SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHODS FOR 1-D LINEAR HYPERBOLIC EQUATIONS WITH DEGENERATE VARIABLE COEFFICIENTS

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Abstract. In this paper, we study the superconvergence behavior of discontinuous Galerkin methods using upwind numerical fluxes for one-dimensional linear hyperbolic equations with degenerate variable coefficients. The study establishes superconvergence results for the flux function approximation as well as for the DG solution itself. To be more precise, we first prove that the DG flux function is superconvergent towards a particular flux function of the exact solution, with an order of $O(h^{k+2})$, when piecewise polynomials of degree k are used. We then prove that the highest superconvergence rate of the DG solution itself is $O(h^{k+\frac{3}{2}})$ as the variable coefficient degenerates or achieves the value zero in the domain. As byproducts, we obtain superconvergence properties for the DG solution and the DG flux function at special points and for cell averages. All theoretical findings are confirmed by numerical experiments.

Key words. Discontinuous Galerkin methods, superconvergence, degenerate variable coefficients, Radau points, upwind fluxes

AMS subject classifications. 65M15, 65M60, 65N30

1. Introduction.

In this paper, we study and analyze the discontinuous Galerkin (DG) method for the following one-dimensional linear hyperbolic equation

$$\begin{aligned} &u_t + (\alpha u)_x = g(x,t), \qquad &(x,t) \in [0,2\pi] \times (0,T], \\ &u(x,0) = u_0(x), \qquad &x \in R, \end{aligned} \tag{1.1}$$

where $u_0(x)$, $\alpha(x)$ and $g(x,t)$ are all smooth functions. Without loss of generality, we only consider periodic boundary conditions. For simplicity, we assume that there are a finite number of zeros of $\alpha(x)$ on the whole domain $[0, 2\pi]$.

The DG method was first introduced by Reed and Hill [19] for solving the neutron transport problem, and was later developed by Cockburn et al. [12, 14, 15, 16] for solving time dependent nonlinear equations. Since then, the DG method has been intensively studied and successfully applied to various problems in a wide range of applications. We refer to [13] and references cited therein for the development of DG methods.

Superconvergence of the DG method has been studied for many years. The first superconvergence result for the DG solution can be found in [1]. Later, in [2], Adjerid and Massey showed a superconvergence rate of $k + 2$ for the DG solution at

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the downwind-biased Radau points for some ordinary differential equations, when piecewise polynomials of degree k are used. In 2007, Celiker and Cockburn proved a $(2k+1)$-th order superconvergence for the numerical traces of the DG and hybridizable DG solutions when applied to steady state problems [9]. Later, Xie and Zhang proved that the same superconvergence rate of $2k+1$ holds for the DG approximation at downwind points for singularly perturbed problems [20]. For time dependent problems, Cheng and Shu in [11] proved that the DG solution for 1-D hyperbolic and convection-diffusion equations is superconvergent towards a particular projection of the exact solution with an order of $k+\frac{3}{2}$, and later this convergence rate was improved to $k+2$ in [21, 22] by Yang and Shu based on the duality arguments. Recently, Cao et al. revisited the 1-D and 2-D hyperbolic problems and proved a $(2k+1)$-th superconvergence rate for the DG solution at downwind points and for cell averages by using the idea of correction functions, see. e.g., [6, 8]. We also refer to [3, 4, 10, 17, 18, 24] as an incomplete list of references on the superconvergence of DG methods. Note that almost all the superconvergence studies in the literature are based on problems with constant or non-degenerate (variable or nonlinear) coefficients. To the best of our knowledge, there has been no superconvergence result of the DG methods when they are applied to problems with general variable coefficients which may degenerate, i.e. the coefficients either change signs or otherwise achieve the value zero in the domain.

This paper is concerned with the superconvergence properties of the DG method for the 1-D linear hyperbolic equation (1.1) with α being any smooth function which may generate or have zeros in the considered domain. As superconvergence has been extensively studied for the problem (1.1) with $\alpha = 1$ (see, e.g., [8, 11, 21]), it is natural to ask whether the superconvergence phenomenon still exists and the same superconvergence results still hold for problems with variable coefficients. As we will see from the result of this paper, as well as the results in [18] for the more complicated nonlinear case, if the variable or nonlinear coefficients are bounded away from zero, we are able to obtain superconvergence results using the same methods as the constant coefficient case. However, the approach used in such analysis breaks down when the coefficient changes signs or otherwise achieves the value zero. In this work, we will prove that the superconvergence phenomenon of the DG methods for (1.1) still exists when the coefficients degenerate, and the superconvergence rate may depend upon specific properties of the variable coefficient function α. The highest superconvergence rate for the DG solution at some special points is obtained, which is a half order higher than the optimal convergence rate. Our numerical examples demonstrate that this highest superconvergence rate for the DG solution approximation is sharp in many situations. As we may recall, these results are very different from the superconvergence results in [8] for the constant coefficient problems. Another important result we establish in this work is the superconvergence for the flux function of the DG methods. For the first time, we prove that the DG flux function is superconvergent towards a particular flux function of the exact solution with an order of $k+2$. This superconvergence result can be viewed as the generalization of the result in [8, 21] for the constant coefficient problems. As byproducts, superconvergence properties for the DG solution and the DG flux function at some special points (Radau points), as well as for cell averages, are also obtained.

The contribution of this paper is to present and reveal the superconvergence behavior of the DG methods for possibly degenerate variable coefficient problems. On the one hand, we uncover the superconvergence phenomenon of the DG methods for hyperbolic equations with variable coefficients, and prove that the superconvergence rate may depend upon the property of the variable coefficient function. On the other hand, we establish the superconvergence results for the flux function of the DG methods, and extend the superconvergence results for constant coefficient problems to the general case. By doing so, we present a full picture for the superconvergence properties and enrich the superconvergence theory of the DG method for linear hyperbolic equations in one dimension. Furthermore, our current work is also part of an ongoing effort to develop DG superconvergence results for degenerate nonlinear hyperbolic equations.

The key step of our superconvergence analysis is still based upon the idea of correction functions. That is, we construct special functions to correct the error between the DG solution and specific projections of the exact solution. However, the correction function here is not the same as that in [8] for constant coefficients, as the correction function in [8] fails to work in cells where the variable coefficient either degenerates to zero or is very close to zero. New techniques are developed in our analysis to guarantee the superconvergence results, especially the superconvergence results for the DG solution itself.

To end this introduction, we would like to emphasize that the superconvergence analysis for linear hyperbolic equations with degenerate variable coefficients is significant and meaningful. First, this is a necessary step towards the analysis of nonlinear hyperbolic problems $u_t + f(u)_x = 0$, for which the wind direction $f'(u)$ changes sign generically and the assumption made in Meng et al [18] for $|f'(u)| \geq \delta > 0$ is quite artificial. Second, the analysis is much more difficult than the case without sonic points, i.e., points at which $f'(u) = 0$, studied before in [8, 21]. The tools developed in this paper are expected to be useful in future analysis, e.g., for the analysis of nonlinear problems. Third, superconvergence for hyperbolic problems with sonic points is important for adaptive computation for such problems, which arises in many applications such as computational fluid dynamics and computational electro-magnetism.

The rest of the paper is organized as follows. In Section 2, we present DG schemes and discuss their stability for linear hyperbolic equations with variable coefficients. Section 3 is dedicated to the superconvergence analysis of the flux function of the DG methods. Using special correction functions, we obtain $(k+2)$-th order superconvergence rate for flux function approximation. In Section 4, we study the superconvergence behavior of the DG approximation itself, and reveal a very important fact that the superconvergence phenomenon exists for general variable coefficient hyperbolic equations, and the superconvergence rate may depend upon specific properties of the variable coefficient function. In Section 5, we provide numerical examples to support our theoretical findings. Finally, concluding remarks and remarks for possible future work are presented in Section 6.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$.

2. The DG methods.

2.1. Numerical schemes. Let $\Omega = [0, 2\pi]$ and $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}} = 2\pi$ be $N+1$ distinct points on the interval Ω. For all positive integers r, we define

$\mathbb{Z}_r = \{1, \dots, r\}$ and denote by

$$\tau_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}), \quad j \in \mathbb{Z}_N$$

the cells and cell centers, respectively. Let $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $\overline{h}_j = h_j/2$ and $h = \max_j h_j$. We assume that the mesh is regular, i.e., the ratio between the maximum and minimum mesh sizes shall stay bounded during mesh refinements.

Define

$$V_h = \{v_h : v_h|_{\tau_j} \in \mathbb{P}_k(\tau_j), \ j \in \mathbb{Z}_N\}$$

to be the finite element space, where $\mathbb{P}_k$ denotes the space of polynomials of degree at most k. The DG scheme for (1.1) reads as: Find $u_h \in V_h$ such that for any $v_h \in V_h$,

$$(u_{ht}, v_h)_j - (\alpha u_h, v_{hx})_j + \alpha \hat{u}_h v_h^-|_{j+\frac{1}{2}} - \alpha \hat{u}_h v_h^+|_{j-\frac{1}{2}} = (g, v_h)_j, \tag{2.1}$$

where $(u, v)_j = \int_{\tau_j} uv dx$, $v_h^-|_{j+\frac{1}{2}}$ and $v_h^+|_{j+\frac{1}{2}}$ denote the left and right limits of v_h at the point $x_{j+\frac{1}{2}}$, respectively, and $\hat{u}_h$ is the numerical flux, which is the discrete approximation to the trace of u on the boundary of each interval. The choice of the numerical flux is of great importance to ensure the stability of the numerical scheme. In this paper, we take the upwind flux defined by

$$\hat{u}_h\big|_{j+\frac{1}{2}} = \begin{cases} u_h(x_{j+\frac{1}{2}}^+), & \text{if } \alpha_{j+\frac{1}{2}} \le 0, \\ u_h(x_{j+\frac{1}{2}}^-), & \text{if } \alpha_{j+\frac{1}{2}} > 0, \end{cases} \quad j \in \mathbb{Z}_N \tag{2.2}$$

where $\alpha_{j+\frac{1}{2}} = \alpha(x_{j+\frac{1}{2}})$.

2.2. Stability. Define

$$H_h^1 = \{v : v|_{\tau_j} \in H^1(\tau_j), \ j \in \mathbb{Z}_N\},$$

and for all $u, v \in H_h^1$, let

$$a_j(u, v) = (u_t, v)_j - (\alpha u, v_x)_j + \alpha \hat{u} v^-|_{j+\frac{1}{2}} - \alpha \hat{u} v^+|_{j-\frac{1}{2}}, \qquad a(u, v) = \sum_{j=1}^{N} a_j(u, v),$$

where $\hat{u}$ is the numerical flux taken as (2.2). Apparently, (2.1) can be rewritten as

$$a_j(u_h, v_h) = (g, v_h)_j, \quad \forall v_h \in V_h, \quad \forall j \in \mathbb{Z}_N.$$

For any periodic boundary condition v, a direct calculation from integration by parts yields

$$a(v, v) = (v_t, v) + \frac{1}{2}(\alpha_x v, v) + \sum_{j=1}^{N} \alpha_{j+\frac{1}{2}} (\{v\} - \hat{v})_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}},$$

where $(u, v) = \sum_{j=1}^{N} (u, v)_j$ and $\{v\} = \frac{v^+ + v^-}{2}$. Thanks to the special choice of the numerical flux, we have

$$\sum_{j=1}^{N} \alpha_{j+\frac{1}{2}} (\{v\} - \hat{v})_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}} \ge 0,$$

which yields

$$(v_t, v) \le a(v,v) - \frac{1}{2}(\alpha_x v, v). \tag{2.3}$$

Especially, by choosing $v = u_h$ in the above inequality, we get

$$(u_{ht}, u_h) \le -\frac{1}{2}(\alpha_x u_h, u_h) + (g, u_h).$$

Then the stability of the numerical scheme is guaranteed by using the Gronwall inequality.

To end this section, we would like to present another stability inequality, which will be used in our superconvergence analysis later. Let

$$A_j(u,v) = (\alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}})a_j(u,v), \quad A(u,v) = \sum_{j=1}^{N} A_j(u,v), \quad u,v \in H_h^1.$$

Recalling the definition of $a_j(\cdot,\cdot)$, we obtain from a direct calculation,

$$\begin{aligned}
A(v,v) &= \sum_{j=1}^{N}(\alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}})((v_t,v)_j - (\alpha v, v_x)_j + \alpha\hat{v}v^-|_{j+\frac{1}{2}} - \alpha\hat{v}v^+|_{j-\frac{1}{2}}) \\
&= \sum_{j=1}^{N}\left((\alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}})((v_t,v)_j + \frac{1}{2}(\alpha_x v, v)_j) + 2\alpha^3_{j+\frac{1}{2}}(\{v\} - \hat{v})_{j+\frac{1}{2}}[v]_{j+\frac{1}{2}}\right) + I,
\end{aligned}$$

where

$$I = \sum_{j=1}^{N}\left(\alpha_{j-\frac{1}{2}}(\alpha^2_{j+\frac{1}{2}} - \alpha^2_{j-\frac{1}{2}})\left(\frac{v^+}{2} - \hat{v}\right)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} + \alpha_{j+\frac{1}{2}}(\alpha^2_{j+\frac{1}{2}} - \alpha^2_{j-\frac{1}{2}})\left(\frac{v^-}{2} - \hat{v}\right)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}}\right),$$

Again, the special choice of the numerical flux in (2.2) ensures that

$$\sum_{j=1}^{N}\alpha^3_{j+\frac{1}{2}}(\{v\} - \hat{v})_{j+\frac{1}{2}}[v]_{j+\frac{1}{2}} \ge 0.$$

Therefore, by defining

$$\|v\|_\alpha = \sum_{j=1}^{N}(\alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}})\int_{\tau_j} v^2(x)dx, \quad \forall v \in H_h^1, \tag{2.4}$$

we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2_\alpha &\le A(v,v) - I - \frac{1}{2}\sum_{j=1}^{N}(\alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}})(\alpha_x v, v)_j \\
&\le |A(v,v)| + \frac{|\alpha|_{1,\infty}}{2}\|v\|^2_\alpha + |I|.
\end{aligned}$$

Noticing that $|\hat{v}_{j-\frac{1}{2}}| \le \|v\|_{0,\infty,\tau_j} + \|v\|_{0,\infty,\tau_{j-1}}, j \in \mathbb{Z}_N$, we get

$$|I| \lesssim h \sum_{j=1}^{N} \Big| (\alpha_{j+\frac{1}{2}} + \alpha_{j-\frac{1}{2}})(\alpha_{j-\frac{1}{2}}(\|v\|_{0,\infty,\tau_j} + \|v\|_{0,\infty,\tau_{j-1}}) + \alpha_{j+\frac{1}{2}}(\|v\|_{0,\infty,\tau_j} + \|v\|_{0,\infty,\tau_{j+1}}))\|v\|_{0,\infty,\tau_j} \Big|$$

$$\lesssim h \sum_{j=1}^{N} (\alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}})\|v\|^2_{0,\infty,\tau_j}.$$

Here and in the following, $A \lesssim B$ denotes that A can be bounded by B multiplied by a constant independent of the mesh size h. Consequently,

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_\alpha \lesssim |A(v,v)| + \|v\|^2_\alpha + h \sum_{j=1}^{N} (\alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}})\|v\|^2_{0,\infty,\tau_j}, \quad \forall v \in H^1_h.$$

By the inverse inequality, we have for all $v_h \in V_h$

$$\|v_h\|_{0,\infty,\tau_j} \lesssim h^{-\frac{1}{2}}\|v_h\|_{0,\tau_j}, \quad j \in \mathbb{Z}_N.$$

Then

$$\frac{d}{dt}\|v_h\|^2_\alpha \lesssim A(v_h, v_h) + \|v_h\|^2_\alpha, \quad \forall v_h \in V_h. \tag{2.5}$$

3. Superconvergence for the DG flux function approximation. This section is dedicated to the superconvergence analysis for the DG flux function approximation. We shall prove that the flux function of the DG method αu_h is superconvergent with an order of $k+2$ towards a particular flux function of the exact solution. It is this supercloseness that gives us the superconvergence properties of the flux function at some special points as well as for the cell average.

We begin with some preliminaries.

3.1. Preliminaries. Let L_m and $L_{j,m}$ be the standard Legendre polynomials of degree m on the interval $[-1,1]$ and τ_j, respectively. That is,

$$L_{j,m}(x) = L_m(s), \quad s = (x - x_j)/\overline{h}_j \in [-1,1].$$

For any function $v \in H^1_h$, we define the primal function $D_x^{-1}v$ of v by

$$D_x^{-1}v\big|_{\tau_j} = \frac{1}{\overline{h}_j}\int_{x_{j-\frac{1}{2}}}^{x} v(x)dx.$$

By the properties of Legendre polynomials and a scaling from τ_j to $[-1,1]$, we have

$$D_x^{-1}L_{j,m}(x) = \int_{-1}^{s} L_m(s)ds = \frac{1}{m(m+1)}(s^2-1)\frac{d}{ds}L_m(s) \tag{3.1}$$

$$= \frac{1}{2m+1}(L_{m+1} - L_{m-1})(s) = \frac{1}{2m+1}(L_{j,m+1} - L_{j,m-1})(x). \tag{3.2}$$

Given a function ψ, we denote by $R_h\psi$ and $P_h^{\pm}\psi$ the traditional L^2 and Gauss-Radau projections of ψ, respectively. That is,

$$(R_h\psi, v_h) = (\psi, v_h), \quad \forall v_h \in \mathbb{P}_k(\tau_j),$$

and

$$(P_h^+\psi, v_h)_j = (\psi, v_h)_j, \quad \forall v_h \in \mathbb{P}_{k-1}(\tau_j), \quad P_h^+\psi(x_{j-\frac{1}{2}}^+) = \psi(x_{j-\frac{1}{2}}^+), \tag{3.3}$$

$$(P_h^-\psi, v_h)_j = (\psi, v_h)_j, \quad \forall v_h \in \mathbb{P}_{k-1}(\tau_j), \quad P_h^-\psi(x_{j+\frac{1}{2}}^-) = \psi(x_{j+\frac{1}{2}}^-). \tag{3.4}$$

In addition, we also define the Gauss-Lobatto projection $Q_h\psi$ (for $k \geq 2$) as follows

$$(Q_h\psi, v_h) = (\psi, v_h), \ \forall v_h \in \mathbb{P}_{k-2}(\tau_j), \ Q_h\psi(x_{j+\frac{1}{2}}^-) = \psi(x_{j+\frac{1}{2}}^-), \ Q_h\psi(x_{j-\frac{1}{2}}^+) = \psi(x_{j-\frac{1}{2}}^+).$$

While for $k = 1$, $Q_h\psi$ is the interpolant of ψ satisfying

$$Q_h\psi(x_{j+\frac{1}{2}}^-) = \psi(x_{j+\frac{1}{2}}^-), \quad Q_h\psi(x_{j-\frac{1}{2}}^+) = \psi(x_{j-\frac{1}{2}}^+).$$

The standard approximation theory gives us

$$\|\psi - P_h^+\psi\|_{0,p} + \|\psi - P_h^-\psi\|_{0,p} + \|\psi - R_h\psi\|_{0,p} + \|\psi - Q_h\psi\|_{0,p} \lesssim h^{k+1}|\psi|_{k+1,p}, \ p \geq 1. \tag{3.5}$$

With the above four projections, we define a particular projection $P_h\psi$ of ψ by

$$P_h\psi = \begin{cases} R_h\psi, & \text{if } \alpha_{j+\frac{1}{2}} \leq 0, \ \alpha_{j-\frac{1}{2}} > 0 \\ P_h^+\psi, & \text{if } \alpha_{j+\frac{1}{2}} \leq 0, \ \alpha_{j-\frac{1}{2}} \leq 0 \\ P_h^-\psi, & \text{if } \alpha_{j+\frac{1}{2}} > 0, \ \alpha_{j-\frac{1}{2}} > 0 \\ Q_h\psi, & \text{if } \alpha_{j+\frac{1}{2}} > 0, \ \alpha_{j-\frac{1}{2}} \leq 0 \end{cases}. \tag{3.6}$$

In each element $\tau_j, j \in \mathbb{Z}_N$, we denote by $r_{j,i}, l_{j,i}, g_{j,i}, gl_{j,i}, i \in \mathbb{Z}_{k+1}$ the right Radau, left Radau, Gauss, and Gauss-Lobatto points of degree $k+1$, respectively, That is, $r_{j,i}$ and $l_{j,i}$ are separately the zeros of the right Radau polynomial $L_{j,k+1} - L_{j,k}$ and the left Radau polynomial $L_{j,k+1} + L_{j,k}$, and $g_{j,i}$ are zeros of the Legendre polynomial $L_{j,k+1}$, and $gl_{j,i}$ are zeros of the Lobatto polynomial $L_{j,k+1} - L_{j,k-1}$. Similarly, we denote by $l^*_{j,m}, m \in \mathbb{Z}_k$ the interior left Radau points (the roots of $L_{j,k+1} + L_{j,k}$ except the point $x = x_{j-\frac{1}{2}}$); $r^*_{j,m}, m \in \mathbb{Z}_k$ the interior right Radau points (the roots of $L_{j,k+1} - L_{j,k}$ except the point $x = x_{j+\frac{1}{2}}$); $g^*_{j,m}, m \in \mathbb{Z}_k$ the Gauss points of degree k (i.e., the roots of $L_{j,k}$); and $gl^*_{j,m}, m \in \mathbb{Z}_k$ the k interior Gauss-Lobatto points (i.e., the roots of $L_{j,k+2} - L_{j,k}$ except the two boundary points $x = x_{j+\frac{1}{2}}, x_{j-\frac{1}{2}}$).

Now we define a class of special points $y_{j,i}, z_{j,m}, i \in \mathbb{Z}_{k+1}, m \in \mathbb{Z}_k$ in each element $\tau_j, j \in \mathbb{Z}_N$ as follows:

$$y_{j,i} = \begin{cases} g_{j,i}, & \text{if } \alpha_{j+\frac{1}{2}} \leq 0, \ \alpha_{j-\frac{1}{2}} > 0 \\ l_{j,i}, & \text{if } \alpha_{j+\frac{1}{2}} \leq 0, \ \alpha_{j-\frac{1}{2}} \leq 0 \\ r_{j,i}, & \text{if } \alpha_{j+\frac{1}{2}} > 0, \ \alpha_{j-\frac{1}{2}} > 0 \\ gl_{j,i}, & \text{if } \alpha_{j+\frac{1}{2}} > 0, \ \alpha_{j-\frac{1}{2}} \leq 0 \end{cases}, \tag{3.7}$$

and

$$z_{j,m} = \begin{cases} gl^*_{j,m}, & \text{if } \alpha_{j+\frac{1}{2}} \leq 0, \ \alpha_{j-\frac{1}{2}} > 0 \\ r^*_{j,m}, & \text{if } \alpha_{j+\frac{1}{2}} \leq 0, \ \alpha_{j-\frac{1}{2}} \leq 0 \\ l^*_{j,m}, & \text{if } \alpha_{j+\frac{1}{2}} > 0, \ \alpha_{j-\frac{1}{2}} > 0 \\ g^*_{j,m}, & \text{if } \alpha_{j+\frac{1}{2}} > 0, \ \alpha_{j-\frac{1}{2}} \leq 0 \end{cases}. \tag{3.8}$$

We have the following approximation properties of $P_h\psi$ at the special points $y_{j,i}, z_{j,m}, (j,i,m) \in \mathbb{Z}_N \times \mathbb{Z}_{k+1} \times \mathbb{Z}_k$.

LEMMA 3.1. *Let $\psi \in W^{k+2,\infty}$, and $P_h\psi \in V_h$ be defined by (3.6). The following approximation properties hold true.*

- *The numerical flux is exact, i.e.,*

$$\hat{P}_h\psi\big|_{j+\frac{1}{2}} = \psi_{j+\frac{1}{2}}, \quad \forall j \in \mathbb{Z}_N. \tag{3.9}$$

  Here the numerical flux is defined by (2.2) with u_h replaced by $P_h\psi$.
- *The function value of $P_h\psi$ is $k+2$-th order superconvergent at the special points $y_{j,i}, (j,i) \in \mathbb{Z}_N \times \mathbb{Z}_{k+1}$ defined by (3.7), i.e.,*

$$|(\psi - P_h\psi)(y_{j,i})| \lesssim h^{k+2}|\psi|_{k+2,\infty,\tau_j}. \tag{3.10}$$

- *The derivative value of $P_h\psi$ is superconvergent at the special points $z_{j,m}, (j,i) \in \mathbb{Z}_N \times \mathbb{Z}_k$, with a convergence rate $k+1$, where $z_{j,m}$ is defined by (3.8). That is,*

$$|\partial_x(\psi - P_h\psi)(z_{j,m})| \lesssim h^{k+1}|\psi|_{k+2,\infty,\tau_j}. \tag{3.11}$$

Proof. For any $j \in \mathbb{Z}_N$, by (2.2) and (3.6), there holds for $\alpha_{j+\frac{1}{2}} > 0$

$$\hat{P}_h\psi\big|_{j+\frac{1}{2}} = P_h\psi(x^-_{j+\frac{1}{2}}) = P^-_h\psi(x^-_{j+\frac{1}{2}}) \ \text{ or } \ Q_h\psi(x^-_{j+\frac{1}{2}}),$$

and for $\alpha_{j+\frac{1}{2}} \le 0$

$$\hat{P}_h\psi\big|_{j+\frac{1}{2}} = P_h\psi(x^+_{j+\frac{1}{2}}) = P^+_h\psi(x^+_{j+\frac{1}{2}}) \ \text{ or } \ Q_h\psi(x^+_{j+\frac{1}{2}}).$$

In both cases, we have, from the properties of $P^{\pm}_h\psi$ and $Q_h\psi$

$$\hat{P}_h\psi\big|_{j+\frac{1}{2}} = \psi(x_{j+\frac{1}{2}}).$$

Now we show (3.10)-(3.11). Using the result in [23] (see Theorem 2.1) and a scaling from $[-1,1]$ to τ_j, we have

$$(\psi - Q_h\psi)\big|_{\tau_j} = \sum_{p=k}^{\infty} \frac{b_{j,p}}{2p+1}(L_{j,p+1} - L_{j,p-1}), \quad \partial_x(\psi - Q_h\psi)\big|_{\tau_j} = \frac{1}{\bar{h}_j}\sum_{p=k}^{\infty}\frac{b_{j,p}}{2p+1}L_{j,p}$$

with

$$b_{j,p} = \frac{2^p p!}{(2p)!}(\bar{h}_j)^{p+1}\partial_x^{p+1}u(\xi_j), \quad \xi_j \in \tau_j.$$

Then

$$|(\psi - Q_h\psi)(gl_{j,i})| \lesssim h^{k+2}|\psi|_{k+2,\infty,\tau_j}, \quad |\partial_x(\psi - Q_h\psi)(g^*_{j,m})| \lesssim h^{k+1}|\psi|_{k+2,\infty,\tau_j}.$$

On the other hand, it has been proved in [7, 8]

$$\begin{aligned}
&|(\psi - P^-_h\psi)(r_{j,i})| + |(\psi - P^+_h\psi)(l_{j,i})| \lesssim h^{k+2}|\psi|_{k+2,\infty,\tau_j},\\
&|\partial_x(\psi - P^-_h\psi)(l^*_{j,m})| + |\partial_x(\psi - P^+_h\psi)(r^*_{j,m})| \lesssim h^{k+1}|\psi|_{k+2,\infty,\tau_j}.
\end{aligned}$$

Therefore, we only need to analyze the superconvergent approximation properties of the L^2 projection $R_h\psi$. To this end, we suppose in each element τ_j,

$$\psi\big|_{\tau_j} = \sum_{p=0}^{\infty} a_{j,p}L_{j,p}, \quad a_{j,p} = \frac{2p+1}{h_j}\int_{\tau_j}\psi L_{j,p}dx. \tag{3.12}$$

By the definition of $R_h\psi$, we easily obtain

$$(\psi - R_h\psi)\big|_{\tau_j} = \sum_{p=k+1}^{\infty} a_{j,p}L_{j,p}, \quad \partial_x(\psi - R_h\psi)\big|_{\tau_j} = \sum_{p=k+1}^{\infty} a_{j,p}\partial_x L_{j,p}.$$

By (3.1)-(3.2)

$$\partial_x L_{j,k+1}(gl^*_{j,m}) = \frac{c_k}{h_j}(L_{j,k+2} - L_{j,k})(gl^*_{j,m}) = 0, \ m \in \mathbb{Z}_k,$$

where c_k is a constant dependent on k. Then

$$(\psi - R_h\psi)(g_{j,i}) = \sum_{p=k+2}^{\infty} a_{j,p}L_{j,p}(g_{j,i}), \quad \partial_x(\psi - R_h\psi)(gl^*_{j,m}) = \sum_{p=k+2}^{\infty} a_{j,p}\partial_x L_{j,p}(gl^*_{j,m}).$$

By denoting $\psi(s,t) = \psi(x,t), s = (x - x_j)/\overline{h}_j \in [-1,1]$, we have, from the integration by parts and the properties of Legendre polynomials,

$$\begin{aligned} a_{j,m} &= \left(\frac{2m+1}{2}\right)\frac{1}{(2m)!}\int_{-1}^{1}\psi\frac{d^m}{ds}(s^2-1)^m ds \\ &= \left(\frac{2m+1}{2}\right)\frac{(-1)^i}{(2m)!}\int_{-1}^{1}\partial_s^i\psi\frac{d^{m-i}}{ds}(s^2-1)^m ds, \quad i \le m. \end{aligned}$$

Noticing that $\partial_s^i\psi = \left(\bar{h}_j\right)^i\partial_x^i\psi$, we have

$$|a_{j,m}| \lesssim h^{i-\frac{1}{p}}|\psi|_{i,p,\tau_j}, \quad 1 \le i \le m. \tag{3.13}$$

Consequently,

$$|(\psi - R_h\psi)(g_{j,i})| + h_j\left|\partial_x(\psi - R_h\psi)(gl^*_{j,m})\right| \lesssim \sum_{m=k+2}^{\infty}|a_{j,m}| \lesssim h^{k+2}|\psi|_{k+2,\infty,\tau_j}.$$

This finishes the proof of (3.10)-(3.11). ☐

We end this subsection with the optimal error estimate for $\|u_h - P_h u\|_0$.

LEMMA 3.2. *Let $u(\cdot,t) \in H^{k+2}, \forall t \in [0,T]$ be the solution of* (1.1)*, and u_h be the solution of* (2.1) *with the initial solution $u_h(x,0) = P_h u_0$. Then*

$$\|(u_h - P_h u)(\cdot,t)\|_0 \lesssim h^{k+1}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+2}. \tag{3.14}$$

Proof. In each element $\tau_j, j \in \mathbb{Z}_N$, if $P_h u|_{\tau_j} = R_h u, P_h^{\pm}u$, we have, from the orthogonality of $R_h u, P_h^{\pm}u$, and (3.9)

$$a_j(u - P_h u, v_h) = (u_t - P_h u_t, v_h)_j + ((\alpha - \alpha_j)(u - P_h u), \partial_x v_h)_j,$$

where $\alpha_j = \alpha(x_j)$ is a constant. If $P_h u = Q_h u$, which indicates $\alpha_{j+\frac{1}{2}}\alpha_{j-\frac{1}{2}} \le 0$, then there exists at least one point $\xi_j \in \tau_j$ such that $\alpha(\xi_j) = 0$, and thus,

$$a_j(u - P_h u, v_h) = (u_t - Q_h u_t, v_h)_j - ((\alpha - \alpha(\xi_j))(u - Q_h u), \partial_x v_h).$$

Combining the above two equations, we have for all $v_h \in V_h$ and $j \in \mathbb{Z}_N$,

$$\begin{aligned} a_j(u - P_h u, v_h) &\lesssim \|u_t - P_h u_t\|_{0,\tau_j}\|v_h\|_{0,\tau_j} + h\|u - P_h u\|_{0,\tau_j}|v_h|_{1,\tau_j} \\ &\lesssim h^{k+1}(\|u\|_{k+1,\tau_j} + \|u_t\|_{k+1,\tau_j})\|v_h\|_{0,\tau_j}, \end{aligned} \tag{3.15}$$

where in the last step, we have used (3.5) and the inverse inequality $|v_h|_{1,\tau_j} \lesssim h_j^{-1}\|v_h\|_{0,\tau_j}$ for all $v_h \in V_h$.

Now we obtain, by choosing $v = u_h - P_h u$ in (2.3), and using (3.9) and the orthogonality $a(u - u_h, v_h) = 0, v_h \in V_h$,

$$\begin{aligned} \frac{d}{dt}\|u_h - P_h u\|_0^2 &\lesssim a(u - P_h u, u_h - P_h u) + \|u_h - P_h u\|_0^2 \\ &= \sum_{j=1}^{N} a_j(u - P_h u, u_h - P_h u) + \|u_h - P_h u\|_0^2. \end{aligned}$$

Then (3.14) follows from (3.15), the Gronwall inequality and the fact that $u_t = -(\alpha u)_x$. ☐

3.2. Analysis. In this subsection, we will study the superconvergence properties of the DG flux function αu_h towards the particular flux function $\alpha P_h u$ of the exact solution in the L^2 norm. In light of (3.14) and the fact that

$$\|\alpha(u_h - P_h u)\|_0 \lesssim \|u_h - P_h u\|_\alpha + h\|u_h - P_h u\|_0 \lesssim h^{k+2}\sup_{t\in[0,T]}\|u(\cdot,t)\|_{k+2} + \|u_h - P_h u\|_\alpha, \tag{3.16}$$

where $\|\cdot\|_\alpha$ is defined by (2.4), our superconvergence goal is reduced to the estimate of the error $\|u_h - P_h u\|_\alpha$.

As indicated by (2.5), the error $\|(u_h - P_h u)(\cdot, t)\|_\alpha$ for any time $t > 0$ is dependent on the initial error and the term $A(u_h - P_h u, u_h - P_h u)$, or equivalently the term $A(u - P_h u, u_h - P_h u)$ due to the orthogonality $A(u - u_h, v_h) = 0, v_h \in V_h$. However, as a direct consequence of (3.15),

$$\begin{aligned} A(u - P_h u, u_h - P_h u) &= \sum_{j=1}^{N}(\alpha_{j+\frac{1}{2}}^2 + \alpha_{j-\frac{1}{2}}^2)a_j(u - P_h u, u_h - P_h u) \\ &\lesssim h^{k+1}\|u\|_{k+2}\|u_h - P_h u\|_\alpha. \end{aligned}$$

Then the standard error estimate only gives us the optimal convergence rate, i.e.,

$$\|(u_h - P_h u)(\cdot, t)\|_\alpha \lesssim h^{k+1}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+2},$$

which is far from our superconvergence goal. Therefore, to improve the convergence rate, some extra term or function is needed to correct the error $A(u - P_h u, v_h), v_h \in V_h$ based on the projection $P_h u$ defined in (3.6). In other words, we need to construct a correction function $w \in V_h$ such that $u_I = P_h u - w$ and

$$A(u - u_I, v_h) = A(u - P_h u, v_h) + A(w, v_h) \lesssim h^{k+1+l}\|v_h\|_\alpha, \quad \forall v_h \in V_h$$

for some $l > 0$. By doing so, we obtain the superconvergence result for $\alpha u_h - \alpha u_I$ (or $\alpha u_h - \alpha P_h u$). Our analysis is along this line. Consequently, the rest of this subsection is dedicated to the construction of the correction function w.

For functions $u, v \in H_h^1$, we define

$$b(u,v) = \sum_{j=1}^{N} b_j(u,v), \quad b_j(u,v) = (u_t, v)_j - (\alpha u, v_x)_j.$$

Apparently, we have from (3.9) and the definition of $a_j(\cdot,\cdot)$,

$$b_j(u - P_h u, v_h) = a_j(u - P_h u, v_h), \quad \forall v_h \in V_h.$$

On the other hand, in light of the definition of $P_h u$ in (3.6), the whole domain Ω can be divided into four parts, that is

$$\Omega = \bigcup_{i=1}^{4} \Omega_i,$$

where

$$\begin{aligned} &\Omega_1 = \{\tau_j : \alpha_{j-\frac{1}{2}} > 0, \alpha_{j+\frac{1}{2}} > 0\}, \quad \Omega_2 = \{\tau_j : \alpha_{j-\frac{1}{2}} \le 0, \alpha_{j+\frac{1}{2}} \le 0\}, \\ &\Omega_3 = \{\tau_j : \alpha_{j-\frac{1}{2}} \le 0, \alpha_{j+\frac{1}{2}} > 0\}, \quad \Omega_4 = \{\tau_j : \alpha_{j-\frac{1}{2}} > 0, \alpha_{j+\frac{1}{2}} \le 0\}. \end{aligned} \tag{3.17}$$

Then

$$\begin{aligned} A(u - P_h u, v_h) = & \sum_{\tau_j \in \Omega_1} \tilde{\alpha}^2_{j+\frac{1}{2}} b_j(u - P_h^- u, v_h) + \sum_{\tau_j \in \Omega_2} \tilde{\alpha}^2_{j+\frac{1}{2}} b_j(u - P_h^+ u, v_h) \\ + & \sum_{\tau_j \in \Omega_3} \tilde{\alpha}^2_{j+\frac{1}{2}} b_j(u - Q_h u, v_h) + \sum_{\tau_j \in \Omega_4} \tilde{\alpha}^2_{j+\frac{1}{2}} b_j(u - R_h u, v_h), \end{aligned} \tag{3.18}$$

where $\tilde{\alpha}^2_{j+\frac{1}{2}} = \alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}}$.

3.2.1. The local correction function for $b_j(u - P_h^- u, v_h)$. Noticing that $(u - P_h^- u) \perp \mathbb{P}_0$, we have

$$D_x^{-1}(u - P_h^- u)(x^-_{j+\frac{1}{2}}) = \frac{1}{\bar{h}_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u - P_h^- u)(x) dx = 0 = D_x^{-1}(u - P_h^- u)(x^+_{j-\frac{1}{2}}).$$

Then a direct calculation from integration by parts yields

$$\begin{aligned} b_j(u - P_h^- u, v_h) &= (u_t - P_h^- u_t, v_h)_j - (\alpha(u - P_h^- u), \partial_x v_h)_j \\ &= -\bar{h}_j (D_x^{-1}(u_t - P_h^- u_t), v_{hx})_j - ((\alpha - \bar{\alpha}_j)(u - P_h^- u), \partial_x v_h)_j, \ v_h \in V_h, \end{aligned} \tag{3.19}$$

where $\bar{\alpha}_j$ is a constant satisfying

$$|\bar{\alpha}_j| = \max_{x \in \tau_j} |\alpha(x)|. \tag{3.20}$$

With $\bar{\alpha}_j$, we now define a special function $w_1 \in V_h$ as follows.

$$\begin{aligned} &\bar{\alpha}_j (w_1, v_h)_j = -\bar{h}_j (D_x^{-1}(u_t - P_h^- u_t), v_h)_j - ((\alpha - \bar{\alpha}_j)(u - P_h^- u), v_h)_j, \quad v_h \in \mathbb{P}_{k-1}, \\ &w_1(x^-_{j+\frac{1}{2}}) = 0, \quad \forall j \in \mathbb{Z}_N. \end{aligned} \tag{3.21}$$

LEMMA 3.3. *Let $w_1 \in V_h$ be defined by* (3.21)*. Then*

$$b_j(u - P_h^- u + w_1, v_h) = (\partial_t w_1, v_h)_j - ((\alpha - \bar{\alpha}_j)w, \partial_x v_h)_j, \forall v_h \in V_h, \tag{3.22}$$

where $\bar{\alpha}_j$ is defined in (3.20)*. Moreover, if $u(\cdot, t) \in W^{k+3,\infty}, t \in [0, T]$, then*

$$\|w_1(\cdot, t)\|_{0,\tau_j} + \|\partial_t w_1(\cdot, t)\|_{0,\tau_j} \lesssim h^{k+\frac{5}{2}} \sum_{l=0}^{k+3} \frac{\|\partial_x^l \alpha\|_{0,\infty,\tau_j}}{\|\alpha\|_{0,\infty,\tau_j}} \|u(\cdot, t)\|_{k+3,\infty}. \tag{3.23}$$

Proof. First, (3.22) follows directly from (3.19) and (3.21). To estimate w_1, we suppose w_1 has the following Legendre expansion in each τ_j

$$w_1\big|_{\tau_j} = \sum_{m=0}^{k} c_m L_{j,m}(x).$$

By choosing $v_h = L_{j,m}, m \le k - 1$ in (3.21), we get

$$\frac{\bar{\alpha}_j h_j}{2m + 1} c_m = -\bar{h}_j (D_x^{-1}(u_t - P_h^- u_t), L_{j,m})_j - ((\alpha - \bar{\alpha}_j)(u - P_h^- u), L_{j,m})_j.$$

Noticing that

$$\|D_x^{-1}(u_t - P_h^- u_t)\|_{0,\infty,\tau_j} \le \|u_t - P_h^- u_t\|_{0,\infty,\tau_j} \lesssim h^{k+1} \|\partial_x^{k+1} u_t\|_{0,\infty,\tau_j} \lesssim h^{k+1} \|\partial_x^{k+2}(\alpha u)\|_{0,\infty,\tau_j},$$

we have for all $m \le k - 1$,

$$\begin{aligned} |c_m| &\lesssim \frac{h^{k+2}}{\bar{\alpha}_j} \left(\|\partial_x^{k+2}(\alpha u)\|_{0,\infty,\tau_j} + \|\partial_x \alpha\|_{0,\infty,\tau_j} \|\partial_x^{k+1} u\|_{0,\infty,\tau_j} \right) \\ &\lesssim \frac{h^{k+2}}{\bar{\alpha}_j} \sum_{l=0}^{k+2} \|\partial_x^l \alpha\|_{0,\infty,\tau_j} \|u\|_{k+2,\infty}. \end{aligned}$$

Since $w_1(x_{j+\frac{1}{2}}^-, t) = 0$, then

$$|c_k| = |\sum_{m=0}^{k-1} c_m| \lesssim \frac{h^{k+2}}{\bar{\alpha}_j} \sum_{l=0}^{k+2} \|\partial_x^l \alpha\|_{0,\infty,\tau_j} \|u\|_{k+2,\infty}.$$

Consequently,

$$\|w_1(\cdot, t)\|_{0,\tau_j}^2 \lesssim h \sum_{m=0}^{k} c_m^2 \lesssim h^{2k+5} \sum_{l=0}^{k+2} \frac{\|\partial_x^l \alpha\|_{0,\infty,\tau_j}^2}{\bar{\alpha}_j^2} \|u(\cdot, t)\|_{k+2,\infty}^2.$$

Similarly, taking time derivative on both sides of (3.21), the identity still holds. Then we follow the same argument to derive

$$\|\partial_t w_1(\cdot, t)\|_{0,\tau_j}^2 \lesssim h^{2k+5} \sum_{l=0}^{k+3} \frac{\|\partial_x^l \alpha\|_{0,\infty,\tau_j}^2}{\bar{\alpha}_j^2} \|u(\cdot, t)\|_{k+3,\infty}^2.$$

Then (3.23) follows. The proof is complete. □

3.2.2. The local correction function for $b_j(u-P_h^+u, v_h)$. Since P_h^+u shares almost the same properties with P_h^-u, the correction function for the term $b_j(u-P_h^+u, v_h)$ is similar to that for $b_j(u-P_h^-u, v_h)$, which is defined as follows. In each element τ_j, let $w_2 \in V_h$ satisfy

$$\begin{aligned} &\bar{\alpha}_j(w_2, v_h)_j = -\bar{h}_j(D_x^{-1}(u_t - P_h^+u_t), v_h)_j - ((\alpha-\bar{\alpha}_j)(u-P_h^+u), v_h)_j, \ v_h \in \mathbb{P}_{k-1}(\tau_j), \\ &w_2(x_{j-\frac{1}{2}}^+) = 0, \quad j \in \mathbb{Z}_N. \end{aligned} \tag{3.24}$$

Here $\bar{\alpha}_j$ is given in (3.20). Following the same argument as in Lemma 3.3, we get

$$b_j(u-P_h^+u+w_2, v_h) = (\partial_t w_2, v_h)_j - ((\alpha-\bar{\alpha}_j)w_2, \partial_x v_h)_j, \tag{3.25}$$

$$\|w_2(\cdot,t)\|_{0,\tau_j} + \|\partial_t w_2(\cdot,t)\|_{0,\tau_j} \lesssim h^{k+\frac{5}{2}} \sum_{l=0}^{k+3} \frac{\|\partial_x^l \alpha\|_{0,\infty,\tau_j}}{\|\alpha\|_{0,\infty,\tau_j}} \|u(\cdot,t)\|_{k+3,\infty}. \tag{3.26}$$

3.2.3. The global correction function for $b(u-P_hu, v_h)$. Now we are ready to define the global correction function on the whole domain. Let $w \in V_h$ be the correction function such that

$$w = \begin{cases} 0, & \alpha_{j+\frac{1}{2}} \le 0, \quad \alpha_{j-\frac{1}{2}} > 0, \\ w_2, & \alpha_{j+\frac{1}{2}} \le 0, \quad \alpha_{j-\frac{1}{2}} \le 0, \\ w_1, & \alpha_{j+\frac{1}{2}} > 0, \quad \alpha_{j-\frac{1}{2}} > 0, \\ 0 & \alpha_{j+\frac{1}{2}} > 0, \quad \alpha_{j-\frac{1}{2}} \le 0, \end{cases} \tag{3.27}$$

where w_1, w_2 are separately defined in (3.21) and (3.24).

We have the following properties of the correction function w.

LEMMA 3.4. *Let $u(\cdot,t) \in W^{k+3,\infty}, \forall t \in [0,T]$ be the solution of* (1.1)*, and $w \in V_h$ be the correction function defined by* (3.27)*. Then*

$$\hat{w}(x_{j+\frac{1}{2}}, t) = 0, \quad j \in \mathbb{Z}_N, \quad \|w(\cdot,t)\|_\alpha + \|\partial_t w(\cdot,t)\|_\alpha \lesssim h^{k+2}\|u(\cdot,t)\|_{k+3,\infty}, \tag{3.28}$$

and

$$A(u-P_hu+w, v_h) \lesssim h^{k+2}\|u\|_{k+3,\infty}\|v_h\|_\alpha, \quad \forall v_h \in V_h. \tag{3.29}$$

Proof. By the definition of w in (3.27), (3.21) and (3.24), we can easily obtain $\hat{w}(x_{j+\frac{1}{2}}, t) = 0, j \in \mathbb{Z}_N$. The second inequality of (3.28) follows from (3.23) and (3.26).

Let $u_I = P_hu - w$. Recall the definition of $A(\cdot,\cdot)$, we immediately get

$$A(w,v) = \sum_{j=1}^{N} \tilde{\alpha}_{j+\frac{1}{2}}^2 \left((w_t, v)_j - (\alpha w, v_x)_j\right) = \sum_{i=1}^{2} \sum_{\tau_j \in \Omega_i} \tilde{\alpha}_{j+\frac{1}{2}} b_j(w_i, v), \tag{3.30}$$

which yields, together with (3.18), (3.22), and (3.25),

$$A(u-u_I, v_h) = \sum_{\tau_j \in \Omega_3 \cup \Omega_4} \tilde{\alpha}_{j+\frac{1}{2}}^2 b_j(u-P_hu, v_h) + \sum_{\tau_j \in \Omega_1 \cup \Omega_2} \tilde{\alpha}_{j+\frac{1}{2}}^2 \big((w_t, v_h)_j - ((\alpha-\bar{\alpha}_j)w, \partial_x v_h)_j\big),$$

where $\tilde{\alpha}^2_{j+\frac{1}{2}}$ and $\bar{\alpha}_j$ is the same as in (3.18) and (3.20) respectively, and $\Omega_i, i \in \mathbb{Z}_4$ are defined by (3.17).

For any $\tau_j \in \Omega_3 \cup \Omega_4$, noticing that $\alpha_{j+\frac{1}{2}}\alpha_{j-\frac{1}{2}} \le 0$, then there exists at least one zero of α on τ_j. Then

$$\|\alpha\|_{0,\infty,\Omega_3\cup\Omega_4} \lesssim h\|\alpha\|_{1,\infty,\Omega_3\cup\Omega_4} \lesssim h,$$

and thus

$$\tilde{\alpha}^2_{j+\frac{1}{2}} = \alpha^2_{j+\frac{1}{2}} + \alpha^2_{j-\frac{1}{2}} \lesssim h^2, \quad \tau_j \in \Omega_3 \cup \Omega_4.$$

Since α is smooth, there are a finite number of zeros of α on the whole domain Ω, which indicates the number of elements in $\Omega_3 \cup \Omega_4$ is upper bounded independent of h. Then by (3.15) and the Cauchy-Schwartz inequality,

$$\begin{aligned}\left|\sum_{\tau_j\in\Omega_3\cup\Omega_4} \tilde{\alpha}^2_{j+\frac{1}{2}} b_j(u-P_hu, v_h)\right| &= \left|\sum_{\tau_j\in\Omega_3\cup\Omega_4} \tilde{\alpha}^2_{j+\frac{1}{2}} a_j(u-P_hu, v_h)\right| \\ &\lesssim h^{k+1}\left(\sum_{\tau_j\in\Omega_3\cup\Omega_4} \tilde{\alpha}^2_{j+\frac{1}{2}}\|u\|^2_{k+2,\tau_j}\right)^{\frac{1}{2}} \|v_h\|_\alpha \lesssim h^{k+2}\|u\|_{k+2}\|v_h\|_\alpha.\end{aligned}$$

Plugging the above inequality into the formula of $A(u-u_I, v_h)$, we get for all $v_h \in V_h$

$$\begin{aligned}|A(u-u_I, v_h)| &\lesssim h^{k+2}\|u\|_{k+2}\|v_h\|_\alpha + (\|w_t\|_\alpha + \|w\|_\alpha)(\|v_h\|_\alpha + h\|\partial_x v_h\|_\alpha) \\ &\lesssim h^{k+2}\|u\|_{k+3,\infty}\|v_h\|_\alpha,\end{aligned}$$

where in the last step, we have used (3.28) and the inverse inequality. $\square$

REMARK 3.5. *Since the number of elements in the domain $\Omega_3 \cup \Omega_4$ is upper bounded independent of h, the terms $\sum_{\tau_j\subset\Omega_3} b_j(u - Q_hu, v_h)$ and $\sum_{\tau_j\subset\Omega_4} b_j(u - R_hu, v_h)$ in the formula (3.18) are of high order, which means the correction function is not necessary in the case $P_hu = Q_hu$ or R_hu. This is why we take $w|_{\tau_j} = 0$ in those elements $\tau_j \in \Omega_3 \cup \Omega_4$.*

3.3. The superconvergence result. Now we are ready to present the superconvergence properties of the DG flux function approximation.

THEOREM 3.6. *Let $u(\cdot,t) \in W^{k+3,\infty}, \forall t \in [0,T]$ be the solution of (1.1), and u_h be the solution of (2.1) with the initial solution $u_h(x,0) = P_hu_0$, where P_hu_0 is defined by (3.6). Then*

$$e_f = \left(\sum_{j=1}^N \int_{\tau_j} \alpha^2(u_h - P_hu)^2(x,t)dx\right)^{\frac{1}{2}} \lesssim h^{k+2}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty}. \tag{3.31}$$

Proof. Let $u_I = P_hu - w$. By choosing $v_h = u_h - u_I$ in (2.5) and using the estimates in (3.28)-(3.29), we have

$$\begin{aligned}\|(u_h-u_I)(\cdot,t)\|^2_\alpha &\lesssim \|(u_h-u_I)(\cdot,0)\|^2_\alpha + \int_0^t A(u-u_I, u_h-u_I)dt \\ &\lesssim \|w(\cdot,0)\|^2_\alpha + th^{k+2}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty}\|(u_h-u_I)(\cdot,\tau)\|_\alpha.\end{aligned}$$

Then

$$\|(u_h - u_I)(\cdot,t)\|_\alpha \lesssim h^{k+2} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty},$$

and thus

$$\|(u_h - P_h u)(\cdot,t)\|_\alpha \lesssim h^{k+2} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty} + \|w(\cdot,t)\|_\alpha \lesssim h^{k+2} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

Then (3.31) follows from (3.16). ◻

REMARK 3.7. *The result in Theorem 3.6 indicates that the flux function αu_h of the DG method is superconvergent with an order of $k+2$ towards the particular flux function $\alpha P_h u$ of the exact solution. When α is a constant, $P_h u$ is reduced to the traditional Gauss-Radau projection $P_h^- u$ (or $P_h^+ u$, dependent on the sign of α). Then* (3.31) *implies that the DG solution is super-close to the Gauss-Radau projection of the exact solution, with a convergence order of $k+2$, which is consistent with the superconvergence result established in [8, 21]. In other words, the result* (3.31) *extends the superconvergence result in [8, 21] for constant coefficient problems to the general case. As we shall demonstrate in our numerical experiments, α in* (3.31) *can not be removed to assure the $(k+2)$-th order superconvergence rate for a general hyperbolic equations with possibly degenerate variable coefficients.*

We next study the superconvergence properties for the flux function of the DG method at some special points and for cell averages. As a direct consequence of (3.31), (3.9)-(3.11), we have the following superconvergence results.

COROLLARY 3.8. *Suppose all the conditions of Theorem 3.6 hold. Then*

$$e_{f,c} + e_{f,r} \lesssim h^{k+2} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}, \quad e_{f,l} \lesssim h^{k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty},$$

where

$$e_{f,c} = \left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_j}\int_{\tau_j}(\alpha u - \alpha u_h)(x,t)dx\right)^2\right)^{\frac{1}{2}}, \quad e_{f,l} = \left(\frac{1}{Nk}\sum_{j=1}^{N}\sum_{i=1}^{k}(\alpha\partial_x(u-u_h))^2(z_{j,m},t)\right)^{\frac{1}{2}},$$

$$e_{f,r} = \left(\frac{1}{N(k+1)}\sum_{j=1}^{N}\sum_{i=1}^{k+1}(\alpha u - \alpha u_h)^2(y_{j,i},t)\right)^{\frac{1}{2}}$$

with $y_{j,i}$ and $z_{j,m}$ are defined by (3.7)-(3.8).

Proof. In each element τ_j, if $P_h u = R_h u, P_h^{\pm} u$, then

$$\int_{\tau_j} \alpha(u - P_h u)dx = \int_{\tau_j} (\alpha - \alpha_j)(u - P_h u)dx \lesssim h^{k+2}\|u\|_{k+1,\infty}.$$

If $P_h u|_{\tau_j} = Q_h u$, there exists a point $\xi_j \in \tau_j$ such that $\alpha(\xi_j) = 0$, then the above inequality still holds. Therefore,

$$\left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_j}\int_{\tau_j}\alpha(u - P_h u)dx\right)^2\right)^{\frac{1}{2}} \lesssim h^{k+2}\|u\|_{k+1,\infty},$$

which yields, together with (3.31) and the triangle inequality,

$$\left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_j}\int_{\tau_j}(\alpha u-\alpha u_h)(x,t)dx\right)^2\right)^{\frac{1}{2}}\lesssim h^{k+2}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty},\quad \forall t\in(0,T].$$

On the other hand, since $P_hu-u_h\in V_h$, the inverse inequality holds true. Then

$$|(P_hu-u_h)(x)|\lesssim\|P_hu-u_h\|_{0,\infty,\tau_j}\lesssim h^{-\frac{1}{2}}\|P_hu-u_h\|_{0,\tau_j},\quad \forall x\in\tau_j,$$

and thus,

$$\begin{aligned}\left(\frac{1}{N(k+1)}\sum_{j=1}^{N}\sum_{i=1}^{k+1}(\alpha P_hu-\alpha u_h)^2(y_{j,i},t)\right)^{\frac{1}{2}}&\lesssim\left(\frac{h^{-1}}{N}\sum_{j=1}^{N}\sum_{i=1}^{k+1}\alpha^2(y_{j,i})\|P_hu-u_h\|^2_{0,\tau_j}\right)^{\frac{1}{2}}\\&\lesssim\|P_hu-u_h\|_\alpha+h\|P_hu-u_h\|_0\\&\lesssim h^{k+2}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left(\frac{1}{Nk}\sum_{j=1}^{N}\sum_{i=1}^{k}(\alpha\partial_x(P_hu-u_h))^2(z_{j,i},t)\right)^{\frac{1}{2}}&\lesssim\|\partial_x(P_hu-u_h)\|_\alpha+h\|\partial_x(P_hu-u_h)\|_0\\&\lesssim h^{-1}\|P_hu-u_h\|_\alpha+\|P_hu-u_h\|_0\\&\lesssim h^{k+1}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty}.\end{aligned}$$

Then the desired results follow from (3.10)-(3.11) and the triangle inequality. ☐

4. Superconvergence for the DG solution. In this section, we will study the superconvergence properties for the DG solution itself. That is, we prove the superconvergence for the DG solution u_h to a particular projection of the exact solution u.

In light of (3.31), the DG solution is superconvergent to the particular projection P_hu with an order of $k+2$ for a class of functions $|\alpha|\geq\delta>0$. However, as $\alpha(x)$ changes signs or otherwise achieves the value zero, we cannot deduce superconvergence result for the error $\|u_h-P_hu\|_0$ directly from (3.31). The standard error estimate also yields only the optimal convergence rate, just as indicated by (3.14). Therefore, new analysis tools or techniques are needed to improve the estimate of the convergence rate.

We have assumed the smooth function $\alpha(x)$ has only a finite number of zeros on the domain Ω. For simplicity, we suppose α has only one zero on Ω. Without loss of generality, we let $\alpha(0)=0$. At the zero $x=0$, we assume there exists a positive integer m such that

$$\alpha(0)=\partial_x\alpha(0)=\ldots=\partial_x^{m-1}\alpha(0)=0,\quad \partial_x^m\alpha(0)\neq 0. \tag{4.1}$$

Let

$$m'=\min(m,k+3), \tag{4.2}$$

and i_0 be a positive integer such that $x_{i_0-\frac{1}{2}} \le h^{\frac{1}{m'}} \le x_{i_0+\frac{1}{2}}$. Now we slightly modify our correction functions by

$$\bar{w}_i\big|_{\tau_j} = \begin{cases} 0, & \tau_j \subset \omega_m^{\perp} = [0, x_{i_0-\frac{1}{2}}], \\ w_i, & \tau_j \subset \omega_m = [x_{i_0+\frac{1}{2}}, 2\pi], \end{cases} \tag{4.3}$$

where $w_i, i \in \mathbb{Z}_2$ are defined by (3.21), (3.24), respectively. Then the global correction function $\bar{w}$ is defined as follows.

$$\bar{w} = \begin{cases} 0, & \alpha_{j+\frac{1}{2}} \le 0, \quad \alpha_{j-\frac{1}{2}} > 0, \\ \bar{w}_2, & \alpha_{j+\frac{1}{2}} \le 0, \quad \alpha_{j-\frac{1}{2}} \le 0, \\ \bar{w}_1, & \alpha_{j+\frac{1}{2}} > 0, \quad \alpha_{j-\frac{1}{2}} > 0, \\ 0 & \alpha_{j+\frac{1}{2}} > 0, \quad \alpha_{j-\frac{1}{2}} \le 0. \end{cases} \tag{4.4}$$

LEMMA 4.1. *Let $u(\cdot, t) \in W^{k+3,\infty}, t \in [0, T]$ be the solution of (1.1), and α be a sufficiently smooth function satisfying (4.1) at its zero point $x = 0$. Let $\bar{w}_i, i \in \mathbb{Z}_2$ be the correction functions defined by (4.3). Then*

$$\sum_{i=1}^{2} (\|\bar{w}_i(\cdot, t)\|_0 + \|\partial_t \bar{w}_i(\cdot, t)\|_0) \lesssim h^{k+1+\frac{1}{2m'}} \|u(\cdot, t)\|_{k+3,\infty}. \tag{4.5}$$

Here m' is given in (4.2). Moreover, there holds

$$b(u - P_h u + \bar{w}, v_h) \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v_h\|_0, \quad v_h \in V_h. \tag{4.6}$$

Proof. Let

$$y_0 = \min\left(2\pi, \frac{(m+1)|\partial_x^m \alpha(0)|}{2|\max_{x\in\Omega} \partial_x^{m+1} \alpha(x)|}\right).$$

For any $x \in \tau_j \subset [x_{i_0+\frac{1}{2}}, y_0]$, by Taylor expansion, there exists a $\bar{x} \in (0, x)$ such that

$$|\alpha(x)| = \left|\frac{\partial_x^m \alpha(0) x^m}{m!} + \frac{\partial_x^{m+1} \alpha(\bar{x}) x^{m+1}}{(m+1)!}\right| \ge \frac{1}{2}|\partial_x^m \alpha(0)| x^m \ge \frac{1}{2}|\partial_x^m \alpha(0)||x_{i_0-\frac{1}{2}}|^m.$$

If $m \le k+3$, we have, from (3.23) and (3.26),

$$\begin{aligned} \|\partial_t w_i\|_{0,\omega_m} + \|w_i\|_{0,\omega_m} &\lesssim \left(\sum_{\tau_j \in \omega_m} \frac{1}{\|\alpha\|_{0,\infty,\tau_j}^2}\right)^{\frac{1}{2}} h^{k+\frac{5}{2}} \|u\|_{k+3,\infty} \\ &\lesssim \left(\sum_{\tau_j \in [x_{i_0+\frac{1}{2}}, y_0]} \frac{1}{|x_{j-\frac{1}{2}}|^{2m}}\right)^{\frac{1}{2}} h^{k+\frac{5}{2}} \|u\|_{k+3,\infty} + c_0 h^{k+2} \|u\|_{k+3,\infty}. \end{aligned}$$

where $c_0 = \max\limits_{x \in [y_0, 2\pi]} \frac{1}{\alpha^2(x)}$ is a constant independent of the mesh size h.

If $m > k+3$, by Taylor expansion and (4.1),

$$\left|\partial_x^l \alpha(x)\right| = \left|\frac{\partial_x^m \alpha(\bar{x}) x^{m-l}}{(m-l)!}\right| \lesssim \left|x_{j+\frac{1}{2}}\right|^{m-l}, \quad \forall x \in \tau_j \subset [x_{i_0+\frac{1}{2}}, y_0], \ l \le k+3 < m,$$

where $\bar{x} \in (0, x)$. By (3.23) and (3.26), we get

$$\|w_i\|_{0,\omega_m} + \|\partial_t w_i\|_{0,\omega_m} \lesssim \left(\sum_{\tau_j \in \omega_m} \sum_{l=0}^{k+3} \frac{\|\partial_x^l \alpha\|^2_{0,\infty,\tau_j}}{\|\alpha\|^2_{0,\infty,\tau_j}} \right)^{\frac{1}{2}} h^{k+\frac{5}{2}} \|u\|_{k+3,\infty}$$

$$\lesssim \left(\sum_{\tau_j \in [x_{i_0+\frac{1}{2}}, y_0]} \sum_{l=0}^{k+3} \frac{|x_{j+\frac{1}{2}}|^{2(m-l)}}{|x_{j-\frac{1}{2}}|^{2m}} \right)^{\frac{1}{2}} h^{k+\frac{5}{2}} \|u\|_{k+3,\infty} + c_0 h^{k+2} \|u\|_{k+3,\infty}$$

$$\lesssim \left(\sum_{\tau_j \in [x_{i_0+\frac{1}{2}}, y_0]} \frac{1}{|x_{j-\frac{1}{2}}|^{2(k+3)}} \right)^{\frac{1}{2}} h^{k+\frac{5}{2}} \|u\|_{k+3,\infty} + h^{k+2} \|u\|_{k+3,\infty}.$$

Then in both cases,

$$\|w_i\|_{0,\omega_m} + \|\partial_t w_i\|_{0,\omega_m} \lesssim \left(\sum_{\tau_j \in [x_{i_0+\frac{1}{2}}, y_0]} \frac{1}{|x_{j-\frac{1}{2}}|^{2m'}} \right)^{\frac{1}{2}} h^{k+\frac{5}{2}} \|u\|_{k+3,\infty} + h^{k+2} \|u\|_{k+3,\infty}.$$

Let $x_{i_0-\frac{1}{2}} \le h^{\frac{1}{m'}} \le x_{i_0+\frac{1}{2}} \le \ldots \le x_{r-\frac{1}{2}} \le y_0 \le x_{r+\frac{1}{2}}$, where $1 \le i_0 \le r \le N$. Since the mesh is quasi-uniform, we get $x_{i+\frac{1}{2}} = cih$, which yields

$$c(i_0 - 1)h \le h^{\frac{1}{m'}} \le c i_0 h.$$

Then

$$\sum_{\tau_j \in [x_{i_0+\frac{1}{2}}, y_0]} \frac{1}{|x_{j-\frac{1}{2}}|^{2m'}} \lesssim \sum_{j=i_0}^{r} \frac{1}{|jh|^{2m'}} \lesssim \frac{1}{h^{2m'}} \int_{i_0}^{N} \frac{1}{x^{2m'}} dx \lesssim \frac{i_0}{(i_0 h)^{2m'}} \lesssim h^{-3+\frac{1}{m'}}.$$

Therefore,

$$\|\partial_t \bar{w}_i\|_0 + \|\bar{w}_i\|_0 = \|\partial_t w_i\|_{0,\omega_m} + \|w_i\|_{0,\omega_m} \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty}, \quad i = 1, 2.$$

This finishes the proof of (4.5).

Recall the definition of $\bar{w}_1$ in (4.3) and the equation (3.22), we have

$$\left| b(u - P_h^- u + \bar{w}_1, v_h) \right| = \left| \sum_{\tau_j \in \omega_m^{\perp}} b_j(u - P_h^- u, v_h) + \sum_{\tau_j \in \omega_m} (\partial_t w_1, v_h)_j - ((\alpha - \bar{\alpha}_j) w_1, v_{hx})_j \right|$$

$$\lesssim h^{k+1} \|u\|_{k+2,\infty} \|v_h\|_{0,1,\omega_m^{\perp}} + (\|w_1\|_{0,\omega_m} + \|\partial_t w_1\|_{0,\omega_m}) \|v_h\|_0$$

$$\lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v_h\|_0.$$

Following the same arguments, we obtain

$$|b(u - P_h^+ u + \bar{w}_2, v_h)| \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v_h\|_0.$$

Noticing that

$$b(u - P_h u + \bar{w}, v_h) = \sum_{\tau_j \in \Omega_1} b_j(u - P_h^- u + \bar{w}_1, v_h) + \sum_{\tau_j \in \Omega_2} b_j(u - P_h^+ u + \bar{w}_2, v_h)$$

$$+ \sum_{\tau_j \in \Omega_3} b_j(u - Q_h u, v_h) + \sum_{\tau_j \in \Omega_4} b_j(u - R_h u, v_h),$$

and the fact that the number of elements in $\Omega_3 \cup \Omega_4$ is upper bounded independent of h, we have (4.6) and the proof is complete. ◻

THEOREM 4.2. *Let $u(\cdot,t) \in W^{k+3,\infty}, t \in [0,T]$ be the solution of (1.1), and u_h be the solution of (2.1) with the initial solution $u_h(x,0) = P_h u_0$, where $P_h u_0$ is defined by (3.6). Suppose α is a sufficiently smooth function satisfying the condition (4.1) at its zero point $x = 0$. Then*

$$e_u = \|(u_h - P_h u)(\cdot,t)\|_0 \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

Here m' is the same as in (4.2).

Proof. Let $u_I = P_h u - \bar{w}$. By choosing $v = u_I - u_h$ in (2.3), and using the initial discretization,

$$\begin{aligned}\|(u_I - u_h)(\cdot,t)\|_0^2 &\lesssim \|(u_I - u_h)(\cdot,0)\|_0^2 + \int_0^t a(u_h - u_I, u_h - u_I)dt \\ &= \|\bar{w}(\cdot,0)\|_0^2 + \int_0^t a(u - u_I, u_h - u_I)dt \\ &\lesssim \left(h^{k+1+\frac{1}{2m'}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}\right)^2 + \int_0^t a(u - u_I, u_h - u_I)dt.\end{aligned}$$

By (3.9) and (3.28), we have

$$\hat{u}_I(x_{j+\frac12}) = \hat{P}_h u(x_{j+\frac12}) = u(x_{j+\frac12}), \quad j \in \mathbb{Z}_N.$$

Then

$$|a(u - u_I, v_h)| = |b(u - P_h u + \bar{w}, v_h)| \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v_h\|_0,$$

which yields

$$\|(u_I - u_h)(\cdot,t)\|_0 \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

Then

$$\|(P_h u - u_h)(\cdot,t)\|_0 \lesssim \|(u_h - P_h u)(\cdot,t)\|_0 + \|\bar{w}(\cdot,t)\|_0 \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

The proof is complete. ◻

REMARK 4.3. *The result of Theorem 4.2 reveals the superconvergence behavior of the DG solution for hyperbolic equations with variable coefficients. As indicated by the above Theorem, the highest superconvergence rate of the DG approximation itself is $h^{k+\frac32}$ when the coefficient $\alpha(x)$ degenerates or has zeros in the domain. This is very different from the constant coefficient problems. As we may recall, the highest superconvergence rate of the DG methods for hyperbolic problems with constant coefficients is $2k+1$. The above Theorem also demonstrates a very important fact that we can always expect the superconvergence phenomenon for the DG approximation no matter what α is, although the local error may not be superconvergent. The worst superconvergence rate is $k+1+\frac{1}{2(k+3)}$, which is $\frac{1}{2(k+3)}$ order higher than the optimal convergence rate $k+1$.*

Denote by $e_{u,c}, e_{u,r}, e_{u,l}$ the errors for the cell average, function value errors at the special points $y_{j,i}$ and derivative errors at points $z_{j,m}$, respectively, where $y_{j,i}$ and $z_{j,i}$ are defined in (3.7)-(3.8). To be more precise,

$$e_{u,c} = \left(\frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{h} \int_{\tau_j} (u - u_h)(x,t) dx \right)^2 \right)^{\frac{1}{2}}, \quad e_{u,l} = \left(\frac{1}{Nk} \sum_{j=1}^{N} \sum_{i=1}^{k} (\partial_x (u - u_h))^2 (z_{j,i}, t) \right)^{\frac{1}{2}},$$

$$e_{u,r} = \left(\frac{1}{N(k+1)} \sum_{j=1}^{N} \sum_{i=1}^{k+1} (u - u_h)^2 (y_{j,i}, t) \right)^{\frac{1}{2}}.$$

We have the following superconvergence results for the above errors.

COROLLARY 4.4. *Suppose all the conditions of Theorem 4.2 hold. Then*

$$e_{u,c} + e_{u,r} \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau \in [0,t]} \|u(\cdot, \tau)\|_{k+3,\infty}, \quad e_{u,l} \lesssim h^{k+\frac{1}{2m'}} \sup_{\tau \in [0,t]} \|u(\cdot, \tau)\|_{k+3,\infty},$$

where m' is defined in (4.2).

REMARK 4.5. *As the number of elements in $\Omega_3 \cup \Omega_4$ is upper bounded independent of h, if we slightly modify the definition of $P_h u$ in the domain $\Omega_3 \cup \Omega_4$, e.g., if we replace $R_h u$ or $Q_h u$ by $P_h^- u$, the superconvergence results in Theorems 3.6 and 4.2 still hold. Similarly, the superconvergent points (Gauss and Gauss-Lobatto points) in $\Omega_3 \cup \Omega_4$ can also be replaced by any other points, for example, the Radau points.*

REMARK 4.6. *One of the main advantages of DG methods is the complete freedom in changing the polynomial degrees in each element independent of that in the neighbors (p adaptivity), which is not shared by the typical finite element methods. As the same with the h-adaptivity, the key ingredient in the procedure of the p adaptivity is the construction of the a posterior error estimates (to determine which elements should be approximated by a higher order polynomial). As demonstrated in [5], the superconvergence results for the DG solution can be used to construct residual-based a posterior DG error estimates. Similarly, we expect that the superconvergence result established in this paper can be applied to the construction of the a posterior DG error estimates for hyperbolic problems with sonic points.*

5. Numerical results.

In this section, we present numerical examples to verify our theoretical findings. In our numerical experiments, we shall test the superconvergence phenomena of the DG flux function approximation as well as the DG solution itself, and measure several errors between the numerical solution and the exact solution, which are given in Theorems 3.6-4.2, and Corollaries 3.8-4.4. All our examples are tested by polynomials $\mathbb{P}_k$ with $k = 1, 2, 3$.

Example 1. We consider the following equation with the periodic boundary condition

$$u_t + (\sin(x)u)_x = g(x,t), \qquad (x,t) \in [0, 2\pi] \times (0, 0.1],$$
$$u(x,0) = e^{\sin(x)}.$$

The function $g(x,t)$ is chosen such that the exact solution to this problem is

$$u(x,t) = e^{\sin(x-t)}.$$

Noticing that, at the zeros $x = 0, \pi$ (the zero at $x = 2\pi$ is the same as that at $x = 0$ because of periodicity) of $\alpha = \sin(x)$,

$$\partial_x \alpha(x) = \cos(x) \neq 0, \quad x = 0, \pi,$$

which indicates that (4.1) holds with $m = 1$. Then

$$m' = \min(m, k+3) = 1.$$

We use the fourth order Runge-Kutta method in time and take $\triangle t = 0.01 h_{min}$ to reduce the time discretization error. Uniform meshes are obtained by equally dividing the interval $[0, 2\pi]$ into N subintervals, $N = 2^j, j = 9, \ldots, 13$. We compute the numerical solution at time $T = 0.1$.

Listed in Table 5.1 are the various errors for the flux function approximation, which are given in Theorem 3.6 and Corollary 3.8. We observe from Table 5.1 a $(k+2)$-th order superconvergence rate for e_f (the error for the flux function approximation), and $e_{f,r}$ (the function value error at the superconvergent points); and a $(k+1)$-th order for the error $e_{f,l}$ (the derivative error at the superconvergent points). These superconvergence results are consistent with the theoretical findings in Theorem 3.6 and Corollary 3.8. As for the cell average ($e_{f,c}$), the convergence rate is $k+2$ for $k = 1$. While for $k = 2, 3$, it seems that the convergence rate can reach $k + \frac{5}{2}$ for $k = 2, 3$, a half order higher than the theoretical estimate.

Table 5.1
Various types of errors for the DG flux function approximation with $\alpha = \sin(x)$.

k	N	e_f	order	$e_{f,c}$	order	$e_{f,r}$	order	$e_{f,l}$	order
	256	6.96e-07	–	2.12e-07	–	2.97e-07	–	6.12e-05	–
	512	8.72e-08	3.00	2.71e-08	2.97	3.71e-08	3.00	1.53e-05	2.00
1	1024	1.09e-08	3.00	3.42e-09	2.98	4.61e-09	3.00	3.82e-06	2.00
	2048	1.37e-09	3.00	4.30e-10	2.99	5.80e-10	3.00	9.54e-07	2.00
	4096	1.71e-10	3.00	5.39e-11	3.00	7.25e-11	3.00	2.39e-07	2.00
	256	2.97e-09	–	5.87e-11	–	2.29e-09	–	4.39e-07	–
	512	1.86e-10	4.00	2.13e-12	4.78	1.43e-10	4.00	5.48e-08	3.00
2	1024	1.16e-11	4.00	8.25e-14	4.69	8.95e-12	4.00	6.85e-09	3.00
	2048	7.28e-13	4.00	3.37e-15	4.61	5.60e-13	4.00	8.57e-10	3.00
	4096	4.55e-14	4.00	1.42e-16	4.56	3.50e-14	4.00	1.07e-10	3.00
	256	8.29e-12	–	2.89e-13	–	7.50e-12	–	2.33e-09	–
	512	2.57e-13	5.01	7.14e-15	5.34	2.35e-13	4.99	1.46e-10	4.00
3	1024	7.92e-15	5.02	1.66e-16	5.43	7.30e-15	5.01	9.06e-12	4.01
	2048	2.45e-16	5.01	3.71e-18	5.48	2.27e-16	5.01	5.64e-13	4.00
	4096	7.62e-18	5.00	8.21e-20	5.50	7.08e-18	5.00	3.52e-14	4.00

In Table 5.2, we compute several types of errors between the DG solution u_h and the exact solution. Table 5.2 demonstrates a superconvergence rate of $k + \frac{3}{2}$ for $e_u, e_{u,c}, e_{u,r}$, and $(k + \frac{1}{2})$-th order for the derivative error ($e_{f,l}$) at the special points $z_{j,m}$, which confirms our theoretical results in Theorem 4.2 and Corollary 4.4.

Example 2. We consider the following equation with the periodic boundary condition

$$u_t + (\sin^2(x)u)_x = g(x,t), \qquad (x,t) \in [0, 2\pi] \times (0, 0.1],$$
$$u(x, 0) = e^{\sin(x)},$$

TABLE 5.2
Various types of errors for the DG solution approximation with $\alpha = \sin(x)$.

k	N	e_u	order	$e_{u,c}$	order	$e_{u,r}$	order	$e_{u,l}$	order
1	256	1.64e-06	–	2.74e-07	–	4.93e-07	–	1.09e-04	–
	512	2.69e-07	2.60	3.70e-08	2.89	7.43e-08	2.73	3.37e-05	1.70
	1024	4.57e-08	2.56	5.11e-09	2.86	1.19e-08	2.65	1.09e-05	1.63
	2048	7.90e-09	2.53	7.37e-10	2.79	1.97e-09	2.59	3.67e-06	1.58
	4096	1.38e-09	2.52	1.12e-10	2.72	3.38e-10	2.55	1.26e-06	1.54
2	256	1.14e-08	–	5.70e-10	–	6.76e-09	–	1.51e-06	–
	512	9.44e-10	3.59	4.87e-11	3.55	5.42e-10	3.64	2.48e-07	2.60
	1024	8.06e-11	3.55	4.27e-12	3.51	4.53e-11	3.58	4.22e-08	2.56
	2048	7.00e-12	3.53	3.77e-13	3.50	3.88e-12	3.54	7.31e-09	2.53
	4096	6.13e-13	3.51	3.33e-14	3.50	3.38e-13	3.52	1.28e-09	2.51
3	256	6.18e-11	–	3.87e-12	–	3.95e-11	–	1.61e-08	–
	512	2.75e-12	4.49	1.72e-13	4.49	1.75e-12	4.49	1.43e-09	3.50
	1024	1.22e-13	4.49	7.60e-15	4.50	7.76e-14	4.50	1.26e-10	3.50
	2048	5.42e-15	4.50	3.36e-16	4.50	3.43e-15	4.50	1.11e-11	3.50
	4096	2.40e-16	4.50	1.48e-17	4.50	1.51e-16	4.50	9.85e-13	3.50

We choose $g(x, t)$ such that the exact solution to this problem is

$$u(x, t) = e^{\sin(x-t)}.$$

In this case,

$$\alpha(x) = \partial_x \alpha(x) = 0, \quad \partial_x^2 \alpha(x) \neq 0, \quad x = 0, \pi.$$

Then (4.1) holds with $m = 2$.

To reduce the time discretization error, we use the fourth order Runge-Kutta method in time and take $\triangle t = 0.001 h_{min}$. We compute the same errors as in Example 1 on the same uniform meshes at time $t = 0.1$. The computational results for the flux function approximation and for the DG solution approximation itself are given in Tables 5.3-5.4, respectively.

From Table 5.3, we observe similar results as in Example 1 for the flux function approximation, which confirms the theoretical results in Theorem 3.6 and Corollary 3.8. Again, we observe that the convergence rate of $e_{f,c}$ (the errors for the cell averages) for $k = 2, 3$ is better than the estimate given in Corollary 3.8, in fact it seems to be one order higher than the theoretical result $k + 2$.

Table 5.4 demonstrates several types of errors and the corresponding convergence rates for the DG solution. The superconvergence phenomena can be observed in this case. Moreover, as indicated by Theorem 4.2 and Corollary 4.4, the superconvergence rate is lower than the one in Example 1 (highest superconvergence rate) due to the different properties of α at its zero point. As we may see from Table 5.4, the superconvergence rate of e_u and $e_{u,r}$ is asymptotically close to our theoretical result $k + 1 + \frac{1}{4}$, and the superconvergence rate of the derivative error $e_{u,l}$ is asymptotically close to $k + \frac{1}{4}$. The convergence rate of the error for the cell average is again better than the theoretical result given in Corollary 4.4.

Table 5.3
Various types of errors for the DG flux function approximation with $\alpha = \sin^2(x)$.

k	N	e_f	order	$e_{f,c}$	order	$e_{f,r}$	order	$e_{f,l}$	order
	256	7.24e-07	–	2.25e-07	–	2.87e-07	–	5.94e-05	–
	512	9.06e-08	3.00	2.88e-08	2.97	3.59e-08	3.00	1.48e-05	2.00
1	1024	1.13e-08	3.00	3.64e-09	2.98	4.49e-09	3.00	3.71e-06	2.00
	2048	1.42e-09	3.00	4.58e-10	2.99	5.61e-10	3.00	9.27e-07	2.00
	4096	1.77e-10	3.00	5.74e-11	3.00	7.01e-11	3.00	2.32e-07	2.00
	256	3.10e-09	–	6.82e-11	–	2.26e-09	–	4.26e-07	–
	512	1.93e-10	4.00	2.20e-12	4.95	1.41e-10	4.00	5.30e-08	3.00
2	1024	1.21e-11	4.00	7.25e-14	4.92	8.85e-12	4.00	6.62e-09	3.00
	2048	7.56e-13	4.00	2.43e-15	4.90	5.53e-13	4.00	8.27e-10	3.00
	4096	4.73e-14	4.00	8.33e-17	4.87	3.46e-14	4.00	1.03e-10	3.00
	256	8.00e-12	–	1.81e-13	–	7.33e-12	–	2.26e-09	–
	512	2.49e-13	5.00	3.50e-15	5.69	2.29e-13	5.00	1.41e-10	4.00
3	1024	7.72e-15	5.00	6.86e-17	5.67	7.13e-15	5.00	8.80e-12	4.00
	2048	2.40e-16	5.00	1.26e-18	5.77	2.22e-16	5.00	5.49e-13	4.00
	4096	7.50e-18	5.00	2.19e-20	5.84	6.94e-18	5.00	3.43e-14	4.00

Table 5.4
Various types of errors for the DG solution approximation with $\alpha = \sin^2(x)$.

k	N	e_u	order	$e_{u,c}$	order	$e_{u,r}$	order	$e_{u,l}$	order
	256	1.51e-06	–	2.78e-07	–	5.08e-07	–	1.13e-04	–
	512	2.15e-07	2.81	3.55e-08	2.97	6.79e-08	2.90	3.08e-05	1.88
1	1024	3.25e-08	2.72	4.48e-09	2.99	9.49e-09	2.84	8.78e-06	1.81
	2048	5.40e-09	2.59	5.62e-10	2.99	1.44e-09	2.72	2.73e-06	1.69
	4096	9.86e-10	2.45	7.04e-11	3.00	2.45e-10	2.56	9.42e-07	1.53
	256	2.26e-08	–	4.34e-10	–	1.24e-08	–	3.15e-06	–
	512	2.13e-09	3.40	2.75e-11	3.98	1.13e-09	3.46	6.00e-07	2.39
2	1024	2.10e-10	3.34	1.87e-12	3.87	1.08e-10	3.38	1.19e-07	2.33
	2048	2.13e-11	3.30	1.33e-13	3.82	1.08e-11	3.33	2.44e-08	2.29
	4096	2.19e-12	3.28	9.65e-15	3.78	1.10e-12	3.29	5.04e-09	2.27
	256	6.09e-11	–	1.25e-12	–	3.82-11	–	1.74e-08	–
	512	3.05e-12	4.32	4.49e-14	4.80	1.88e-12	4.35	1.75e-09	3.31
3	1024	1.56e-13	4.30	1.60e-15	4.81	9.42e-14	4.32	1.80e-10	3.29
	2048	8.01e-15	4.28	5.72e-17	4.80	4.81e-15	4.29	1.86e-11	3.27
	4096	4.17e-16	4.27	2.07e-18	4.79	2.48e-16	4.27	1.94e-12	3.26

6. Conclusion remarks. We demonstrated the superconvergence behavior of the DG method for 1-D linear hyperbolic equations with variable coefficients when upwind fluxes are used. We first prove that the flux function αu_h of the DG method is superconvergent with an order of $k+2$ towards the flux function $\alpha P_h u$ of the exact solution under the L^2 norm, with a suitably chosen projection $P_h u$ of the exact solution u. We then established superconvergence for the flux function approximation at some special points and for cell averages. These results extend the superconvergence results for constant coefficients to a more general situation. We then study the superconvergence behavior of the DG solution itself and reveal that the superconvergence phenomena do exist for the DG methods applied to hyperbolic equations with possibly degenerate variable coefficients. Contrary to the constant coefficient problems, the convergence rate may change according to the specific property of α. The highest superconvergence rate that can be achieved is $k+\frac{3}{2}$, half an order higher than the optimal convergence rate. Numerical examples are provided to validate the theoretical findings and to show their sharpness in many situations. Finally, we would like to mention that the superconvergence techniques established in this work can extend to the Dirichlet boundary condition; all superconvergence results still hold true in that case. Our on-going work includes the superconvergence study of the DG methods for possibly degenerate nonlinear hyperbolic equations.

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