OPTIMAL ERROR ESTIMATES OF THE SEMIDISCRETE CENTRAL DISCONTINUOUS GALERKIN METHODS FOR LINEAR HYPERBOLIC EQUATIONS

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Abstract. We analyze the central discontinuous Galerkin (DG) method for time-dependent linear conservation laws. In one dimension, optimal a priori L^2 error estimates of order $k+1$ are obtained for the semidiscrete scheme when piecewise polynomials of degree at most k ($k \geq 0$) are used on overlapping uniform meshes. We then extend the analysis to multidimensions on uniform Cartesian meshes when piecewise tensor product polynomials are used on overlapping meshes. Numerical experiments are given to demonstrate the theoretical results.

Key words. Optimal error estimate; central DG; superconvergence points

AMS subject classifications. 65M60, 65M12

1. Introduction.

We study the central discontinuous Galerkin (DG) method for solving hyperbolic equations [14]. Even though central DG methods give optimal convergence rate in numerical tests, previous error estimates can only provide sub-optimal results [14]. In this paper we prove optimal error estimates of the central DG approximation based on tensor-product polynomials for solving linear hyperbolic conservation laws

$$u_t + \sum_{i=1}^{d} c_i u_{x_i} = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T] \tag{1.1a}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{1.1b}$$

where Ω is a bounded rectangular domain in $\mathbb{R}^d$, and $\mathbf{x} = (x_1, \ldots, x_d)$. Here c_i are constants and $u_0(\mathbf{x})$ is a given smooth function. We assume periodic boundary condition for simplicity, although this is not essential for the analysis, inflow-outflow boundary conditions can also be considered along the same lines.

The central scheme of Nessyahu and Tadmor [17] solves hyperbolic conservation laws on a staggered mesh and avoids solving Riemann problems across cell boundaries. The excessive numerical dissipation for small time steps is considered by Kurganov and Tadmor [10] and is controlled by a variable control volume, which in turn yields a semidiscrete nonstaggered central scheme. Liu [12] uses another coupling technique to avoid the excessive numerical dissipation for small time steps. The staggered meshes can be viewed as a collection of overlapping cells and the numerical solution is realized by its overlapping cell averages. Overlapping cells lend themselves to the development of the central DG method [13], following the series of work by Cockburn and Shu on DG methods [5, 7]. Besides the advantage of avoiding Riemann solvers, another advantage of central DG schemes is that they allow much larger time steps (proportional to $O(h/k)$ where h is the the spatial mesh size and k is the polynomial degree) for

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linear stability than that for regular DG schemes (which are proportional to $O(h/k^2)$), particularly for higher spatial orders [18]. Liu et al in [14] give L^2 stability analysis and sub-optimal error estimates of the central DG method for linear hyperbolic conservation laws. They use the standard L^2 projection in the error estimate, resulting in a sub-optimal k-th order accuracy. The difficulty leading to this loss of optimality is the lack of a suitable projection, similar to the Gauss-Radau projection used for regular DG schemes, to eliminate the trouble-some cell boundary terms belonging to the approximation error in the error estimates. Later, central DG schemes have been extended to diffusion equations in [15], again with stability analysis and sub-optimal error estimates for the linear heat equation.

For smooth solutions of linear conservation laws, optimal a priori error estimates of order $k+1$ for one-dimensional and some multidimensional cases [6, 22, 4, 11, 19, 20] can be obtained for regular DG schemes when upwind fluxes are used. Similar optimal a priori error estimates can also be obtained when upwind-biased fluxes are used [16]. Xu and Shu [21] introduced a general approach for proving optimal error estimates by utilizing the local DG (LDG) scheme and its time derivatives with different test functions and fully making use of the so-called Gauss-Radau projections. We should also mention works on superconvergence of DG schemes [2, 1]. A key ingredient of all the optimal error estimates mentioned above is a suitable projection, to deal with the troublesome intercell terms in the approximation error. For upwind fluxes, this suitable projection is the Gauss-Radau projection; for the upwind-biased fluxes, a special global projection is introduced in [16] for this purpose. Unfortunately, up to now a suitable projection is elusive for central DG schemes, thus leading to only sub-optimal error estimates in [14]. The main contribution of the present paper is a careful construction and analysis of special projections, suitable for central DG schemes yielding optimal error estimates. In one dimension, we find the superconvergence points of the central DG scheme and successfully construct a proper local projection $\mathbb{P}_h^*$ according to duality of overlapping cells. The existence and optimal approximation properties of this projection are proved by standard finite element techniques. Moreover, the projection $\mathbb{P}_h^*$ can eliminate the space-discrete terms involving $u-\mathbb{P}_h^* u$ when $u \in P^{k+1}(\Omega)$, here $P^{k+1}(\Omega)$ is the polynomial space of degree at most $k+1$ over each cell. This superconvergence property leads to the derivation of optimal convergence rate. The proof of optimal convergence results is valid for uniform meshes and for polynomials of arbitrary degree $k \geq 0$.

For multidimensional Cartesian meshes, we follow the same arguments as in the one-dimensional case to construct a suitable projection $\mathbb{P}_h^*$ and analyze its existence and approximation properties. We would like to remark that this new projection utilizes Q^k, the space of tensor-product polynomials of degree at most k in each variable. The superconvergence result of $\mathbb{P}_h^*$ on Cartesian meshes helps to yield optimal convergence results.

The organization of the paper is as follows. In section 2, we first recall the central discontinuous Galerkin method for linear hyperbolic equations. Then, we construct a special projection, and study its existence, uniqueness and optimal approximation properties. This projection is used to prove optimal error estimate for the semi-discrete central DG scheme for the linear hyperbolic equation in one dimension in this section. We extend the analysis to multidimensions in section 3, in which optimal error estimates are proved, following the same lines of the one dimensional case. We provide numerical examples to show our theoretical results in section 4. In section 5 we give a few concluding remarks and perspectives for future work. Finally, in the appendix we provide proofs for some of the more technical results of the error estimates.

2. The central DG method in one dimension. In this section, we consider the one-dimensional scalar linear conversation law equation

$$\begin{cases} u_t + u_x = 0, \ x \in [a,b], \ t \geq 0 \\ u(x,0) = u_0(x), \ x \in [a,b] \end{cases} \tag{2.1}$$

with periodic boundary condition.

Let $\{x_j\}$ be a partition of $[a,b]$ with $h_{j+\frac{1}{2}} = x_{j+1} - x_j$ and $h = \max_j h_{j+\frac{1}{2}}$. Denote $x_{j+\frac{1}{2}} = \frac{1}{2}(x_{j+1} + x_j)$, $I_j = (x_{j+\frac{1}{2}}, x_{j-\frac{1}{2}})$, and $I_{j+\frac{1}{2}} = (x_j, x_{j+1})$. V_h is the set of piecewise polynomials of degree at most k over the subintervals $\{I_j\}$ with no continuity assumed across the subinterval boundaries. Likewise, W_h is the set of piecewise polynomials of degree at most k over the subintervals $I_{j+\frac{1}{2}}$ with no continuity assumed across the subinterval boundaries.

The central DG method is defined on overlapping cells and uses both spaces V_h and W_h. The semi-discrete version of the central DG scheme is defined as follows. Find $u_h(\cdot,t) \in V_h$ and $v_h(\cdot,t) \in W_h$, such that for any $\varphi_h \in V_h$ and $\psi_h \in W_h$,

$$\begin{aligned} \int_{I_j} \partial_t u_h \varphi_h dx \quad &= \quad \frac{1}{\tau_{max}} \int_{I_j} (v_h - u_h)\varphi_h dx + \int_{I_j} v_h \partial_x \varphi_h dx \\ &\quad - v_h(x_{j+\frac{1}{2}}, t)\varphi_h(x^-_{j+\frac{1}{2}}) + v_h(x_{j-\frac{1}{2}}, t)\varphi_h(x^+_{j-\frac{1}{2}}) \end{aligned} \tag{2.2a}$$

$$\begin{aligned} \int_{I_{j+\frac{1}{2}}} \partial_t v_h \psi_h dx \quad &= \quad \frac{1}{\tau_{max}} \int_{I_{j+\frac{1}{2}}} (u_h - v_h)\psi_h dx + \int_{I_{j+\frac{1}{2}}} u_h \partial_x \psi_h dx \\ &\quad - u_h(x_{j+1}, t)\psi_h(x^-_{j+1}) + u_h(x_j, t)\psi_h(x^+_j) \end{aligned} \tag{2.2b}$$

where τ_{max} is an upper bound for the time step size due to the CFL restriction, that is, $\tau_{max} = c\,h$ with a given constant CFL number c dictated by stability. In [14], the following stability result is proved for this scheme.

THEOREM 2.1. *(Liu et al [14]). The numerical solutions u_h and v_h of the central DG scheme (2.2) for the equation (2.1) have the following L^2 stability property*

$$\frac{1}{2}\frac{d}{dt}\int_a^b (u_h^2 + v_h^2)dx = -\frac{1}{\tau_{max}}\int_a^b (v_h - u_h)^2 dx \leq 0. \tag{2.3}$$

REMARK 2.1. *The sketch of the proof is taking the test functions $\varphi_h = u_h$ and $\psi_h = v_h$ in (2.2) respectively, summing up over j, and using the periodic boundary condition. The details are given in [14]. The proof of Theorem 2.1 is similar to the proof of the cell entropy inequality for the regular DG method in [9].*

2.1. Optimal L^2 error estimate. In this subsection, we show the optimal a priori error estimate of the scheme (2.2). The main idea is to establish a special projection to facilitate the proof of the optimal L^2 error estimate. We first state our main result in the following theorem.

THEOREM 2.2. *The numerical solutions u_h and v_h of the central DG scheme (2.2) using uniform meshes for the equation (2.1) with a smooth initial condition $u(\cdot,0) \in H^{k+2}$ satisfies the following L^2 error estimate*

$$\|u - u_h\|^2 + \|u - v_h\|^2 \leq C h^{2k+2} \tag{2.4}$$

where u is the exact solution of (2.1), k is the polynomial degree in the finite element spaces V_h and W_h, and the constant C depends on the $(k+2)$-th order Sobolev norm of the initial condition $\|u(\cdot,0)\|_{H^{k+2}}$ as well as on the final time t, but is independent of the mesh size h.

Let us first introduce a few notations. We define

$$\begin{aligned} B_j(u_h,v_h;\varphi_h,\psi_h) = &\int_{I_j} \partial_t u_h \varphi_h dx - \frac{1}{\tau_{max}}\int_{I_j}(v_h-u_h)\varphi_h dx - \int_{I_j} v_h \partial_x \varphi_h dx \\ &+ v_h(x_{j+\frac{1}{2}},t)\varphi_h(x^-_{j+\frac{1}{2}}) - v_h(x_{j-\frac{1}{2}},t)\varphi_h(x^+_{j-\frac{1}{2}}) \\ &+ \int_{I_{j+\frac{1}{2}}} \partial_t v_h \psi_h dx - \frac{1}{\tau_{max}}\int_{I_{j+\frac{1}{2}}}(u_h-v_h)\psi_h dx - \int_{I_{j+\frac{1}{2}}} u_h \partial_x \psi_h dx \\ &+ u_h(x_{j+1},t)\psi_h(x^-_{j+1}) - u_h(x_j,t)\psi_h(x^+_j) \end{aligned} \tag{2.5}$$

We also define

$$\begin{aligned} \widetilde{B}_j(u_h,v_h;\varphi_h) = &\frac{1}{\tau_{max}}\int_{I_j}(v_h-u_h)\varphi_h dx + \int_{I_j} v_h \partial_x \varphi_h dx \\ &- v_h(x_{j+\frac{1}{2}},t)\varphi_h(x^-_{j+\frac{1}{2}}) + v_h(x_{j-\frac{1}{2}},t)\varphi_h(x^+_{j-\frac{1}{2}}) \end{aligned} \tag{2.6}$$

$$\begin{aligned} \widehat{B}_{j+\frac{1}{2}}(u_h,v_h;\psi_h) = &\frac{1}{\tau_{max}}\int_{I_{j+\frac{1}{2}}}(u_h-v_h)\psi_h dx + \int_{I_{j+\frac{1}{2}}} u_h \partial_x \psi_h dx \\ &- u_h(x_{j+1},t)\psi_h(x^-_{j+1}) + u_h(x_j,t)\psi_h(x^+_j) \end{aligned} \tag{2.7}$$

Thus

$$\begin{aligned} B_j(u_h,v_h;\varphi_h,\psi_h) = &\int_{I_j} \partial_t u_h \varphi_h dx + \int_{I_{j+\frac{1}{2}}} \partial_t v_h \psi_h dx \\ &- \widetilde{B}_j(u_h,v_h;\varphi_h) - \widehat{B}_{j+\frac{1}{2}}(u_h,v_h;\psi_h) \end{aligned} \tag{2.8}$$

Clearly, we have:

$$B_j(u_h,v_h;\varphi_h,\psi_h) = 0 \tag{2.9}$$

for all j and all $\varphi_h \in V_h$ and $\psi_h \in W_h$. It is also clear that the exact solution u of the PDE (2.1) satisfies

$$B_j(u,u;\varphi_h,\psi_h) = 0 \tag{2.10}$$

for all j and all $\varphi_h \in V_h$ and $\psi_h \in W_h$. Subtracting (2.9) from (2.10), we obtain the error equation

$$B_j(u-u_h,u-v_h;\varphi_h,\psi_h) = 0 \tag{2.11}$$

for all j and all $\varphi_h \in V_h$ and $\psi_h \in W_h$.

We now define $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ as the following projections into V_h and W_h respectively. That is, for each j,

$$\int_{I_j} \mathbb{P}_h^* \omega(x) dx = \int_{I_j} \omega(x) dx \tag{2.12a}$$

(2.12b) $$\widetilde{P_h}(\mathbb{P}_h^*\omega;\varphi_h)_j = \widetilde{P_h}(\omega;\varphi_h)_j \qquad \forall \varphi_h \in \mathbb{P}^k(I_j)$$

where $\widetilde{P_h}(\omega;\varphi_h)_j$ is defined as follows

$$\begin{aligned}\widetilde{P_h}(\omega;\varphi_h)_j = \frac{1}{\tau_{max}}\Bigg(&\int_{x_{j-\frac{1}{2}}}^{x_j}\omega(x+\frac{h}{2})\varphi_h(x)dx + \int_{x_j}^{x_{j+\frac{1}{2}}}\omega(x-\frac{h}{2})\varphi_h(x)dx \\ &- \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\omega(x)\varphi_h(x)dx\Bigg) + \int_{x_{j-\frac{1}{2}}}^{x_j}\omega(x+\frac{h}{2})(\varphi_h(x))_x dx \\ &+ \int_{x_j}^{x_{j+\frac{1}{2}}}\omega(x-\frac{h}{2})(\varphi_h(x))_x dx - \omega(x_j)\left(\varphi_h(x_{j+\frac{1}{2}}^-) - \varphi_h(x_{j-\frac{1}{2}}^+)\right)\end{aligned}$$

and, similarly,

(2.13a) $$\int_{I_{j+\frac{1}{2}}}\mathbb{Q}_h^*\omega(x)dx = \int_{I_{j+\frac{1}{2}}}\omega(x)dx$$

(2.13b) $$\widetilde{Q_h}(\mathbb{Q}_h^*\omega;\psi_h)_{j+\frac{1}{2}} = \widetilde{Q_h}(\omega;\psi_h)_{j+\frac{1}{2}} \qquad \forall \psi_h(x) \in \mathbb{P}^k(I_{j+\frac{1}{2}})$$

where $\widetilde{Q_h}(\omega;\psi_h)_{j+\frac{1}{2}}$ is defined as follows

$$\begin{aligned}\widetilde{Q_h}(\omega;\psi_h)_{j+\frac{1}{2}} = \frac{1}{\tau_{max}}\Bigg(&\int_{x_j}^{x_{j+\frac{1}{2}}}\omega(x+\frac{h}{2})\psi_h(x)dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}}\omega(x-\frac{h}{2})\psi_h(x)dx \\ &- \int_{x_j}^{x_{j+1}}\omega(x)\psi_h(x)dx\Bigg) + \int_{x_j}^{x_{j+\frac{1}{2}}}\omega(x+\frac{h}{2})(\psi_h(x))_x dx \\ &+ \int_{x_{j+\frac{1}{2}}}^{x_{j+1}}\omega(x-\frac{h}{2})(\psi_h(x))_x dx - \omega(x_{j+\frac{1}{2}})\left(\psi_h(x_{j+1}^-) - \psi_h(x_j^+)\right)\end{aligned}$$

Here $\mathbb{P}^k(I_j)$ and $\mathbb{P}^k(I_{j+\frac{1}{2}})$ denote the spaces of polynomials of degree up to k in the cell I_j and the cell $I_{j+\frac{1}{2}}$ respectively. Next, we prove the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ are well defined. Note the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ are local projections, so we only consider the projections defined on the reference interval $[-1,1]$. In this case, $h=2, \tau_{max}=2c$. Without loss of generality we will only consider $\mathbb{P}_h^*$.

Remark 2.2. *The equation (2.12a) is required by the conservation laws. Note that $\widetilde{P_h}(\omega,\varphi_h)_j = 0, \forall\omega$ when $\varphi_h = 1$, so (2.12b) alone misses one condition which is provided by (2.12a). The projection is a local one on cell I_j. Therefore we only need to prove the existence as well as uniqueness and boundedness in the L^∞-norm. Classical approximation results then imply optimal approximation of the projections.*

Lemma 2.1. *The projection $\mathbb{P}_h^*$ defined by (2.12) on the interval $[-1,1]$ exists and is unique for any smooth function ω, and the projection is bounded in the L^∞ norm, i.e.*

(2.14) $$\|\mathbb{P}_h^*\omega\|_\infty \le C(k)\|\omega\|_\infty$$

where $C(k)$ is a constant that only depends on k but is independent of ω.

Proof. The proof of this lemma is provided in the Appendix; see section A.1.

Since the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ are k-th degree polynomial preserving local projections, standard approximation theory [3] implies, for a smooth function ω,

$$\|\mathbb{P}_h^*\omega(x) - \omega(x)\| + h^{\frac{1}{2}}\|\mathbb{P}_h^*\omega(x) - \omega(x)\|_\Gamma \le Ch^{k+1} \tag{2.15}$$

and

$$\|\mathbb{Q}_h^*\omega(x) - \omega(x)\| + h^{\frac{1}{2}}\|\mathbb{Q}_h^*\omega(x) - \omega(x)\|_\Gamma \le Ch^{k+1} \tag{2.16}$$

where Γ denotes the set of boundary points of all elements I_j or $I_{j+\frac{1}{2}}$ respectively, the norm $\|\cdot\|_\Gamma$ is the standard L^2 norm, and the positive constant C, here and below, solely depending on ω and its derivatives, is independent of h.

We also recall that [3], for any $\omega_h \in V_h$ or $\omega_h \in W_h$, there exists a positive constant C independent of ω_h and h, such that

$$\|\partial_x \omega_h\| \le Ch^{-1}\|\omega_h\|; \qquad \|\omega_h\|_\Gamma \le Ch^{-1/2}\|\omega_h\| \tag{2.17}$$

where Γ is the set of boundary points of all elements I_j or $I_{j+\frac{1}{2}}$.

Besides the standard approximation results (2.15) and (2.16), the special projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ also have the following superconvergence result.

PROPOSITION 2.1. *Assume that u is a $(k+1)$-th degree polynomial function in $\mathbb{P}^{k+1}([a,b])$. For a uniform partition on the interval $[a,b]$, set $u_I = \mathbb{P}_h^* u \in V_h$ and $v_I = \mathbb{Q}_h^* u \in W_h$ where $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ are defined by (2.12) and (2.13). Then we have*

$$\widetilde{B}_j(u_I, v_I; \varphi_h) = \widetilde{B}_j(u, u; \varphi_h) \quad \forall \varphi_h \in \mathbb{P}^k(I_j) \tag{2.18a}$$

$$\widehat{B}_{j+\frac{1}{2}}(u_I, v_I; \psi_h) = \widehat{B}_{j+\frac{1}{2}}(u, u; \psi_h) \quad \forall \psi_h \in \mathbb{P}^k(I_{j+\frac{1}{2}}) \tag{2.18b}$$

where $\widetilde{B}$ and $\widehat{B}$ are defined by (2.6) and (2.7).

Proof. The proof of this proposition is provided in the Appendix; see section A.2.

We now take:

$$\varphi_h = \mathbb{P}_h^* u - u_h, \qquad \psi_h = \mathbb{Q}_h^* u - u_h \tag{2.19}$$

in the error equation (2.11), and define

$$\varphi^e = \mathbb{P}_h^* u - u, \qquad \psi^e = \mathbb{Q}_h^* u - u \tag{2.20}$$

to obtain

$$B_j(\varphi_h, \psi_h; \varphi_h, \psi_h) = B_j(\varphi^e, \psi^e; \varphi_h, \psi_h). \tag{2.21}$$

For the left-hand side of (2.21), we use Theorem 2.1 to conclude

$$\sum_j B_j(\varphi_h, \psi_h; \varphi_h, \psi_h) = \frac{1}{2}\frac{d}{dt}\int_a^b (\varphi_h^2 + \psi_h^2)dx + \frac{1}{\tau_{max}}\int_a^b (\varphi_h - \psi_h)^2 dx \tag{2.22}$$

We then write the right-hand side of (2.21) as a sum of three terms

$$B_j(\varphi^e, \psi^e; \varphi_h, \psi_h) = B_j^1 + B_j^2 + B_j^3 \tag{2.23}$$

where

$$B_j^1 = -\widetilde{B}_j(\varphi^e, \psi^e; \varphi_h)$$
$$B_j^2 = -\widehat{B}_{j+\frac{1}{2}}(\varphi^e, \psi^e; \psi_h)$$
$$B_j^3 = \int_{I_j} \partial_t \varphi^e \varphi_h dx + \int_{I_{j+\frac{1}{2}}} \partial_t \psi^e \psi_h dx$$

and we will estimate each term separately.

By using the simple inequality

$$\mu\nu \leq \frac{1}{2}(\mu^2 + \nu^2) \tag{2.24}$$

and the special projection properties (2.15)-(2.16) for $\partial_t \varphi^e$ and $\partial_t \psi^e$, we have:

$$\sum_j B_j^3 \leq \int_a^b (\varphi_h)^2 dx + \int_a^b (\psi_h)^2 dx + Ch^{2k+2}. \tag{2.25}$$

For B_j^1, let $\widehat{u_I}^j$ be the Taylor expansion polynomial of order $k+1$ of u over the interval I_j, i.e. $\widehat{u_I}^j = \sum_{i=0}^{k+1} \frac{1}{i!} u^{(i)}(x_j)(x - x_j)^i, \quad x \in (x_{j-1}, x_{j+1})$. Let r_u denote the residual term i.e. $r_u^j = u - \widehat{u_I}^j$. Recalling the Bramble-Hilbert lemma in [3], we have

$$\|r_u^j\|_{L^\infty(I_j)} \leq Ch^{k+\frac{3}{2}} |u|_{H^{k+2}(I_j)} \tag{2.26}$$

Then we rewrite φ^e and ψ^e

$$\begin{aligned} \varphi^e = \mathbb{P}_h^* u - u &= \mathbb{P}_h^*(\widehat{u_I}^j + r_u^j) - \widehat{u_I}^j - r_u \\ &= \mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{P}_h^* r_u^j - r_u^j \end{aligned} \tag{2.27}$$

$$\begin{aligned} \psi^e = \mathbb{Q}_h^* u - u &= \mathbb{Q}_h^*(\widehat{u_I}^j + r_u^j) - \widehat{u_I}^j - r_u^j \\ &= \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{Q}_h^* r_u^j - r_u^j \end{aligned} \tag{2.28}$$

Thus, using Proposition 2.1 we have

$$\begin{aligned} B_j^1 &= -\widetilde{B}_j(\varphi^e, \psi^e; \varphi_h) \\ &= -\widetilde{B}_j(\mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h) \\ &= -\widetilde{B}_j(\mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j, \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j; \varphi_h) - \widetilde{B}_j(\mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h) \\ &= -\widetilde{B}_j(\mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h) \end{aligned} \tag{2.29}$$

Therefore, by using the simple inequality (2.24), the special projection property (2.15) and (2.16), the property (2.26) for r_u, and the inequality in (2.17) for φ_h, we have:

$$\sum_j B_j^1 \leq Ch^{2k+2} |u|^2_{H^{k+2}([a,b])} + C \int_a^b (\varphi_h)^2 dx \tag{2.30}$$

Similarly, for B_j^2 we have

$$\sum_j B_j^2 \leq Ch^{2k+2} |u|^2_{H^{k+2}([a,b])} + C \int_a^b (\psi_h)^2 dx \tag{2.31}$$

Combining (2.25), (2.30) and (2.31) with (2.22), we obtain from (2.21)

$$\frac{1}{2}\frac{d}{dt}\int_a^b \left((\varphi_h)^2+(\psi_h)^2\right)dx \le C\int_a^b \left((\varphi_h)^2+(\psi_h)^2\right)dx + Ch^{2k+2}|u|^2_{H^{k+2}([a,b])}.$$

This, together with the approximation results (2.15) and (2.16), implies the desired error estimate (2.4).

REMARK 2.3. *Notice that, the first term on the right side of (2.2) is a numerical dissipation term. This is directly related to the difference of the two duplicative representations* u_h *and* v_h *of the solution in overlapping cells. It is important for the uniqueness as well as existence of the special projections* $\mathbb{P}_h^*$ *defined in (2.12) and* $\mathbb{Q}_h^*$ *defined in (2.13). In fact, if we eliminate this term, the stability of central scheme is still preserved, however the optimal convergence accuracy is lost and the special projections do not exist for some* k.

REMARK 2.4. *The special projections* $\mathbb{P}_h^*$ *defined in (2.12) and* $\mathbb{Q}_h^*$ *defined in (2.13) are related to the superconvergence points of the central DG method for one-dimensional linear scalar hyperbolic equation. In fact, we find that the zeros of the polynomial* $x^{k+1}-\mathbb{P}_h^*(x^{k+1})$ *are the superconvergence points of the central DG scheme. We will display the numerical results in section 4. This is the reason that the special projection can help us to obtain the optimal a prior error estimates.*

3. The central DG method in multidimensions. In this section, we consider the semidiscrete central DG method for multidimensional linear conservation laws. Without loss of generality, we describe our central DG scheme and prove the optimal a priori error estimates in two dimensions ($d=2$); all the arguments we present in our analysis depend on the tensor product structure of the mesh and finite element space and can be easily extended to the more general cases $d>2$. Hence, from now on, we shall restrict ourselves to the following two-dimensional problem

$$\begin{cases} u_t+u_x+u_y=0, & (x,y,t)\in\Omega\times(0,T] \\ u(x,y,0)=u_0(x,y), & (x,y)\in\Omega \end{cases} \tag{3.1}$$

again with periodic boundary conditions.

We recall the 2D formulation of the central DG scheme in [13]. Let $\{C_I=[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]\times[y_{j-\frac{1}{2}},y_{j+\frac{1}{2}}]\}, I=(i,j)$, be a partition of Ω into uniform square cells, depicted by the solid lines in Figure 3.1, and tagged by their cell centroid at the $\mathbf{x}_I := I\Delta x$. Let $V_h := \{v\in L^2(\Omega): v|_{C_I}\in Q^k(C_I)\ \forall I\}$, where $Q^k(C_I)$ denotes the space of tensor-product polynomials of degrees at most k in each variable defined on C_I; no continuity is assumed across cell boundaries. Let $\{D_{I+\frac{1}{2}}=[x_i,x_{i+1}]\times[y_j,y_{j+1}]\}$ be the dual mesh which consists of a $\Delta x/2$ shift of the C_I's, depicted by the dashed lines in Figure 3.1. Let $\mathbf{x}_{I+\frac{1}{2}}=(I+\frac{1}{2})\Delta x$ be the cell centroid of the cell $D_{I+\frac{1}{2}}$ and let $W_h := \{v\in L^2(\Omega): v|_{D_{I+\frac{1}{2}}}\in Q^k(D_{I+\frac{1}{2}})\ \forall I\}$, where $Q^k(D_{I+\frac{1}{2}})$ denotes the space of tensor-product polynomials of degrees at most k in each variable defined on $\{D_{I+\frac{1}{2}}\}$; again, no continuity is assumed across the cell boundary.

3.1. The central DG method and optimal error estimate.

3.1.1. The central DG scheme. The semidiscrete central DG approximations $u_h\in V_h$ and $v_h\in W_h$ of (3.1) are defined such that for all admissible test functions φ_h and ψ_h and all I's,

$$\frac{d}{dt}\int_{C_I} u_h\varphi_h dxdy = \frac{1}{\tau_{max}}\left(\int_{C_I}(v_h-u_h)\varphi_h dxdy\right)$$

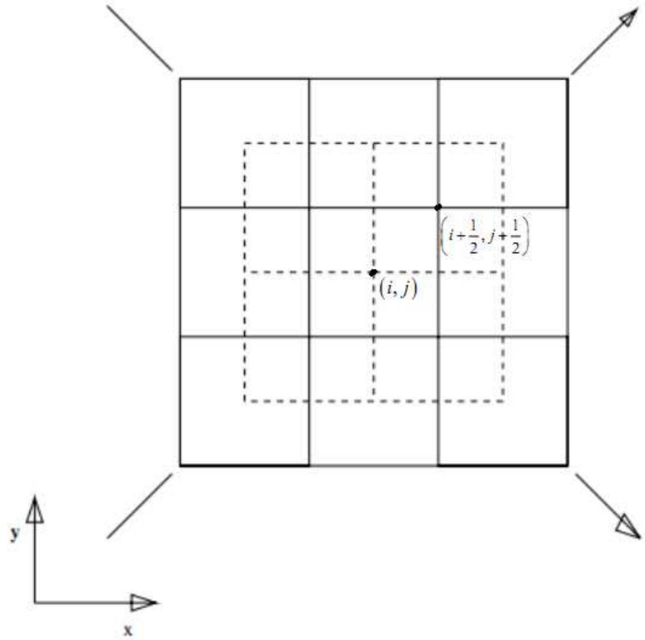

FIG. 3.1. *2D overlapping cells formed by collapsing the staggered dual cells on two adjacent time levels to one time level.*

$$
\begin{aligned}
&+\int_{C_I} v_h\left(\partial_x\varphi_h+\partial_y\varphi_h\right)dxdy\\
&-\int_{y_{j-\frac12}}^{y_{j+\frac12}}\Big(v_h(x_{i+\frac12},y)\varphi_h(x^-_{i+\frac12},y)-v_h(x_{i-\frac12},y)\varphi_h(x^+_{i-\frac12},y)\Big)\,dy\\
&-\int_{x_{i-\frac12}}^{x_{i+\frac12}}\Big(v_h(x,y_{j+\frac12})\varphi_h(x,y^-_{j+\frac12})-v_h(x,y_{j-\frac12})\varphi_h(x,y^+_{j-\frac12})\Big)\,dx\\
&=:\widetilde{B}(u_h,v_h;\varphi_h)_{i,j}\quad\forall\varphi_h\in V_h \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\int_{D_{I+\frac12}} v_h\psi_h dxdy =&\frac{1}{\tau_{max}}\left(\int_{D_{I+\frac12}}(u_h-v_h)\psi_h dxdy\right)\\
&+\int_{D_{I+\frac12}} u_h\left(\partial_x\psi_h+\partial_y\psi_h\right)dxdy\\
&-\int_{y_j}^{y_{j+1}}\big(u_h(x_{i+1},y)\psi_h(x^-_{i+1},y)-u_h(x_i,y)\psi_h(x^+_i,y)\big)\,dy\\
&-\int_{x_i}^{x_{i+1}}\big(u_h(x,y_{j+1})\psi_h(x,y^-_{j+1})-u_h(x,y_j)\psi_h(x,y^+_j)\big)\,dx\\
&=:\widehat{B}(v_h,u_h;\psi_h)_{i+\frac12,j+\frac12}\quad\forall\psi_h\in W_h \qquad (3.3)
\end{aligned}
$$

where τ_{max} is the max step size, determined by τ_{max} =(CFL factor)$\times h$/(maximum characteristic speed), in which the CFL factor should be less than 1/2. For the initial condition, we simply take $u_h(0) = \mathbb{P}_h u_0$ and $v_h(0) = \mathbb{Q}_h u_0$, where $\mathbb{P}_h$ and $\mathbb{Q}_h$ are the L^2 projections into V_h and W_h, respectively, and we have

$$\|u_0 - \mathbb{P}_h u_0\|_{L^2(C_I)} \le Ch^{k+1}\|u_0\|_{H^{k+1}(C_I)}$$
$$\|u_0 - \mathbb{Q}_h u_0\|_{L^2(D_{I+\frac{1}{2}})} \le Ch^{k+1}\|u_0\|_{H^{k+1}(D_{I+\frac{1}{2}})}.$$

The L^2-stability for the central DG scheme (3.2)-(3.3) is proved in [14].

Proposition 3.1. *[14]. The numerical solutions u_h and v_h of the semidiscrete central DG method (3.2)-(3.3) for the equation (3.1) has the following L^2 stability property*

$$\|u_h(T)\|^2_{L^2(\Omega)} + \|v_h(T)\|^2_{L^2(\Omega)} \le \|u_h(0)\|^2_{L^2(\Omega)} + \|v_h(0)\|^2_{L^2(\Omega)} \tag{3.4}$$

3.2. A priori error estimates. Let us state a priori error estimate for the two-dimensional case, whose proof will be given in the next subsection.

Theorem 3.1. *The numerical solutions u_h and v_h of the central DG scheme (3.2)-(3.3) using uniform meshes for the equation (3.1) with a smooth initial condition $u(\cdot, 0) \in H^{k+2}$ satisfies the following L^2 error estimate*

$$\|u(T) - u_h(T)\|^2_{L^2(\Omega)} + \|u(T) - v_h(T)\|^2_{L^2(\Omega)} \le Ch^{2k+2} \tag{3.5}$$

where u is the exact solution of (3.1), k is the degree of the piecewise tensor product polynomials in the finite element spaces V_h and W_h, and the constant C depends on the $(k+2)$-th order Sobolev norm of the initial condition $\|u(\cdot, 0)\|_{H^{k+2}}$ as well as on the final time T, but is independent of the mesh size h.

3.3. Proof of the error estimates. In this subsection we prove Theorem 3.1 stated in the previous subsection. To do that, we proceed as follows. First, in subsection 3.3.1 we establish the existence as well as uniqueness of suitably defined special local projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$. In addition, the optimal approximation properties of $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ are derived. We prove a few propositions and superconvergence results of the special projections in subsection 3.3.2. Finally, we complete the proof Theorem 3.1 in subsection 3.3.3.

3.3.1. The special projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$. Prior to giving the definition of the special projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$, we would like to recall the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ introduced in (2.12)-(2.13) for the one dimensional case. We note that we right shift by $\frac{h}{2}$ when integrating on the subinterval $[x_{i-\frac{1}{2}}, x_i]$ and left shift by $\frac{h}{2}$ when integrating on the subinterval $[x_i, x_{i+\frac{1}{2}}]$. We now continue to apply this technique in the two dimensional case, for x and y variables respectively. We define the projection $\mathbb{P}_h^*$ from $u \in L^\infty(C_I)$ into $\mathbb{P}_h^* u \in Q^k(C_I)$ over C_I satisfying the following two equations.

$$\int_{C_I} \mathbb{P}_h^* u dxdy = \int_{C_I} u dxdy \tag{3.6}$$

$$\widetilde{P_h}(\mathbb{P}_h^* u, \varphi_h)_{i,j} = \widetilde{P_h}(u, \varphi_h)_{i,j}, \quad \forall \varphi_h \in Q^k(C_I) \tag{3.7}$$

where $\widetilde{P_h}(u,\varphi_h)_{i,j}$ is defined as follows:

$$
\begin{aligned}
\widetilde{P_h}(u,\varphi_h)_{i,j} =& \frac{1}{\tau_{max}} \Bigg(\int_{y_{j-\frac12}}^{y_j} \int_{x_{i-\frac12}}^{x_i} u(x+\frac{h}{2}, y+\frac{h}{2})\varphi_h \, dxdy \\
&+ \int_{y_{j-\frac12}}^{y_j} \int_{x_i}^{x_{i+\frac12}} u(x-\frac{h}{2}, y+\frac{h}{2})\varphi_h \, dxdy \\
&+ \int_{y_j}^{y_{j+\frac12}} \int_{x_{i-\frac12}}^{x_i} u(x+\frac{h}{2}, y-\frac{h}{2})\varphi_h \, dxdy \\
&+ \int_{y_j}^{y_{j+\frac12}} \int_{x_i}^{x_{i+\frac12}} u(x-\frac{h}{2}, y-\frac{h}{2})\varphi_h \, dxdy \\
&- \int_{y_{j-\frac12}}^{y_{j+\frac12}} \int_{x_{i-\frac12}}^{x_{i+\frac12}} u(x,y)\varphi_h \, dxdy \Bigg) \\
&+ \int_{y_{j-\frac12}}^{y_j} \int_{x_{i-\frac12}}^{x_i} u(x+\frac{h}{2}, y+\frac{h}{2}) \left(\partial_x\varphi_h + \partial_y\varphi_h\right) dxdy \\
&+ \int_{y_{j-\frac12}}^{y_j} \int_{x_i}^{x_{i+\frac12}} u(x-\frac{h}{2}, y+\frac{h}{2}) \left(\partial_x\varphi_h + \partial_y\varphi_h\right) dxdy \\
&+ \int_{y_j}^{y_{j+\frac12}} \int_{x_{i-\frac12}}^{x_i} u(x+\frac{h}{2}, y-\frac{h}{2}) \left(\partial_x\varphi_h + \partial_y\varphi_h\right) dxdy \\
&+ \int_{y_j}^{y_{j+\frac12}} \int_{x_i}^{x_{i+\frac12}} u(x-\frac{h}{2}, y-\frac{h}{2}) \left(\partial_x\varphi_h + \partial_y\varphi_h\right) dxdy \\
&- \int_{x_{i-\frac12}}^{x_i} u(x+\frac{h}{2}, y_j) \Big(\varphi_h(x, y^-_{j+\frac12}) - \varphi_h(x, y^+_{j-\frac12})\Big) \, dx \\
&- \int_{x_i}^{x_{i+\frac12}} u(x-\frac{h}{2}, y_j) \Big(\varphi_h(x, y^-_{j+\frac12}) - \varphi_h(x, y^+_{j-\frac12})\Big) \, dx \\
&- \int_{y_{j-\frac12}}^{y_j} u(x_i, y+\frac{h}{2}) \Big(\varphi_h(x^-_{i+\frac12}, y) - \varphi_h(x^+_{i-\frac12}, y)\Big) \, dy \\
&- \int_{y_j}^{y_{j+\frac12}} u(x_i, y-\frac{h}{2}) \Big(\varphi_h(x^-_{i+\frac12}, y) - \varphi_h(x^+_{i-\frac12}, y)\Big) \, dy
\end{aligned} \tag{3.8}
$$

Similarly, we can define the projection $\mathbb{Q}_h^*$ from $u \in L^\infty(D_{I+\frac12})$ into $Q^k(D_{I+\frac12})$ over $D_{I+\frac12}$. The equation (3.6) is required by the conservation laws. Note that $\widetilde{P_h}(u,\varphi_h)_{i,j} = 0, \forall u$ when $\varphi_h = 1$, so (3.7) alone misses one condition which is provided by (3.6).

Existence and optimal approximate property of the projection $\mathbb{P}_h^*$ are established in the following lemma.

Lemma 3.1. *Assume u is sufficiently smooth. Then, there exists a unique $\mathbb{P}_h^* u \in V_h$ satisfying (3.6) and (3.7). Moreover, there holds the following approximating property*

$$
\|u - \mathbb{P}_h^* u\|_{L^2(C_I)} \le C h^{k+1} \|u\|_{H^{k+1}(C_I)} \tag{3.9}
$$

where C=C(k) is independent of the element C_I and the mesh size h.

Proof. The proof of this lemma is provided in the Appendix; see section A.3.

3.3.2. Properties of the projection $\mathbb{P}_h^*$. To obtain the optimal L^2 error estimate, we need the following lemmas.

LEMMA 3.2. *Assume that $u = x^{k+1}$ or y^{k+1}, let $u_I = \mathbb{P}_h^* u$ and $v_I = \mathbb{Q}_h^* u$, then $\forall (x,y) \in C_I$ we have following relations:*

$$\begin{aligned} x^{k+1} - u_I(x,y) =& (x+\frac{h}{2})^{k+1} - v_I(x+\frac{h}{2}, y+\frac{h}{2}) \\ =& (x+\frac{h}{2})^{k+1} - v_I(x+\frac{h}{2}, y-\frac{h}{2}) \\ =& (x-\frac{h}{2})^{k+1} - v_I(x-\frac{h}{2}, y+\frac{h}{2}) \\ =& (x-\frac{h}{2})^{k+1} - v_I(x-\frac{h}{2}, y-\frac{h}{2}) \end{aligned} \tag{3.10}$$

or

$$\begin{aligned} y^{k+1} - u_I(x,y) =& (y+\frac{h}{2})^{k+1} - v_I(x+\frac{h}{2}, y+\frac{h}{2}) \\ =& (y+\frac{h}{2})^{k+1} - v_I(x-\frac{h}{2}, y+\frac{h}{2}) \\ =& (y-\frac{h}{2})^{k+1} - v_I(x+\frac{h}{2}, y-\frac{h}{2}) \\ =& (y-\frac{h}{2})^{k+1} - v_I(x-\frac{h}{2}, y-\frac{h}{2}) \end{aligned} \tag{3.11}$$

Proof. The details of the proof of this lemma are provided in Appendix; see section A.4.

Again, the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ satisfy the following superconvergence result.

LEMMA 3.3. *Assume that $u = x^{k+1}$ or y^{k+1}, let $u_I = \mathbb{P}_h^* u$ and $v_I = \mathbb{Q}_h^* u$, then*

$$\widetilde{B}(u_I, v_I; \varphi_h)_{i,j} = \widetilde{B}(u, u; \varphi_h)_{i,j} \tag{3.12}$$

$$\widehat{B}(v_I, u_I; \psi)_{i+\frac{1}{2},j+\frac{1}{2}} = \widehat{B}(u, u; \psi_h)_{i+\frac{1}{2},j+\frac{1}{2}} \tag{3.13}$$

where $\widetilde{B}$ and $\widehat{B}$ are defined by (3.2) and (3.3).

Proof. We will only give the details of the proof for one case, namely $\widetilde{B}(u_I, v_I; \varphi)_{i,j} = \widetilde{B}(u, u; \varphi_h)_{i,j}$ is true when $u = x^{k+1}$, as the other cases can be handled similarly. We can use lemma 3.2 to replace v_I by u_I. After simplification, we have

$$\begin{aligned} \widetilde{B}(u_I, v_I; \varphi_h)_{i,j} =& \widetilde{P_h}(u_I - x^{k+1}, \varphi_h)_{i,j} + \widetilde{B}(x^{k+1}, x^{k+1}; \varphi_h)_{i,j} \\ =& \widetilde{B}(u, u; \varphi_h)_{i,j} \end{aligned} \tag{3.14}$$

where we have used the definition of the special projection $\mathbb{P}_h^*$ in (3.7). ∎

Next, we will use these lemmas to prove our finial result, Theorem 3.1.

3.3.3. Proof of Theorem 3.1. We take

$$\varphi_h = \mathbb{P}_h^* u - u_h; \quad \psi_h = \mathbb{Q}_h^* u - v_h \tag{3.15}$$

$$\varphi^e = \mathbb{P}_h^* u - u; \quad \psi^e = \mathbb{Q}_h^* u - u \tag{3.16}$$

The sum of the (3.2) and (3.3) gives

$$\begin{aligned} B_{i,j}(u_h, v_h; \varphi_h, \psi_h) = &\int_{C_I} \partial_t u_h \varphi_h \, dxdy + \int_{D_{I+\frac{1}{2}}} \partial_t v_h \psi_h \, dxdy \\ &- \widetilde{B}(u_h, v_h; \varphi_h)_{i,j} - \widehat{B}(v_h, u_h; \psi_h)_{i+\frac{1}{2},j+\frac{1}{2}} \\ =&0 \end{aligned} \tag{3.17}$$

The exact solution of the PDE (3.1) also satisfies the above scheme, hence we have

$$B_{i,j}(\varphi_h, \psi_h; \varphi_h, \psi_h) = B_{i,j}(\varphi^e, \psi^e; \varphi_h, \psi_h) \tag{3.18}$$

For the left-hand side of (3.18), we can use the stability results in [14] to obtain

$$\sum_{i,j} B_{i,j}(\varphi_h, \psi_h; \varphi_h, \psi_h) = \frac{1}{2}\frac{d}{dt}\int_\Omega \varphi_h^2 + \psi_h^2 \, dxdy + \frac{1}{\tau_{max}} \int_\Omega (\varphi_h - \psi_h)^2 \, dxdy \tag{3.19}$$

From Lemma 3.3, we know that on an arbitrary element C_I, we have the following results

$$\widetilde{B}(\mathbb{P}_h^* u, \mathbb{Q}_h^* u; \varphi_h)_{i,j} = \widetilde{B}(u, u; \varphi_h)_{i,j} \quad \forall u \in P^{k+1}([x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]) \quad \forall \varphi_h \in Q^k(C_I) \tag{3.20}$$

Next, on each element C_I, we consider the Taylor expansion of u around (x_i, y_j),

$$u = Tu + Ru$$

where

$$Tu = \sum_{l=0}^{k+1} \sum_{m=0}^{l} \frac{1}{(l-m)!m!} \frac{\partial^l u(x_i, y_j)}{\partial x^{l-m} \partial y^m} (x - x_i)^{l-m} (y - y_j)^m,$$

$$Ru = (k+2) \sum_{m=0}^{k+2} \frac{(x - x_i)^{k+2-m} (y - y_j)^m}{(k+2-m)!m!} \int_0^1 (1 - s)^{k+1} \frac{\partial^{k+2} u(x_i^s, y_j^s)}{\partial x^{k+2-m} \partial y^m} \, ds$$

with $x_i^s = x_i + s(x - x_i), y_j^s = y_j + s(y - y_j)$. Clearly, $Tu \in P^{k+1}([x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}])$. Note that the operator $\mathbb{P}_h^*$ is linear and thus $\mathbb{P}_h^* u = \mathbb{P}_h^* Tu + \mathbb{P}_h^* Ru$. From (3.20) we then get

$$\begin{aligned} \widetilde{B}(\varphi^e, \psi^e; \varphi_h)_{i,j} =&\widetilde{B}(\mathbb{P}_h^* Tu - Tu, \mathbb{Q}_h^* Tu - Tu; \varphi_h)_{i,j} + \widetilde{B}(\mathbb{P}_h^* Ru - Ru, \mathbb{Q}_h^* Ru - Ru, \varphi_h)_{i,j} \\ =&\widetilde{B}(\mathbb{P}_h^* Ru - Ru, \mathbb{Q}_h^* Ru - Ru, \varphi_h)_{i,j} \end{aligned} \tag{3.21}$$

Again recalling the Bramble-Hilbert lemma in [3], we have

$$\|Ru\|_{L^\infty(C_I)} \leq C h^{k+1} |u|_{H^{k+2}(C_I)} \tag{3.22}$$

Next using the same derivation as in the one dimensional case, we get

$$\sum_{i,j} |B_1(\mathbb{P}_h^* Ru - Ru, \mathbb{Q}_h^* Ru - Ru, \varphi_h)_{i,j}| \le Ch^{k+2} \|u\|^2_{H^{k+2}(\Omega)} + \|\varphi_h\|^2_{L^2(\Omega)} \tag{3.23}$$

Similarly, we get

$$\sum_{i,j} |B_2(\mathbb{Q}_h^* Ru - Ru, \mathbb{P}_h^* Ru - Ru, \psi_h)_{i,j}| \le Ch^{k+2} \|u\|^2_{H^{k+2}(\Omega)} + \|\psi_h\|^2_{L^2(\Omega)} \tag{3.24}$$

We can easily use the approximation properties of the projections $\mathbb{P}_h^*$, $\mathbb{Q}_h^*$ to show that

$$\sum_{i,j} \int_{C_I} \partial_t \varphi^e \varphi_h \, dxdy \le Ch^{2k+2} \|u\|^2_{H^{k+1}(\Omega)} + \|\varphi_h\|^2_{L^2(\Omega)} \tag{3.25}$$

$$\sum_{i,j} \int_{D_{I+\frac{1}{2}}} \partial_t \psi^e \psi_h \, dxdy \le Ch^{2k+2} \|u\|^2_{H^{k+1}(\Omega)} + \|\psi_h\|^2_{L^2(\Omega)} \tag{3.26}$$

From (3.23)-(3.26) and (3.19) we have

$$\|\varphi_h\|^2_{L^2(\Omega)} + \|\psi_h\|^2_{L^2(\Omega)} \le Ch^{2k+2} \|u\|^2_{H^{k+2}(\Omega)} \tag{3.27}$$

Together with the approximation properties of the projection (3.9) we have finished the proof of Theorem 3.1.

4. Numerical examples. In this section, we present numerical examples to verify our theoretical findings. In our numerical experiments, we measure the maximum errors at the zeros of the polynomial $x^{k+1} - \mathbb{P}_h^*(x^{k+1})$ in each cell, and the L^1, L^∞ and L^2 errors respectively. They are defined by

$$E_{super} = \max_{j \in Z_N, x_{ij} \in G_j} |(u - u_h)(x_{ij}, T)| \tag{4.1}$$

$$E_1 = \int_0^{2\pi} |(u - u_h)(x, T)| \; dx \tag{4.2}$$

$$E_2 = \left(\int_0^{2\pi} (u - u_h)^2 (x, T) \; dx \right)^{\frac{1}{2}} \tag{4.3}$$

$$E_\infty = \max_x |(u - u_h)(x, T)| \tag{4.4}$$

where G_j is a set of zeros of the polynomial $x^{k+1} - \mathbb{P}_h^*(x^{k+1})$ in cell I_j.

EXAMPLE 4.1. *We consider the following equation with periodic boundary condition:*

$$\begin{aligned} & u_t + u_x = 0, \quad (x,t) \in [0, 2\pi] \times (0, 2\pi) \\ & u(x, 0) = \sin(x) \\ & u(0, t) = u(2\pi, t) \end{aligned} \tag{4.5}$$

The exact solution to this problem is

$$u(x, t) = \sin(x - t). \tag{4.6}$$

The problem is solved by the central DG scheme (2.2) with $k = 1, 2, 3$, respectively. Uniform meshes are used in our experiments, which are constructed by equally dividing the interval, $[0, 2\pi]$, into N subintervals with $N = 10, 20, 40, 80, 160$. The seventh order strong-stability preserving Runge-Kutta method [8] with the time step $\Delta t = 0.05h$, $h = \frac{2\pi}{N}$ is used to reduce the time discretization error. Table 4.1 shows that the order of convergence of the error achieves the expected $(k+1)$-th order of accuracy. Also the superconvergence phenomenon is found. According to this numerical example, we find that the central DG scheme for the liner hyperbolic equation has $(k+2)$-th order superconvergence accuracy at the zeros of the special polynomial $x^{k+1} - \mathbb{P}_h^*(x^{k+1})$ in each element.

Table 4.1
Errors and the corresponding convergence rates. $T = 1$, $\tau_{max} = \frac{1}{2k+1}h$

	N	E_{super}	Rate	E_1	Rate	E_2	Rate	E_∞	Rate
$k=1$	10	5.63E-03	–	1.20E-02	–	1.34E-02	–	1.85E-02	–
	20	7.29E-04	2.95	2.85E-03	2.07	3.15E-03	2.09	4.43E-03	2.07
	40	9.06E-05	3.01	6.85E-04	2.06	7.59E-04	2.06	1.07E-03	2.05
	80	1.13E-05	3.00	1.67E-04	2.03	1.86E-04	2.03	2.63E-04	2.03
	160	1.41E-06	3.00	4.14E-05	2.01	4.59E-05	2.01	6.49E-05	2.01
$k=2$	10	1.12E-04	–	1.35E-04	–	1.52E-04	–	2.09E-04	–
	20	6.90E-06	4.01	1.78E-05	2.92	1.98E-05	2.94	2.79E-05	2.91
	40	4.26E-07	4.01	2.25E-06	2.99	2.50E-06	2.99	3.53E-06	2.99
	80	2.64E-08	4.01	2.82E-07	3.00	3.13E-07	3.00	4.43E-07	2.99
	160	1.65E-09	4.00	3.53E-08	3.00	3.92E-08	3.00	5.54E-08	3.00
$k=3$	10	7.94E-06	–	1.77E-05	–	1.93E-05	–	2.73E-05	–
	20	2.51E-07	4.99	1.08E-06	4.03	1.21E-06	4.00	1.70E-06	4.00
	40	7.79E-09	5.01	6.78E-08	3.99	7.53E-08	4.00	1.06E-07	4.00
	80	2.43E-10	5.00	4.23E-09	4.00	4.70E-09	4.00	6.65E-09	4.00
	160	7.55E-12	5.00	2.64E-10	4.00	2.94E-10	4.00	4.15E-10	4.00

Example 4.2. *We consider the following equation with periodic boundary condition:*

$$
\begin{aligned}
&u_t + u_x + u_y = 0, \quad (x, y, t) \in [0, 2\pi] \times [0, 2\pi] \times (0, 2\pi) \\
&u(x, y, 0) = 2 + \sin(x + y) \\
&u(0, y, t) = u(2\pi, y, t) \quad u(x, 0, t) = u(x, 2\pi, t)
\end{aligned} \tag{4.7}
$$

The exact solution to this problem is

$$
u(x, y, t) = 2 + \sin(x + y - 2t) \tag{4.8}
$$

We test this example using Q^k polynomials with $0 \le k \le 2$ on a uniform mesh with $N \times N$ cells. Again the seventh order strong-stability preserving Runge-Kutta method [8] with time step $\Delta t = 0.05h$, $h = \frac{2\pi}{N}$ is used to reduce the time discretization errors. The results in Table 4.2 show that the order of convergence of the error, $\|u - u_h\|_{L^2(\Omega)}$, achieves the expected $(k+1)$-th order of accuracy. We have also observed the superconvergence phenomenon of E_{super}.

5. Concluding remarks. In this paper, optimal L^2 error estimates to central DG methods applied to linear conservation laws are proved. Our analysis is carried out both in one dimension and in multidimensions for uniform Cartesian meshes and

Table 4.2
L^2 errors and corresponding convergence rates in cases $k=1,2$. $T=1$, $\tau_{max}=\frac{1}{2k+1}h$

	$N\times N$	E_{super}	Rate	E_1	Rate	E_2	Rate	E_∞	Rate
$k=1$	10×10	8.43E-03	–	5.14	–	3.55E-01	–	5.11E-02	–
	20×20	1.00E-03	3.07	1.35	1.92	9.24E-02	1.94	1.41E-02	1.86
	40×40	1.19E-04	3.07	3.44E-01	1.97	2.35E-02	1.98	3.66E-03	1.94
	80×80	1.44E-05	3.04	8.66E-02	1.99	5.92E-03	1.99	9.29E-04	1.97
	160×160	1.77E-06	3.02	2.17E-02	2.00	1.48E-03	2.00	2.34E-04	1.99
	320×320	2.20E-07	3.01	5.43E-03	2.00	3.72E-04	2.00	5.87E-05	2.00
$k=2$	10×10	2.85E-04	–	1.33E-01	–	8.82E-03	–	1.37E-03	–
	20×20	1.75E-05	4.02	1.72E-02	2.96	1.13E-03	2.96	1.77E-04	2.95
	40×40	1.10E-06	4.00	2.16E-03	2.99	1.43E-04	2.99	2.25E-05	2.97
	80×80	6.81E-08	4.01	2.70E-04	3.00	1.79E-05	3.00	2.82E-06	3.00
	160×160	4.25E-09	4.00	3.38E-05	3.00	2.23E-06	3.00	3.53E-07	3.00
	320×320	2.65E-10	4.00	4.23E-06	3.00	2.80E-07	3.00	4.41E-08	3.00

tensor-product polynomial spaces, and is valid for arbitrary polynomial degree $k \geq 0$. The main ingredients in the proof is the construction and analysis of special projections. We find that the projections are closely related to the superconvergence points of the scheme. We also give numerical examples to verify the results of our theoretical analysis. Extension of this work to non-uniform meshes, nonlinear equations and to general superconvergence results is interesting and challenging, and constitutes our future work.

Appendix A. Proof of a few technical lemmas and propositions.
In this appendix, we collect the proof of some of the technical lemmas and propositions in the error estimates.

A.1. Proof of Lemma 2.1. *Proof.* Note that the procedure to find $\mathbb{P}_h^*\omega \in \mathbb{P}^k([-1,1])$ is to solve a linear system, so the existence and uniqueness are equivalent. Thus, we only prove the uniqueness of the projection $\mathbb{P}_h^*$. We set $\omega_I(x) = \mathbb{P}_h^*\omega(x)$ with $\omega(x)=0$, and would like to prove $\omega_I(x)=0$. By the definition of the projection $\mathbb{P}_h^*$, we have:

$$\begin{aligned}\widetilde{P_h}(\omega_I;\varphi_h) =& \frac{1}{2c}\left(\int_{-1}^0 \omega_I(x+1)\varphi_h(x)dx + \int_0^1 \omega_I(x-1)\varphi_h(x)dx - \int_{-1}^1 \omega_I(x)\varphi_h(x)dx\right)\\ &+ \int_{-1}^0 \omega_I(x+1)(\varphi_h(x))_x dx + \int_0^1 \omega_I(x-1)(\varphi_h(x))_x dx\\ &- \omega_I(0)\left(\varphi_h(1)-\varphi_h(-1)\right) = 0 \quad \forall \varphi_h(x)\in\mathbb{P}^k([-1,1])\end{aligned} \tag{A.1}$$

$$\int_{-1}^1 \omega_I(x)dx = 0 \tag{A.2}$$

Specially, we set $\varphi_h(x)=\omega_I(x)\in\mathbb{P}^k([-1,1])$ to get

$$\begin{aligned}\widetilde{P_h}(\omega_I;\omega_I) =& \frac{1}{2c}\left(\int_{-1}^0 \omega_I(x+1)\omega_I(x)dx + \int_0^1 \omega_I(x-1)\omega_I(x)dx - \int_{-1}^1 \omega_I(x)^2 dx\right)\\ &+ \int_{-1}^0 \omega_I(x+1)(\omega_I(x))_x dx + \int_0^1 \omega_I(x-1)(\omega_I(x))_x dx\end{aligned}$$

$$-\,\omega_I(0)\,(\omega_I(1)-\omega_I(-1))=0 \tag{A.3}$$

We rewrite $\widetilde{P_h}(\omega_I;\omega_I)$ by a change of variable $x\to x+1$ for the integrations over $[-1,0]$ to get

$$\begin{aligned}\widetilde{P_h}(\omega_I;\omega_I)=&\frac{1}{2c}\left(2\int_0^1\omega_I(x-1)\omega_I(x)dx-\int_0^1\omega_I(x-1)^2dx-\int_0^1\omega_I(x)^2dx\right)\\&+\int_0^1\omega_I(x)(\omega_I(x-1))_xdx+\int_0^1\omega_I(x-1)(\omega_I(x))_xdx\\&-\omega_I(0)(\omega_I(1)-\omega_I(-1))\\=&\frac{-1}{2c}\int_0^1(\omega_I(x)-\omega_I(x-1))^2dx=0\end{aligned} \tag{A.4}$$

Thus

$$\omega_I(x)=\omega_I(x-1),\quad \forall x\in(0,1) \tag{A.5}$$

$$\omega_I(x+1)=\omega_I(x),\quad \forall x\in(-1,0) \tag{A.6}$$

We can put (A.5) into (A.1) to obtain

$$\widetilde{P_h}(\omega_I;\varphi_h)=\int_{-1}^1\omega_I(x)(\varphi_h(x))_xdx-\omega_I(0)(\varphi(1)-\varphi(-1))=0\quad \forall\varphi_h(x)\in\mathbb{P}^k([-1,1]) \tag{A.7}$$

Specially, we set $\varphi_h(x)=x$, and use (A.2) to obtain $\omega_I(0)=0$. Therefore, we have:

$$\int_{-1}^1\omega_I(x)(\varphi_h(x))_xdx=0,\quad \forall\varphi_h(x)\in\mathbb{P}^k([-1,1]) \tag{A.8}$$

This, together with (A.5), implies $\omega_I(x)\equiv 0$. We have now finished the proof of uniqueness.

We now move to the proof of the second part (2.14). We denote $\mathbb{P}_h^*\omega(x)=\omega_I(x)=\sum_{i=0}^k a_ix^i$, and set the test function $\varphi_h(x)=x,x^2,\dots,x^k$. Thus:

$$\widetilde{P_h}(\omega_I;x^l)=\sum_{i=0}^k\alpha_{il}a_i,\quad 1\le l\le k$$

$$\int_{-1}^1\omega_I(x)dx=\sum_{i=0}^k\frac{1^{i+1}-(-1)^{i+1}}{i+1}a_i=\sum_{i=0}^k\alpha_{i0}a_i \tag{A.9}$$

It is easy to prove $|\widetilde{P_h}(\omega;x^l)|\le C\|\omega(x)\|_\infty$, and the coefficients $\alpha_{il},0\le i\le k,0\le l\le k$ are independent of $\omega(x)$. We denote $\beta=(a_0,a_1,\dots,a_k)^T$, $A(i,l)=\alpha_{il},0\le i\le k,0\le l\le k$, and $b_0=\int_{-1}^1\omega(x)dx$, $b_l=\widetilde{P_h}(\omega;x^l),l=1,\dots,k$, $\gamma=(b_0,b_1,\dots,b_k)^T$. We can solve the following linear system:

$$A^T\beta=\gamma \tag{A.10}$$

to get $\beta=(A^T)^{-1}\gamma$. Each component of β is bounded by $\|\omega\|_\infty$, i.e. $|a_i|\lesssim\|\omega\|_\infty,i=0,1,\dots,k$. Thus $\|\mathbb{P}_h^*\omega(x)\|_\infty\lesssim\|\omega(x)\|_\infty$.

□

A.2. Proof of Proposition 2.1. Note that $u_I = u$ and $v_I = u$ when $u(x) \in \mathbb{P}^k([a,b])$, by the uniqueness of the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$. Therefore, we just need to prove one case that $u(x) = x^{k+1}$. Before we prove Proposition 2.1, we show a simple claim as following.

Claim A.1. *When the notations are the same as those in Proposition 2.1, and $u(x) = x^{k+1}$, then*

(A.11)
$$(x+\frac{h}{2})^{k+1}-v_I(x+\frac{h}{2}) = x^{k+1}-u_I(x) = (x-\frac{h}{2})^{k+1}-v_I(x-\frac{h}{2}), \quad x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \forall j$$

Proof. We just need to prove $x^{k+1}-v_I(x) = (x-\frac{h}{2})^{k+1}-u_I(x-\frac{h}{2}), \forall x \in (x_j, x_{j+1})$. We set $\widetilde{v}_I(x) = u_I(x-\frac{h}{2})-(x-\frac{h}{2})^{k+1}+x^{k+1}$, then we just need to prove $v_I(x) = \widetilde{v}_I(x)$. By the uniqueness of the projection $\mathbb{Q}_h^*$, we just need to check the following equations:

$$\int_{x_j}^{x_{j+1}} \widetilde{v}_I(x)dx = \int_{x_j}^{x_{j+1}} u(x)dx$$

$$\widetilde{Q_h}(\widetilde{v}_I;\psi_h)_{j+\frac{1}{2}} = \widetilde{Q_h}(u;\psi_h)_{j+\frac{1}{2}} \quad \forall \psi_h \in \mathbb{P}^k(I_{j+\frac{1}{2}}) \tag{A.12}$$

The first equation can be checked as follows

$$\begin{aligned}\int_{x_j}^{x_{j+1}} \widetilde{v}_I(x)dx &= \int_{x_j}^{x_{j+1}} u_I(x-\frac{h}{2}) - (x-\frac{h}{2})^{k+1} + x^{k+1}\,dx \\ &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I(x) - x^{k+1}dx + \int_{x_j}^{x_{j+1}} x^{k+1}\,dx \\ &= \int_{x_j}^{x_{j+1}} x^{k+1}dx = \int_{x_j}^{x_{j+1}} u(x)\,dx\end{aligned}$$

where we have used the definition of the projection $\mathbb{P}_h^*$ in (2.12). The second equation can be checked as follows

$$\begin{aligned}\widetilde{Q_h}(\widetilde{v}_I(x);\psi_h(x))_{j+\frac{1}{2}} &= \widetilde{P_h}(u_I(x)-u(x);\psi_h(x+\frac{h}{2}))_j + \widetilde{Q_h}(x^{k+1};\psi_h(x))_{j+\frac{1}{2}} \\ &= \widetilde{Q_h}(x^{k+1};\psi_h(x))_{j+\frac{1}{2}} \qquad \forall \psi_h(x) \in \mathbb{P}^k(I_{j+\frac{1}{2}})\end{aligned}$$

where we have used the fact $\psi_h(x+\frac{h}{2}) \in V_h$. Therefore the uniqueness of the projection $\mathbb{Q}_h^*$ implies that $v_I(x) = \widetilde{v}_I(x)$.
□

Now, we begin to prove Proposition 2.1. We will just prove one case $\widetilde{B}_j(u_I, v_I;\varphi_h) = \widetilde{B}_j(u,u;\varphi_h)$, as the other one follows the same lines. We use the claim to $\widetilde{B}_j(u_I, v_I;\varphi_h)$:

$$\begin{aligned}\widetilde{B}_j(u_I, v_I;\varphi_h) = &\frac{1}{\tau_{max}}\left(\int_{x_{j-\frac{1}{2}}}^{x_j} v_I(x)\varphi_h dx + \int_{x_j}^{x_{j+\frac{1}{2}}} v_I(x)\varphi_h dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I(x)\varphi_h dx\right) \\ &+ \int_{x_{j-\frac{1}{2}}}^{x_j} v_I(x)\partial_x\varphi_h dx + \int_{x_j}^{x_{j+\frac{1}{2}}} v_I(x)\partial_x\varphi_h dx \\ &- v_I(x_{j+\frac{1}{2}})\varphi_h(x^-_{j+\frac{1}{2}}) + v_I(x_{j-\frac{1}{2}})\varphi_h(x^+_{j-\frac{1}{2}}) \\ = &\frac{1}{\tau_{max}}\left(\int_{x_{j-\frac{1}{2}}}^{x_j}\left(u_I(x+\frac{h}{2}) - (x+\frac{h}{2})^{k+1} + x^{k+1}\right)\varphi_h dx\right.\end{aligned}$$

$$+\int_{x_j}^{x_{j+\frac{1}{2}}}\left(u_I(x-\frac{h}{2})-(x-\frac{h}{2})^{k+1}+x^{k+1}\right)\varphi_h dx-\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}u_I(x)\varphi_h dx\Bigg)$$
$$+\int_{x_{j-\frac{1}{2}}}^{x_j}\left(u_I(x+\frac{h}{2})-(x+\frac{h}{2})^{k+1}+x^{k+1}\right)\partial_x\varphi_h dx$$
$$+\int_{x_j}^{x_{j+\frac{1}{2}}}\left(u_I(x-\frac{h}{2})-(x-\frac{h}{2})^{k+1}+x^{k+1}\right)\partial_x\varphi_h dx$$
$$-\left(u_I(x_j)-x_j^{k+1}+x_{j+\frac{1}{2}}^{k+1}\right)\varphi_h(x_{j+\frac{1}{2}}^-)+\left(u_I(x_j)-x_j^{k+1}+x_{j-\frac{1}{2}}^{k+1}\right)\varphi_h(x_{j-\frac{1}{2}}^+)$$
$$=\widetilde{P_h}(u_I(x)-x^{k+1};\varphi_h(x))_j+\widetilde{B}_j(x^{k+1},x^{k+1};\varphi_h)$$
$$=\widetilde{B}_j(u(x),u(x);\varphi_h) \tag{A.13}$$

A.3. Proof of Lemma 3.1. *Proof.* Let u_I denote $\mathbb{P}_h^*u$. Assume that $u\equiv 0$. Take $\varphi_h=u_I$ in (3.7), we get

$$0=\widetilde{P_h}(u_I,u_I)=\frac{1}{\tau_{max}}\Bigg(\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i}2u_I(x+\frac{h}{2},y+\frac{h}{2})u_I(x,y)+2u_I(x+\frac{h}{2},y)u_I(x,y+\frac{h}{2})\,dxdy$$
$$-\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i}u_I(x,y)^2+u_I(x,y+\frac{h}{2})^2+u_I(x+\frac{h}{2},y)^2+u_I(x+\frac{h}{2},y+\frac{h}{2})^2\,dxdy\Bigg)$$
$$=\frac{-1}{\tau_{max}}\Bigg(\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i}\left(u_I(x+\frac{h}{2},y+\frac{h}{2})-u_I(x,y)\right)^2dxdy$$
$$+\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i}\left(u_I(x+\frac{h}{2},y)-u_I(x,y+\frac{h}{2})\right)^2dxdy\Bigg) \tag{A.14}$$

where we have again used change of variable to shift all the integration regions to the same subcell $(x_{i-\frac{1}{2}},x_i)\times(y_{j-\frac{1}{2}},y_j)$ to simplify the formulation. Then

$$u_I(x,y)=u_I(x+\frac{h}{2},y+\frac{h}{2}),\quad u_I(x+\frac{h}{2},y)=u_I(x,y+\frac{h}{2}),\forall(x,y)\in(x_{i-\frac{1}{2}},x_i)\times(y_{j-\frac{1}{2}},y_j)$$

Thus $u_I(x,y)\equiv c_0$ on C_I, c_0 is a constant. By (3.6) we immediately get $u_I\equiv 0$, and we have finished the proof of uniqueness, hence also existence. We note that this projection is a local projection, hence we can make a change of variables to the reference element $[-1/2,1/2]\times[-1/2,1/2]$ by taking $\xi=\frac{x-x_i}{h}$ and $\eta=\frac{y-y_j}{h}$. Taking a similar derivation as in the proof of (A.1), we obtain

$$\|u_I\|_{L^\infty(C_I)}\le C(k)\|u\|_{L^\infty(C_I)} \tag{A.15}$$

Again standard approximation theory [3] implies the optimal approximating estimates. ∎

A.4. Proof of Lemma 3.2. *Proof.* We will only show that $x^{k+1}-u_I(x,y)=(x+\frac{h}{2})^{k+1}-v_I(x+\frac{h}{2},y+\frac{h}{2})$ when $u=x^{k+1}$, as the other cases are similar. Note that we know $u_I = \mathbb{P}_h^* u$ is unique. We let $\tilde{u}_I(x,y) = v_I(x+\frac{h}{2}, y+\frac{h}{2}) - (x+\frac{h}{2})^{k+1} + x^{k+1}$, and clearly we have $\tilde{u}_I(x,y) \in Q^k(C_I)$. Next to prove $\tilde{u}_I(x,y)$ satisfies (3.6)-(3.7).

$$\begin{aligned}
\int_{C_I} \tilde{u}_I(x,y)\,dxdy &= \int_{C_I} \left(v_I(x+\frac{h}{2}, y+\frac{h}{2}) - (x+\frac{h}{2})^{k+1} + x^{k+1} \right) dxdy \\
&= \int_{D_{I+\frac{1}{2}}} \left(v_I(x,y) - x^{k+1} \right) dxdy + \int_{C_I} x^{k+1}\,dxdy \\
&= \int_{C_I} x^{k+1}\,dxdy \qquad \text{(A.16)}
\end{aligned}$$

$$\widetilde{P_h}(\tilde{u}_I, \varphi_h)_{i,j} = ① + ② + ③ \qquad \text{(A.17)}$$

$$\begin{aligned}
① =& \frac{1}{\tau_{max}} \Bigg(\int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_i}^{x_{i+\frac{1}{2}}} \left(v_I(x+\frac{h}{2}, y+\frac{h}{2}) - (x+\frac{h}{2})^{k+1} \right) \varphi_h(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy \\
&+ \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(v_I(x-\frac{h}{2}, y+\frac{h}{2}) - (x-\frac{h}{2})^{k+1} \right) \varphi_h(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy \\
&+ \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \int_{x_i}^{x_{i+\frac{1}{2}}} \left(v_I(x+\frac{h}{2}, y-\frac{h}{2}) - (x+\frac{h}{2})^{k+1} \right) \varphi_h(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy \\
&+ \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(v_I(x-\frac{h}{2}, y-\frac{h}{2}) - (x-\frac{h}{2})^{k+1} \right) \varphi_h(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy \\
&- \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} \left(v_I(x,y) - x^{k+1} \right) \varphi_h(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy \Bigg) \\
&+ \frac{1}{\tau_{max}} \Bigg(\int_{y_{j-\frac{1}{2}}}^{y_j} \int_{x_{i-\frac{1}{2}}}^{x_i} (x+\frac{h}{2})^{k+1} \varphi_h(x,y)\,dxdy + \int_{y_{j-\frac{1}{2}}}^{y_j} \int_{x_i}^{x_{i+\frac{1}{2}}} (x-\frac{h}{2})^{k+1} \varphi_h(x,y)\,dxdy \\
&+ \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_i} (x+\frac{h}{2})^{k+1} \varphi_h(x,y)\,dxdy + \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_i}^{x_{i+\frac{1}{2}}} (x-\frac{h}{2})^{k+1} \varphi_h(x,y)\,dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} x^{k+1} \varphi_h(x,y)\,dxdy \Bigg)
\end{aligned}$$

$$\begin{aligned}
② =& \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_i}^{x_{i+\frac{1}{2}}} \left(v_I(x+\frac{h}{2}, y+\frac{h}{2}) - (x+\frac{h}{2})^{k+1} \right) (\partial_x \varphi_h + \partial_y \varphi_h)(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy \\
&+ \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(v_I(x-\frac{h}{2}, y+\frac{h}{2}) - (x-\frac{h}{2})^{k+1} \right) (\partial_x \varphi_h + \partial_y \varphi_h)(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy \\
&+ \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \int_{x_i}^{x_{i+\frac{1}{2}}} \left(v_I(x+\frac{h}{2}, y-\frac{h}{2}) - (x+\frac{h}{2})^{k+1} \right) (\partial_x \varphi_h + \partial_y \varphi_h)(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy \\
&+ \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(v_I(x-\frac{h}{2}, y-\frac{h}{2}) - (x-\frac{h}{2})^{k+1} \right) (\partial_x \varphi_h + \partial_y \varphi_h)(x-\frac{h}{2}, y-\frac{h}{2})\,dxdy
\end{aligned}$$

$$+\int_{y_{j-\frac12}}^{y_j}\int_{x_{i-\frac12}}^{x_i}(x+\frac{h}{2})^{k+1}(\partial_x\varphi_h+\partial_y\varphi_h)(x,y)\,dxdy$$
$$+\int_{y_{j-\frac12}}^{y_j}\int_{x_i}^{x_{i+\frac12}}(x-\frac{h}{2})^{k+1}(\partial_x\varphi_h+\partial_y\varphi_h)(x,y)\,dxdy$$
$$+\int_{y_j}^{y_{j+\frac12}}\int_{x_{i-\frac12}}^{x_i}(x+\frac{h}{2})^{k+1}(\partial_x\varphi_h+\partial_y\varphi_h)(x,y)\,dxdy$$
$$+\int_{y_j}^{y_{j+\frac12}}\int_{x_i}^{x_{i+\frac12}}(x-\frac{h}{2})^{k+1}(\partial_x\varphi_h+\partial_y\varphi_h)(x,y)\,dxdy$$

$$\begin{aligned}
③ =&-\int_{y_j}^{y_{j+\frac12}}\left(v_I(x_{i+\frac12},y+\frac{h}{2})-x_{i+\frac12}^{k+1}\right)\left(\varphi_h(x^-_{i+\frac12},y-\frac{h}{2})-\varphi_h(x^+_{i-\frac12},y-\frac{h}{2})\right)dy\\
&-\int_{y_{j+\frac12}}^{y_{j+1}}\left(v_I(x_{i+\frac12},y-\frac{h}{2})-x_{i+\frac12}^{k+1}\right)\left(\varphi_h(x^-_{i+\frac12},y-\frac{h}{2})-\varphi_h(x^+_{i-\frac12},y-\frac{h}{2})\right)dy\\
&-\int_{x_i}^{x_{i+\frac12}}\left(v_I(x+\frac{h}{2},y_{j+\frac12})-(x+\frac{h}{2})^{k+1}\right)\left(\varphi_h(x-\frac{h}{2},y^-_{j+\frac12})-\varphi_h(x-\frac{h}{2},y^+_{j-\frac12})\right)dx\\
&-\int_{x_{i+\frac12}}^{x_{i+1}}\left(v_I(x-\frac{h}{2},y_{j+\frac12})-(x-\frac{h}{2})^{k+1}\right)\left(\varphi_h(x-\frac{h}{2},y^-_{j+\frac12})-\varphi_h(x-\frac{h}{2},y^+_{j-\frac12})\right)dx\\
&-\int_{x_{i-\frac12}}^{x_i}(x+\frac{h}{2})^{k+1}\left(\varphi_h(x,y^-_{j+\frac12})-\varphi_h(x,y^+_{j-\frac12})\right)dx\\
&-\int_{x_i}^{x_{i+\frac12}}(x-\frac{h}{2})^{k+1}\left(\varphi_h(x,y^-_{j+\frac12})-\varphi_h(x,y^+_{j-\frac12})\right)dx\\
&-\int_{y_{j-\frac12}}^{y_j}x_i^{k+1}\left(\varphi_h(x^-_{i+\frac12},y)-\varphi_h(x^+_{i-\frac12},y)\right)dy\\
&-\int_{y_j}^{y_{j+\frac12}}x_i^{k+1}\left(\varphi_h(x^-_{i+\frac12},y)-\varphi_h(x^+_{i-\frac12},y)\right)dy
\end{aligned}$$

Thus

$$\begin{aligned}
\widetilde{P_h}(\tilde u_I,\varphi_h)_{i,j}&=\widetilde{Q_h}(v_I(x,y)-x^{k+1},\varphi_h(x-\frac{h}{2},y-\frac{h}{2}))_{i+\frac12,j+\frac12}+\widetilde{P_h}(x^{k+1},\varphi_h(x,y))_{i,j}\\
&=\widetilde{P_h}(x^{k+1},\varphi_h(x,y))_{i,j} \qquad \text{(A.18)}
\end{aligned}$$

where we have used the definition of $v_I=\mathbb{Q}_h^*u$ over $D_{I+\frac12}$. Therefore (A.16)-(A.18) imply $\tilde u_I=u_I$. □

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