A phase-based interior penalty discontinuous Galerkin method for the Helmholtz equation with spatially varying wavenumber*

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Abstract

This paper is concerned with an interior penalty discontinuous Galerkin (IPDG) method based on a flexible type of non-polynomial local approximation space for the Helmholtz equation with varying wavenumber. The local approximation space consists of multiple polynomial-modulated phase functions which can be chosen according to the phase information of the solution. We obtain some approximation properties for this space and *a prior* L^2 error estimates for the h-convergence of the IPDG method using duality argument. We also provide ample numerical examples to show that, building phase information into the local spaces often gives more accurate results comparing to using the standard polynomial spaces.

1 Introduction

In this paper, we consider the following Helmholtz equation on a bounded convex polygonal domain $\Omega \subset \mathbb{R}^2$ with a Robin boundary condition:

$$-\Delta u - \kappa^2 u = f \quad \text{in } \Omega \tag{1}$$

$$\nabla u \cdot n + i\kappa u = g \quad \text{on } \partial\Omega, \tag{2}$$

where κ is the wavenumber, which is a smooth real-valued function, f is the source term in $L^2(\Omega)$, g is the boundary data in $L^2(\partial\Omega)$ and n is the outward normal on $\partial\Omega$.

The Helmholtz equation with high and varying wavenumber arises from many areas, including seismology, electromagnetics, underwater acoustics [4], plasma physics [18] and medical imaging [28]. Standard finite element methods

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based on low-order polynomials do not perform well for the Helmholtz equation at high wavenumber. On the one hand, low-order polynomials do not well resolve the solution unless several grid points per wavelength are used. On the other hand, such methods suffer from the so-called *pollution effect*: for a fixed number of grid points per wavelength, the numerical error grows with the wavenumber [3,16]. Indeed, it was suggested in [21,24] that the pollution effect can be suppressed by using higher order polynomials for problems with higher wavenumber.

It is natural to consider oscillatory non-polynomial basis to overcome the shortcomings of low-order polynomials, in the hope that such basis can resolve the solution using significantly fewer degrees of freedom than polynomials. For Helmholtz equation with homogeneous media, plane waves are extensively used as the basis in the literature. Examples using such basis include the partition of unity finite element method [2,23,27,31], the least-square methods [25,30] and the discrete enrichment method [1,8,9]. A good survey of such methods can be found in [13]. Besides, Cessenat and Després proposed the ultra-weak variational formulation (UWVF) for the Helmholtz equation in [7]. The UWVF approximates the exact solution of the Helmholtz equation by a linear combination of free space solutions of the homogeneous Helmholtz equations, including plane waves. Later, the UWVF was recast as a discontinuous Galerkin method in [6] and generalized into the plane wave discontinuous Galerkin (PWDG) method in [12], where plane waves with uniformly-spaced directions are used as basis. The convergence of the PWDG method regarding refining the mesh, namely the h-version, is analyzed in [12] using the Schatz' duality argument [29]. It was shown that the h-version is still afflicted by the pollution effect. A detailed quantitative study of such effect for this method can be found in [11]. Later, it was suggested in [14] that the convergence of the PWDG method regarding using more plane waves, namely the p-version, is immune to the pollution effect. Recently, the exponential convergence of a strategy in choosing h and p locally, namely the hp-version, has been established theoretically in [15].

For problems with heterogeneous media or nonzero source function f, plane waves are no longer the free space solutions of the Helmholtz equation. In [12], the PWDG method was shown to obtain only second order accuracy for a generic source term. It was also shown numerically in [5] that using plane waves with uniformly-spaced directions for varying wavenumber did not outperform polynomials significantly in their experiments. Instead of using plane waves as basis, Imbert-Gérard and Després [18] proposed an extension of the UWVF method with high order accuracy for smooth varying wavenumber. Based on the idea of generalized plane waves proposed by Melenk [20], they considered an adapted basis of the form $e^{P(x)}$ where P is a polynomial with complex coefficients constructed locally according to an approximation to the wavenumber. Further analysis of the adapted basis in two dimensions can be found in [17].

The above mentioned methods do not assume the knowledge of phase values of the solution. However, it has been shown that taking advantage of the knowledge of the phase values, the accuracy could be greatly enhanced with the same mesh size. For example, Betcke and Philips [5] compared uniform plane wave basis to low-order polynomial modulated plane waves with dominant directions under a DG formulation similar to the original PWDG method, and showed numerically the latter outperformed the former for a Helmholtz problem with varying wavenumber. In [26], Nguyen et al. proposed a hybridizable discontinuous Galerkin (HDG) method using local approximation spaces consisting of $\sum p_\ell(x)e^{i\psi_\ell(x)}$, where $p_\ell(x)$ are polynomials and $\psi_\ell(x)$ are solutions of the eikonal equation. They showed that their phase-based HDG method can obtain high-order accuracy with several orders of magnitude fewer degrees of freedom comparing to the HDG method with standard polynomial basis. More examples of using such polynomial-modulated basis can be found in [10, 19].

In this paper, we will consider a general type of local space consisting of $\sum p_\ell(x)e^{iq_\ell(x)}$ where $p_\ell(x)$ are polynomials and $q_\ell(x)$ are real-valued functions. We prove some approximation properties when q_ℓ are polynomials satisfying certain additional assumptions. We will use these local spaces in an interior penalty discontinuous Galerkin (IPDG) framework to solve the Helmholtz equation with varying wavenumber, and give error estimates in terms of the wavenumber $\kappa(x)$ using Schatz's argument. We call our method a phase-based method following [26], due to its flexibility in choosing the basis related to the phase of the exact solution.

This paper is organized as follows. In section 2, we give some preliminaries and then state some notations we use in this paper. In section 3, we discuss the properties of the local bases and the formulation of the IPDG method. In section 4, we use Schatz's argument to prove the convergence of the IPDG method for the Helmholtz equation with varying wavenumber. In section 5, we give numerical examples to show the h-convergence of this method, and numerically verify a few properties of the local spaces. Finally, conclusions are drawn in section 6.

2 Preliminaries

We assume $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain. Let $\mathcal{T}_h$ be a conforming triangulation of Ω, where the index h is a constant multiple of the maximum diameter of the triangles. For any triangle $K \in \mathcal{T}_h$, we denote the the diameter of K by h_K. We also define the skeleton $\mathcal{F}$ to be the union of boundaries of all triangles K in $\mathcal{T}_h$, the boundary skeleton $\mathcal{F}^B := \mathcal{F} \cap \partial\Omega$, and the interior skeleton $\mathcal{F}^I := \partial\Omega \setminus \mathcal{F}^B$.

Assumption (M)**.** *We assume the triangulation $\mathcal{T}_h$ satisfies the following properties:*

1. ***Shape regularity**. There is a positive constant α independent of h such that for any $K \in \mathcal{T}_h$,*
$$h_K/\rho_K \leq \alpha,$$
where ρ_K is the radius of the inscribed circle of K.

2. ***Quasi-uniformity**. There is a positive constant τ independent of h such that for any $K \in \mathcal{T}_h$,*
$$h_K \geq \tau h.$$

Let $K^\pm$ be two triangles sharing an edge e. Let $n^\pm$ be the outward normal of $K^\pm$ on e and $u^\pm$ be two smooth scalar functions on $K^\pm$, respectively. We define the average operator $\{\cdot\}$ and jump operator $[\![\cdot]\!]$ by

$$\{u\} := \frac{1}{2}(u^+ + u^-) \quad \text{and} \quad [\![u]\!] := u^+n^+ + u^-n^-.$$

Similarly, for smooth vector field $\boldsymbol{\sigma}^{\pm}$ defined on $K^{\pm}$, respectively, we define the average operator $\{\cdot\}$ and jump operator $[\![\cdot]\!]$ by

$$\{\boldsymbol{\sigma}\} := \frac{1}{2}(\boldsymbol{\sigma}^{+} + \boldsymbol{\sigma}^{-}) \quad \text{and} \quad [\![\boldsymbol{\sigma}]\!] := \boldsymbol{\sigma}^{+} \cdot n^{+} + \boldsymbol{\sigma}^{-} \cdot n^{-}.$$

On the standard Sobolev space $W^{s,p}(S)$, we denote the Sobolev norm by $\|\cdot\|_{s,p,S}$ and the semi-norm by $|\cdot|_{s,p,S}$. When $p = 2$, we denote the Sobolev norm by $\|\cdot\|_{s,S}$ and the semi-norm by $|\cdot|_{s,S}$. When $S = \Omega$, we simplify the notation by writing $\|\cdot\|_{s}$ and $|\cdot|_{s}$ for the Sobolev norm and semi-norm respectively. We also denote the $L^2(\Omega)$ inner product by $(\cdot,\cdot)$.

3 The phase-based IPDG method

3.1 The local approximation spaces and their properties

On each $K \in \mathcal{T}_h$, we denote the *local space of polynomials with degree* r by $P^r(K)$. We call a function of the form $e^{iq(x)}$ a phase function. For a real vector-valued function $\mathbf{q} := (q_1, \dots, q_m) \in (H^{r+1}(K))^m$, we define *the local space of polynomial-modulated phase functions with degree* r by

$$Q^r(K,\mathbf{q}) := \left\{ \sum_{\ell=1}^{m} p_\ell(x) e^{iq_\ell(x)} : p_\ell \in P^r(K) \right\}.$$

We define two L^2 projection operators onto $P^r(K)$ and $Q^r(K,\mathbf{q})$, respectively,

$$\mathsf{P}_r : H^{r+1}(K) \to P^r(K), \quad (\mathsf{P}_r u, v) = (u, v) \quad \text{for any } v \in P^r(K),$$

$$\mathsf{Q}_r : H^{r+1}(K) \to Q^r(K,\mathbf{q}), \quad (\mathsf{Q}_r u, v) = (u, v) \quad \text{for any } v \in Q^r(K,\mathbf{q}).$$

In this section we will discuss certain special choices of $\mathbf{q}$.

Assumption (Q)**.** *We assume* $\mathbf{q} := (q_1, \dots q_m)$ *satisfies:*

1. *There is an integer* N *independent of* h *such that for any* $K \in \mathcal{T}_h$ *and* $1 \le \ell \le m$,
$$q_\ell \in P^N(K);$$

2. *There is a constant* $C > 0$ *independent of* h *such that for any* $K \in \mathcal{T}_h$, $1 \le \ell \le m+1$, $|\alpha| \le N$ *and* $x \in K$,
$$\left| \frac{\partial^\alpha q_\ell}{\partial x^\alpha}(x) \right| \le C;$$

3. *There is a constant* $C > 0$ *and integer* $k_0 \ge 0$ *both independent of* h *such that for any* $K \in \mathcal{T}_h$ *and* $1 \le \ell \le m$ *and* $p_\ell \in P^r(K)$,
$$\sum_{\ell=1}^{m} \|p_\ell\|_{0,K} \le C h^{-k_0} \left\| \sum_{\ell=1}^{m} p_\ell e^{iq_\ell} \right\|_{0,K}.$$

Remark 3.1. When $m = 1$, the third assumption in (Q) holds with $k_0 = 0$ since

$$\|p_1\|_{0,K} = \left\|p_1 e^{iq_1}\right\|_{0,K}.$$

For $m > 1$, we note that the spectral radius of the matrix P corresponding to $\sum_{\ell=1}^m \|p_\ell\|_{0,K}$ is of order h. This assumption says that the smallest eigenvalue of the mass matrix for $Q^r(K)$ is of order h^{k_0+1}. In section 5.5 of this paper, we will show numerically that this assumption holds for some choices of bases.

We will repeatedly use the following lemma in this section, which is a direct consequence of the second assumption in (Q):

Lemma 3.2. *There is a constant $C > 0$ independent of h and $K \in \mathcal{T}_h$ such that for any $1 \leq \ell \leq m$ and any positive integer j, there is a polynomial $\tilde{q}_{\ell,j}$ of degree at most j such that for any $x \in K$,*

$$\left|e^{iq_\ell(x)} - \tilde{q}_{\ell,j}(x)\right| \leq Ch^{j+1}.$$

Proposition 3.3 (Trace inverse inequality)**.** *There is $C > 0$ independent of h such that for any $K \in \mathcal{T}_h$ and $v \in Q^r(K, \mathbf{q})$,*

$$|v|_{1,\partial K} \leq Ch^{-1/2} |v|_{1,K}.$$

Proof. For $v \in Q^r(K, \mathbf{q})$, we write $v := \sum_{\ell=1}^m p_\ell e^{iq_\ell}$ and $g_\ell := ip_\ell \nabla q_\ell + \nabla p_\ell$, where $p_\ell \in P^r(K)$. We note that $\nabla v = \sum_{\ell=1}^m g_\ell e^{iq_\ell}$. Let j be a positive integer.

Firstly, we consider the reference triangle $\widehat{K}$. By the equivalency of L^1 and L^2 norms, triangle inequality and Hölder's inequality,

$$|\widehat{v}|_{1,\partial\widehat{K}} \leq C \left(\sum_{\ell=1}^m \|\widehat{g}_\ell\|_{0,\partial\widehat{K}} \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\partial\widehat{K}} + \left\| \sum_{\ell=1d}^m \widehat{g}_\ell \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\partial\widehat{K}} \right), \tag{3}$$

where the hat notation $(\widehat{\cdot})$ denotes the pullback associated with the affine map from $\widehat{K}$ to K. Applying trace inverse inequality for polynomials,

$$|\widehat{v}|_{1,\partial\widehat{K}} \leq C \left(\sum_{\ell=1}^m \|\widehat{g}_\ell\|_{0,\widehat{K}} \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\partial\widehat{K}} + \left\| \sum_{\ell=1}^m \widehat{g}_\ell \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\widehat{K}} \right). \tag{4}$$

Also, using the equivalency of L^1 and L^2 norms on $P^{N+r+j-1}(\widehat{K})$, triangle inequality, and Hölder's inequality,

$$\left\| \sum_{\ell=1}^m \widehat{g}_\ell \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\widehat{K}} \leq C \left(\sum_{\ell=1}^m \|\widehat{g}_\ell\|_{0,\widehat{K}} \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\widehat{K}} + |\widehat{v}|_{1,\widehat{K}} \right). \tag{5}$$

Combining equations (4) and (5),

$$|\widehat{v}|_{1,\partial\widehat{K}} \leq C \left(\sum_{\ell=1}^m \|\widehat{g}_\ell\|_{0,\widehat{K}} \left(\left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\widehat{K}} + \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\partial\widehat{K}} \right) + |\widehat{v}|_{1,\widehat{K}} \right). \tag{6}$$

Secondly, we consider $K \in \mathcal{T}_h$. By the definition of $\tilde{q}_{\ell,j}$ in Lemma 3.2, we have

$$\left\|e^{iq_\ell} - \tilde{q}_{\ell,j}\right\|_{0,K} \leq \quad Ch^{j+2} \text{ and } \left\|e^{iq_\ell} - \tilde{q}_{\ell,j}\right\|_{0,\partial K} \leq Ch^{j+\frac{3}{2}}. \tag{7}$$

Applying the scaling argument, equation (6), and equation (7),

$$|v|_{1,\partial K} \leq Ch^{-\frac{1}{2}} \left(\left(h^{j+\frac{1}{2}} + h^{j+1} \right) \sum_{\ell=1}^{m} \|g_\ell\|_{0,K} + |v|_{1,K} \right). \tag{8}$$

Applying the second assumption in (Q),

$$|v|_{1,\partial K} \leq Ch^{-\frac{1}{2}} \left(1 + h^{j-k_0+\frac{1}{2}} + h^{j-k_0+1} \right) |v|_{1,K}. \tag{9}$$

The result follows from taking $j = k_0$. □

Proposition 3.4 (Inverse estimates)**.** *For any $1 \leq k \leq r$ there is $C_{inv,k} > 0$ independent of h such that for any $K \in \mathcal{T}_h$ and $v \in Q^r(K, \mathbf{q})$,*

$$|v|_{k,K} \leq C_{inv,k} h^{-k} \|v\|_{0,K}. \tag{10}$$

Proof. For $v \in Q^r(K, \mathbf{q})$, we write $v := \sum_{\ell=1}^{m} p_\ell e^{iq_\ell}$, where $p_\ell \in P^r(K)$. Let j be a positive integer.

Firstly, we consider the reference triangle $\widehat{K}$. Using Leibniz's rule and Hölder's inequality, we have

$$\sum_{\ell=1}^{m} \left| \widehat{p}_\ell (e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j}) \right|_{k,1,\widehat{K}} \leq C \sum_{\ell=1}^{m} \|\widehat{p}_\ell\|_{k,\widehat{K}} \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{k,\widehat{K}}. \tag{11}$$

By the equivalency of L^1 and L^2 norms, the triangle inequality, and equation (11),

$$|\widehat{v}|_{k,\widehat{K}} \leq C \left(\sum_{\ell=1}^{m} \|\widehat{p}_\ell\|_{k,\widehat{K}} \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{k,\widehat{K}} + \left| \sum_{\ell=1}^{m} \widehat{p}_\ell \widehat{\tilde{q}}_{\ell,j} \right|_{k,1,\widehat{K}} \right). \tag{12}$$

Using norm equivalence on $P^{r+j}(\widehat{K})$,

$$|\widehat{v}|_{k,\widehat{K}} \leq C \left(\sum_{\ell=1}^{m} \|\widehat{p}_\ell\|_{0,\widehat{K}} \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{k,\widehat{K}} + \left\| \sum_{\ell=1}^{m} \widehat{p}_\ell \widehat{\tilde{q}}_{\ell,j} \right\|_{0,1,\widehat{K}} \right). \tag{13}$$

Also, using the triangle inequality and Hölder's inequality,

$$\left\| \sum_{\ell=1}^{m} \widehat{p}_\ell \widehat{\tilde{q}}_{\ell,j} \right\|_{0,1,\widehat{K}} \leq \sum_{\ell=1}^{m} \|\widehat{p}_\ell\|_{0,\widehat{K}} \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{0,\widehat{K}} + \|\widehat{v}\|_{0,\widehat{K}}. \tag{14}$$

Combining equations (13) and (14),

$$|\widehat{v}|_{k,\widehat{K}} \leq C \left(\sum_{\ell=1}^{m} \|\widehat{p}_\ell\|_{0,\widehat{K}} \left\| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right\|_{k,\widehat{K}} + \|\widehat{v}\|_{0,\widehat{K}} \right). \tag{15}$$

Secondly, we consider $K \in \mathcal{T}_h$. By the definition of $\tilde{q}_{\ell,j}$ in Lemma 3.2, for any $0 \leq s \leq k$,

$$\left| e^{i\widehat{q}_\ell} - \widehat{\tilde{q}}_{\ell,j} \right|_{s,\widehat{K}} \leq Ch^{s-1} \left| e^{iq_\ell} - \tilde{q}_{\ell,j} \right|_{s,K} \leq Ch^{j+1}. \tag{16}$$

Applying the scaling argument, equation (15), and equation (16),

$$|v|_{k,K} \leq Ch^{-k}\left(h^{j+1}\sum_{\ell=1}^{m}\|p_\ell\|_{0,K} + \|v\|_{0,K}\right). \tag{17}$$

The result follows from taking $j = k_0 - 1$ and the third assumption in (Q). $\square$

Theorem 3.5 (Projection error estimates)**.** *For any $0 \leq k \leq r$, there is a constant $C > 0$ independent of h such that for any $K \in \mathcal{T}_h$, and $v \in H^{r+1}(K)$,*

$$|v - \mathsf{Q}_r v|_{k,K} \leq Ch^{r+1-k}\|v\|_{r+1,K}. \tag{18}$$

Proof. It suffices to show for any $v \in H^{r+1}(K)$ and q to be one of the q_ℓ,

$$|v - \mathsf{Q}_r v|_{k,K} \leq Ch^{r+1-k}\left|e^{-iq}v\right|_{r+1,K}. \tag{19}$$

The results will follows from Leibniz's rule and the second assumption in (Q).

First of all we consider $k = 0$. We observe that $e^{iq}\mathsf{P}_r(e^{-iq}v) \in Q^r(K, \mathbf{q})$,

$$\|v - \mathsf{Q}_r v\|_{0,K} \leq \left\|v - e^{iq}\mathsf{P}_r(e^{-iq}v)\right\|_{0,K} = \left\|e^{-iq}v - \mathsf{P}_r(e^{-iq}v)\right\|_{0,K}. \tag{20}$$

Applying the projection error estimates for $P^r(K)$,

$$\|v - \mathsf{Q}_r v\|_{0,K} \leq Ch^{r+1}\left|e^{-iq}v\right|_{r+1,K}. \tag{21}$$

For $k > 0$, using the triangle inequality,

$$|v - \mathsf{Q}_r v|_{k,K} \leq \left|v - e^{iq}\mathsf{P}_r(e^{-iq}v)\right|_{k,K} + \left|e^{iq}\mathsf{P}_r(e^{-iq}v) - \mathsf{Q}_r v\right|_{k,K}. \tag{22}$$

Applying Leibniz's rule and the third assumption in (Q) to the first term,

$$\begin{aligned}\left|v - e^{iq}\mathsf{P}_r(e^{-iq}v)\right|_{k,K} &\leq C\left|e^{-iq}v - \mathsf{P}_r(e^{-iq}v)\right|_{k,K} \\ &\leq Ch^{r+1-k}\left|e^{-iq}v\right|_{r+1,K}.\end{aligned} \tag{23}$$

For the second term in (22), applying the inverse estimates of $Q^r(K)$ in Proposition 3.4, and the projection error estimates for $P^r(K)$,

$$\begin{aligned}\left|e^{iq}\mathsf{P}_r(e^{-iq}v) - \mathsf{Q}_r v\right|_{k,K} &\leq Ch^{-k}\left(\left\|v - e^{iq}\mathsf{P}_r(e^{-iq}v)\right\|_{0,K} + \|v - \mathsf{Q}_r v\|_{0,K}\right) \\ &\leq Ch^{-k}\left\|v - e^{iq}\mathsf{P}_r(e^{-iq}v)\right\|_{0,K} \\ &\leq Ch^{r+1-k}\left|e^{-iq}v\right|_{r+1,K}.\end{aligned} \tag{24}$$

The result follows from equations (22), (23) and (24). $\square$

3.2 Derivation of the discrete scheme

Multiplying equation (1) by the complex conjugate of a smooth test function v on each K and applying integration by parts twice on the first term, we get

$$-\int_K u\overline{\Delta v}\,dx + \int_{\partial K} u\overline{\nabla v \cdot n}\,ds - \int_{\partial K}\nabla u \cdot n\overline{v}\,ds - \int_K \kappa^2 u\overline{v}\,dx = \int_K f\overline{v}\,dx, \tag{25}$$

where n is the outward normal on K. We approximate u, v in (25) by discrete functions u_h, v_h, and the boundary values of u and ∇u by numerical fluxes $\widehat{u}_h$ and $\widehat{\sigma}_h$,

$$-\int_K u_h \overline{\Delta v_h}\, dx + \int_{\partial K} \widehat{u}_h \overline{\nabla v_h \cdot n}\, ds - \int_{\partial K} \widehat{\sigma}_h \cdot n \overline{v_h}\, ds - \int_K \kappa^2 u_h \overline{v_h}\, dx = \int_K f \overline{v_h}\, dx, \tag{26}$$

Then we reverse the second integration by parts and get

$$\int_K \nabla u_h \cdot \overline{\nabla v_h}\, dx - \int_{\partial K} (u_h - \widehat{u}_h) \overline{\nabla v_h \cdot n}\, ds - \int_{\partial K} \widehat{\sigma}_h \cdot n \overline{v_h}\, ds - \int_K \kappa^2 u_h \overline{v_h}\, dx = \int_K f \overline{v_h}\, dx. \tag{27}$$

Next, we choose the numerical fluxes. On interior edges, we take

$$\widehat{\boldsymbol{\sigma}}_h = \{\nabla_h u_h\} - i\frac{\mathbf{a}}{h}[\![u_h]\!] \quad \text{and} \quad \widehat{u}_h = \{u_h\}.$$

On boundary edges, we take

$$\widehat{\boldsymbol{\sigma}}_h = (g - i\kappa u_h)n \quad \text{and} \quad \widehat{u}_h = u_h.$$

Here ∇_h is the piecewise gradient and $\mathbf{a}$ is the penalty parameter to be chosen. Now, we define the finite element space

$$V_h := \{v|_K \in Q^r(K, \mathbf{q}) : K \in \mathcal{T}_h\},$$

where $\mathbf{q}$ can be depending on K. Substituting the numerical fluxes into equation (27), summing over $K \in \mathcal{T}_h$ and applying the "magic formula":

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} v\overline{\sigma} \cdot n\, ds = \int_{\mathcal{F}^I} [\![v]\!] \cdot \{\overline{\sigma}\}\, ds + \int_{\mathcal{F}^I} \{v\} \cdot [\![\overline{\sigma}]\!]\, ds + \int_{\partial\Omega} v\sigma \cdot n\, ds, \tag{28}$$

we obtain the following discrete scheme: Find $u_h \in V_h$ such that for any $v_h \in V_h$,

$$a_h(u_h, v_h) - (\kappa^2 u_h, v_h) = (f, v_h) + \int_{\mathcal{F}_h^B} g\overline{v}_h\ ds, \tag{29}$$

$$a_h(u, v) := \int \nabla_h u \cdot \nabla_h \overline{v}\, dx - \int_{\mathcal{F}_h^I} [\![u]\!] \cdot \{\nabla_h \overline{v}\}\ ds - \int_{\mathcal{F}_h^I} \{\nabla_h u\} \cdot [\![\overline{v}]\!]\ ds + i\int_{\mathcal{F}_h^I} \frac{\mathbf{a}}{h}[\![u]\!] \cdot [\![\overline{v}]\!]\ ds + i\int_{\mathcal{F}_h^B} \kappa u\overline{v}\ ds.$$

Remark 3.6. The plane wave discontinuous Galerkin method of Gittelson and Hiptmair [12] for constant wavenumber $\kappa(x) = \omega$ is obtained by choosing the interior fluxes as

$$\boldsymbol{\sigma}_h = \{\nabla_h u_h\} - i\frac{\mathbf{a}}{h}[\![u_h]\!] \quad \text{and} \quad \widehat{u}_h = \{u_h\} + i\mathbf{b}h[\![u_h]\!], \tag{30}$$

and the boundary fluxes as

$$\begin{aligned}\widehat{\boldsymbol{\sigma}}_h &= \nabla u_h - (1 - \mathbf{d}\omega h)(\nabla u_h + i\omega u_h n - gn) \quad \text{and} \\ \widehat{u}_h &= u_h + i\mathbf{d}h(\nabla u_h \cdot n + i\omega u_h - g),\end{aligned} \tag{31}$$

in equation (27), with $\mathbf{a} \geq \mathbf{a}_{\min} > 0$, $\mathbf{b} \geq 0$ and $\mathbf{d} > 0$. Our choice of fluxes is equivalent to taking $\mathbf{b} = \mathbf{d} = 0$. This choice of fluxes is valid as we will see in Proposition 4.2.

4 Convergence analysis

In this section, we replace the assumption (Q) in the previous section by a weaker one:

Assumption (Q*)**.** *We assume the space $Q^r(K, \mathbf{q})$ satisfies:*

1. ***Trace inverse inequality**. There is $C_{tinv} > 0$ independent of h such that for any $K \in \mathcal{T}_h$, $v \in Q^r(K, \mathbf{q})$,*

$$|v|_{1,\partial K} \leq C_{tinv} h^{-1/2} |v|_{1,K}\,; \tag{32}$$

2. ***Projection error estimates**. For $1 \leq k \leq r$, there is a constant $C_{proj,k} > 0$ independent of h such that for any $K \in \mathcal{T}_h$, $v \in H^{r+1}(K)$,*

$$|v - \mathsf{Q}_r v|_{k,K} \leq C_{proj,k} h^{r+1-k} \|v\|_{r+1}\,. \tag{33}$$

Let $V \subseteq H^2(\Omega)$ to be the space of all possible solution of the model problem (1) and (2). We extend the definition of a_h to the space $(V + V_h) \times (V + V_h)$. Note that by the definition our discrete scheme (29) is consistent, meaning that for any $v_h \in V_h$,

$$a_h(u - u_h, v_h) - (\kappa^2(u - u_h), v_h) = 0, \tag{34}$$

where u is the analytic solution of the model problem and u_h is the discrete solution of (1) and (2).

Assumption (P)**.** *We assume there is a constant $\mathbf{a}_{min} > C_{tinv}^2$ such that the penalty parameter $\mathbf{a}$ satisfies*

$$\mathbf{a} \geq \mathbf{a}_{min} \quad \text{on } \mathcal{F}_h^I.$$

Proposition 4.1. *The discrete problem* (29) *has a unique solution.*

Proof. Suppose $a_h(u_h, u_h) = 0$. The imaginary part of the equation gives $u_h = 0$ on $\mathcal{F}_h^B$ and $[\![u_h]\!] = 0$ on $\mathcal{F}_h^I$ while the real part gives $\nabla u_h = 0$ on every $K \in \mathcal{T}_h$. □

Next, we define an auxiliary bilinear form b_h and a mesh-dependent norm $\|\cdot\|_{DG}$ on $V + V_h$:

$$b_h(u, v) := a_h(u, v) + (\kappa^2 u, v),$$

$$
\begin{aligned}
\|v\|_{DG}^2 \quad := \quad & \|\nabla_h v\|_{0,\Omega}^2 + \left\|\mathbf{a}^{1/2} h^{-1/2} [\![v]\!]\right\|_{0,\mathcal{F}_h^I}^2 + \left\|\mathbf{a}^{-1/2} h^{1/2} \{\nabla v\}\right\|_{\mathcal{F}_h^I}^2 + \\
& \left\|\kappa^{1/2} v\right\|_{0,\mathcal{F}_h^B}^2 + \|\kappa v\|_{0,\Omega}^2 .
\end{aligned}
$$

Note that by applying Cauchy's inequality, we have

$$
\left|a_h(u,v) \pm (\kappa^2 u, v)\right| \le 2 \|u\|_{DG} \|v\|_{DG} . \tag{35}
$$

for any $u, v \in V + V_h$.

Proposition 4.2 (Coercivity of b_h)**.** *There is a constant $C_{coer} > 0$ independent of h such that for any $v \in V_h$,*

$$
b_h(v,v) \ge C_{coer} \|v\|_{DG}^2 .
$$

Proof. Let $v \in V_h$. By definition we have

$$
\begin{aligned}
b_h(v,v) \quad = \quad & \|\nabla v\|_0^2 - 2\mathrm{Re} \int_{\mathcal{F}_h^I} [\![v]\!] \cdot \{\nabla_h \bar{v}\} \; dS + i \left\|\mathbf{a}^{1/2} h^{-1/2} [\![v]\!]\right\|_{0,\mathcal{F}_h^I}^2 \\
& + i \left\|\kappa^{1/2} v\right\|_{0,\mathcal{F}_h^B}^2 + \|\kappa v\|_0^2 .
\end{aligned} \tag{36}
$$

Using the Cauchy's inequality and Young's inequality, for any $s > 0$,

$$
\left| 2\mathrm{Re} \int_{\mathcal{F}_h^I} [\![v]\!] \cdot \{\nabla_h v\} \; dS \right| \le \frac{s}{\mathbf{a}_{\min}} \left\|\mathbf{a}^{1/2} h^{-1/2} [\![v]\!]\right\|_{0,\mathcal{F}_h^I}^2 + \frac{1}{s} \left\|h^{1/2} \{\nabla_h v\}\right\|_{0,\mathcal{F}_h^I}^2 .
$$

Inserting this into equation (36),

$$
\begin{aligned}
\sqrt{2}\,|b_h(v,v)| \quad \ge \quad & \mathrm{Re}\;\, b_h(v,v) + \mathrm{Im}\;\, b_h(v,v) \\
\ge \quad & \|\nabla v\|_0^2 - \frac{s}{\mathbf{a}_{\min}} \left\|\mathbf{a}^{1/2} h^{-1/2} [\![v]\!]\right\|_{0,\mathcal{F}_h^I}^2 - \frac{1}{s} \left\|h^{1/2} \{\nabla v\}\right\|_{0,\mathcal{F}_h^I}^2 \\
& + \left\|\mathbf{a}^{1/2} h^{-1/2} [\![v]\!]\right\|_{0,\mathcal{F}_h^I}^2 + \left\|\kappa^{1/2} v\right\|_{0,\mathcal{F}_h^B}^2 + \|\kappa v\|_0^2 \\
\ge \quad & (1-t) \|\nabla v\|_0^2 + \left(\frac{t}{C_{\mathrm{tinv}}^2} - \frac{1}{s}\right) \mathbf{a}_{\min} \left\|\mathbf{a}^{-1/2} h^{1/2} \{\nabla v\}\right\|_{0,\mathcal{F}_h^I}^2 \\
& + \left(1 - \frac{s}{\mathbf{a}_{\min}}\right) \left\|\mathbf{a}^{1/2} h^{-1/2} [\![v]\!]\right\|_{0,\mathcal{F}_h^I}^2 \\
& + \left\|\kappa^{1/2} v\right\|_{0,\mathcal{F}_h^B}^2 + \|\kappa v\|_0^2
\end{aligned} \tag{37}
$$

Pick s, t and $\mathbf{a}_{\min}$ such that $\mathbf{a}_{\min} > s > C_{\mathrm{tinv}}^2$ and $1 > t > C_{\mathrm{tinv}}^2 / s$, the result follows. □

We will use Schatz's duality argument to show the convergence of our method.

Assumption (K)**.** *We assume the wavenumber $\kappa(x)$ satisfies:*

1. *There are constants κ_* and κ^* such that for any $x \in \Omega$,*

$$
0 < \kappa_* \le \kappa(x) \le \kappa^* < \infty;
$$

2. $\eta_{\kappa,\Omega} := 1 - \min_{x_0\in\Omega} \max_{x\in\Omega} \kappa^{-1} \left|\nabla\kappa \cdot (x - x_0)^T\right| > 0$;

3. κ_* *is bounded away from zero.*

Proposition 4.3. *Let $u \in H^2(\Omega)$ be the analytic solution to the model problem* (1)*,*(2)*, and $u_h \in V_h$ be the discrete solution of* (29)*, then there is $C_{abs} > 0$ independent of h such that*

$$\|u - u_h\|_{DG} \le C_{abs} \left(\inf_{v_h\in V_h} \|u - v_h\|_{DG} + \kappa^*(\kappa^*\kappa_*^{-1}) \|u - u_h\|_0 \right).$$

Proof. Let $v_h \in V_h$. By triangle inequality,

$$\|u - u_h\|_{DG} \quad \le \quad \|u - v_h\|_{DG} + \|v_h - u_h\|_{DG} \tag{38}$$

For the second term, from the coercivity of b_h (Proposition 4.2), scheme consistency (34) and equation (35),

$$\begin{aligned} \|v_h - u_h\|_{DG}^2 \le \frac{1}{C_{\text{coer}}} b_h(v_h - u_h, v_h - u_h) \\ = \quad \frac{1}{C_{\text{coer}}} b_h(v_h - u, v_h - u_h) + \frac{2}{C_{\text{coer}}} \left|(\kappa^2(u - u_h), v_h - u_h)\right| \\ \le \quad \frac{2}{C_{\text{coer}}} \Big(\|u - v_h\|_{DG} \|v_h - u_h\|_{DG} + \left|(\kappa^2(u - u_h), v_h - u_h)\right| \Big). \end{aligned} \tag{39}$$

Using Cauchy's inequality and comparing the L^2 norm to the DG norm,

$$\left|(\kappa^2(u - u_h), v_h - u_h)\right| \quad \le \quad \kappa^*(\kappa^*\kappa_*^{-1}) \|u - u_h\|_0 \|v_h - u_h\|_{DG} \tag{40}$$

Since v_h is arbitrary, the result follows by combining (38), (39) and (40). □

Next we modify the proof of the regularity theorem given by Melenk [22] in order to provide regularity for the adjoint problem with variable wavenumber. We define the following weighted norm on $H^1(\Omega)$:

$$\|v\|_{1,\kappa}^2 = |v|_{1,\Omega}^2 + \|\kappa v\|_0^2 .$$

Theorem 4.4. *Let Ω be a bounded convex domain. Consider the following adjoint problem of* (1) *and* (2)*:*

$$-\Delta\varphi - \kappa^2\varphi = w \qquad \text{in } \Omega, \tag{41}$$

$$-\nabla\varphi \cdot n + i\kappa\varphi = 0 \qquad \text{on } \partial\Omega, \tag{42}$$

where $w \in L^2(\Omega)$. Let $\lambda_\kappa := \left\|\kappa^{-1}\nabla\kappa\right\|_0$. Then the solution $\varphi \in H^2(\Omega)$ and there is a constant $C_1, C_2 > 0$ only depending on Ω such that

$$\|\varphi\|_{1,\kappa} \le C_1\, \eta_{\kappa,\Omega}^{-1}(\kappa^*\kappa_*^{-1}) \|w\|_0 .$$

$$|\varphi|_2 \le C_2 \eta_{\kappa,\Omega}^{-1}(\kappa^*\kappa_*^{-1})(1 + \kappa^* + \lambda_\kappa) \|w\|_0 .$$

Proof. Without loss of generality, assume the domain Ω contains the origin and the origin is the maximizer of x_0 in the definition of $\eta_{k,\Omega}$. Consider the weak form of the problem:

$$B_h(\varphi, \psi) := \int_\Omega \nabla\varphi \cdot \overline{\nabla\psi}\, dx - \int_\Omega \kappa^2\varphi\overline{\psi}\, dx + i \int_{\partial\Omega} \kappa\varphi\overline{\psi}\, ds = \int_\Omega w\overline{\psi}\, dx. \tag{43}$$

Letting $\psi = \varphi$ in (43) and consider the real part and imaginary part separately,

$$\int_\Omega \nabla\varphi \cdot \overline{\nabla\varphi}\, dx - \int_\Omega \kappa^2 \varphi\overline{\varphi}\, dx + i\int_{\partial\Omega} \kappa\varphi\overline{\varphi}\, ds = \int_\Omega w\overline{\varphi}\, dx. \tag{44}$$

The real part of (44) gives

$$\left\|\kappa^{1/2}\varphi\right\|_{0,\partial\Omega}^2 \quad \leq \quad \left|\int_\Omega w\overline{\varphi}\, dx\right| \leq \frac{\varepsilon}{2}\left\|\kappa^{1/2}\varphi\right\|_0^2 + \frac{1}{2\varepsilon\kappa_*}\|w\|_0^2 \tag{45}$$

and hence

$$\|\kappa\varphi\|_{0,\partial\Omega}^2 \leq \kappa^*\kappa_*^{-1}\left(\frac{\varepsilon}{2}\|\kappa\varphi\|_0^2 + \frac{1}{2\varepsilon}\|w\|_0^2\right). \tag{46}$$

The imaginary part of (44) gives

$$|\varphi|_1^2 \leq \|\kappa\varphi\|_0^2 + \int_\Omega w\overline{\varphi}\, dx \leq 2\|\kappa\varphi\|_0^2 + \frac{1}{4\kappa_*^2}\|w\|_0^2. \tag{47}$$

By the regularity theory, $\varphi \in H^2(\Omega)$. Therefore it is valid to substitute $\psi = \nabla\varphi \cdot x^T$ in (43). We apply integration by parts and the following identities for smooth function v to the imaginary part of equation (47):

$$\begin{aligned} \mathrm{Re}\,(v\overline{\nabla v}) &= \frac{1}{2}\nabla|v|^2, &(48)\\ \nabla v \cdot \overline{\nabla(x^T\cdot\nabla v)} &= \frac{1}{2}\nabla\cdot(|\nabla v|^2 x^T), &(49)\end{aligned}$$

we obtain the following identity:

$$\begin{aligned}\int_\Omega (\kappa^2 + \kappa\nabla\kappa\cdot x^T)|\varphi|^2\, dx + \frac{1}{2}\int_{\partial\Omega}|\nabla\varphi|^2 x^T\cdot n\, dx - \frac{1}{2}\int_{\partial\Omega}|\kappa\varphi|^2 x^T\cdot n\, dx\\ +\mathrm{Re}\; i\int_{\partial\Omega}\kappa\varphi x^T\cdot\overline{\nabla\varphi}\, ds = \mathrm{Re}\int_\Omega w x^T\cdot\overline{\nabla\varphi}\, dx. \quad (50)\end{aligned}$$

For the first term in (50), we have

$$\begin{aligned}\int_\Omega (\kappa^2 + \kappa\nabla\kappa\cdot x^T)|\varphi|^2\, dx &= \int_\Omega \left(1 + \kappa^{-1}\nabla\kappa\cdot x^T\right)|\kappa\varphi|^2\, dx\\ &\geq \eta_{\kappa,\Omega}\|\kappa\varphi\|_0^2. \quad (51)\end{aligned}$$

We also note that since Ω is a bounded convex domain, there is $\gamma > 0$ such that on $\partial\Omega$, $x^T\cdot n \geq \gamma$ By combining (47), (50) and using $x^T\cdot n \geq \gamma$, there is a constant $C > 0$ such that

$$\eta_{\kappa,\Omega}\|\kappa\varphi\|_0^2 + \frac{\gamma}{2}|\varphi|_{1,\partial\Omega}^2 \leq C\left(\|\kappa\varphi\|_0^2 + \|\kappa\varphi\|_{0,\partial\Omega}|\varphi|_{1,\partial\Omega} + \|w\|_0|\varphi|_1\right), \tag{52}$$

where the constant C here and hereafter only depends on Ω. Applying Cauchy's inequality to (52),

$$\eta_{\kappa,\Omega}\|\kappa\varphi\|_0^2 \leq C\left(\|\kappa\varphi\|_{0,\partial\Omega}^2 + \|w\|_0|\varphi|_1\right). \tag{53}$$

Applying (46) and (47) and Cauchy's inequality with suitable weight to this equation,

$$\|\kappa\varphi\|_0^2 \leq C\eta_{\kappa,\Omega}^{-2}(\kappa^*\kappa_*^{-1})^2\left(1+\kappa_*^{-2}\right)\|w\|_0^2 . \tag{54}$$

The first estimate follows from (47) and the assumption that κ_* is bounded away from zero. For the second estimate, using the regularity theory for $-\Delta$,

$$\begin{aligned} |\varphi|_{H^2(\Omega)} &\leq C\left(\|\Delta\varphi\|_0 + |\partial_n\varphi|_{1/2,\partial\Omega}\right) \\ &\leq C\left(\left\|w+\kappa^2\varphi\right\|_0 + |\kappa\varphi|_{1/2,\partial\Omega}\right) \end{aligned} \tag{55}$$

Also, using the trace theorem for Sobolev space,

$$\begin{aligned} |\kappa\varphi|_{1/2,\partial\Omega} &\leq C\,\|\kappa\varphi\|_1 \\ &\leq C\left(\|\kappa\varphi\|_0 + \lambda_\kappa\,\|\kappa\varphi\|_0 + \kappa^*\,|\varphi|_1\right) \end{aligned} \tag{56}$$

Combining (55) and (56),

$$\begin{aligned} |\varphi|_{H^2(\Omega)} &\leq C(\|w\|_0 + \kappa^*\,\|\kappa\varphi\|_0 + \left(1+\left\|\kappa^{-1}\nabla\kappa\right\|_0\right)\|\kappa\varphi\|_0 + \kappa^*\,|\varphi|_1) \\ &\leq C\left(\|w\|_0 + (1+\kappa^*+\lambda_\kappa)\,\|\varphi\|_{1,\kappa}\right), \end{aligned}$$

and the result following from the first estimate. □

Lemma 4.5. *There is* $C>0$ *depending only on the parameter* $\mathbf{a}$ *and* Ω *such that for any* $u \in H^{r+1}(\Omega)$,

$$\|u-\mathsf{Q}_r u\|_{DG} \leq C\,h^r(1+\kappa^* h)\,\|u\|_{r+1} .$$

Proof. Here we use the *multiplicative trace inequality*: there is $C>0$ independent of h such that for any $K \in \mathcal{T}_h$, $v \in H^1(K)$,

$$\|v\|_{0,\partial K}^2 \leq C\,\|v\|_{0,K}\,(h^{-1}\,\|v\|_{0,K} + |v|_{1,K}), \tag{57}$$

which implies,

$$h\,|u-\mathsf{Q}_r u|_{1,\partial K}^2 \leq C\left(|u-\mathsf{Q}_r u|_{1,K}^2 + h\,|u-\mathsf{Q}_r u|_{1,K}\,|u-\mathsf{Q}_r u|_{2,K}\right), \tag{58}$$

$$\begin{aligned} h^{-1}\,\|u-\mathsf{Q}_r u\|_{0,\partial K}^2 \leq\; & C\Big(h^{-2}\,\|u-\mathsf{Q}_r u\|_{0,K}^2 + \\ & h^{-1}\,|u-\mathsf{Q}_r u|_{1,K}\,\|u-\mathsf{Q}_r u\|_{0,K}\Big). \end{aligned} \tag{59}$$

Also for a boundary edge e,

$$\left\|\kappa^{1/2}(u-\mathsf{Q}_r u)\right\|_{0,e}^2 \leq (\kappa^* h)\left(h^{-1}\,\|u-\mathsf{Q}_r u\|_{0,e}^2\right). \tag{60}$$

Applying equation (59), (60) and the projection error estimate in Assumption (Q*) to the definition of $\|\cdot\|_{DG}$,

$$\|u-\mathsf{Q}_r u\|_{DG}^2 \leq Ch^{2r}(1+\kappa^* h + (\kappa^* h)^2)\,\|u\|_{r+1}^2 , \tag{61}$$

and the result follows. □

Lemma 4.6. *Let φ be the solution to the adjoint problem* (41), (42) *with right hand side $w \in L^2(\Omega)$. Then there is a constant $C_{dual} > 0$ depending only on the parameter* $\mathbf{a}$, $\mathbf{q}$ *and* Ω *such that*

$$\|\varphi - \mathsf{Q}_1\varphi\|_{DG} \leq C_{dual}\,\eta_{\kappa,\Omega}^{-1}(1+\kappa^* h)(1+\kappa^*+\kappa_*+\lambda_\kappa)h\,\|w\|_0\,.$$

Proof. By Lemma 4.5 and definition of $\|\cdot\|_{1,\kappa}$,

$$\begin{aligned}\|\varphi - \mathsf{Q}_1\varphi\|_{DG} &\leq Ch(1+\kappa^* h)\,\|\varphi\|_2 \\ &\leq Ch(1+\kappa^* h)\left((1+\kappa_*)\,\|\varphi\|_{1,\kappa} + |\varphi|_2\right). \end{aligned} \quad (62)$$

The result follows from Theorem 4.4. □

Theorem 4.7. *Let $u \in H^{r+1}(\Omega)$ be the analytic solution of* (1), (2). *Let $u_h \in V_h$ be the the discrete solution of the DG method* (29). *Let*

$$\theta_\kappa := \eta_{\kappa,\Omega}^{-1}\kappa^*(\kappa^*\kappa_*^{-1})(1+\kappa^*+\kappa_*+\lambda_\kappa).$$

Provided that the following threshold condition holds:

$$2C_{abs}\,C_{dual}\,\theta_\kappa(1+\kappa^* h)h < 1,$$

then there is a constant C only depends on $\Omega, \alpha, \tau, \mathbf{a}$, and $\mathbf{q}$ such that

$$\|u - u_h\|_{DG} \leq Ch^r\,\|u\|_{r+1}\,,$$

$$\|u - u_h\|_0 \leq C\theta_\kappa(1+\kappa^* h)h^{r+1}\,\|u\|_{r+1}\,.$$

Proof. Let φ be the solution to the adjoint problem (41) and (42) with right hand side $w \in L^2(\Omega)$. By consistency of the adjoint problem,

$$(u-u_h, w_h) = a_h(u-u_h, \varphi - \mathsf{Q}_1\varphi) - (\kappa^2(u-u_h), \varphi - \mathsf{Q}_1\varphi). \quad (63)$$

Hence by equation (35),

$$|(u-u_h, w_h)| \leq 2\,\|u-u_h\|_{DG}\,\|\varphi - \mathsf{Q}_1\varphi\|_{DG}\,. \quad (64)$$

Applying Lemma 4.6,

$$\|u-u_h\|_0 \leq 2\,C_{\mathrm{dual}}\,\eta_{\kappa,\Omega}^{-1}(1+\kappa^* h)(1+\kappa^*+\kappa_*+\lambda_\kappa)h\,\|u-u_h\|_{DG}\,. \quad (65)$$

Applying Proposition 4.3,

$$\begin{aligned}\|u-u_h\|_{DG} \leq\ & C_{\mathrm{abs}}\Big(\inf_{v_h\in V_h}\|u-v_h\|_{DG} + \\ & 2C_{\mathrm{dual}}\,\theta_\kappa(1+\kappa^* h)h\,\|u-u_h\|_{DG}\Big), \end{aligned} \quad (66)$$

The threshold condition implies

$$\|u-u_h\|_{DG} \leq C\inf_{v_h\in V_h}\|u-v_h\|_{DG}\,. \quad (67)$$

The error estimate in DG norm follows from taking $v_h = \mathsf{Q}_r u$ in equation (67) and applying Lemma 4.5, Then the error estimate in L^2 norm follows from equation (65). □

5 Numerical experiments

5.1 A study of the convergence

In the following examples, we will only consider two square domains $\Omega_1 = [0,1]^2$ and $\Omega_2 = [0.5, 1.5]^2$. To form the triangulation $\mathcal{T}_h$, we subdivide Ω into $n \times n$ squares and further subdivide each square into two triangles by the diagonal. The mesh size h is taken as $1/n$. The penalty parameter is taken as $\mathbf{a} = 10$ for all of our examples, which may not satisfy the assumption (P) but we have observed the expected h-convergence with this choice of parameter in all of our test cases. We will consider examples with settings shown in Table 1, where the source terms f and g are taken accordingly. We show the real part of the solutions in Figure 1. We will use the following local spaces:

$$
\begin{aligned}
Q^r_{1,\omega} &:= Q^r\left(K, \omega \left|x - y^1\right|\right), \\
Q^r_{2,\omega} &:= Q^r\left(K, \left(\omega \left|x - y^2\right|, \omega \left|x - y^3\right|\right)\right), \\
Q^r_{3,\omega} &:= Q^r\left(K, \left(\omega \left|x - y^2\right|, \omega \left|x - y^3\right|, \omega \left|x - y^4\right|\right)\right), \\
Q^r_{4,\omega} &:= Q^r\left(K, \omega x_1^2\right), \\
Q^r_{5,\omega} &:= Q^r\left(K, \left(\omega x_1^2, \omega x_2\right)\right).
\end{aligned}
$$

We will compare the numerical results using these spaces to those using polynomials with closest number of basis. We also note that the phase functions we use in the local spaces for the first three examples are not polynomials, but numerical results in section 5.2 suggest that these spaces satisfy (Q*).

Firstly, we consider Example 1 with $\omega = 1, 10$ and 100, respectively. In Table 2, we compare the relative L^2-error (i.e. the L^2-error divided by the L^2 norm of the solution) of using the local space $Q^r_{1,\omega}(K)$ with those of polynomial space $P^r(K)$, for $r = 1$ and 2. The results suggest that, for $r = 1, 2$, the numerical solutions are second and third order accurate in the L^2-norm, respectively.

Secondly, we consider Example 2 and 3 with $\omega = 100$. We solve these problems using the local spaces $Q^1_{2,100}(K)$ and $Q^1_{3,100}(K)$, and compare the relative L^2-error to $P^2(K)$ and $P^3(K)$, respectively. The results suggest that, the numerical solutions for using $Q^1_{2,100}(K)$ and $Q^1_{3,100}(K)$ are second and third order accurate in the L^2-norm, respectively.

Thirdly, we consider Example 4 with $\omega = 1, 10$, and 100. Note that we have

$$
\max_{x \in \Omega} \kappa^{-1} \left| \nabla \kappa \cdot (x - (1/2, 1/2))^T \right| = 2/3,
$$

which shows that the second assumption in (K) is satisfied. For $r = 1, 2$, we solve the problem using the local space $Q^r_{4,100}(K)$. In Table 5, we compare the relative L^2-error from using these local space to $P^1(K)$ and $P^2(K)$, respectively. The results suggest that, for $r = 1, 2$, the numerical solutions are second order and third order accurate in the L^2-norm, respectively.

Finally, we consider Example 5 with $\omega = 100$. We solve the problem using the local space $Q^1_{5,100}(K)$ and compare the relative L^2-error to those using $P^2(K)$ in Table 6. The results suggest that, the numerical solutions are second order accurate in the L^2-norm.

In all of the examples above, our choice of basis outperforms polynomials with closest number of degrees of freedom.

Remark 5.1. In our examples, we have used the exact phases in the solutions to help us forming the basis. For an unknown solution, instead, one should approximate the phases of the solution, for example, via geometric optics, to form the local basis.

5.2 A study of some constants

In this section we compute the constants in the trace inverse inequality (32) and the inverse estimates in Proposition 3.4. We will also compute the constant in the third assumption of (Q) for a given k_0 in order to justify that the spaces in our examples satisfy the assumption.

For the trace inverse inequality, we consider the stiffness matrix S and the matrix S_∂ given by

$$(S_\partial)_{ij} := \int_{\partial K} \overline{\nabla v_i} \cdot \nabla v_j \, dx.$$

We solve the following generalized eigenvalue problem:

$$(hS_\partial)x = \lambda_1 Sx,$$

and take $C_{\text{tinv},h}$ as the maximum of $\sqrt{\lambda_1}$. We note that the stiffness matrix S is not necessarily invertible, e.g. the stiffness matrices for $P^r(K)$. For local spaces with single phase, the stiffness matrix is singular only if e^{-iq_1} is a polynomial, which is not the case of our local spaces. For our choices of local space with multiple phases, it can be checked that all the stiffness matrices are invertible.

Similarly, for the inverse estimates we consider the stiffness matrix S, the matrix S_2 corresponding to the $|\cdot|_2^2$, and the mass matrix M. We solve the following generalized eigenvalue problems:

$$(h^2 S)x = \lambda_2 Mx,$$

$$(h^4 S_2)x = \lambda_3 Mx,$$

and take $C_{\text{inv},1,h}$ and $C_{\text{inv},2,h}$ as the maximum of $\sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$, respectively.

Also, in order to show the third assumption in (Q) is satisfied for the local spaces with multiple phases, that is $Q^1_{1,100}, Q^1_{2,100}$ and $Q^1_{4,100}$, we consider

$$Px = \lambda_4 Mx,$$

and take $C_{Q,h}$ as the maximum of $\sqrt{\lambda_4}$. Here P is the matrix associate with $\sum_{\ell=1}^m \|p_\ell\|_{0,K}^2$, $p_\ell \in P^1(K)$. The results are shown in Tables 7.

6 Conclusion

In this paper, we prove some properties of a type of local approximation spaces consisting of polynomial-modulated phase functions and derive an *a priori* estimates for an IPDG method incorporating these spaces. We provide numerical results to demonstrate the efficiency of the local spaces of polynomial-modulated phase function with the knowledge of the phase values. The error levels obtained with such local spaces is usually several order of magnitudes lower than those with standard polynomial spaces. In the future, we would like to further investigate the method by looking for more concrete classes of local approximation spaces satisfying our assumptions.

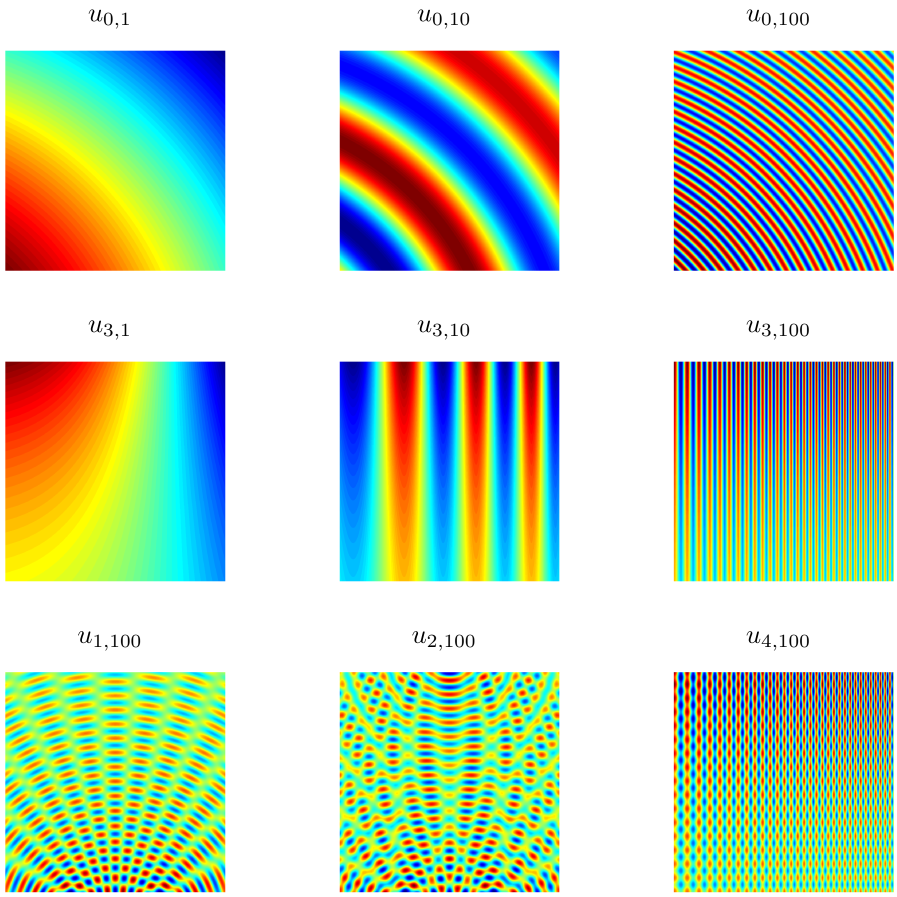

Figure 1: Plots for the real part of the exact solution in the examples.

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Example	Domain	κ	Solution
1	Ω_1	ω	$u_{1,\omega} := H_0^{(1)}(\omega \lvert x - y^1 \rvert)$
2	Ω_1	ω	$u_{2,\omega} := \sum_{k=2}^{3} H_0^{(1)}(\omega \lvert x - y^k \rvert)$
3	Ω_1	ω	$u_{3,\omega} := \sum_{k=2}^{4} H_0^{(1)}(\omega \lvert x - y^k \rvert)$
4	Ω_2	$2\omega x_1$	$u_{4,\omega} := e^{x_2} e^{i\omega x_1^2}$
5	Ω_2	$2\omega x_1$	$u_{5,\omega} := e^{x_2} e^{i\omega x_1^2} + e^{i\omega x_2}$

Table 1: Problem settings for different examples. Here, $\Omega_1 = [0,1]^2$, $\Omega_2 = [0.5, 1.5]^2$, ω is a constant. $H_0^{(1)}$ is the zeroth order Hankel function of the first kind. The variables y^1,y^2,y^3 and y^4 are $(-1,-1), (0.3,-0.1), (0.7,-0.1)$ and $(0.5, 1.1)$, respectively. The variables x_1 and x_2 are the two coordinates of x.

$1/h$	Local space	$\omega = 1$		$\omega = 10$		$\omega = 100$	
		Error	Order	Error	Order	Error	Order
8		4.353e-03	-	4.566e-03	-	2.757e-03	-
11		2.595e-03	1.62	3.016e-03	1.30	1.612e-03	1.68
16		1.356e-03	1.73	1.808e-03	1.37	8.686e-04	1.65
23	$Q^1_{1,\omega}$	7.015e-04	1.82	1.019e-03	1.58	4.708e-04	1.69
32		3.776e-04	1.88	5.718e-04	1.75	2.673e-04	1.71
45		1.965e-04	1.92	3.041e-04	1.85	1.471e-04	1.75
64		9.909e-05	1.94	1.550e-04	1.91	7.899e-05	1.77
8		6.643e-03	-	1.646e-01	0.00	9.986e-01	-
11		3.827e-03	1.73	9.286e-02	1.80	9.985e-01	0.00
16		1.945e-03	1.81	4.565e-02	1.90	9.868e-01	0.03
23	P^1	9.877e-04	1.87	2.251e-02	1.95	1.023e+00	-0.10
32		5.259e-04	1.91	1.173e-02	1.97	1.014e+00	0.03
45		2.716e-04	1.94	5.961e-03	1.99	1.023e+00	-0.03
64		1.363e-04	1.96	2.955e-03	1.99	1.094e+00	0.19
8		2.904e-04	-	4.687e-04	0.00	3.567e-04	-
11		1.266e-04	2.61	2.090e-04	2.54	1.751e-04	2.23
16		4.561e-05	2.72	7.647e-05	2.68	8.914e-05	1.80
23	$Q^2_{1,\omega}$	1.641e-05	2.82	2.776e-05	2.79	4.261e-05	2.03
32		6.345e-06	2.88	1.080e-05	2.86	1.504e-05	3.15
45		2.347e-06	2.92	4.011e-06	2.90	4.895e-06	3.29
64		8.319e-07	2.95	1.425e-06	2.94	1.640e-06	3.10
8		3.481e-04	-	7.268e-03	0.00	1.009e+00	-
11		1.498e-04	2.65	2.751e-03	3.05	1.018e+00	-0.03
16		5.346e-05	2.75	8.840e-04	3.03	1.037e+00	-0.05
23	P^2	1.912e-05	2.83	2.968e-04	3.01	1.110e+00	-0.19
32		7.365e-06	2.89	1.103e-04	3.00	6.224e-01	1.75
45		2.718e-06	2.92	3.971e-05	3.00	2.284e-01	2.94
64		9.617e-07	2.95	1.383e-05	3.00	6.562e-02	3.54

Table 2: The relative L^2-error for Example 1.

$1/h$	Local space	Error	Order	Local space	Error	Order
8	$Q^1_{2,100}$	7.448e-03	-	P^2	8.636e-03	-
11		4.312e-03	1.72		1.016e+00	-0.03
16		2.091e-03	1.93		1.028e+00	-0.03
23		8.712e-04	2.41		1.018e+00	0.03
32		3.927e-04	2.41		8.810e-01	0.44
45		1.782e-04	2.31		5.893e-01	1.18
64		8.199e-05	2.20		2.275e-01	2.70

Table 3: The relative L^2-error for Example 2.

$1/h$	Local space	Error	Order	Local space	Error	Order
8	$Q^1_{3,100}$	1.006e+00	-	P^3	1.029e+00	-
11		5.107e-03	1.65		1.037e+00	-0.02
16		2.878e-03	1.53		9.511e-01	0.23
23		9.378e-04	3.09		6.546e-01	1.03
32		2.181e-04	4.42		2.254e-01	3.23
45		6.422e-05	3.58		4.020e-02	5.06
64		2.150e-05	3.11		5.820e-03	5.48

Table 4: The relative L^2-error for Example 3.

1/h	Local space	$\omega = 1$		$\omega = 10$		$\omega = 100$	
		Error	Order	Error	Order	Error	Order
8		1.040e-03	-	8.770e-04	-	7.100e-04	-
11		5.550e-04	1.96	4.710e-04	1.95	3.740e-04	2.02
16		2.640e-04	1.98	2.260e-04	1.96	1.790e-04	1.96
23	$Q^1_{4,\omega}$	1.280e-04	1.99	1.110e-04	1.96	8.750e-05	1.97
32		6.640e-05	1.99	5.790e-05	1.97	4.560e-05	1.97
45		3.360e-05	2.00	2.950e-05	1.98	2.320e-05	1.98
64		1.660e-05	2.00	1.470e-05	1.98	1.160e-05	1.98
8		6.609e-03	0.00	1.183e+00	-	1.000e+00	-
11		3.535e-03	1.97	1.159e+00	0.06	1.000e+00	0.00
16		1.682e-03	1.98	8.450e-01	0.84	1.000e+00	0.00
23	P^1	8.168e-04	1.99	5.160e-01	1.36	1.000e+00	0.00
32		4.227e-04	1.99	2.992e-01	1.65	1.003e+00	-0.01
45		2.140e-04	2.00	1.612e-01	1.81	9.983e-01	0.01
64		1.058e-04	2.00	1.612e-01	1.90	9.956e-01	0.01
8		1.087e-05	-	1.098e-05	-	1.169e-05	-
11		4.206e-06	2.98	4.233e-06	2.99	4.469e-06	3.02
16		1.373e-06	2.99	1.378e-06	2.99	1.425e-06	3.05
23	$Q^2_{4,\omega}$	4.636e-07	2.99	4.645e-07	3.00	4.718e-07	3.05
32		1.724e-07	2.99	1.726e-07	3.00	1.740e-07	3.02
45		6.209e-08	3.00	6.213e-08	3.00	6.260e-08	3.00
64		2.171e-08	2.98	2.161e-08	3.00	2.184e-08	2.99
8		1.690e-04	-	4.346e-01	-	1.000e+00	-
11		6.546e-05	2.98	1.794e-01	2.78	1.001e+00	0.00
16		2.139e-05	2.99	5.125e-02	3.34	1.002e+00	0.00
23	P^2	7.225e-06	2.99	1.378e-02	3.62	9.999e-01	0.01
32		2.688e-06	2.99	4.096e-03	3.67	1.001e+00	0.00
45		9.676e-07	3.00	1.196e-03	3.61	1.029e+00	-0.08
64		3.366e-07	3.00	3.509e-04	3.48	1.090e+00	-0.16

Table 5: The relative L^2-error for Example 4.

1/h	Local space	Error	Order	Local space	Error	Order
8		6.744e-04	-		9.946e-01	0.00
11		3.581e-04	1.99		9.868e-01	0.02
16		1.759e-04	1.90		1.039e+00	-0.14
23	$Q^1_{5,100}$	8.594e-05	1.97	P^2	9.676e-01	0.20
32		4.397e-05	1.97		9.552e-01	0.04
45		1.913e-05	2.44		9.760e-01	-0.06
64		7.799e-06	2.55		1.032e+00	-0.16

Table 6: The relative L^2-error for Example 5.

$1/h$	Local space	$\omega = 1$		$\omega = 10$		$\omega = 100$	
		$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$	$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$	$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$
8	$Q^1_{1,\omega}$	3.716	6.003	3.722	6.250	3.722	15.854
16		3.716	6.001	3.722	6.064	3.722	9.933
32		3.715	6.000	3.722	6.016	3.722	7.332
64		3.715	6.000	3.722	6.004	3.722	6.383
8	$Q^2_{1,\omega}$	4.856	12.535	4.886	12.672	4.971	20.628
16		4.826	12.534	4.881	12.568	4.927	15.333
32		4.808	12.533	4.880	12.542	4.894	13.350
64		4.771	12.533	4.880	12.535	4.887	12.748

$1/h$	Local space	$\omega = 1$	$\omega = 10$	$\omega = 100$
		$C_{\text{inv},2,h}$	$C_{\text{inv},2,h}$	$C_{\text{inv},2,h}$
8	$Q^2_{1,\omega}$	69.856	77.962	401.012
16		69.789	71.928	188.971
32		69.773	70.316	110.651
64		69.769	69.905	82.130

$1/h$	Local space	$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$	$h^3 C_{Q,h}$	Local space	$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$	$h^4 C_{Q,h}$
8	$Q^1_{2,100}$	5.515	23.056	0.1181	$Q^1_{3,100}$	5.631	23.512	1.510e-02
16		5.508	20.498	0.2360		6.073	22.822	2.179e-02
32		5.257	19.911	0.4107		6.051	21.244	3.212e-02
64		5.135	19.695	0.4224		6.023	21.093	3.755e-02

$1/h$	Local space	$\omega = 1$		$\omega = 10$		$\omega = 100$	
		$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$	$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$	$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$
8	$Q^1_{4,\omega}$	3.726	6.007	3.783	6.677	3.885	26.661
16		3.714	6.002	3.741	6.185	3.835	15.090
32		3.708	6.000	3.721	6.048	3.763	9.459
64		3.705	6.000	3.712	6.012	3.732	7.085
8	$Q^2_{4,\omega}$	5.152	12.537	5.177	12.904	5.631	23.512
16		5.145	12.534	5.177	12.631	6.073	22.822
32		4.866	12.534	5.142	12.558	6.051	21.244
64		4.194	12.533	4.889	12.540	6.023	21.093

$1/h$	Local space	$\omega = 1$	$\omega = 10$	$\omega = 100$
		$C_{\text{inv},2,h}$	$C_{\text{inv},2,h}$	$C_{\text{inv},2,h}$
8	$Q^2_{4,\omega}$	69.995	89.934	998.594
16		69.827	75.480	363.086
32		69.783	71.264	168.749
64		69.771	70.148	101.619

$1/h$	Local space	$C_{\text{tinv},h}$	$C_{\text{inv},1,h}$	$h^3 C_{Q,h}$
8	$Q^1_{5,100}$	3.952	26.705	2.574e-03
16		4.508	15.552	1.167e-03
32		4.784	13.911	1.134e-03
64		4.854	14.044	1.136e-03

Table 7: The numerical values of the constants for different local spaces.