + d_j(x)(x - x_j)/h_j, \qquad \bar{e}_q = b_j + s_j(x)(x - x_j)/h_j,$$

$$\bar{e}_p = v_j + w_j(x)(x - x_j)/h_j, \qquad \bar{e}_r = l_j + g_j(x)(x - x_j)/h_j,$$

where r_j, b_j, v_j, l_j are constants and $d_j(x), s_j(x), w_j(x), g_j(x) \in P^{k-1}$. First, taking $\psi = d_j(x)\,(x - x_{j-\frac{1}{2}})/h_j$ in equation (2.26d), and using the definition of $\mathcal{B}_j^-$, we have

$$\int_{I_j} e_q d_j(x)(x - x_{j-\frac{1}{2}})/h_j \,\mathrm{d}x - \mathcal{B}_j^-(d_j) = 0.$$

By the property of $\mathcal{B}_j^-$ in Lemma 2.3 we obtain

$$\int_{I_j} d_j^2(x)\,\mathrm{d}x \leqslant 4\int_{I_j} e_q d_j(x)(x - x_{j-\frac{1}{2}})\,\mathrm{d}x.$$

Defining piecewise polynomials $d(x)$ and $\phi_1(x)$, such that $d(x) = d_j(x)$ and $\phi_1(x) = x - x_{j-\frac{1}{2}}$ on I_j, and summing the above inequality over j, we get

$$\|d\|_{L^2} \leqslant 4\|e_q\|_{L^2}\|\phi_1\|_{L^\infty} \leqslant 4h\|e_q\|_{L^2}, \tag{2.27}$$

where we have used the fact that $\|\phi_1\|_{L^\infty} = h$. Similarly, taking $\xi = s_j(x)(x - x_{j+\frac{1}{2}})/h_j$ in equation (2.26c), $\eta = w_j(x)(x - x_{j-\frac{1}{2}})/h_j$ in equation (2.26b) and using the definition of $\mathcal{B}_j^-$ and $\mathcal{B}_j^+$, we have

$$\int_{I_j} e_r w_j(x)(x - x_{j-\frac{1}{2}})/h_j\,\mathrm{d}x - \mathcal{B}_j^-(w_j) = 0,$$

$$\int_{I_j} e_p s_j(x)(x - x_{j+\frac{1}{2}})/h_j\,\mathrm{d}x - \mathcal{B}_j^+(s_j) = 0.$$

By the properties of $\mathcal{B}_j^-$ and $\mathcal{B}_j^+$ in Lemma 2.3 we obtain

$$\int_{I_j} w_j^2(x)\,\mathrm{d}x \leqslant 4\int_{I_j} e_r w_j(x)(x - x_{j-\frac{1}{2}})\,\mathrm{d}x,$$

$$\int_{I_j} s_j^2(x)\,\mathrm{d}x \leqslant -4\int_{I_j} e_p s_j(x)(x - x_{j+\frac{1}{2}})\,\mathrm{d}x.$$

Defining piecewise polynomials $w(x), s(x)$ and $\phi_2(x)$, such that $w(x) = w_j(x), s(x) = s_j(x)$ and $\phi_2(x) = x - x_{j+\frac{1}{2}}$ on I_j, and summing the above inequality over j, we get

$$\|w\|_{L^2} \leqslant 4\|e_r\|_{L^2}\|\phi_1\|_{L^\infty} \leqslant 4h\|e_r\|_{L^2}, \tag{2.28}$$

$$\|s\|_{L^2} \leqslant 4\|e_p\|_{L^2}\|\phi_2\|_{L^\infty} \leqslant 4h\|e_p\|_{L^2}, \tag{2.29}$$

where we have used the fact that $\|\phi_1\|_{L^\infty} = \|\phi_2\|_{L^\infty} = h$. Then, letting $\rho = (g_j(x) + \beta s_j(x))(x - x_{j+\frac{1}{2}})/h_j$ in equation (2.26a), we have

$$\int_{I_j} (e_u)_t \rho\,\mathrm{d}x - \alpha\left[\int_{I_j} \bar{e}_u \rho_x\,\mathrm{d}x + \bar{e}_u^- \rho^+|_{j-\frac{1}{2}}\right] + \int_{I_j} (\bar{e}_r + \beta\bar{e}_q)_x \rho\,\mathrm{d}x = 0,$$

which can be written as

$$\int_{I_j} (e_u)_t (g_j(x) + \beta s_j(x))(x - x_{j+\frac{1}{2}})/h_j\,\mathrm{d}x - \alpha R_j^1 + R_j^2 = 0,$$

where

$$\begin{aligned} R_j^1 &= \int_{I_j} \bar{e}_u((g_j(x)+\beta s_j(x))(x-x_{j+\frac{1}{2}})/h_j)_x \,\mathrm{d}x \\ &\quad -\left[r_{j-1}+\frac{1}{2}d_{j-1}\left(x_{j-\frac{1}{2}}\right)\right]\times\left(g_j\left(x_{j-\frac{1}{2}}\right)+\beta s_j\left(x_{j-\frac{1}{2}}\right)\right) \\ &= \int_{I_j} d_j(x)\frac{x-x_j}{h_j}((g_j(x)+\beta s_j(x))(x-x_{j+\frac{1}{2}})/h_j)_x \,\mathrm{d}x \\ &\quad +\left[r_j-r_{j-1}-\frac{1}{2}d_{j-1}\left(x_{j-\frac{1}{2}}\right)\right]\left(g_j\left(x_{j-\frac{1}{2}}\right)+\beta s_j\left(x_{j-\frac{1}{2}}\right)\right) \end{aligned}$$

and

$$R_j^2 = \mathcal{B}_j^+(g_j+\beta s_j) = -\frac{1}{4h_j}\int_{I_j}(g_j(x)+\beta s_j(x))^2\,\mathrm{d}x - \frac{1}{4}\left(g_j\left(x_{j-\frac{1}{2}}\right)+\beta s_j\left(x_{j-\frac{1}{2}}\right)\right)^2.$$

Therefore,

$$\int_{I_j}(g_j(x)+\beta s_j(x))^2\,\mathrm{d}x \leqslant 4\int_{I_j}(e_u)_t(g_j(x)+\beta s_j(x))(x-x_{j+\frac{1}{2}})\,\mathrm{d}x - 4\alpha h_j R_j^1.$$

Summing the above inequality over all j we arrive at

$$\|g+\beta s\|_{L^2}^2 \leqslant 4\|(e_u)_t\|_{L^2}\|g+\beta s\|_{L^2}\|\phi_2\|_{L^\infty} + 4\alpha\left|\sum_j h_j R_j^1\right|. \tag{2.30}$$

Taking $\psi = 1$ in equation (2.26d) we get

$$r_j - r_{j-1} - \frac{1}{2}d_{j-1}\left(x_{j-\frac{1}{2}}\right) = \int_{I_j} e_q\,\mathrm{d}x - \frac{1}{2}d_j\left(x_{j+\frac{1}{2}}\right).$$

So, the term $h_j R_j^1$ can be formulated as

$$\begin{aligned} h_j R_j^1 &= \int_{I_j} d_j(x)(x-x_j)(g_j(x)+\beta s_j(x))/h_j\,\mathrm{d}x \\ &\quad + \int_{I_j} d_j(x)(x-x_j)(g_j'(x)+\beta s_j'(x))(x-x_{j+\frac{1}{2}})/h_j\,\mathrm{d}x \\ &\quad + h_j\left[\int_{I_j} e_q\,\mathrm{d}x - \frac{1}{2}d_j\left(x_{j+\frac{1}{2}}\right)\right]\left(g_j\left(x_{j-\frac{1}{2}}\right)+\beta s_j\left(x_{j-\frac{1}{2}}\right)\right). \end{aligned}$$

By the inverse properties (2.10) and (2.11) we have the estimate

$$\left|\sum_j h_j R_j^1\right| \leqslant C(k)\|g+\beta s\|_{L^2}(\|d\|_{L^2} + h\|e_q\|_{L^2}), \tag{2.31}$$

where k is the degree of polynomials in the finite element space V_h^k. Combining equations (2.27), (2.30) and (2.31) together and recalling that $\|\phi_2\|_{L^\infty} = \|x - x_{j+\frac{1}{2}}\|_{L^\infty} = h$, we conclude that

$$\|g + \beta s\|_{L^2} \leqslant C(\alpha, k)h(\|(e_u)_t\|_{L^2} + \|e_q\|_{L^2}). \tag{2.32}$$

Thus,

$$\|g\|_{L^2} \leqslant \|g + \beta s\|_{L^2} + |\beta|\|s\|_{L^2} \leqslant C(\alpha, \beta, k)h(\|(e_u)_t\|_{L^2} + \|e_q\|_{L^2} + \|e_p\|_{L^2}). \tag{2.33}$$

Now, we return to the error equation (2.25). Note that $(\varepsilon_u)_t, \varepsilon_q, \varepsilon_p$ and ε_r are orthogonal to any piecewise constant functions, then

$$\left|\int_I (\varepsilon_u)_t \bar{e}_u \,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} (\varepsilon_u)_t d_j(x)(x - x_j)/h_j \,\mathrm{d}x\right| \leqslant \|(\varepsilon_u)_t\|_{L^2}\|d\|_{L^2}\|\phi\|_{L^\infty},$$

$$\left|\int_I \varepsilon_p \bar{e}_p \,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_p w_j(x)(x - x_j)/h_j \,\mathrm{d}x\right| \leqslant \|\varepsilon_p\|_{L^2}\|w\|_{L^2}\|\phi\|_{L^\infty},$$

$$\left|\int_I \varepsilon_r \bar{e}_q \,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_r s_j(x)(x - x_j)/h_j \,\mathrm{d}x\right| \leqslant \|\varepsilon_r\|_{L^2}\|s\|_{L^2}\|\phi\|_{L^\infty},$$

$$\left|\int_I \varepsilon_q \bar{e}_r \,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_q g_j(x)(x - x_j)/h_j \,\mathrm{d}x\right| \leqslant \|\varepsilon_q\|_{L^2}\|g\|_{L^2}\|\phi\|_{L^\infty},$$

$$\left|\int_I \varepsilon_p \bar{e}_u \,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_p d_j(x)(x - x_j)/h_j \,\mathrm{d}x\right| \leqslant \|\varepsilon_p\|_{L^2}\|d\|_{L^2}\|\phi\|_{L^\infty},$$

where $\phi = (x - x_j)/h_j$. Then, equation (2.25) becomes

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_p\|_{L^2}^2 \leqslant \|\phi\|_{L^\infty}\big[&\|(\varepsilon_u)_t\|_{L^2}\|d\|_{L^2} + \|\varepsilon_p\|_{L^2}\|w\|_{L^2} + \|\varepsilon_r\|_{L^2}\|s\|_{L^2}\\ &+ \|\varepsilon_q\|_{L^2}\|g\|_{L^2} + |\beta|\|\varepsilon_p\|_{L^2}\|d\|_{L^2}\big] + \frac{\beta^2}{2}\|\bar{e}_u\|_{L^2}^2.\end{aligned}$$

Using the approximation property of the projections (2.9) and the fact that $\|\phi\|_{L^\infty} = \frac{1}{2}$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_p\|_{L^2}^2 \leqslant Ch^{k+1}(\|d\|_{L^2} + \|s\|_{L^2} + \|w\|_{L^2} + \|g\|_{L^2}) + \frac{\beta^2}{2}\|\bar{e}_u\|_{L^2}^2, \tag{2.34}$$

where $C = C(\beta, \|u\|_{k+4}, \|u_t\|_{k+1})$. Substituting equations (2.27), (2.28), (2.29) and (2.33) into equation (2.34), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_p\|_{L^2}^2 \leqslant Ch^{k+2}(\|(e_u)_t\|_{L^2} + \|e_q\|_{L^2} + \|e_p\|_{L^2} + \|e_r\|_{L^2}) + \frac{\beta^2}{2}\|\bar{e}_u\|_{L^2}^2, \tag{2.35}$$

where $C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+1})$. Integrating the above inequality with respect to time and using the bound for initial error (2.17), we obtain

$$\frac{1}{2}\|\bar{e}_u(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\bar{e}_p(t)\|_{L^2}^2\,\mathrm{d}t \leqslant Ch^{k+2}\int_0^t (\|(e_u)_t\|_{L^2} + \|e_q\|_{L^2} + \|e_p\|_{L^2} + \|e_r\|_{L^2})\,\mathrm{d}t$$
$$+ \frac{\beta^2}{2}\int_0^t \|\bar{e}_u(t)\|_{L^2}^2\,\mathrm{d}t + Ch^{2k+3}, \tag{2.36}$$

where $C = C(\alpha, \beta, \lambda, \|u\|_{k+4}, \|u_t\|_{k+1})$.

To get the superconvergence result we need the following lemma.

LEMMA 2.8 Under the same condition as in Theorem 2.5 we have

$$\|e_u(t)\|_{L^2} + \|e_q(t)\|_{L^2} \leqslant C\mathrm{e}^{C_1 t}h^{k+1}, \tag{2.37}$$

$$\int_0^t (\|e_p\|_{L^2} + \|e_r\|_{L^2})\,\mathrm{d}t \leqslant C\mathrm{e}^{C_1 t}h^{k+1}, \tag{2.38}$$

where $C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+3})$. Moreover, we have

$$\int_0^t \|(e_u)_t\|_{L^2}\,\mathrm{d}t \leqslant C\mathrm{e}^{C_1 t}h^{k+1}, \tag{2.39}$$

where $C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1})$.

The proof of this lemma is given in Appendix. Using Lemma 2.8 we get from equation (2.36) that

$$\frac{1}{2}\|\bar{e}_u(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\bar{e}_p(t)\|_{L^2}^2\,\mathrm{d}t \leqslant \frac{\beta^2}{2}\int_0^t \|\bar{e}_u(t)\|_{L^2}^2\,\mathrm{d}t + C\mathrm{e}^{C_1 t}h^{2k+3}.$$

Gronwall's inequality gives us the desired result:

$$\|\bar{e}_u(t)\|_{L^2}^2 + \int_0^t \|\bar{e}_p(t)\|_{L^2}^2\,\mathrm{d}t \leqslant C\mathrm{e}^{2C_1 t}h^{2k+3}$$

and, in particular,

$$\|\bar{e}_u(t)\|_{L^2} \leqslant C\mathrm{e}^{C_1 t}h^{k+3/2},$$

where $C = C(\alpha, \beta, \lambda, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1})$ and $C_1 = C_1(\alpha, \beta) > 0$.

3. Numerical examples

In this section we provide some numerical experiments to demonstrate the superconvergence of the LDG method for fourth-order problems. We do not pay attention to the efficiency of time discretizations, thus either the third-order explicit total variation diminishing (TVD) Runge–Kutta method (Shu & Osher, 1988) or the second-order implicit Crank–Nicholson method can be used in our calculation.

EXAMPLE 3.1 To demonstrate superconvergence as well as the long-time behaviour of the error, we consider the linear fourth-order problem

$$\begin{cases} u_t + u_x + u_{xx} + u_{xxxx} = 0, \\ u(x, 0) = \sin x, \end{cases} \tag{3.1}$$

with periodic boundary conditions. The exact solution to this problem is

$$u(x, t) = \sin(x - t). \tag{3.2}$$

Note that for problems containing high-order derivatives, such as problem (3.1), the popular explicit nonlinearly stable high-order TVD Runge–Kutta methods (Shu & Osher, 1988) will suffer from extremely small time step restriction due to the stiffness of the LDG spatial discretization operator. Thus, the second-order implicit Crank–Nicholson time discretization method is used to perform long-time simulations in this example. We consider both the special projection P_h^* and the usual L^2 projection of the initial condition as our numerical initial conditions and get similar results. Uniform meshes and numerical fluxes (2.3) are used in the calculation.

Table 1 lists the numerical errors and their orders for $k = 1$ at different final times T when the special projection $P_h^* u$ is used as the initial condition. From the table we conclude that, at any time, we can observe third-order accuracy for $\bar{e}_u$ and $\bar{e}_p$, indicating that the error estimate obtained in equation (2.18) is not optimal. Even though we have derived an exponential growth result for $\bar{e}_u$ and $\bar{e}_p$, we

TABLE 1 *P^1 polynomials for Example* 3.1 *on a uniform mesh of N cells at different times T; P_h^* projection of the initial condition*

		$T = 1$		$T = 10$		$T = 100$	
	N	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\bar{e}_u$	20	4.68×10^{-04}	—	3.04×10^{-03}	—	2.92×10^{-02}	—
	40	6.18×10^{-05}	2.92	3.87×10^{-04}	2.97	3.75×10^{-03}	2.96
	80	7.93×10^{-06}	2.96	4.90×10^{-05}	2.98	4.73×10^{-04}	2.99
	160	1.00×10^{-06}	2.98	6.15×10^{-06}	3.00	5.92×10^{-05}	3.00
e_u	20	4.27×10^{-03}	—	5.25×10^{-03}	—	2.96×10^{-02}	—
	40	1.06×10^{-03}	2.01	1.13×10^{-03}	2.22	3.90×10^{-03}	2.93
	80	2.65×10^{-04}	2.00	2.70×10^{-04}	2.07	5.42×10^{-04}	2.85
	160	6.64×10^{-05}	2.00	6.66×10^{-05}	2.02	8.89×10^{-05}	2.61
$\bar{e}_p$	20	4.66×10^{-04}	—	3.06×10^{-03}	—	2.93×10^{-02}	—
	40	6.16×10^{-05}	2.92	3.88×10^{-04}	2.98	3.75×10^{-03}	2.96
	80	7.92×10^{-06}	2.96	4.90×10^{-05}	2.98	4.73×10^{-04}	2.99
	160	1.00×10^{-06}	2.98	6.15×10^{-06}	3.00	5.92×10^{-05}	3.00
e_p	20	4.27×10^{-03}	—	5.26×10^{-03}	—	2.96×10^{-02}	—
	40	1.06×10^{-03}	2.01	1.13×10^{-03}	2.22	3.90×10^{-03}	2.93
	80	2.65×10^{-04}	2.00	2.70×10^{-04}	2.07	5.42×10^{-04}	2.85
	160	6.64×10^{-05}	2.00	6.66×10^{-05}	2.02	8.89×10^{-05}	2.61

can clearly observe that they actually grow linearly with respect to time for this particular example, which guarantees that the errors for e_u and e_p do not grow much with respect to time for a long time, $t = \mathcal{O}(h^{-1})$. This is especially prominent for fine grids.

Table 2 lists the numerical errors and their orders for $k = 2$ at different final times T when the special projection $P_h^* u$ is used as the initial condition. We can clearly see that both $\bar{e}_u$ and $\bar{e}_p$ achieve fourth-order accuracy at $T = 1$. For longer time, for example $T = 10$ and $T = 50$, the orders seem also to converge to four, if we keep on refining the mesh. We also observe that the errors for both $\bar{e}_u$ and $\bar{e}_p$ do not grow much until the final time we have run ($T = 50$), especially for fine grids. For the case of $k = 3$ the results in Table 3 also demonstrate the superconvergence of $\bar{e}_u$ and $\bar{e}_p$.

If we use the L^2 projection of the initial condition as our numerical initial condition instead, we also obtain the superconvergence results for $\bar{e}_u$ and $\bar{e}_p$ and observe little difference compared to the case when $P_h^* u$ is used as the numerical initial condition, indicating that the definition of the operator P_h^* is only for a technical purpose in the proof and not essential to the computation; see Tables 4–6.

We would like to mention that, apart from the superconvergence results for $\bar{e}_u$ and $\bar{e}_p$, we have also obtained similar superconvergence results for $\bar{e}_q$ and $\bar{e}_r$ in our numerical experiments, which are not listed here to save space.

EXAMPLE 3.2 We consider problem (3.1) with exact solution (3.2) and the boundary conditions

$$u(0,t) = g_1(t), \quad u_x(2\pi,t) = g_2(t), \quad u_{xx}(0,t) = g_3(t), \quad u_{xxx}(2\pi,t) = g_4(t), \tag{3.3}$$

where $g_i(t)$ corresponds to the data from the exact solution.

TABLE 2 *P^2 polynomials for Example 3.1 on a uniform mesh of N cells at different time T. P_h^* projection of the initial condition*

		$T = 1$		$T = 10$		$T = 50$	
	N	L^2 error	Order	L^2 error	Order	L^2 error	Order
$\bar{e}_u$	10	6.93×10^{-05}	—	1.17×10^{-04}	—	4.72×10^{-04}	—
	20	4.23×10^{-06}	4.03	5.20×10^{-06}	4.49	1.54×10^{-05}	4.93
	40	2.63×10^{-07}	4.01	2.79×10^{-07}	4.22	5.30×10^{-07}	4.86
	80	1.64×10^{-08}	4.00	1.64×10^{-08}	4.08	1.90×10^{-08}	4.80
e_u	10	8.56×10^{-04}	—	8.62×10^{-04}	—	9.76×10^{-04}	—
	20	1.07×10^{-04}	3.00	1.07×10^{-04}	3.01	1.08×10^{-04}	3.18
	40	1.34×10^{-05}	3.00	1.34×10^{-05}	3.00	1.34×10^{-05}	3.01
	80	1.67×10^{-06}	3.00	1.67×10^{-06}	3.00	1.67×10^{-06}	3.00
$\bar{e}_p$	10	5.69×10^{-05}	—	1.10×10^{-04}	—	4.71×10^{-04}	—
	20	3.81×10^{-06}	3.90	4.87×10^{-06}	4.50	1.53×10^{-05}	4.94
	40	2.49×10^{-07}	3.94	2.66×10^{-07}	4.19	5.24×10^{-07}	4.87
	80	1.60×10^{-08}	3.96	1.60×10^{-08}	4.06	1.87×10^{-08}	4.81
e_p	10	8.56×10^{-04}	—	8.62×10^{-04}	—	9.76×10^{-04}	—
	20	1.07×10^{-04}	3.00	1.07×10^{-04}	3.01	1.08×10^{-04}	3.18
	40	1.34×10^{-05}	3.00	1.34×10^{-05}	3.00	1.34×10^{-05}	3.01
	80	1.67×10^{-06}	3.00	1.67×10^{-06}	3.00	1.67×10^{-06}	3.00

TABLE 3 *P^3 polynomials for Example* 3.1 *on a uniform mesh of N cells; $T = 1$; P_h^* projection of the initial condition*

	$\bar{e}_u$		e_u		$\bar{e}_p$		e_p	
N	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
5	6.60×10^{-05}	—	5.26×10^{-04}	—	5.63×10^{-05}	—	5.27×10^{-04}	—
10	1.73×10^{-06}	5.25	3.30×10^{-05}	4.00	1.52×10^{-06}	5.22	3.30×10^{-05}	4.00
20	5.39×10^{-08}	5.01	2.06×10^{-06}	4.00	5.02×10^{-08}	4.91	2.06×10^{-06}	4.00
40	1.71×10^{-09}	4.98	1.29×10^{-07}	4.00	1.69×10^{-09}	4.89	1.29×10^{-07}	4.00

TABLE 4 *P^1 polynomials for Example* 3.1 *on a uniform mesh of N cells; $T = 1$; L^2 projection of the initial condition*

	$\bar{e}_u$		e_u		$\bar{e}_p$		e_p	
N	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
20	4.36×10^{-04}	—	4.26×10^{-03}	—	4.38×10^{-04}	—	4.26×10^{-03}	—
40	5.63×10^{-05}	2.95	1.06×10^{-03}	2.00	5.64×10^{-05}	2.96	1.06×10^{-03}	2.00
80	7.15×10^{-06}	2.98	2.66×10^{-04}	2.00	7.16×10^{-06}	2.98	2.66×10^{-04}	2.00
160	9.00×10^{-07}	2.99	6.64×10^{-05}	2.00	9.01×10^{-07}	2.99	6.64×10^{-05}	2.00

TABLE 5 *P^2 polynomials for Example* 3.1 *on a uniform mesh of N cells; $T = 1$; L^2 projection of the initial condition*

	$\bar{e}_u$		e_u		$\bar{e}_p$		e_p	
N	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
10	6.90×10^{-05}	—	8.56×10^{-04}	—	5.66×10^{-05}	—	8.56×10^{-04}	—
20	4.23×10^{-06}	4.03	1.07×10^{-04}	3.00	3.81×10^{-06}	3.89	1.07×10^{-04}	3.00
40	2.62×10^{-07}	4.01	1.34×10^{-05}	3.00	2.49×10^{-07}	3.93	1.34×10^{-05}	3.00
80	1.65×10^{-08}	3.99	1.67×10^{-06}	3.00	1.61×10^{-08}	3.95	1.67×10^{-06}	3.00

TABLE 6 *P^3 polynomials for Example* 3.1 *on a uniform mesh of N cells; $T = 1$; L^2 projection of the initial condition*

	$\bar{e}_u$		e_u		$\bar{e}_p$		e_p	
N	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
5	5.58×10^{-05}	—	5.25×10^{-04}	—	4.43×10^{-05}	—	5.25×10^{-04}	—
10	1.73×10^{-06}	5.01	3.30×10^{-05}	3.99	1.51×10^{-06}	4.88	3.30×10^{-05}	3.99
20	5.39×10^{-08}	5.00	2.06×10^{-06}	4.00	5.02×10^{-08}	4.91	2.06×10^{-06}	4.00
40	1.95×10^{-09}	4.79	1.29×10^{-07}	4.00	1.80×10^{-09}	4.80	1.29×10^{-07}	4.00

Note that the above boundary conditions are matched with the alternating numerical fluxes (2.3). Both here and below, we use the third-order explicit TVD Runge–Kutta method and the L^2 projection of the initial condition as our numerical initial condition. Table 7 lists the results for both P^1 and P^2

TABLE 7 *P^1 and P^2 polynomials for Example* 3.2 *with boundary conditions* (3.3) *on a uniform mesh of N cells; $T = 0.5$*

P^k	$k=1$				$k=2$			
	$\bar{e}_u$		e_u		$\bar{e}_u$		e_u	
N	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
10	3.07×10^{-03}	—	1.71×10^{-02}	—	6.85×10^{-05}	—	8.56×10^{-04}	—
20	3.95×10^{-04}	2.96	4.25×10^{-03}	2.01	4.22×10^{-06}	4.02	1.07×10^{-04}	3.00
40	5.03×10^{-05}	2.98	1.06×10^{-03}	2.00	2.62×10^{-07}	4.01	1.34×10^{-05}	3.00
80	6.34×10^{-06}	2.99	2.65×10^{-04}	2.00	1.71×10^{-08}	3.94	1.67×10^{-06}	3.00

polynomials at $T = 0.5$ when fluxes (2.3) are used. To impose the given boundary conditions (3.3), the corresponding boundary fluxes are defined as

$$(u_h)^-_{\frac{1}{2}} = g_1(t), \quad (q_h)^+_{N+\frac{1}{2}} = g_2(t), \quad (p_h)^-_{\frac{1}{2}} = g_3(t), \quad (r_h)^+_{N+\frac{1}{2}} = g_4(t).$$

From the table we can clearly see that $\bar{e}_u$ achieves $(k+2)$th-order superconvergence and the error e_u achieves the expected $(k+1)$th order of accuracy.

EXAMPLE 3.3 We consider problem (3.1) with exact solution (3.2) and the boundary conditions

$$u(0,t) = h_1(t), \quad u(2\pi,t) = h_2(t), \quad u_x(0,t) = h_3(t), \quad u_x(2\pi,t) = h_4(t), \tag{3.4}$$

where $h_i(t)$ corresponds to the data from the exact solution.

In this example the minimal dissipation LDG method is used to deal with the Dirichlet boundary conditions (3.4). The distinctive feature of this method is that the stabilization parameters associated with the numerical fluxes are taken to be identically zero on all interior cell interfaces (that is, only the numerical fluxes at boundaries are penalized) and this is why its dissipation is said to be minimal; see, e.g. Castillo *et al.* (2002). More precisely, the numerical fluxes based on equation (2.3) for u_h, q_h, p_h, r_h are chosen as

$$(\hat{u}_h, \hat{q}_h, \hat{p}_h, \hat{r}_h)_{j+\frac{1}{2}} = \begin{cases} (u_h^-, q_h^+, p_h^-, r_h^+)_{j+\frac{1}{2}}, & j = 1, \ldots, N-1, \\ (u_h^-, q_h^-, \hat{p}_h, r_h^+)_{\frac{1}{2}}, & j = 0, \\ (u_h^+, q_h^+, p_h^-, \hat{r}_h)_{N+\frac{1}{2}}, & j = N, \end{cases} \tag{3.5}$$

where

$$(u_h)^-_{\frac{1}{2}} = h_1(t), \quad (u_h)^+_{N+\frac{1}{2}} = h_2(t), \quad (q_h)^-_{\frac{1}{2}} = h_3(t), \quad (q_h)^+_{N+\frac{1}{2}} = h_4(t),$$

and

$$(\hat{p}_h)_{\frac{1}{2}} = (p_h)^+_{\frac{1}{2}} + \kappa_1 [q_h]_{\frac{1}{2}}, \quad (\hat{r}_h)_{N+\frac{1}{2}} = (r_h)^-_{N+\frac{1}{2}} - \kappa_2 [u_h]_{N+\frac{1}{2}},$$

with κ_1 and κ_2 positive constants that are $\mathcal{O}(h^{-1})$ and $\mathcal{O}(h^{-3})$, respectively. The numerical errors and their orders obtained by using P^1 and P^2 polynomials at $T = 0.5$ are listed in Table 8. From the table we can clearly see that $\bar{e}_u$ achieves $(k+2)$th-order superconvergence and the error e_u achieves the

TABLE 8 P^1 *and* P^2 *polynomials for Example* 3.3 *with boundary conditions* (3.4) *on a uniform mesh of N cells;* $T = 0.5$*;* $\kappa_1 = 30/h$, $\kappa_2 = 10/h^3$

P^k	$k=1$				$k=2$			
	$\bar{e}_u$		e_u		$\bar{e}_u$		e_u	
N	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
10	2.82×10^{-03}	—	1.71×10^{-02}	—	7.00×10^{-05}	—	8.55×10^{-04}	—
20	3.53×10^{-04}	3.00	4.25×10^{-03}	2.01	4.22×10^{-06}	4.05	1.07×10^{-04}	3.00
40	4.43×10^{-05}	2.99	1.06×10^{-03}	2.00	2.62×10^{-07}	4.01	1.34×10^{-05}	3.00
80	5.58×10^{-06}	2.99	2.65×10^{-04}	2.00	1.70×10^{-08}	3.95	1.67×10^{-06}	3.00

expected $(k+1)$th order of accuracy. Based on the results in Examples 3.2 and 3.3, we conclude that the superconvergence property also holds true for initial boundary value problems.

EXAMPLE 3.4 To test the validity of the superconvergence property for solutions with singularities, we solve the equation $u_t + u_x + u_{xx} + u_{xxxx} = 0$ on the interval $[-1, 1]$ with discontinuous initial data

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \leqslant 0.5, \\ 0 & \text{otherwise,} \end{cases} \tag{3.6}$$

and periodic boundary conditions.

By Fourier analysis we can derive the exact solution of Example 3.4 in the form

$$u(x,t) = \frac{1}{2} + 2\sum_{\omega=1}^{\infty} \mathrm{e}^{(\omega^2\pi^2-\omega^4\pi^4)t} \frac{\sin\left(\frac{\omega\pi}{2}\right)}{\omega\pi} \cos(\omega\pi(x-t)).$$

In our computation the exact solution is taken as

$$u(x,t) = \frac{1}{2} + 2\sum_{\omega=1}^{5} \mathrm{e}^{(\omega^2\pi^2-\omega^4\pi^4)t} \frac{\sin\left(\frac{\omega\pi}{2}\right)}{\omega\pi} \cos(\omega\pi(x-t)) \tag{3.7}$$

with negligible error. The numerical errors and their orders for Example 3.4 using both P^1 and P^2 polynomials at $T = 0.05$ are given in Table 9 from which we can clearly observe $(k+2)$th and $(k+1)$th orders of accuracy for $\bar{e}_u$ and e_u, respectively; that is, the conclusions also hold true for solutions with singularities in the initial condition, provided the singularities are located at cell boundaries.

EXAMPLE 3.5 In this example we solve the linear time-dependent biharmonic equation

$$\begin{cases} u_t + u_{xxxx} = 0, \\ u(x,0) = \sin x, \end{cases} \tag{3.8}$$

with periodic boundary conditions. The exact solution to this problem is

$$u(x,t) = \mathrm{e}^{-t} \sin x. \tag{3.9}$$

TABLE 9 *P^1 and P^2 polynomials for Example* 3.4 *with discontinuous initial data* (3.6) *on a uniform mesh of N cells; $T = 0.05$*

P^k	$k=1$				$k=2$			
	$\bar{e}_u$		e_u		$\bar{e}_u$		e_u	
N	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
4	1.08×10^{-03}	—	1.18×10^{-03}	—	5.26×10^{-05}	—	1.08×10^{-04}	—
8	5.93×10^{-05}	4.19	2.21×10^{-04}	2.42	3.01×10^{-06}	4.13	1.34×10^{-05}	3.02
16	4.89×10^{-06}	3.60	5.34×10^{-05}	2.05	1.85×10^{-07}	4.02	1.67×10^{-06}	3.00
32	5.40×10^{-07}	3.18	1.33×10^{-05}	2.01	1.15×10^{-08}	4.00	2.09×10^{-07}	3.00

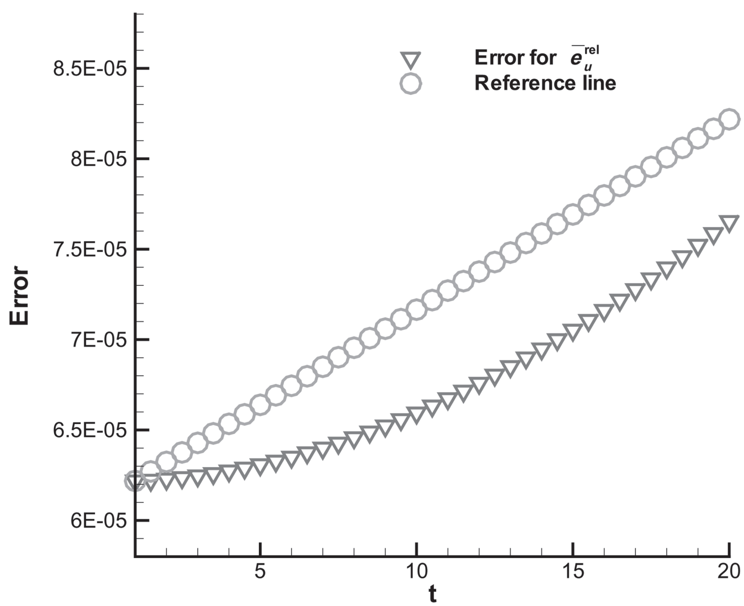

FIG. 1. The growth of the relative error versus time using P^1 polynomials.

Note that the exact solution u is exponentially decaying with respect to time; let us denote by $\bar{e}_u^{\text{rel}}$ a measure of the relative error between $\bar{e}_u$ and u, namely, $\|\bar{e}_u^{rel}\|_{L^2} = \frac{\|\bar{e}_u\|_{L^2}}{\|u\|_{L^2}}$. The growth of the relative error in the L^2 norm, $\|\bar{e}_u^{rel}\|_{L^2}$, versus time, obtained by using P^1 polynomials on a uniform mesh of 40 cells, is plotted in Fig. 1. We can see that the relative error grows essentially linearly with respect to time from $T = 8$ to the final time $T = 20$ that we have run. This example demonstrates that not only the absolute error grows at most linearly with time, as proved in Theorem 2.7, but also the relative error grows only linearly in time.

TABLE 10 *P^1 and P^2 polynomials for Example* 3.6 *solving the Kuramoto–Sivashinsky equation* (3.10) *on a uniform mesh of N cells; $T = 0.5$*

P^k	$k=1$				$k=2$			
	$\bar{e}_u$		e_u		$\bar{e}_u$		e_u	
N	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
80	2.32×10^{-02}	—	7.04×10^{-02}	—	2.16×10^{-03}	—	5.30×10^{-03}	—
160	4.26×10^{-03}	2.44	1.64×10^{-02}	2.11	1.24×10^{-04}	4.12	6.59×10^{-04}	3.01
320	5.65×10^{-04}	2.91	3.99×10^{-03}	2.03	7.56×10^{-06}	4.04	8.23×10^{-05}	3.00
640	7.19×10^{-05}	2.97	9.92×10^{-04}	2.01	4.74×10^{-07}	4.00	1.03×10^{-05}	3.00

EXAMPLE 3.6 In order to see the superconvergence of the method for nonlinear problems, we consider the Kuramoto–Sivashinsky equation

$$u_t + \left(\frac{u^2}{2}\right)_x + u_{xx} + \sigma u_{xxx} + u_{xxxx} = 0, \tag{3.10}$$

with the exact solution given by

$$\begin{aligned} u(x,t) = c + 9 - 15(&\tanh(k(x-ct-x_0)) + \tanh^2(k(x-ct-x_0)) \\ &- \tanh^3(k(x-ct-x_0))), \end{aligned} \tag{3.11}$$

where $\sigma = 4$, $c = 6$, $k = \frac{1}{2}$ and $x_0 = -10$.

The computational domain is $[-30, 30]$. Although the exact solution is not periodic, we can still use periodic boundary conditions in our computation since the exact solution is negligibly small at the boundary of the domain for short-time simulations, for example, $T = 2$, due to the large size of the computational domain. We use the Godunov flux, which is an upwind flux, for the nonlinear convection part, and alternating fluxes (2.3) for other parts. The projection is defined element by element as follows. If $u(x_j, t)$ is positive we choose P_h^- on the cell I_j; otherwise, we use P_h^+. We test this example using both P^1 and P^2 polynomials at $T = 0.5$. The numerical errors and orders of accuracy for $\bar{e}_u$ and e_u are given in Table 10. From the table we can see that the error $\bar{e}_u$ achieves $(k+2)$th-order superconvergence and the error e_u achieves the expected $(k+1)$th order of accuracy. This example shows that the superconvergence property also holds true for some nonlinear equations.

4. Concluding remarks

In this paper we have studied the superconvergence of the LDG method for linear fourth-order time-dependent problems. We prove that the error between the numerical solution and a particular projection of the exact solution achieves $(k + \frac{3}{2})$th-order superconvergence when polynomials of degree k ($k \geqslant 1$) are used. Various numerical experiments, including linear equations, nonlinear equations, initial boundary value problems and solutions having singularities, are shown to demonstrate that the conclusions hold true for very general cases. Even though we consider only the one-dimensional case in this paper, similar results should hold for certain tensor product two-dimensional cases; see Cheng & Shu (2010) for related discussion for convection and second-order diffusion equations.

The theoretical study of the superconvergence of the DG or LDG method for nonlinear equations and for nonperiodic boundary conditions is more challenging and this will be carried out in the future.

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References

Adjerid, S. & Issaev, D. (2005) Superconvergence of the local discontinuous Galerkin method applied to diffusion problems. *Proceedings of the Third MIT Conference on Computational Fluid and Solid Mechanics* (K.-J. Bathe ed.). Boston, MA: Elsevier, pp. 1040–1044.

Adjerid, S. & Klauser, A. (2005) Superconvergence of discontinuous finite element solutions for transient convection–diffusion problems. *J. Sci. Comput.*, **22**, 5–24.

Bassi, F. & Rebay, S. (1997) A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier–Stokes equations. *J. Comput. Phys.*, **131**, 267–279.

Castillo, P., Cockburn, B., Schötzau, D. & Schwab, C. (2002) Optimal a priori error estimates for the *hp*-version of the local discontinuous Galerkin method for convection–diffusion problems. *Math. Comp.*, **71**, 455–478.

Cheng, Y. & Shu, C. W. (2008) Superconvergence and time evolution of discontinuous Galerkin finite element solutions. *J. Comput. Phys.*, **227**, 9612–9627.

Cheng, Y. & Shu, C. W. (2009) Superconvergence of local discontinuous Galerkin methods for one-dimensional convection–diffusion equations. *Comput. Struct.,* **87**, 630–641.

Cheng, Y. & Shu, C. W. (2010) Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection–diffusion equations in one space dimension. *SIAM J. Numer. Anal.*, **47**, 4044–4072.

Ciarlet, P. G. (1978) *The Finite Element Method for Elliptic Problems*. Amsterdam, The Netherlands: North-Holland.

Cockburn, B. (1999) Discontinuous Galerkin methods for convection-dominated problems. *High-Order Methods for Computational Physics* (T. J. Barth and H. Deconinck eds). Lecture Notes in Computational Science and Engineering. Berlin, Germany: Springer, pp. 69–224.

Cockburn, B., Hou, S. & Shu, C. W. (1990) The Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: the multidimensional case. *Math. Comp.*, **54**, 545–581.

Cockburn, B., Lin, S. Y. & Shu, C. W. (1989) TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws III: one dimensional systems. *J. Comput. Phys.*, **84**, 90–113.

Cockburn, B. & Shu, C. W. (1989) TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework. *Math. Comput.*, **52**, 411–435.

Cockburn, B. & Shu, C. W. (1998a) The local discontinuous Galerkin method for time-dependent convection–diffusion systems. *SIAM J. Numer. Anal.*, **35**, 2440–2463.

Cockburn, B. & Shu, C. W. (1998b) The Runge–Kutta discontinuous Galerkin method for conservation laws V: multidimensional systems. *J. Comput. Phys.*, **141**, 199–224.

Cockburn, B. & Shu, C. W. (2001) Runge–Kutta discontinuous Galerkin methods for convection-dominated problems. *J. Sci. Comput.*, **16**, 173–261.

Dong, B. & Shu, C. W. (2009) Analysis of a local discontinuous Galerkin method for linear time-dependent fourth-order problems. *SIAM J. Numer. Anal.*, **47**, 3240–3268.

Elliott, C. M. & Zheng, S. (1986) On the Cahn–Hilliard equation. *Arch. Rational Mech. Anal.*, **96**, 339–357.

FENG, X. & PROHL, A. (2004) Error analysis of a mixed finite element method for the Cahn–Hilliard equation. *Numer. Math.*, **99**, 47–84.

REED, W. H. & HILL, T. R. (1973) Triangular mesh methods for the neutron transport equation. *Technical Report LA-UR-73-479*. Los Alamos, NM: Los Alamos Scientific Laboratory.

SHU, C. W. & OSHER, S. (1988) Efficient implementation of essentially non-oscillatory shock-capturing schemes. *J. Comput. Phys.*, **77**, 439–471.

XIA, Y., XU, Y. & SHU, C. W. (2007) Local discontinuous Galerkin methods for the Cahn–Hilliard type equations. *J. Comput. Phys.*, **227**, 472–491.

XU, Y. & SHU, C. W. (2004) Local discontinuous Galerkin methods for three classes of nonlinear wave equations. *J. Comput. Math.*, **22**, 250–274.

XU, Y. & SHU, C. W. (2006) Local discontinuous Galerkin methods for the Kuramoto–Sivashinsky equations and the Ito-type coupled KdV equations. *Comput. Methods Appl. Mech. Eng.*, **195**, 3430–3447.

XU, Y. & SHU, C. W. (2010) Local discontinuous Galerkin methods for high-order time-dependent partial differential equations. *Commun. Comput. Phys.*, **7**, 1–46.

XU, Y. & SHU, C. W. Optimal error estimates of the semi-discrete local discontinuous Galerkin methods for high order wave equations. *SIAM J. Numer. Anal.* (in press).

YAN, J. & SHU, C. W. (2002a) A local discontinuous Galerkin method for KdV type equations. *SIAM J. Numer. Anal.*, **40**, 769–791.

YAN, J. & SHU, C. W. (2002b) Local discontinuous Galerkin methods for partial differential equations with higher order derivatives. *J. Sci. Comput.*, **17**, 27–47.

ZHANG, Q. & SHU, C. W. (2010) Stability analysis and a priori error estimates to the third order explicit Runge–Kutta discontinuous Galerkin method for scalar conservation laws. *SIAM J. Numer. Anal.*, **48**, 1038–1063.

Appendix

In this appendix we give the proofs for some of the technical lemmas and theorems.

A.1 *The proof of Lemma* 2.4

We will first prove the existence and uniqueness of $P_h^* u$.
When using the DG discretization operator $\mathcal{D}$, equation (2.14) can be written as

$$\int_{I_j} r_h \eta \, \mathrm{d}x - \mathcal{D}_{I_j}(p_h, \eta; p_h^-) = 0, \tag{A.1a}$$

$$\int_{I_j} p_h \xi \, \mathrm{d}x - \mathcal{D}_{I_j}(q_h, \xi; q_h^+) = 0, \tag{A.1b}$$

$$\int_{I_j} q_h \psi \, \mathrm{d}x - \mathcal{D}_{I_j}(P_h^* u, \psi; (P_h^* u)^-) = 0 \tag{A.1c}$$

for any $\psi, \xi, \eta \in V_h^k$. Since the exact solutions $u, q = u_x, p = u_{xx}, r = u_{xxx}$ also satisfy scheme (A.1), we thus have the error equations

$$\int_{I_j} (r - r_h)\eta \, \mathrm{d}x - \mathcal{D}_{I_j}(p - p_h, \eta; (p - p_h)^-) = 0, \tag{A.2a}$$

$$\int_{I_j} (p - p_h)\xi \, \mathrm{d}x - \mathcal{D}_{I_j}(q - q_h, \xi; (q - q_h)^+) = 0, \tag{A.2b}$$

$$\int_{I_j} (q - q_h)\psi \,\mathrm{d}x - \mathcal{D}_{I_j}(u - P_h^* u, \psi; (u - P_h^* u)^-) = 0, \tag{A.2c}$$

for any $\psi, \xi, \eta \in V_h^k$. Denote

$$u - P_h^* u = (u - P_h^- u) + (P_h^- u - P_h^* u) = \varepsilon_u + E_u,$$

$$q - q_h = (q - P_h^+ q) + (P_h^+ q - q_h) = \varepsilon_q + E_q,$$

$$p - p_h = (p - P_h^- p) + (P_h^- p - p_h) = \varepsilon_p + E_p,$$

$$r - r_h = (r - P_h^+ r) + (P_h^+ r - r_h) = \varepsilon_r + E_r.$$

By virtue of the properties of the projections P_h^+ and P_h^- given in equations (2.7) and (2.8), equation (A.2) becomes

$$\int_{I_j} (\varepsilon_r + E_r)\eta \,\mathrm{d}x - \mathcal{D}_{I_j}(E_p, \eta; E_p^-) = 0, \tag{A.3a}$$

$$\int_{I_j} (\varepsilon_p + E_p)\xi \,\mathrm{d}x - \mathcal{D}_{I_j}(E_q, \xi; E_q^+) = 0, \tag{A.3b}$$

$$\int_{I_j} (\varepsilon_q + E_q)\psi \,\mathrm{d}x - \mathcal{D}_{I_j}(E_u, \psi; E_u^-) = 0. \tag{A.3c}$$

Also, conditions (2.15) and (2.16) are equivalent to

$$\int_{I_j} (E_u - E_q + E_r)\rho \,\mathrm{d}x = 0 \tag{A.4}$$

for any $\rho \in P^{k-1}$ on I_j and

$$E_u^- = (E_q - E_r)^+ \quad \text{at } x_{j-\frac{1}{2}}. \tag{A.5}$$

Note that equations (A.3), (A.4) and (A.5) are a linear system for $E_u, E_q, E_p, E_r \in V_h^k$. To prove the existence and uniqueness of $P_h^* u$, we need only to prove the uniqueness of E_u, then $P_h^* u = P_h^- u - E_u$ will exist and is unique.

Plugging conditions (A.4) and (A.5) into (A.3), we obtain

$$\int_{I_j} (\varepsilon_r + E_r)\eta \,\mathrm{d}x - \mathcal{D}_{I_j}(E_p, \eta; E_p^-) = 0, \tag{A.6a}$$

$$\int_{I_j} (\varepsilon_p + E_p)\xi \,\mathrm{d}x - \mathcal{D}_{I_j}(E_q, \xi; E_q^+) = 0, \tag{A.6b}$$

$$\int_{I_j} (\varepsilon_q + E_q)\psi \,\mathrm{d}x - \mathcal{D}_{I_j}(E_q - E_r, \psi; (E_q - E_r)^+) = 0, \tag{A.6c}$$

which is

$$\int_{I_j} E_r \eta \, dx - \mathcal{D}_{I_j}(E_p, \eta; E_p^-) = -\int_{I_j} \varepsilon_r \eta \, dx, \tag{A.7a}$$

$$\int_{I_j} E_p \xi \, dx - \mathcal{D}_{I_j}(E_q, \xi; E_q^+) = -\int_{I_j} \varepsilon_p \xi \, dx, \tag{A.7b}$$

$$\int_{I_j} E_q \psi \, dx - \mathcal{D}_{I_j}(E_q - E_r, \psi; (E_q - E_r)^+) = -\int_{I_j} \varepsilon_q \psi \, dx, \tag{A.7c}$$

for any $\psi, \xi, \eta \in V_h^k$. Note that equation (A.7) is a linear system, hence the existence of (E_q, E_p, E_r) follows by the uniqueness.

We claim that the solution (E_q, E_p, E_r) to equation (A.7) is unique. Suppose both (E_q^1, E_p^1, E_r^1) and (E_q^2, E_p^2, E_r^2) satisfy equation (A.7) and denote $g_q = E_q^1 - E_q^2$, $g_p = E_p^1 - E_p^2$, $g_r = E_r^1 - E_r^2$, then equation (A.7) yields

$$\int_{I_j} g_r \eta \, dx - \mathcal{D}_{I_j}(g_p, \eta; g_p^-) = 0, \tag{A.8a}$$

$$\int_{I_j} g_p \xi \, dx - \mathcal{D}_{I_j}(g_q, \xi; g_q^+) = 0, \tag{A.8b}$$

$$\int_{I_j} g_q \psi \, dx - \mathcal{D}_{I_j}(g_q - g_r, \psi; (g_q - g_r)^+) = 0, \tag{A.8c}$$

for any $\psi, \xi, \eta \in V_h^k$. Now taking $(\psi, \xi, \eta) = (g_q - g_r, g_p, g_q)$ in equation (A.8), adding them up and summing over all j, we get

$$\|g_q\|_{L^2}^2 + \|g_p\|_{L^2}^2 - \mathcal{D}(g_p, g_q; g_p^-) - \mathcal{D}(g_q, g_p; g_q^+) - \mathcal{D}(g_q - g_r, g_q - g_r; (g_q - g_r)^+) = 0.$$

By the property of the operator $\mathcal{D}$ in Lemma 2.1, we have

$$\|g_q\|_{L^2}^2 + \|g_p\|_{L^2}^2 + \frac{1}{2}\sum_j [g_q - g_r]_{j+\frac{1}{2}}^2 = 0,$$

which implies $g_q = g_p = 0$ and further, $g_r = 0$. We have thus proved the existence and uniqueness of E_q and E_r, then conditions (A.4) and (A.5) lead to the existence and uniqueness of E_u and thus $P_h^* u$.

We obtain the error estimate (2.17) in three steps.

Step 1. By Lemma 2.2 equation (A.6) can be rewritten as

$$\int_{I_j} (\varepsilon_r + E_r)\eta \, dx - \int_{I_j} (E_p)_x \eta \, dx - [E_p]\eta^+|_{j-\frac{1}{2}} = 0, \tag{A.9a}$$

$$\int_{I_j} (\varepsilon_p + E_p)\xi \, dx - \int_{I_j} (E_q)_x \xi \, dx - [E_q]\xi^-|_{j+\frac{1}{2}} = 0, \tag{A.9b}$$

$$\int_{I_j} (\varepsilon_q + E_q)\psi \, dx - \int_{I_j} (E_q - E_r)_x \psi \, dx - [E_q - E_r]\psi^-|_{j+\frac{1}{2}} = 0. \tag{A.9c}$$

Define $E_q = b_j + s_j(x)(x - x_j)/h_j$, $E_p = v_j + w_j(x)(x - x_j)/h_j$, $E_r = l_j + g_j(x)(x - x_j)/h_j$ on I_j, where b_j, v_j, l_j are constants and $s_j(x), w_j(x), g_j(x) \in P^{k-1}$. First, we let $\psi = (s_j(x) - g_j(x))(x - x_{j+\frac{1}{2}})/h_j$ in equation (A.9c) and get, by the definition of $\mathcal{B}_j^+$,

$$\int_{I_j} (\varepsilon_q + E_q)(s_j(x) - g_j(x))(x - x_{j+\frac{1}{2}})/h_j \,\mathrm{d}x - \mathcal{B}_j^+(s_j - g_j) = 0.$$

Using the property of $\mathcal{B}_j^+$ in Lemma 2.3, we have

$$\int_{I_j} (\varepsilon_q + E_q)(s_j(x) - g_j(x))(x - x_{j+\frac{1}{2}})/h_j \,\mathrm{d}x + \frac{1}{4h_j}\int_{I_j} (s_j(x) - g_j(x))^2 \,\mathrm{d}x$$
$$+ \frac{1}{4}(s_j(x_{j-\frac{1}{2}}) - g_j(x_{j-\frac{1}{2}}))^2 = 0.$$

Thus,

$$\int_{I_j} (s_j(x) - g_j(x))^2 \,\mathrm{d}x \leqslant -4\int_{I_j} (\varepsilon_q + E_q)(s_j(x) - g_j(x))(x - x_{j+\frac{1}{2}}) \,\mathrm{d}x. \tag{A.10}$$

Define piecewise polynomials $s(x)$, $g(x)$ and $\phi_2(x)$, such that $s(x) = s_j(x)$, $g(x) = g_j(x)$, $\phi_2(x) = x - x_{j+\frac{1}{2}}$ on I_j, then summing equation (A.10) over all j,

$$\|s - g\|_{L^2} \leqslant 4\|\varepsilon_q + E_q\|_{L^2}\|\phi_2\|_{L^\infty}.$$

By approximation results (2.9) and the fact that $\|\phi_2\|_{L^\infty} = h$ we get

$$\|s - g\|_{L^2} \leqslant 4h\|\varepsilon_q + E_q\|_{L^2} \leqslant Ch^{k+2} + Ch\|E_q\|_{L^2}, \tag{A.11}$$

where $C = C(\|u\|_{k+2})$. Similarly, letting $\xi = s_j(x)(x - x_{j+\frac{1}{2}})/h_j$ in equation (A.9b) and $\eta = w_j(x)(x - x_{j-\frac{1}{2}})/h_j$ in equation (A.9a), and using the definition of $\mathcal{B}_j^-$ and $\mathcal{B}_j^+$, we get

$$\int_{I_j} (\varepsilon_r + E_r)w_j(x)(x - x_{j-\frac{1}{2}})/h_j \,\mathrm{d}x - \mathcal{B}_j^-(w_j) = 0,$$

$$\int_{I_j} (\varepsilon_p + E_p)s_j(x)(x - x_{j+\frac{1}{2}})/h_j \,\mathrm{d}x - \mathcal{B}_j^+(s_j) = 0.$$

Using the properties of $\mathcal{B}_j^-$ and $\mathcal{B}_j^+$ in Lemma 2.3, we have

$$\int_{I_j} (\varepsilon_r + E_r)w_j(x)(x - x_{j-\frac{1}{2}})/h_j \,\mathrm{d}x - \frac{1}{4h_j}\int_{I_j} w_j^2(x) \,\mathrm{d}x - \frac{w_j^2(x_{j+\frac{1}{2}})}{4} = 0,$$

$$\int_{I_j} (\varepsilon_p + E_p)s_j(x)(x - x_{j+\frac{1}{2}})/h_j \,\mathrm{d}x + \frac{1}{4h_j}\int_{I_j} s_j^2(x) \,\mathrm{d}x + \frac{s_j^2(x_{j-\frac{1}{2}})}{4} = 0.$$

Thus,

$$\int_{I_j} w_j^2(x)\,\mathrm{d}x \leqslant 4\int_{I_j} (\varepsilon_r + E_r) w_j(x)(x - x_{j-\frac{1}{2}})\,\mathrm{d}x,$$

$$\int_{I_j} s_j^2(x)\,\mathrm{d}x \leqslant -4\int_{I_j} (\varepsilon_p + E_p) s_j(x)(x - x_{j+\frac{1}{2}})\,\mathrm{d}x.$$

Define piecewise polynomials $w(x)$ and $\phi_1(x)$, such that $w(x) = w_j(x)$, $\phi_1(x) = x - x_{j-\frac{1}{2}}$ on I_j; thus, $\|\phi_1\|_{L^\infty} = h$, and finally, we get

$$\|w\|_{L^2} \leqslant 4\|\varepsilon_r + E_r\|_{L^2}\|\phi_1\|_{L^\infty} \leqslant 4h\|\varepsilon_r + E_r\|_{L^2} \leqslant Ch^{k+2} + Ch\|E_r\|_{L^2}, \tag{A.12}$$

$$\|s\|_{L^2} \leqslant 4\|\varepsilon_p + E_p\|_{L^2}\|\phi_2\|_{L^\infty} \leqslant 4h\|\varepsilon_p + E_p\|_{L^2} \leqslant Ch^{k+2} + Ch\|E_p\|_{L^2}, \tag{A.13}$$

where $C = C(\|u\|_{k+4})$.

Step 2. On the one hand, taking $(\psi, \xi, \eta) = (E_q - E_r, E_p, E_q)$ in equation (A.7), adding them up and summing over all j, we obtain

$$\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 - \mathcal{D}(E_p, E_q; E_p^-) - \mathcal{D}(E_q, E_p; E_q^+) - \mathcal{D}(E_q - E_r, E_q - E_r; (E_q - E_r)^+)$$

$$= -\int_I \varepsilon_r E_q\,\mathrm{d}x - \int_I \varepsilon_p E_p\,\mathrm{d}x - \int_I \varepsilon_q (E_q - E_r)\,\mathrm{d}x.$$

By the property of the operator $\mathcal{D}$ in Lemma 2.1, we have

$$\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 + \frac{1}{2}\sum_j [E_q - E_r]_{j+\frac{1}{2}}^2 = -\int_I \varepsilon_r E_q\,\mathrm{d}x - \int_I \varepsilon_p E_p\,\mathrm{d}x - \int_I \varepsilon_q (E_q - E_r)\,\mathrm{d}x,$$

and thus

$$\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 \leqslant \left|\int_I \varepsilon_r E_q\,\mathrm{d}x\right| + \left|\int_I \varepsilon_p E_p\,\mathrm{d}x\right| + \left|\int_I \varepsilon_q (E_q - E_r)\,\mathrm{d}x\right|.$$

Note that ε_q, ε_p and ε_r are orthogonal to any piecewise constant functions, then

$$\left|\int_I \varepsilon_r E_q\,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_r s_j(x)(x - x_j)/h_j\,\mathrm{d}x\right| \leqslant \|\varepsilon_r\|_{L^2}\|s\|_{L^2}\|\phi\|_{L^\infty},$$

$$\left|\int_I \varepsilon_p E_p\,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_p w_j(x)(x - x_j)/h_j\,\mathrm{d}x\right| \leqslant \|\varepsilon_p\|_{L^2}\|w\|_{L^2}\|\phi\|_{L^\infty},$$

$$\left|\int_I \varepsilon_q (E_q - E_r)\,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_q (s_j(x) - g_j(x))(x - x_j)/h_j\,\mathrm{d}x\right| \leqslant \|\varepsilon_q\|_{L^2}\|s - g\|_{L^2}\|\phi\|_{L^\infty},$$

where $\phi = (x - x_j)/h_j$. Therefore,

$$\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 \leqslant \|\phi\|_{L^\infty}(\|\varepsilon_r\|_{L^2}\|s\|_{L^2} + \|\varepsilon_p\|_{L^2}\|w\|_{L^2} + \|\varepsilon_q\|_{L^2}\|s - g\|_{L^2}).$$

From the approximation results (2.9) and employing $\|\phi\|_{L^\infty} = \frac{1}{2}$, we conclude that

$$\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 \leqslant Ch^{k+1}(\|s\|_{L^2} + \|w\|_{L^2} + \|s - g\|_{L^2}). \tag{A.14}$$

Collecting (A.11)–(A.13) into (A.14) we arrive at

$$\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 \leqslant Ch^{2k+3} + Ch^{k+2}(\|E_q\|_{L^2} + \|E_p\|_{L^2} + \|E_r\|_{L^2}), \tag{A.15}$$

where $C = C(\|u\|_{k+4})$.

On the other hand, taking $(\psi, \xi, \eta) = (-E_p, E_q, E_r - E_q)$ in equation (A.7), adding them up and summing over all j, we obtain

$$\|E_r\|_{L^2}^2 + \mathcal{D}(E_p, E_q - E_r; E_p^-) + \mathcal{D}(E_q - E_r, E_p; (E_q - E_r)^+) - \mathcal{D}(E_q, E_q; E_q^+)$$

$$= -\int_I \varepsilon_r(E_r - E_q)\,\mathrm{d}x - \int_I \varepsilon_p E_q\,\mathrm{d}x + \int_I \varepsilon_q E_p\,\mathrm{d}x + \int_I E_q E_r\,\mathrm{d}x.$$

By the property of the operator $\mathcal{D}$ in Lemma 2.1, we have

$$\|E_r\|_{L^2}^2 + \frac{1}{2}\sum_j [E_q]_{j+\frac{1}{2}}^2 = -\int_I \varepsilon_r(E_r - E_q)\,\mathrm{d}x - \int_I \varepsilon_p E_q\,\mathrm{d}x + \int_I \varepsilon_q E_p\,\mathrm{d}x + \int_I E_q E_r\,\mathrm{d}x,$$

and thus

$$\frac{1}{2}\|E_r\|_{L^2}^2 \leqslant \left|\int_I \varepsilon_r(E_r - E_q)\,\mathrm{d}x\right| + \left|\int_I \varepsilon_p E_q\,\mathrm{d}x\right| + \left|\int_I \varepsilon_q E_p\,\mathrm{d}x\right| + \frac{1}{2}\|E_q\|_{L^2}^2.$$

Note that ε_q, ε_p and ε_r are orthogonal to any piecewise constant functions, then

$$\left|\int_I \varepsilon_r(E_r - E_q)\,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_r(g_j(x) - s_j(x))(x - x_j)/h_j\,\mathrm{d}x\right| \leqslant \|\varepsilon_r\|_{L^2}\|g - s\|_{L^2}\|\phi\|_{L^\infty},$$

$$\left|\int_I \varepsilon_p E_q\,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_p s_j(x)(x - x_j)/h_j\,\mathrm{d}x\right| \leqslant \|\varepsilon_p\|_{L^2}\|s\|_{L^2}\|\phi\|_{L^\infty},$$

$$\left|\int_I \varepsilon_q E_p\,\mathrm{d}x\right| = \left|\sum_j \int_{I_j} \varepsilon_q w_j(x)(x - x_j)/h_j\,\mathrm{d}x\right| \leqslant \|\varepsilon_q\|_{L^2}\|w\|_{L^2}\|\phi\|_{L^\infty},$$

we recall that $\phi = (x - x_j)/h_j$. Therefore,

$$\frac{1}{2}\|E_r\|_{L^2}^2 \leqslant \|\phi\|_{L^\infty}(\|\varepsilon_r\|_{L^2}\|g - s\|_{L^2} + \|\varepsilon_p\|_{L^2}\|s\|_{L^2} + \|\varepsilon_q\|_{L^2}\|w\|_{L^2}) + \frac{1}{2}\|E_q\|_{L^2}^2.$$

From the approximation results (2.9) and employing $\|\phi\|_{L^\infty} = \frac{1}{2}$, we conclude that

$$\frac{1}{2}\|E_r\|_{L^2}^2 \leqslant Ch^{k+1}(\|g - s\|_{L^2} + \|s\|_{L^2} + \|w\|_{L^2}) + \frac{1}{2}\|E_q\|_{L^2}^2. \tag{A.16}$$

Collecting (A.11)–(A.13) into (A.16), we arrive at

$$\frac{1}{2}\|E_r\|_{L^2}^2 \leqslant Ch^{2k+3} + Ch^{k+2}(\|E_q\|_{L^2} + \|E_p\|_{L^2} + \|E_r\|_{L^2}) + \frac{1}{2}\|E_q\|_{L^2}^2, \tag{A.17}$$

where $C = C(\|u\|_{k+4})$. Then, equations (A.15) and (A.17) produce

$$\frac{1}{2}\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 + \frac{1}{2}\|E_r\|_{L^2}^2 \leqslant Ch^{2k+3} + Ch^{k+2}(\|E_q\|_{L^2} + \|E_p\|_{L^2} + \|E_r\|_{L^2}),$$

which implies

$$\|E_q\|_{L^2} + \|E_p\|_{L^2} + \|E_r\|_{L^2} \leqslant C(\|u\|_{k+4})h^{k+3/2}. \tag{A.18}$$

Step 3. Suppose that

$$E_u = \sum_{n=0}^{k} a_n^j P_n\left(\frac{2(x-x_j)}{h_j}\right), \quad E_q - E_r = \sum_{n=0}^{k} b_n^j P_n\left(\frac{2(x-x_j)}{h_j}\right)$$

on I_j, where $P_n(\cdot)$ denotes the nth-order Legendre polynomial. By using the technique in Cheng & Shu (2010), conditions (A.4) and (A.5) yield the relationship

$$\|E_u\|_{L^2} \leqslant C(\lambda)\|E_q - E_r\|_{L^2}. \tag{A.19}$$

We recall that λ is the maximum of two different two different arbitrary mesh sizes. A combination of equations (A.18) and (A.19) gives us a bound for E_u,

$$\|E_u\|_{L^2} \leqslant C(\lambda)(\|E_q\|_{L^2} + \|E_r\|_{L^2}) \leqslant C(\lambda, \|u\|_{k+4})h^{k+3/2}.$$

A.2 *The proof of Lemma* 2.8

If we can prove

$$\|\bar{e}_u(t)\|_{L^2} + \|\bar{e}_q(t)\|_{L^2} + \int_0^t (\|\bar{e}_p\|_{L^2} + \|\bar{e}_r\|_{L^2})\,\mathrm{d}t \leqslant C\mathrm{e}^{C_1 t}h^{k+1}, \tag{A.20}$$

with $C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+3})$, then the estimates (2.37) and (2.38) in Lemma 2.8 will follow by the approximation error estimates (2.9) and the triangle inequality. To this end, on the one hand, we rewrite equation (2.24) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|_{L^2}^2 + \|\bar{e}_p\|_{L^2}^2 \leqslant -\int_I (\varepsilon_u)_t \bar{e}_u\,\mathrm{d}x - \int_I \varepsilon_p \bar{e}_p\,\mathrm{d}x - \int_I \varepsilon_r \bar{e}_q\,\mathrm{d}x + \int_I \varepsilon_q \bar{e}_r\,\mathrm{d}x$$
$$- \beta\int_I \varepsilon_p \bar{e}_u\,\mathrm{d}x + \beta^2\|\bar{e}_u\|_{L^2}^2 + \frac{1}{4}\|\bar{e}_p\|_{L^2}^2. \tag{A.21}$$

On the other hand, taking the time derivative in equation (2.21d), letting $(\rho, \psi, \xi, \eta) = (-\bar{e}_p, \bar{e}_q, (\bar{e}_u)_t, \bar{e}_r)$ in equation (2.21), adding them up and summing over all j, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_q\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2 + \int_I (\varepsilon_q)_t \bar{e}_q \,\mathrm{d}x + \int_I \varepsilon_r \bar{e}_r \,\mathrm{d}x - \int_I (\varepsilon_u)_t \bar{e}_p \,\mathrm{d}x + \int_I \varepsilon_p (\bar{e}_u)_t \,\mathrm{d}x - \alpha \mathcal{D}(\bar{e}_u, \bar{e}_p; \bar{e}_u^-)$$
$$- \beta \mathcal{D}(\bar{e}_q, \bar{e}_p; \bar{e}_q^+) - \mathcal{D}(\bar{e}_r, \bar{e}_p; \bar{e}_r^+) - \mathcal{D}(\bar{e}_p, \bar{e}_r; \bar{e}_p^-) - \mathcal{D}(\bar{e}_q, (\bar{e}_u)_t; \bar{e}_q^+) - \mathcal{D}((\bar{e}_u)_t, \bar{e}_q; (\bar{e}_u)_t^-) = 0.$$

Using the property of the operator $\mathcal{D}$ in Lemma 2.1, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_q\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2 + \int_I (\varepsilon_q)_t \bar{e}_q \,\mathrm{d}x + \int_I \varepsilon_r \bar{e}_r \,\mathrm{d}x - \int_I (\varepsilon_u)_t \bar{e}_p \,\mathrm{d}x$$
$$+ \int_I \varepsilon_p (\bar{e}_u)_t \,\mathrm{d}x - \alpha \mathcal{D}(\bar{e}_u, \bar{e}_p; \bar{e}_u^-) - \beta \mathcal{D}(\bar{e}_q, \bar{e}_p; \bar{e}_q^+) = 0. \tag{A.22}$$

By taking $\psi = \bar{e}_p$ in equation (2.21d) and summing over all j we get

$$\mathcal{D}(\bar{e}_u, \bar{e}_p; \bar{e}_u^-) = \int_I e_q \bar{e}_p \,\mathrm{d}x. \tag{A.23}$$

Using the property of the operator $\mathcal{D}$ in Lemma 2.1, and then taking $\eta = \bar{e}_q$ in equation (2.21b) and summing over all j, we obtain

$$\mathcal{D}(\bar{e}_q, \bar{e}_p; \bar{e}_q^+) = -\mathcal{D}(\bar{e}_p, \bar{e}_q; \bar{e}_p^-) = -\int_I e_r \bar{e}_q \,\mathrm{d}x. \tag{A.24}$$

Plugging equations (A.23) and (A.24) into (A.22), then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_q\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_r\|_{L^2}^2 \leqslant -\int_I (\varepsilon_q)_t \bar{e}_q \,\mathrm{d}x - \int_I \varepsilon_r \bar{e}_r \,\mathrm{d}x + \int_I (\varepsilon_u)_t \bar{e}_p \,\mathrm{d}x - \int_I \varepsilon_p (\bar{e}_u)_t \,\mathrm{d}x$$
$$+ \alpha \int_I \varepsilon_q \bar{e}_p \,\mathrm{d}x - \beta \int_I \varepsilon_r \bar{e}_q \,\mathrm{d}x + \frac{1}{4}\|\bar{e}_p\|_{L^2}^2 + \left(\alpha^2 + \frac{\beta^2}{2}\right)\|\bar{e}_q\|_{L^2}^2. \tag{A.25}$$

Combining equations (A.21) and (A.25) we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_q\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_p\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_r\|_{L^2}^2 \leqslant \Lambda + \Theta + \beta^2\|\bar{e}_u\|_{L^2}^2 + \left(\alpha^2 + \frac{\beta^2}{2}\right)\|\bar{e}_q\|_{L^2}^2, \tag{A.26}$$

where

$$\Lambda = -\int_I \varepsilon_p (\bar{e}_u)_t \,\mathrm{d}x$$

and

$$\Theta = -\int_I (\varepsilon_u)_t \bar{e}_u \, dx - \int_I \varepsilon_p \bar{e}_p \, dx - \int_I \varepsilon_r \bar{e}_q \, dx + \int_I \varepsilon_q \bar{e}_r \, dx - \beta \int_I \varepsilon_p \bar{e}_u \, dx$$

$$- \int_I (\varepsilon_q)_t \bar{e}_q \, dx - \int_I \varepsilon_r \bar{e}_r \, dx + \int_I (\varepsilon_u)_t \bar{e}_p \, dx + \alpha \int_I \varepsilon_q \bar{e}_p \, dx - \beta \int_I \varepsilon_r \bar{e}_q \, dx.$$

Using the approximation property of the projections (2.9), we get

$$|\Theta| \leqslant Ch^{k+1}(\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2} + \|\bar{e}_p\|_{L^2} + \|\bar{e}_r\|_{L^2}).$$

Integrating Λ with respect to time, we get, after a simple integration by parts,

$$\int_0^t \Lambda \, dt = \int_0^t \int_I (\varepsilon_p)_t \bar{e}_u \, dx \, dt - \int_I (\bar{e}_u(t)\varepsilon_p(t) - \bar{e}_u(0)\varepsilon_p(0)) \, dx.$$

Thus, by the approximation results (2.9) and the choice of initial condition in Lemma 2.4, we conclude that

$$\left| \int_0^t \Lambda \, dt \right| \leqslant Ch^{k+1} \int_0^t \|\bar{e}_u\|_{L^2} \, dt + \frac{1}{4}\|\bar{e}_u(t)\|_{L^2}^2 + Ch^{2k+2},$$

where $C = C(\|u\|_{k+4}, \|u_t\|_{k+3})$. Integrating equation (A.26) with respect to time, we obtain

$$\frac{1}{4}\|\bar{e}_u(t)\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_q(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t (\|\bar{e}_p\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2) \, dt$$

$$\leqslant \frac{1}{2}\|\bar{e}_u(0)\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_q(0)\|_{L^2}^2 + Ch^{k+1}\int_0^t (\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2} + \|\bar{e}_p\|_{L^2} + \|\bar{e}_r\|_{L^2}) dt$$

$$+ \beta^2 \int_0^t \|\bar{e}_u\|_{L^2}^2 \, dt + \left(\alpha^2 + \frac{\beta^2}{2}\right)\int_0^t \|\bar{e}_q\|_{L^2}^2 \, dt + Ch^{2k+2}, \tag{A.27}$$

where $C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+3})$. Gronwall's inequality and the estimates of the initial condition (2.17) and (A.18) give us the error estimate (A.20).

To prove the estimate (2.39), we first need to get a bound for $(\bar{e}_u)_t(\cdot, 0)$. Using conditions (2.15) and (2.16), we have, at $t = 0$,

$$\mathcal{D}_{I_j}(\bar{e}_r, \rho; \bar{e}_r^+) = \mathcal{D}_{I_j}(\bar{e}_q, \rho; \bar{e}_q^+) - \mathcal{D}_{I_j}(\bar{e}_u, \rho; \bar{e}_u^-)$$

for any $\rho \in V_h^k$. Then, equation (2.21a) becomes

$$\int_{I_j} (e_u)_t \rho \, dx + (\alpha - 1)\mathcal{D}_{I_j}(\bar{e}_u, \rho; \bar{e}_u^-) + (\beta + 1)\mathcal{D}_{I_j}(\bar{e}_q, \rho; \bar{e}_q^+) = 0.$$

It follows from equations (2.21c) and (2.21d) that, at $t = 0$,

$$\int_{I_j} (e_u)_t \rho \, dx + (\alpha - 1)\int_{I_j} e_q \rho \, dx + (\beta + 1)\int_{I_j} e_p \rho \, dx = 0$$

for any $\rho \in V_h^k$. Taking $\rho = (\bar{e}_u)_t(\cdot,0)$ and summing the above equality over all j, we get, at $t=0$,

$$\|(\bar{e}_u)_t(\cdot,0)\|_{L^2} \leqslant \|(\varepsilon_u)_t(\cdot,0)\|_{L^2} + |\alpha-1|\|e_q(\cdot,0)\|_{L^2} + |\beta+1|\|e_p(\cdot,0)\|_{L^2}.$$

Then, the approximation results (2.9) and estimates for the initial data in equation (A.18) give us

$$\|(\bar{e}_u)_t(\cdot,0)\|_{L^2} \leqslant Ch^{k+1}, \tag{A.28}$$

where $C = C(\alpha,\beta,\|u\|_{k+4},\|u_t\|_{k+1})$. Then, taking the time derivative in equation (2.21), letting $(\rho,\psi,\xi,\eta) = ((\bar{e}_u)_t, -(\bar{e}_r)_t, (\bar{e}_p)_t, (\bar{e}_q)_t)$, adding them up and summing over all j, we obtain

$$\begin{aligned}&\int_I (\bar{e}_u)_{tt}(\bar{e}_u)_t\,\mathrm{d}x + \int_I (\bar{e}_p)_t^2\,\mathrm{d}x + \int_I (\varepsilon_u)_{tt}(\bar{e}_u)_t\,\mathrm{d}x + \int_I (\varepsilon_p)_t(\bar{e}_p)_t\,\mathrm{d}x + \int_I (\varepsilon_r)_t(\bar{e}_q)_t\,\mathrm{d}x\\ &\quad - \int_I (\varepsilon_q)_t(\bar{e}_r)_t\,\mathrm{d}x + \alpha\mathcal{D}((\bar{e}_u)_t,(\bar{e}_u)_t;(\bar{e}_u)_t^-) + \beta\mathcal{D}((\bar{e}_q)_t,(\bar{e}_u)_t;(\bar{e}_q)_t^+) + \mathcal{D}((\bar{e}_r)_t,(\bar{e}_u)_t;(\bar{e}_r)_t^+)\\ &\quad + \mathcal{D}((\bar{e}_u)_t,(\bar{e}_r)_t;(\bar{e}_u)_t^-) - \mathcal{D}((\bar{e}_q)_t,(\bar{e}_p)_t;(\bar{e}_q)_t^+) - \mathcal{D}((\bar{e}_p)_t,(\bar{e}_q)_t;(\bar{e}_p)_t^-) = 0.\end{aligned}$$

Using the property of the operator $\mathcal{D}$ in Lemma 2.1, we have

$$\begin{aligned}&\int_I (\bar{e}_u)_{tt}(\bar{e}_u)_t\,\mathrm{d}x + \int_I (\bar{e}_p)_t^2\,\mathrm{d}x + \int_I (\varepsilon_u)_{tt}(\bar{e}_u)_t\,\mathrm{d}x + \int_I (\varepsilon_p)_t(\bar{e}_p)_t\,\mathrm{d}x + \int_I (\varepsilon_r)_t(\bar{e}_q)_t\,\mathrm{d}x\\ &\quad - \int_I (\varepsilon_q)_t(\bar{e}_r)_t\,\mathrm{d}x + \frac{\alpha}{2}\sum_j [(\bar{e}_u)_t]^2_{j+\frac{1}{2}} + \beta\mathcal{D}((\bar{e}_q)_t,(\bar{e}_u)_t;(\bar{e}_q)_t^+) = 0.\end{aligned} \tag{A.29}$$

Note that

$$\mathcal{D}((\bar{e}_q)_t,(\bar{e}_u)_t;(\bar{e}_q)_t^+) = \int_I (e_p)_t(\bar{e}_u)_t\,\mathrm{d}x. \tag{A.30}$$

Combining equations (A.29) and (A.30) we arrive at

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\bar{e}_u)_t\|_{L^2}^2 + \|(\bar{e}_p)_t\|_{L^2}^2 \leqslant &- \int_I (\varepsilon_u)_{tt}(\bar{e}_u)_t\,\mathrm{d}x - \int_I (\varepsilon_p)_t(\bar{e}_p)_t\,\mathrm{d}x - \int_I (\varepsilon_r)_t(\bar{e}_q)_t\,\mathrm{d}x\\ &+ \int_I (\varepsilon_q)_t(\bar{e}_r)_t\,\mathrm{d}x - \beta\int_I ((\varepsilon_p)_t + (\bar{e}_p)_t)(\bar{e}_u)_t\,\mathrm{d}x.\end{aligned} \tag{A.31}$$

Integrating the above inequality with respect to time,

$$\frac{1}{2}\|(\bar{e}_u)_t(t)\|_{L^2}^2 + \int_0^t \|(\bar{e}_p)_t(t)\|_{L^2}^2\,\mathrm{d}t \leqslant \frac{1}{2}\|(\bar{e}_u)_t(0)\|_{L^2}^2 + \Upsilon + \Xi, \tag{A.32}$$

where

$$\Upsilon = -\int_I\int_0^t (\varepsilon_p)_t(\bar{e}_p)_t\,\mathrm{d}t\,\mathrm{d}x - \beta\int_I\int_0^t (\bar{e}_p)_t(\bar{e}_u)_t\,\mathrm{d}t\,\mathrm{d}x$$

and

$$\Xi = -\int_I \int_0^t (\varepsilon_u)_{tt} (\bar{e}_u)_t \,\mathrm{d}t\,\mathrm{d}x - \int_I \int_0^t (\varepsilon_r)_t (\bar{e}_q)_t \,\mathrm{d}t\,\mathrm{d}x$$

$$+ \int_I \int_0^t (\varepsilon_q)_t (\bar{e}_r)_t \,\mathrm{d}t\,\mathrm{d}x - \beta \int_I \int_0^t (\varepsilon_p)_t (\bar{e}_u)_t \,\mathrm{d}t\,\mathrm{d}x.$$

By the Cauchy–Schwarz inequality and approximation results (2.9) we obtain

$$|\Upsilon| \leqslant \int_0^t \|(\bar{e}_p)_t(t)\|_{L^2}^2 \,\mathrm{d}t + \frac{1}{2}\int_0^t \|(\varepsilon_p)_t\|_{L^2}^2 \,\mathrm{d}t + \frac{\beta^2}{2}\int_0^t \|(\bar{e}_u)_t(t)\|_{L^2}^2 \,\mathrm{d}t$$

$$\leqslant \int_0^t \|(\bar{e}_p)_t(t)\|_{L^2}^2 \,\mathrm{d}t + \frac{\beta^2}{2}\int_0^t \|(\bar{e}_u)_t(t)\|_{L^2}^2 \,\mathrm{d}t + Ch^{2k+2}. \tag{A.33}$$

Using integration by parts with respect to time, we get

$$|\Xi| \leqslant \int_0^t \|(\varepsilon_u)_{ttt}\|_{L^2}\|\bar{e}_u\|_{L^2}\,\mathrm{d}t + \|(\varepsilon_u)_{tt}(t)\|_{L^2}\|\bar{e}_u(t)\|_{L^2} + \|(\varepsilon_u)_{tt}(0)\|_{L^2}\|\bar{e}_u(0)\|_{L^2}$$

$$+ \int_0^t \|(\varepsilon_r)_{tt}\|_{L^2}\|\bar{e}_q\|_{L^2}\,\mathrm{d}t + \|(\varepsilon_r)_t(t)\|_{L^2}\|\bar{e}_q(t)\|_{L^2} + \|(\varepsilon_r)_t(0)\|_{L^2}\|\bar{e}_q(0)\|_{L^2}$$

$$+ \int_0^t \|(\varepsilon_q)_{tt}\|_{L^2}\|\bar{e}_r\|_{L^2}\,\mathrm{d}t + \|(\varepsilon_q)_t(t)\|_{L^2}\|\bar{e}_r(t)\|_{L^2} + \|(\varepsilon_q)_t(0)\|_{L^2}\|\bar{e}_r(0)\|_{L^2}$$

$$+ |\beta|\left[\int_0^t \|(\varepsilon_p)_{tt}\|_{L^2}\|\bar{e}_u\|_{L^2}\,\mathrm{d}t + \|(\varepsilon_p)_t(t)\|_{L^2}\|\bar{e}_u(t)\|_{L^2} + \|(\varepsilon_p)_t(0)\|_{L^2}\|\bar{e}_u(0)\|_{L^2}\right]$$

$$\leqslant Ch^{k+1}\int_0^t (\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2} + \|\bar{e}_r\|_{L^2})\,\mathrm{d}t$$

$$+ Ch^{k+1}(\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2} + \|\bar{e}_r\|_{L^2}) + Ch^{2k+5/2} \tag{A.34}$$

with $C = C(\beta, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1})$, where we have used the approximation results (2.9) and estimates for the initial data (A.18) and (2.17) to obtain the last inequality. Plugging equations (A.33) and (A.34) into (A.32) and using estimates (A.20), (A.28), we have

$$\frac{1}{2}\|(\bar{e}_u)_t(t)\|_{L^2}^2 \leqslant C\mathrm{e}^{C_1 t}h^{2k+2} + Ch^{k+1}\|\bar{e}_r(t)\|_{L^2} + \frac{\beta^2}{2}\int_0^t \|(\bar{e}_u)_t(t)\|_{L^2}^2\,\mathrm{d}t. \tag{A.35}$$

Denote $\mathbb{E}(t) = \int_0^t \|(\bar{e}_u)_t(t)\|_{L^2}^2\,\mathrm{d}t$ and integrate over equation (A.35) with respect to time,

$$\frac{1}{2}\mathbb{E}(t) \leqslant C\mathrm{e}^{C_1 t}h^{2k+2} + \frac{\beta^2}{2}\int_0^t \mathbb{E}(s)\,\mathrm{d}s.$$

Gronwall's inequality gives us

$$\mathbb{E}(t) = \int_0^t \|(\bar{e}_u)_t(t)\|_{L^2}^2\,\mathrm{d}t \leqslant C\mathrm{e}^{2C_1 t}h^{2k+2}.$$

Therefore,

$$\int_0^t \|(\bar{e}_u)_t(t)\|_{L^2}\,\mathrm{d}t \leqslant C\mathrm{e}^{C_1 t} h^{k+1},$$

where $C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1})$ and $C_1 = C_1(\alpha, \beta) > 0$. Then, estimate (2.39) follows by taking into account the approximation error estimates (2.9) and the triangle inequality. This finishes the proof for Lemma 2.8.

A.3 *The proof of Theorem* 2.7

To get the linear growth result for the case $\alpha = \beta = 0$, similarly to Lemma 2.8, we need to prove the following error estimates:

$$\|\bar{e}_u(t)\|_{L^2} + \|\bar{e}_q(t)\|_{L^2} \leqslant C(1+t)h^{k+1}, \tag{A.36}$$

$$\int_0^t (\|\bar{e}_p\|_{L^2} + \|\bar{e}_r\|_{L^2})\,\mathrm{d}t \leqslant C(1+t)^{3/2}h^{k+1}, \tag{A.37}$$

where $C = C(\|u\|_{k+4}, \|u_t\|_{k+3})$. Note that, if equations (A.36) and (A.37) hold, we can easily get a bound for $(\bar{e}_u)_t(t)$,

$$\int_0^t \|(\bar{e}_u)_t(t)\|_{L^2}\,\mathrm{d}t \leqslant C(1+t)^2 h^{k+1}, \tag{A.38}$$

and thus equation (2.36) will give us the desired result in Theorem 2.7 by combining with the approximation error estimates (2.9). First, let us prove the error estimate (A.36).

If $\alpha = \beta = 0$, then equation (A.27) reduces to

$$\frac{1}{4}\|\bar{e}_u(t)\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_q(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t (\|\bar{e}_p\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2)\,\mathrm{d}t$$
$$\leqslant \frac{1}{2}\|\bar{e}_u(0)\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_q(0)\|_{L^2}^2 + Ch^{k+1}\int_0^t (\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2} + \|\bar{e}_p\|_{L^2} + \|\bar{e}_r\|_{L^2})\,\mathrm{d}t + Ch^{2k+2},$$

which, by using the bounds for the initial error (2.17) and (A.18), is

$$\frac{1}{4}\|\bar{e}_u(t)\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_q(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t (\|\bar{e}_p\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2)\,\mathrm{d}t$$
$$\leqslant Ch^{k+1}\int_0^t (\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2})\,\mathrm{d}t + Ch^{k+1}\int_0^t (\|\bar{e}_p\|_{L^2} + \|\bar{e}_r\|_{L^2})\,\mathrm{d}t + Ch^{2k+2}. \tag{A.39}$$

By using the Cauchy–Schwarz inequality we get

$$(\|\bar{e}_u(t)\|_{L^2} + \|\bar{e}_q(t)\|_{L^2})^2 \leqslant Ch^{k+1}\int_0^t (\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2})\,\mathrm{d}t + Ch^{2k+2},$$

where $C = C(\|u\|_{k+4}, \|u_t\|_{k+3})$. Since $\|u\|_{k+4}$ and $\|u_t\|_{k+3}$ are bounded uniformly for any time $t \in [0, T]$, thus we can assume that $C = C(\|u\|_{k+4}, \|u_t\|_{k+3}) \leqslant \tilde{C}$ with $\tilde{C}$ a positive constant independent of time. Denote $\tilde{\mathbb{E}}(t) = \|\bar{e}_u(t)\|_{L^2} + \|\bar{e}_q(t)\|_{L^2}$, then we have

$$\tilde{\mathbb{E}}^2(t) \leqslant \tilde{C}h^{k+1}\int_0^t \tilde{\mathbb{E}}(s)\,\mathrm{d}s + \tilde{C}h^{2k+2}.$$

Define $z(t) = \tilde{C}h^{k+1}\int_0^t \tilde{\mathbb{E}}(s)\,\mathrm{d}s + \tilde{C}h^{2k+2}$, thus $\sqrt{z(0)} \leqslant \tilde{C}h^{k+1}$, and the above inequality gives us

$$\tilde{\mathbb{E}}(t) \leqslant \sqrt{z(t)}.$$

Therefore,

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \tilde{C}h^{k+1}\tilde{\mathbb{E}}(t) \leqslant \tilde{C}h^{k+1}\sqrt{z(t)}.$$

We integrate the above inequality with respect to time between 0 and t and use the estimate for $\sqrt{z(0)}$ to obtain a bound on $\tilde{\mathbb{E}}(t)$,

$$\tilde{\mathbb{E}}(t) \leqslant \sqrt{z(t)} \leqslant \sqrt{z(0)} + \tilde{C}h^{k+1}t \leqslant \tilde{C}h^{k+1}(1+t),$$

that is,

$$\|\bar{e}_u(t)\|_{L^2} + \|\bar{e}_q(t)\|_{L^2} \leqslant \tilde{C}h^{k+1}(1+t).$$

Finally, the error estimate (A.37) follows by combining equations (A.36) and (A.39). This completes the proof of Theorem 2.7.