Numerical study on the convergence to steady state solutions of a new class of high order WENO schemes

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Abstract

A new class of high order weighted essentially non-oscillatory (WENO) schemes [J. Comput. Phys., 318 (2016), 110-121] is applied to solve Euler equations with steady state solutions. It is known that the classical WENO schemes [J. Comput. Phys., 126 (1996), 202-228] might suffer from slight post-shock oscillations. Even though such post-shock oscillations are small enough in magnitude and do not visually affect the essentially non-oscillatory property, they are truly responsible for the residue to hang at a truncation error level instead of converging to machine zero. With the application of this new class of WENO schemes, such slight post-shock oscillations are essentially removed and the residue can settle down to machine zero in steady state simulations. This new class of WENO schemes uses a convex combination of a quartic polynomial with two linear polynomials on unequal size spatial stencils in one dimension and is extended to two dimensions in a dimension-by-dimension fashion. By doing so, such WENO schemes use the same information as the classical WENO schemes in [J. Comput. Phys., 126 (1996), 202-228] and yield the same formal order of accuracy in smooth regions, yet they could converge to steady state solutions with very tiny residue close to machine zero for our extensive list of test problems including shocks, contact discontinuities, rarefaction waves or their interactions, and with these complex waves passing through the boundaries of the computational domain.

Key Words: WENO scheme, unequal size spatial stencil, steady state solution.AMS (MOS) subject classification: 65M06, 35L65

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1 Introduction

Steady state solutions are often computed in computational fluid dynamics, which in the inviscid compressible case is governed by the Euler equations

$$\begin{cases} f(u)_x + g(u)_y = 0, \\ u(x,y) = u_0(x,y), \end{cases}$$
(1.1)

where $u = (\rho, \rho\mu, \rho\nu, E)^T$, $f(u) = (\rho\mu, \rho\mu^2 + p, \rho\mu\nu, \mu(E+p))^T$ and $g(u) = (\rho\nu, \rho\mu\nu, \rho\nu^2 + p, \nu(E+p))^T$. Here ρ is density, μ and ν are the velocities in the x and y directions, p is pressure and $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho(\mu^2 + \nu^2)$ is total energy where $\gamma = 1.4$ for air at ordinary temperature. For the purpose of obtaining a numerical solution of (1.1), one possibility is to solve the following unsteady Euler equations

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0, \\ u(x, y, 0) = u_0(x, y), \end{cases}$$
(1.2)

with a suitable time discretization method. If the time derivative or the residue of the unsteady Euler equations (1.2) is small enough, ideally when it reaches machine zero, the numerical solution of the steady state equations (1.1) is obtained. The main difficulty in computing (1.1) and (1.2) is the possible existence of strong discontinuities. When the solution contains strong shocks or contact discontinuities, the physical variables might change abruptly. Many numerical schemes will produce nonphysical oscillations near such strong discontinuities. In the literature, many schemes have been designed with the aim of eliminating or at least reducing such nonphysical oscillations by applying artificial viscosities [20, 21] or applying limiters [15, 20, 39]. The application of artificial viscosity results in a method which is easy to be implemented, and its residue can often converge to machine zero without problems. Jameson et al. [19, 22] proposed a combination of a finite volume discretization method together with dissipative terms of third order and a Runge-Kutta time stepping scheme to yield an effective method for solving the Euler equations in arbitrary geometric domains, and this method was used by them to determine the steady transonic flow past an airfoil using an O-type mesh. However, this method has the drawback that one often needs to adjust certain parameters in the artificial viscosity to obtain good performance. When applying limiters such as the total variation diminishing (TVD) limiters, one obtains a method which is very efficient in computing supersonic flows including strong shocks [15]. However the application of TVD type limiters will degenerate the accuracy of the numerical scheme to first order near smooth extrema [28], and (more relevant to our current paper) the lack of sufficient smoothness of the numerical fluxes with such limiters often prevents the residue from settling down to machine zero. Yee et al. [43] proposed an implicit unconditionally stable high resolution TVD scheme and applied it to compute steady state solutions for the compressible inviscid equations. It was found that the improvement of computational efficiency and convergence rate is required for practical application. Thereafter, Yee and Harten [42] promoted such TVD scheme to the multidimensional hyperbolic conservation laws in curvilinear coordinates, and they improved the computational efficiency of the implicit algorithm for steady state airfoil calculations.

Over the past years, many numerical schemes had been designed to improve the first order methods [12] to arbitrary numerical order of accuracy for unsteady problems. Harten et al. relaxed on the TVD criterion [15] and introduced the basis for the reconstruction of high order essentially non-oscillatory (ENO) type schemes in order to obtain uniform high order accuracy, resulting in the finite volume ENO schemes to simulate unsteady equations [17]. For the two dimensional extensions, see [4, 5, 16]. Such ENO schemes apply the locally smoothest stencil and abandon others when approximating the variables at cell boundaries, resulting in high order accuracy in smooth regions and simultaneously avoiding nonphysical oscillations near discontinuities. Later, Shu and Osher designed finite difference ENO schemes with TVD Runge-Kutta time discretizations [35, 36], which are more efficient for multidimensional computation. Liu et al. [27] proposed a finite volume weighted ENO (WENO) scheme using the same candidate stencils of an r-th order ENO scheme, such that (r+1)-th order of accuracy is obtained in smooth regions. In [23], Jiang and Shu proposed a new framework of constructing WENO schemes, and gave a finite difference WENO scheme from the the same candidate stencils of an r-th order ENO scheme to obtain (2r + 1)-th order of accuracy in smooth regions. Thereafter, extensive followup work has been performed, for example, two dimensional finite volume WENO schemes [10, 18] and three dimensional finite volume WENO schemes [38, 48] were proposed on unstructured meshes. All these WENO schemes perform very well in numerous numerical simulations of unsteady Euler equations, leading to the conclusion that a WENO scheme together with a high order TVD (also called strong stability preserving, or SSP) Runge-Kutta time discretization [13, 33, 35] could yield good results in simulating unsteady problems. For example, WENO schemes can successfully capture shocklets and sound waves [46], and the detailed structure of the contact discontinuities for strong Mach reflections [32, 47].

However, when the classical WENO schemes [23, 34] coupled with a third order TVD Runge-Kutta time discretization [13, 33, 35] are used to solve for the steady state problems, the residue often stops diminishing at the truncation error level which is far above the machine zero, even if the physical variables do not change much (at least visually) with further time iteration. In [30], Serna and Marquina proposed a new limiter to reconstruct the numerical flux, and the application of this limiter could improve the convergence of the numerical solution to steady states. Zhang and Shu [45] analyzed that slight post-shock oscillations would propagate from the region near the shocks downstream to the smooth region and result in the residue hanging up at a high truncation error level rather than settling down to machine zero. The upwind-biased interpolation technique [44] was developed to improve the convergence of fifth order accurate finite difference WENO scheme for solving Euler systems with steady state problems. However, the residue still could not converge to machine zero for many two dimensional test cases. An alternative approach is to abandon time marching and use Newton iterations or a more robust method such as the homotopy method [14] to directly solve the nonlinear system derived from high order WENO spatial discretizations. The problem with such an approach is that one must be very careful in order to obtain the correct physical solution, since the nonlinear system could have multiple solutions. Recently, a novel high order fixed-point sweeping WENO method was proposed to simulate hyperbolic conservation laws with steady state solutions and the associated convergence property was researched in [41]. However, the problem of the failure of the residue to settle down to machine zero for some two dimensional test cases still exists.

Recently, a new class of fifth order WENO schemes has been designed to solve time dependent hyperbolic conservation laws [50, 51]. For related work we refer to [1, 3, 7, 11, 24, 25, 26, 29]. This new class of WENO schemes uses a convex combination of a quartic polynomial with two linear polynomials on unequal sized spatial stencils in one dimension and is extended to two dimensions in a dimension-by-dimension manner. Such WENO schemes use the same information as the classical WENO schemes and yield the same formal order of accuracy in smooth regions, yet they have a few advantages over the classical WENO schemes, one of them being that the new WENO scheme can provide a unified polynomial approximation in the whole cell with the same nonlinear weights, comparing with the classical WENO schemes which involve different linear and nonlinear weights for different points inside the cell. Computational results in [50, 51] indicate that this new class of WENO schemes works well for time-dependent test problems.

In this paper, we study the performance of this new class of WENO schemes for steady state computation. Because of the global nature (the whole cell is approximated by the same WENO polynomial) and the smoothness of nonlinear weights (which is a signature advantage of WENO schemes versus ENO schemes and was already identified as the reason for better steady state computation in [23]), as well as the fact that two of the three substencils in the WENO reconstruction correspond to low order linear polynomials, the new class of WENO schemes could converge to steady state solutions with close to machine zero residue for an extensive list of the standard test problems, including shocks, contact discontinuities, rarefaction waves or their interactions, and with these complex waves passing through the boundaries of the computational domain. To the best of our knowledge, this appears to be the first class of high order WENO schemes whose residue could settle down to machine zero for such a large class of two dimensional test cases with standard Runge-Kutta time discretization. Of course, other time marching methods as well as special tools such as preconditioning to speed up steady state convergence could make the steady state convergence more efficient, however this is not the focus of the current paper and hence will not be further explored.

The organization of this paper is as follows. In Section 2, we briefly review the construction of the new class of WENO schemes [50, 51], using its fifth order version as an example. In Section 3, several standard steady state test problems including sophisticated wave structures, both inside the computational fields and passing through the boundaries of the computational domain, are presented to demonstrate the good performance of residue convergence to machine zero. Concluding remarks are given in Section 4.

2 Fifth order WENO scheme for steady state computation

2.1 Fifth order finite difference WENO scheme

We will use the two dimensional hyperbolic conservation laws as an example to explain the new fifth order finite difference schemes [50]. Thus we have the associated semidiscretization of (1.2) and reformulate it as

$$\frac{du}{dt} = L(u), \tag{2.1}$$

where L(u) is the high order spatial discretization of $-f(u)_x - g(u)_y$. For simplicity, we use a uniform mesh (x_i, y_k) with $h = x_{i+1} - x_i = y_{k+1} - y_k$ as the mesh size. We also denote the half points as $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$, $y_{k+1/2} = \frac{1}{2}(y_k + y_{k+1})$. $u_{i,k}(t)$ is the numerical approximation to the nodal point value $u(x_i, y_k, t)$ of the exact solution. For conservation, the right hand side of (2.1) is written as

$$\frac{du_{i,k}(t)}{dt} = L(u)_{i,k} = -\frac{1}{h}(\hat{f}_{i+1/2,k} - \hat{f}_{i-1/2,k}) - \frac{1}{h}(\hat{g}_{i,k+1/2} - \hat{g}_{i,k-1/2}), \quad (2.2)$$

where $\hat{f}_{i+1/2,k}$ and $\hat{g}_{i,k+1/2}$ are the numerical fluxes. We use the fifth order version as an example in this paper, thus we require $\frac{1}{h}(\hat{f}_{i+1/2,k} - \hat{f}_{i-1/2,k})$ to be a fifth order approximation to $f(u)_x$ and $\frac{1}{h}(\hat{g}_{i,k+1/2} - \hat{g}_{i,k-1/2})$ to be a fifth order approximation to $g(u)_y$ at (x_i, y_k) , respectively. For upwinding and stability, we often split the flux f(u) into $f(u) = f^+(u) + f^-(u)$ with $\frac{df^+(u)}{du} \ge 0$ and $\frac{df^-(u)}{du} \le 0$ and then approximate each of them separately using its own wind direction in the x-direction. A simple Lax-Friedrichs flux splitting $f^{\pm}(u) = \frac{1}{2}(f(u) \pm \alpha u)$ is used in this paper, in which α is set as $\max_u |f'(u)|$ over the whole range of u in the x-line. Likewise for g(u) in the y-direction. Thus the numerical fluxes are also split accordingly

$$\hat{f}_{i+1/2,k} = \hat{f}_{i+1/2,k}^{+} + \hat{f}_{i+1/2,k}^{-}, \quad \hat{g}_{i,k+1/2} = \hat{g}_{i,k+1/2}^{+} + \hat{g}_{i,k+1/2}^{-}.$$
(2.3)

All the one-dimensional and two-dimensional finite difference and finite volume WENO schemes are based on the simple reconstruction procedure detailed below. Suppose we are given the cell averages $\bar{w}_j = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} w(x) dx$ for all j and would like to obtain a fifth order WENO polynomial approximation $w_i(x)$ defined on $I_i = (x_{i-1/2}, x_{i+1/2})$, based on an upwind-biased stencil consisting of $I_j = (x_{j-1/2}, x_{j+1/2})$ with j = i-2, i-1, i, i+1, i+2. The procedure in [50, 51], with similar ideas in earlier references [3, 25, 26] as well, is summarized as follows.

Reconstruction Algorithm.

Step 1. We choose the following big spatial stencil $T_1 = \{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\}$ and reconstruct a quartic polynomial $p_1(x)$ satisfying

$$\frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} p_1(x) dx = \bar{w}_j, \qquad j = i-2, \dots, i+2.$$
(2.4)

We also choose two smaller spatial stencils $T_2 = \{I_{i-1}, I_i\}$ and $T_3 = \{I_i, I_{i+1}\}$, and reconstruct two linear polynomials satisfying

$$\frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} p_2(x) dx = \bar{w}_j, \qquad j = i - 1, i,$$
(2.5)

and

$$\frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} p_3(x) dx = \bar{w}_j, \qquad j = i, i+1.$$
(2.6)

Step 2. We rewrite the quartic polynomial $p_1(x)$ as:

$$p_1(x) = \gamma_1 \left(\frac{1}{\gamma_1} p_1(x) - \frac{\gamma_2}{\gamma_1} p_2(x) - \frac{\gamma_3}{\gamma_1} p_3(x) \right) + \gamma_2 p_2(x) + \gamma_3 p_3(x).$$
(2.7)

Clearly, the equality (2.7) holds for arbitrarily chosen positive linear weights γ_1 , γ_2 and γ_3 satisfying $\gamma_1 + \gamma_2 + \gamma_3 = 1$. Based on a balance between the sharp shock transitions and essentially nonoscillatory property in nonsmooth region, following the practice in [9, 50, 51, 52], we set the linear weights as $\gamma_1 = 0.98$ and $\gamma_2 = \gamma_3 = 0.01$ in this paper.

Step 3. We compute the smoothness indicators β_n , n = 1, 2, 3, which measure how smooth the functions $p_n(x)$, n = 1, 2, 3, are in the interval $[x_{i-1/2}, x_{i+1/2}]$, by using the classical recipe for the smoothness indicators as in [23, 34]

$$\beta_n = \sum_{\ell=1}^r \int_{x_{i-1/2}}^{x_{i+1/2}} h^{2\ell-1} \left(\frac{d^\ell p_n(x)}{dx^\ell}\right)^2 dx, \qquad n = 1, 2, 3,$$
(2.8)

where r = 4 for n = 1 and r = 1 for n = 2, 3, respectively.

Step 4. We calculate the nonlinear weights, by adopting the strategy in WENO-Z as specified in [2, 6, 8]

$$\tau = \left(\frac{|\beta_1 - \beta_2| + |\beta_1 - \beta_3|}{2}\right)^2, \tag{2.9}$$

and

$$\omega_n = \frac{\bar{\omega}_n}{\sum_{\ell=1}^3 \bar{\omega}_\ell}, \qquad \bar{\omega}_n = \gamma_n \left(1 + \frac{\tau}{\varepsilon + \beta_n} \right), \qquad n = 1, 2, 3.$$
(2.10)

Here ε is a small positive number to avoid the denominator to become zero, and we take $\varepsilon = 10^{-6}$ in our computation.

Step 5. We replace the linear weights in (2.7) with the nonlinear weights (2.10), and the final reconstruction polynomial $w_i(x)$ is given by

$$w_i(x) = \omega_1 \left(\frac{1}{\gamma_1} p_1(x) - \frac{\gamma_2}{\gamma_1} p_2(x) - \frac{\gamma_3}{\gamma_1} p_3(x) \right) + \omega_2 p_2(x) + \omega_3 p_3(x).$$
(2.11)

For our finite difference scheme (2.2), the numerical flux $\hat{f}_{i+1/2,k}^+$ in (2.3) is obtained by using the *Reconstruction Algorithm* with $\bar{w}_j = f^+(u_{j,k})$ to obtain $w_i(x)$, and then by setting $\hat{f}_{i+1/2,k}^+ = w_i(x_{i+1/2})$. The construction of the numerical flux $\hat{f}_{i+1/2,k}^-$ is mirror-symmetric with respect to $x_{i+1/2}$, and the procedure for $g(u)_y$ in the y-direction is similar. Finally, the semidiscrete scheme (2.2) is discretized in time by the third order TVD Runge-Kutta method [35]

$$\begin{cases} u^{(1)} = u^{n} + \Delta t L(u^{n}), \\ u^{(2)} = \frac{3}{4}u^{n} + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}), \\ u^{n+1} = \frac{1}{3}u^{n} + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}). \end{cases}$$
(2.12)

As mentioned before, we do not discuss efficiency issues of reaching steady states, hence we do not consider other types of time discretization methods as well as strategies such as preconditioning to speed up convergence towards steady states.

For more details of the finite difference WENO schemes, we refer to [23, 34]. For the Euler equations, all of the reconstruction procedures are performed in the local characteristic directions. We do not give further details and again refer the readers to [23, 34].

2.2 Fifth order finite volume WENO scheme

We use the same two dimensional hyperbolic conservation laws (1.2) once again as an example to explain the new fifth order finite volume WENO schemes [51]. For simplicity, the grid meshes are uniformly divided into cells, and the cell sizes are $h = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}}$, with the cell centers $(x_i, y_k) = (\frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), \frac{1}{2}(y_{k-\frac{1}{2}} + y_{k+\frac{1}{2}}))$. We denote the two dimensional cells by $I_{i,k} = I_i \times J_k = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}]$ and define the two dimensional cell averages as $\tilde{u}_{i,k}(t) = \frac{1}{h^2} \int_{y_{k-1/2}}^{y_{k+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, y, t) dxdy$, where we use \bar{w} to denote the cell averaging operator in the x-direction and \tilde{w} to denote the cell averaging operator in the x-direction and \tilde{w} to denote the cell averaging operator in the grid cell $I_{i,k}$ and obtain the integral

formulation

$$\frac{d\tilde{u}_{i,k}(t)}{dt} = -\frac{1}{h^2} \left(\int_{y_{k-1/2}}^{y_{k+1/2}} f(u(x_{i+1/2}, y, t)) dy - \int_{y_{k-1/2}}^{y_{k+1/2}} f(u(x_{i-1/2}, y, t)) dy + \int_{x_{i-1/2}}^{x_{i+1/2}} g(u(x, y_{k+1/2}, t)) dx - \int_{x_{i-1/2}}^{x_{i+1/2}} g(u(x, y_{k-1/2}, t)) dx \right). \quad (2.13)$$

We approximate (2.13) by the following semi-discrete conservative scheme

$$\frac{d\tilde{\bar{u}}_{i,k}(t)}{dt} = L(u)_{i,k} = -\frac{1}{h}(\hat{f}_{i+1/2,k} - \hat{f}_{i-1/2,k}) - \frac{1}{h}(\hat{g}_{i,k+1/2} - \hat{g}_{i,k-1/2}), \quad (2.14)$$

where the numerical fluxes $\hat{f}_{i+1/2,k}$ and $\hat{g}_{i,k+1/2}$ are defined as

$$\hat{f}_{i+1/2,k} = \sum_{\ell=1}^{3} \varpi_{\ell} \hat{f}(u_{i+1/2,k+\sigma_{\ell}}^{-}, u_{i+1/2,k+\sigma_{\ell}}^{+}), \\ \hat{g}_{i,k+1/2} = \sum_{\ell=1}^{3} \varpi_{\ell} \hat{g}(u_{i+\sigma_{\ell},k+1/2}^{-}, u_{i+\sigma_{\ell},k+1/2}^{+}),$$
(2.15)

to approximate $\frac{1}{h} \int_{y_{k-1/2}}^{y_{k+1/2}} f(u(x_{i+1/2}, y, t)) dy$ and $\frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} g(u(x, y_{k+1/2}, t)) dx$, respectively, with $\varpi_{1,2} = \frac{5}{18}$, $\varpi_3 = \frac{8}{18}$, $\sigma_{1,2} = \pm \frac{\sqrt{15}}{10}$ and $\sigma_3 = 0$ being the three-point Gaussian quadrature weights and nodes in the cell $\left[-\frac{1}{2}, \frac{1}{2}\right]$. The numerical fluxes $\hat{f}(a, b)$ and $\hat{g}(a, b)$ are defined as the simple Lax-Friedrichs flux. $u_{i+1/2,k+\sigma_{\ell}}^{\pm}$ and $u_{i+\sigma_{\ell},k+1/2}^{\pm}$ are the fifth order approximations of $u(x_{i+1/2}^{\pm}, y_{k+\sigma_{\ell}}, t)$ and $u(x_{i+\sigma_{\ell}}, y_{k+1/2}^{\pm}, t)$, respectively, which will be reconstructed by the WENO procedure.

Unfortunately, even though the building block of finite volume WENO schemes is still the simple *Reconstruction Algorithm* as given in the previous section, this algorithm must be used multiple times in order to "de-cell average" from the two dimensional cell averages $\tilde{u}_{i,k}$ to the point values $u_{i+1/2,k+\sigma_{\ell}}^{\pm}$ etc. at the Gauss quadrature points along cell boundaries. For example, to obtain $u_{i+1/2,k+\sigma_{\ell}}^{-}$ and $u_{i-1/2,k+\sigma_{\ell}}^{+}$, the following procedure is followed:

(1) For each k, take $\bar{w}_j = \tilde{\bar{u}}_{j,k}$ and use the *Reconstruction Algorithm* to obtain $w_i(x)$, then identify $\tilde{u}_{i+1/2,k}^- = w_i(x_{i+1/2})$ and $\tilde{u}_{i-1/2,k}^+ = w_i(x_{i-1/2})$, respectively, which are the y-cell averages of u at $x = x_{i+1/2}^-$ and at $x = x_{i-1/2}^+$, respectively;

(2) For each i + 1/2, take $\tilde{w}_j = \tilde{u}_{i+1/2,j}^-$ and use the *Reconstruction Algorithm* to obtain the polynomial $w_k(y)$, then identify $u_{i+1/2,k+\sigma_\ell}^- = w_k(y_k + \sigma_\ell h)$ at the three Gaussian points for $\ell = 1, 2, 3$. Similarly, take $\tilde{w}_j = \tilde{u}_{i-1/2,j}^+$ and use the *Reconstruction Algorithm* to obtain the polynomial $w_k(y)$, then identify $u_{i-1/2,k+\sigma_\ell}^+ = w_k(y_k + \sigma_\ell h)$ at the three Gaussian points for $\ell = 1, 2, 3$.

After these point values at the Gaussian points are obtained, we can use the scheme (2.14) with the fluxes defined by (2.15), together with the Runge-Kutta time discretization (2.12), to advance in time. In the reconstruction above, local characteristic decomposition is used to avoid spurious oscillations. We again refer to [34, 31] for more details.

3 Numerical tests

In this section we perform numerical experiments to test the steady state computation performance of the new class of WENO schemes [50, 51] and term them as the WENO-FD scheme and WENO-FV scheme for the finite difference and finite volume versions, respectively. We also make a comparison with the classical fifth order WENO schemes [23, 34] and term them as the WENO-JS-FD scheme and WENO-JS-FV scheme for the finite difference and finite volume versions, respectively. The CFL number is set as 0.6. The average residue is defined as

$$Res_A = \sum_{i=1}^{N} \frac{|R1_i| + |R2_i| + |R3_i| + |R4_i|}{4 \times N},$$
(3.1)

where R_{*_i} are local residuals of different conservative variables, that is, $R1_i = \frac{\partial \rho}{\partial t}|_i = \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t}$, $R2_i = \frac{\partial(\rho\mu)}{\partial t}|_i = \frac{(\rho\mu)_i^{n+1} - (\rho\mu)_i^n}{\Delta t}$, $R3_i = \frac{\partial(\rho\nu)}{\partial t}|_i = \frac{(\rho\nu)_i^{n+1} - (\rho\nu)_i^n}{\Delta t}$, $R4_i = \frac{\partial E}{\partial t}|_i = \frac{E_i^{n+1} - E_i^n}{\Delta t}$, and N is the total number of grid points or cells. Notice that here we are using a single index *i* to list all the two-dimensional grid points or cells.

Example 3.1. As an accuracy test we consider the following two dimensional Euler equations with source terms

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho \mu \\ \rho \nu \\ E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho \mu \\ \rho \mu^2 + p \\ \rho \mu \nu \\ \mu(E+p) \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \rho \nu \\ \rho \mu \nu \\ \rho \nu^2 + p \\ \nu(E+p) \end{pmatrix} = \begin{pmatrix} 0.4\cos(x+y) \\ 0.6\cos(x+y) \\ 0.6\cos(x+y) \\ 1.8\cos(x+y) \end{pmatrix}.$$
(3.2)

The source terms are chosen so that the exact steady state solutions are given by $\rho(x, y, \infty) = 1 + 0.2 \sin(x+y), \ \mu(x, y, \infty) = 1, \ \nu(x, y, \infty) = 1 \text{ and } p(x, y, \infty) = 1 + 0.2 \sin(x+y)$. We take

	WENO-FD scheme				WENO-FV scheme			
grid points/cells	L^1 error	order	L^{∞} error	order	L^1 error	order	L^{∞} error	order
10×10	7.60E-4		2.66E-3		5.02E-4		1.06E-3	
20×20	1.30E-5	5.87	3.58E-5	6.21	1.53E-5	5.03	3.51E-5	4.93
30×30	1.84E-6	4.83	4.76E-6	4.97	2.05E-6	4.96	4.70E-6	4.96
40×40	4.49E-7	4.90	1.13E-6	5.00	4.89E-7	4.99	1.13E-6	4.94
50×50	1.49E-7	4.92	3.73E-7	4.96	1.60E-7	5.00	3.75E-7	4.96
60×60	6.07E-8	4.94	1.51E-7	4.95	6.44E-8	5.00	1.51E-7	4.97
70×70	2.83E-8	4.95	7.03E-8	4.97	2.97E-8	5.00	7.03E-8	4.97
80×80	1.46E-8	4.96	3.61E-8	4.99	1.52E-8	5.01	3.61E-8	4.99

Table 3.1: 2D Euler equations with source terms. WENO-FD scheme and WENO-FV scheme. Steady state. L^1 and L^{∞} errors.

the numerical initial conditions as $\rho(x, y, 0) = 1 + 0.2 \sin(x+y)$, $\mu(x, y, 0) = 1$, $\nu(x, y, 0) = 1$, $p(x, y, 0) = 1 + 0.2 \sin(x+y)$, and then march to numerical steady states. The computational domain is set as $(x, y) \in [0, 2\pi] \times [0, 2\pi]$, and the exact steady state solution is applied as boundary conditions in both directions. The history of the residue (3.1) as a function of time is shown in Figure 3.1, in which we can see that the residue settles down to tiny numbers close to machine zero. The L^1 and L^{∞} errors and orders of accuracy at steady states are listed in Table 3.1, from which we can see that the designed fifth order accuracy is achieved for both the WENO-FD and WENO-FV schemes.

Example 3.2. Shock reflection problem. The computational domain is a rectangle of length 4 and height 1. The boundary conditions are that of a reflection condition along the bottom boundary, supersonic outflow along the right boundary and Dirichlet conditions on the other two sides:

$$(\rho, \mu, \nu, p)^{T}) = \begin{cases} (1.0, 2.9, 0, 1.0/1.4)^{T}|_{(0,y,t)^{T}}, \\ (1.69997, 2.61934, -0.50632, 1.52819)^{T}|_{(x,1,t)^{T}}, \end{cases}$$

Initially, we set the solution in the entire domain to be that at the left boundary. We show the density contours with 15 equally spaced contour lines from 1.10 to 2.58 and the density distribution along the line y = 0.5, after numerical steady state is reached, in Figure 3.2 and



Figure 3.1: 2D Euler equations with source terms. The evolution of the average residue. Left: the results of the WENO-FD scheme; right: the results of the WENO-FV scheme. Different numbers indicate different mesh levels from 10×10 to 80×80 points or cells.

Figure 3.7, for the finite difference and finite volume schemes respectively. The history of the residue (3.1) as a function of time is also shown in these figures. The difference between the two types of WENO schemes can be clearly observed from the residue history. It can be observed that the average residue of the WENO-JS-FD scheme and the WENO-JS-FV scheme can only settle down to a value around 10^{-1} , while the average residue of the WENO-FV scheme can settle down to a value around $10^{-12.5}$, close to machine zero.

In order to explore the reason of such difference in residue time history between the two different types of WENO schemes, we plot the nonlinear weights in each of the four local characteristic fields in the x and y directions at different points along the line y = 0.5, in the last time step when numerical steady state has been reached, in Figure 3.3 to Figure 3.6 for the finite difference schemes. Similar results are also obtained for the finite volume schemes, the figures for them are omitted to save space. We can clearly observe that the nonlinear weights of the WENO-JS-FD schemes are still widely oscillatory and do not settle down to the linear weights after the first shock, even in smooth regions, while the nonlinear weights of the WENO-FD schemes can settle down to the linear weights except for a narrow region

near the shocks.

Example 3.3. A Mach 3 wind tunnel with a forward-facing step. The setup of the problem is as follows: the wind tunnel is 1 length unit wide and 3 length units long. The step is 0.2 length units high and is located 0.6 length units from the right going Mach 3 flow. Reflective boundary conditions are applied along the walls of the tunnel and inflow and outflow boundary conditions are applied at the entrance and the exit. There is a singularity point at the corner of the step. We apply the same procedure as specified in [40] and use an assumption of nearly steady flow in the region near the corner to fix the singularity. The results are shown when the numerical solution has settled down to steady state. We show 30 equally spaced density contours from 0.32 to 6.15 computed by the two different types of WENO schemes in the computational domain in Figure 3.8 and Figure 3.9. It can be observed that the results of the WENO schemes are less oscillatory than that of the WENO-JS schemes. The bigger difference is in the time history of residue. It can be observed in Figure 3.8 and Figure 3.9 that the average residue of the WENO-JS-FD scheme and the WENO-JS-FV scheme can only settle down to a value around $10^{-0.7}$, while the average residue of the WENO-FD scheme and the WENO-FV scheme settles down to a value around $10^{-12.6}$, close to machine zero.

Example 3.4. This benchmark example is a supersonic flow past a two dimensional plate with an attack angle of $\alpha = 15^{\circ}$. The free stream Mach number is $M_{\infty} = 3$. The ideal gas goes from the left toward the plate. The initial condition is set as $p = \frac{1}{\gamma M_{\infty}^2}$, $\rho = 1$, $\mu = \cos(\alpha)$ and $\nu = \sin(\alpha)$. The computational field is set as $[0, 10] \times [-5, 5]$. The plate is set at $x \in [1, 2]$ with y = 0. The slip boundary condition is imposed on the plate. The physical values of the inflow and outflow boundary conditions are applied in different directions. The results are shown when the numerical solutions reach their steady states. We show 30 equally spaced pressure contours from 0.02 to 0.23 computed by different types of WENO and WENO-JS schemes in the computational domain, and the time history of the residue (3.1) in Figure



Figure 3.2: The shock reflection problem. From top to bottom: 15 equally spaced density contours from 1.10 to 2.58 of the WENO-FD scheme; of the WENO-JS-FD scheme; the density distribution along the line y = 0.5 for the WENO-FD scheme (plus signs) and the WENO-JS-FD scheme (squares); and the evolution of the average residue. 121×31 points.



Figure 3.3: The shock reflection problem. The nonlinear weights of $f_{i+1/2,k}^+$ along the line y = 0.5 of the first, second, third and fourth local characteristic components in the *x*-direction from top to bottom. Different signs and lines correspond to nonlinear weights of different stencils. Left: the results of the WENO-FD scheme; Right: the results of the WENO-JS-FD scheme. 121×31 points.



Figure 3.4: The shock reflection problem. The nonlinear weights of $f_{i-1/2,k}^-$ along the line y = 0.5 of the first, second, third and fourth local characteristic components in the x-direction from top to bottom. Different signs and lines correspond to nonlinear weights of different stencils. Left: the results of the WENO-FD scheme; Right: the results of the WENO-JS-FD scheme. 121×31 points.



Figure 3.5: The shock reflection problem. The nonlinear weights of $g_{i,k+1/2}^+$ along the line y = 0.5 of the first, second, third and fourth local characteristic components in the y-direction from top to bottom. Different signs and lines correspond to nonlinear weights of different stencils. Left: the results of the WENO-FD scheme; Right: the results of the WENO-JS-FD scheme. 121×31 points.



Figure 3.6: The shock reflection problem. The nonlinear weights of $g_{i,k-1/2}^-$ along the line y = 0.5 of the first, second, third and fourth local characteristic components in the y-direction from top to bottom. Different signs and lines correspond to nonlinear weights of different stencils. Left: the results of the WENO-FD scheme; Right: the results of the WENO-JS-FD scheme. 121×31 points.



Figure 3.7: The shock reflection problem. From top to bottom: 15 equally spaced density contours from 1.10 to 2.58 of the WENO-FV scheme; of the WENO-JS-FV scheme; the density distribution along the line y = 0.5 for the WENO-FV scheme (plus signs) and the WENO-JS-FV scheme (squares); and the evolution of the average residue. 120×30 cells.



Figure 3.8: A Mach 3 wind tunnel with a forward-facing step. From top to bottom: 30 equally spaced density contours from 0.32 to 6.15 of the WENO-FD scheme; of the WENO-JS-FD scheme; and the evolution of the average residue. 91×31 points.



Figure 3.9: A Mach 3 wind tunnel with a forward-facing step. From top to bottom: 30 equally spaced density contours from 0.32 to 6.15 of the WENO-FV scheme; of the WENO-JS-FV scheme; and the evolution of the average residue. 90×30 cells.



Figure 3.10: A supersonic flow past a plate with an attack angle. From top left to top right to bottom: 30 equally spaced pressure contours from 0.02 to 0.23 of the WENO-FD scheme; of the WENO-JS-FD scheme; and the evolution of the average residue. 201×201 points.

3.10 and Figure 3.11. Again, the contours of the WENO schemes appear to be cleaner. More noticeably, the average residue of the WENO-JS-FD scheme and the WENO-JS-FV scheme settles down to a value around $10^{-2.5}$, while the average residue of the WENO-FD scheme and the WENO-FV scheme can settle down to a value around $10^{-13.5}$, close to machine zero. Although the boundary is very far away from the plate, all the waves including the shocks and the rarefaction waves propagate to the far field boundaries. This usually causes difficulties for the residue of numerical schemes such as the finite difference and finite volume WENO-JS schemes to settle down to machine zero, while it does not seem to cause much trouble for the WENO schemes in this paper.



Figure 3.11: A supersonic flow past a plate with an attack angle. From top left to top right to bottom: 30 equally spaced pressure contours from 0.02 to 0.23 of the WENO-FV scheme; of the WENO-JS-FV scheme; and the evolution of the average residue. 200×200 cells.

Example 3.5. This example is a supersonic flow past two plates with an attack angle of $\alpha = 15^{\circ}$. The free stream Mach number is set as $M_{\infty} = 3$. The ideal gas goes from the left toward the plates. The initial condition is set as $p = \frac{1}{\gamma M_{\infty}^2}$, $\rho = 1$, $\mu = \cos(\alpha)$ and $\nu = \sin(\alpha)$. The computational field is set as $[0, 10] \times [-5, 5]$. Two plates are set at $x \in [2, 3]$ with y = -2and at $x \in [2,3]$ with y = 2. The slip boundary condition is imposed on these plates. The physical values of the inflow and outflow boundary conditions are applied on the left, right, top and bottom boundaries, respectively. The results are shown when the numerical solutions have settled down to their steady states. We show 30 equally spaced pressure contours from 0.02 to 0.23 computed by two types of WENO and WENO-JS schemes and the time history of the residue (3.1) in Figure 3.12 and Figure 3.13. The main purpose of this example is to verify the performance of the finite difference and finite volume WENO schemes in the presence of strong shocks, rarefaction waves and their interactions among each other. It can be observed that the average residue of the WENO-JS-FD scheme settles down to a value around $10^{-2.5}$ and the average residue of the WENO-JS-FV scheme settles down to a value around 10^{-2} , while the average residue of the WENO-FD scheme and the WENO-FV scheme can settle down to a value around $10^{-13.5}$, close to machine zero.

Example 3.6. A supersonic flow past a square column. This example is a supersonic flow past a square column with a zero attack angle. The free stream Mach number is $M_{\infty} = 4$. The ideal gas goes from the left toward the square column. The initial condition is set as $p = \frac{1}{\gamma M_{\infty}^2}$, $\rho = 1$, $\mu = 1$ and $\nu = 0$. The computational field is set as $[-5,5] \times [-9,9]$. The square column region is set as $(x, y) \in [1, 5] \times [-0.5, 0.5]$. The slip boundary condition is imposed on the square column. The physical values of the inflow and outflow boundary conditions are applied at the outer boundaries. The results are shown when the numerical solutions have settled down to their steady states. We show 30 equally spaced pressure contours from 0.05 to 0.87 computed by the WENO and the WENO-JS schemes and the time history of the residue (3.1) in Figure 3.14 and Figure 3.15. It can be observed that the average residue of the WENO-JS-FD scheme and the WENO-JS-FV scheme can only



Figure 3.12: A supersonic flow past two plates with an attack angle. From top left to top right to bottom: 30 equally spaced pressure contours from 0.02 to 0.23 of the WENO-FD scheme; of the WENO-JS-FD scheme; and the evolution of the average residue. 201×201 points.



Figure 3.13: A supersonic flow past two plates with an attack angle. From top left to top right to bottom: 30 equally spaced pressure contours from 0.02 to 0.23 of the WENO-FV scheme; of the WENO-JS-FV scheme; and the evolution of the average residue. 200×200 cells.

settle down to a value around $10^{-3.7}$ and 10^{-3} , respectively, while the average residue of the WENO-FD scheme and the WENO-FV scheme settles down to a value around 10^{-14} , close to machine zero. In this case, the shocks and the rarefaction waves pass through the right boundary. This is usually one reason that residue for high order schemes has difficulty settling down to machine zero, but it does not seem to affect the WENO-FD and WENO-FV schemes in this paper very much.

Example 3.7. A supersonic flow past two square columns. This example is a supersonic flow past two square columns with a zero attack angle. The free stream Mach number is $M_{\infty} = 4$. The ideal gas goes from the left toward the square columns. The initial condition is set as $p = \frac{1}{\gamma M_{\infty}^2}$, $\rho = 1$, $\mu = 1$ and $\nu = 0$. The computational domain is set as $[-5, 5] \times [-9, 9]$. The square columns are located at $(x, y) \in [1, 5] \times [-4.5, -3.5]$ and $(x, y) \in [1, 5] \times [3.5, 4.5]$. The slip boundary condition is imposed on the square columns. The physical values of the inflow and outflow boundary conditions are applied at the outer boundaries. The results are shown when the numerical solutions have converged to their steady states. We show 30 equally spaced pressure contours from 0.05 to 0.87 computed by the WENO and the WENO-JS schemes and the time history of the residue (3.1) in Figure 3.16 and Figure 3.17. We can conclude that the average residues of the WENO-JS-FD scheme and the WENO-JS-FV scheme only settle down to a value around $10^{-3.5}$ and $10^{-2.7}$, respectively, yet the average residue of the WENO-FD scheme and the WENO-FV scheme settles down to a value around 10^{-14} , close to machine zero. In this case, the shocks, the rarefaction waves and their interactions all pass through the right boundary. It is one of the reasons that residues for many high order schemes such as the WENO-JS-FD and WENO-JS-FV schemes do not converge to machine zero, however this does not seem to be the case for the WENO-FD and WENO-FV schemes in this paper.



Figure 3.14: A supersonic flow past a square column problem. From top left to top right to bottom: 30 equally spaced pressure contours from 0.06 to 0.87 of the WENO-FD scheme; of the WENO-JS-FD scheme; and the evolution of the average residue. 61×109 points.

Figure 3.15: A supersonic flow past a square column problem. From top left to top right to bottom: 30 equally spaced pressure contours from 0.06 to 0.87 of the WENO-FV scheme; of the WENO-JS-FV scheme; and the evolution of the average residue. 60×108 cells.

Figure 3.16: A supersonic flow past two square columns problem. From top left to top right to bottom: 30 equally spaced pressure contours from 0.06 to 0.87 of the WENO-FD scheme; of the WENO-JS-FD scheme; and the evolution of the average residue. 61×109 points.

Figure 3.17: A supersonic flow past two square columns problem. From top left to top right to bottom: 30 equally spaced pressure contours from 0.06 to 0.87 of the WENO-FV scheme; of the WENO-JS-FV scheme; and the evolution of the average residue. 60×108 cells.

4 Concluding remarks

In this paper we apply a new class of fifth order finite difference and finite volume WENO schemes developed in [50, 51] to solve steady state problems for Euler equations. It seems that residue for such new WENO schemes can settle down to machine zero for all standard test problems that we have tried, including some problems containing strong shocks, contact discontinuities, rarefaction waves, their interactions, and associated compound sophisticated waves passing through computational boundaries. When plotting the nonlinear weights of such finite difference and finite volume WENO schemes at steady state, we observe that they are close to the linear weights except for a very small region near discontinuities, while they are still very oscillatory and far from the linear weights for the classical WENO schemes in [23, 34] downstream of the first shock, even in smooth regions. The results in this paper indicate that the WENO schemes in [50, 51] have a good potential in steady state computation of computational fluid dynamics (CFD) problems. In the future, we plan to study whether this good property still holds on unstructured meshes, in three-dimensions, in the presence of more complex boundary conditions such as the inverse Lax-Wendroff type boundary conditions [37], and for discontinuous Galerkin methods with WENO limiters [49, 52].

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