# SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHOD FOR SCALAR NONLINEAR HYPERBOLIC EQUATIONS

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**Abstract.** In this paper, we study the superconvergence behavior of the semi-discrete discontinuous Galerkin (DG) method for scalar nonlinear hyperbolic equations in one spatial dimension. Superconvergence results for problems with fixed and alternating wind directions are established. On the one hand, we prove that, if the wind direction is fixed (i.e., the derivative of the flux function is bounded away from zero), both the cell average error and numerical flux error at cell interfaces converge at a rate of $2k+1$ when upwind fluxes and piecewise polynomials of degree $k$ are used. Moreover, we also prove that the function value approximation of the DG solution is superconvergent at interior right Radau points, and the derivative value approximation is superconvergent at interior left Radau points, with an order of $k+2$ and $k+1$, respectively. As a byproduct, we show a $k+2$-th order superconvergence of the DG solution towards the Gauss-Radau projection of the exact solution. On the other hand, superconvergence results for problems with alternating wind directions (i.e., the derivative of the flux function either changes sign or otherwise achieves the value zero in the domain) are also established. To be more precise, we first prove that the DG flux function is superconvergent towards a particular flux function of the exact solution, with an order of $k+2$, when Godunov fluxes are used. We then prove that the highest superconvergence rate of the DG solution itself is $k+\frac{3}{2}$ when sonic points (i.e., the derivative of the flux function achieves zero) appear in the computational domain. As byproducts, we obtain superconvergence properties for the DG solution and the DG flux function at special points and for cell average. Numerical experiments demonstrate that most of our results are optimal, i.e., the superconvergence rates are sharp.

**Key words.** Discontinuous Galerkin methods, superconvergence, nonlinear hyperbolic equations

**AMS subject classifications.** 65M15, 65M60, 65N30

## 1. Introduction.

In this paper, we investigate the superconvergence behavior of the discontinuous Galerkin (DG) method for the following one-dimensional nonlinear hyperbolic equation

$$
\begin{aligned}
&u_t + f(u)_x = 0, \qquad (x,t) \in [a,b] \times [0,T], \\
&u(x,0) = u_0(x), \qquad x \in \bar{\Omega} = [a,b],
\end{aligned} \tag{1}
$$

where $u_0$ is sufficiently smooth. We assume that the nonlinear flux function $f(u)$ is sufficiently smooth with respect to $u$ and the final time $T$ is not too large so that the exact solutions are smooth. In this paper, for simplicity, we only consider the periodic boundary condition.

The superconvergence behavior of the DG (see, e.g., [12, 13, 14, 15, 16, 17]) method has been studied for many years, and has been a hot research topic in recent

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years. We refer to [1, 2, 3, 10, 11, 18, 23, 24, 4, 19] for an incomplete list of references on the superconvergence of DG methods for hyperbolic problems. However, all the studies in the literature are based on linear equations. Only recently, Meng et al. studied the superconvergence of semi-discrete DG methods for (1) with fixed wind direction, i.e., $|f'(u)|$ possesses a uniform positive lower bound [20]. They proved that the order of the error between the DG solution and a particular projection of the exact solution can achieve $k + \frac{3}{2}$ when upwind fluxes were used. Compared with the linear case, where the highest superconvergence rate can reach $2k + 1$ (see, e.g., [7, 6]), the superconvergence rate $k + \frac{3}{2}$ is far from optimal. Furthermore, to our best knowledge, no superconvergence results of DG methods applied to nonlinear hyperbolic equations with sonic points (i.e., points at which $f'(u) = 0$) are available in the literature.

The main purpose of this paper is to study superconvergence properties of DG methods for (1) with general flux functions. If the wind direction is fixed, our analysis indicates a $2k + 1$-th order convergence rate of the numerical flux at mesh nodes and for the cell average, and a $k + 2$-th order of the error between the DG solution and the Gauss-Radau projection of the exact solution as well as the function value error at interior right Radau points, and a $k + 1$-th order of the derivative error at interior left Radau points. As we may recall, these superconvergence results are the same as for non-degenerate linear hyperbolic problems (see, [7, 6]). While if the wind direction is changing in the computational domain, we proved that the superconvergence phenomenon still exists and the convergence rate may depend upon the specific property of the flux functions. Specifically, we first prove that the DG flux function is superconvergent towards a particular flux function of the exact solution, with an error bound $O(h^{k+2})$, when Godunov fluxes are used. Then we establish the supercloseness result between the DG solution itself and a particular projection of the exact solution, and reveal that the highest superconvergence rate for nonlinear hyperbolic equations with sonic points is $O(h^{k+\frac{3}{2}})$. As byproducts, we obtain superconvergence properties for the DG solution and the DG flux function at special points and for cell averages. These superconvergence results are similar to linear problems with degenerate variable coefficients, see [5].

The contribution of this paper is to provide a symmetric method to study the superconvergence behavior of the DG methods for nonlinear problems. On the one hand, we establish the superconvergence results for the monotone flux, and prove that all the superconvergence results for the linear case in [7, 6] still hold true for nonlinear problems. Furthermore, the superconvergence results established in this paper improve those of [20] to the possible optimal superconvergence rates. On the other hand, we uncover the superconvergence phenomenon of the DG methods for (1) with the degenerate flux function, and extend the superconvergence result for linear problems to a more general nonlinear case. By doing so, we present a full picture for the superconvergence properties of DG methods applied to the (degenerate) nonlinear hyperbolic problems, and enrich the superconvergence theory of the DG method for linear hyperbolic equations in one dimension.

To end the introduction, we would like to emphasize that the main difficulty in the superconvergence analysis for nonlinear problems is how to deal with the nonlinear terms in the error equation. Our analysis is along the following lines: we first use a Taylor expansion to linearize the error equation; subsequently, we make an a priori error assumption to deal with the nonlinearity of the flux and other high-order terms in the linearization, then the superconvergence analysis for nonlinear problems is reduced to a linear one; finally, we introduce a correction function to deal with the linear part to obtain higher-order accuracy.

The rest of the paper is organized as follows. In Section 2, we present DG schemes for the one-dimensional nonlinear hyperbolic conservation laws, and discuss the choice of numerical fluxes. Section 3 is dedicated to the superconvergence analysis of the DG methods for problems with fixed wind direction. Superconvergence results are established by using the idea of correction function, Taylor expansion, and an a priori error assumption. In Section 4, we study the superconvergence behavior of the DG approximation for problems with sonic points, and prove that the DG flux function is superconvergent with an order of $O(h^{k+2})$ to a particular flux function of the exact solution. Superconvergence behavior of the DG solution itself is also discussed in this case. We reveal a very important fact that the superconvergence phenomena are also valid for general fluxes, and the superconvergence rates may depend upon specific properties of the flux functions. In Section 5, we provide numerical examples to support our theoretical findings. Finally, concluding remarks and some possible future works are presented in Section 6.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index $D$; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$. Notation $A \lesssim B$ implies that $A$ can be bounded by $B$ multiplied by a constant independent of the mesh size $h$.

**2. DG schemes and the energy inequality.** Let $\Omega = [0, 2\pi]$ and $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}}$ be $N+1$ distinct points on the interval $\Omega$. For all positive integers $r$, we define $\mathbb{Z}_r = \{1, \ldots, r\}$, and denote by

$$\tau_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}),\ j \in \mathbb{Z}_N$$

the cells and cell centers, respectively. Let $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $\overline{h}_j = h_j/2$ and $h = \max_j h_j$. We assume that the mesh is regular, i.e., the ratio between the maximum and minimum mesh sizes shall stay bounded during mesh refinements.

Define the discontinuous finite element space

$$V_h = \{v : \ v|_{\tau_j} \in \mathbb{P}_k(\tau_j),\ j \in \mathbb{Z}_N\},$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree at most $k$ with coefficients as functions of $t$. The DG scheme for (1) reads as: Find $u_h \in V_h$ such that for any $v \in V_h$

$$((u_h)_t, v)_j - (f(u_h), v_x)_j + \hat{f}(u_h)v^-\big|_{j+\frac{1}{2}} - \hat{f}(u_h)v^+\big|_{j-\frac{1}{2}} = 0, \quad j \in \mathbb{Z}_N, \tag{2}$$

where $(u, v)_j = \int_{\tau_j} uvdx$, $v^-_{j+\frac{1}{2}}$ and $v^+_{j+\frac{1}{2}}$ separately denote the left and right limits of $v$ at the point $x_{j+\frac{1}{2}}$, and $\hat{f}(u_h)$ is the numerical flux, which is a single-valued function defined at each cell interface and in general depends on the values of the numerical solution $u_h$ from both sides of the interface, i.e.,

$$\hat{f}(u_h)\big|_{j+\frac{1}{2}} = \hat{f}(u_h(x^-_{j+\frac{1}{2}}), u_h(x^+_{j+\frac{1}{2}})), \quad \forall j \in \mathbb{Z}_N.$$

The choice of the numerical flux is of great significance in assuring the stability of the DG scheme. If $f'(u) \geq \delta > 0$, for all $x \in \Omega$, we define the numerical flux as

$$\hat{f}(u_h) = f(u_h^-).$$

Similarly, if $f'(u) \le -\delta < 0$, we choose

$$\hat{f}(u_h) = f(u_h^+).$$

However, if $f'(u)$ changes its sign in the computational domain, we take the Godunov flux as our numerical flux, i.e.,

$$\hat{f}(u_h) = \hat{f}(u_h^-, u_h^+) = \begin{cases} \min_{u_h^- \le \mu \le u_h^+} f(\mu), & \text{if} \quad u_h^- < u_h^+, \\ \max_{u_h^+ \le \mu \le u_h^-} f(\mu), & \text{if} \quad u_h^- \ge u_h^+. \end{cases} \tag{3}$$

Note that the upwind flux for problems with alternating wind direction is defined as follows.

$$\tilde{f}(u_h) = \begin{cases} f(u_h^-), & \text{if} \quad f'(u) > 0, \\ f(u_h^+) & \text{if} \quad f'(u) \le 0. \end{cases} \tag{4}$$

The above flux is an essential part in our later energy estimate and error analysis for problems with alternating wind direction.

Throughout this paper, we will use the following notation

$$e = u - u_h, \quad \xi = u_I - u_h, \quad \eta = u - u_I. \tag{5}$$

Here $u_I$ is some special interpolation function of $u$ to be determined.

To end with this section, we would like to present the following energy inequality, which will be frequently used in our later superconvergence analysis.

THEOREM 1. *Let $u$ be the solution of (1) and $u_I \in V_h$ be some special interpolation function of $u$. Assume that $u_h$ is the solution of (2) with the numerical fluxes chosen as the upwind flux (for fixed wind directions) or Godunov flux (for alternating wind directions) . Then for both numerical fluxes,*

$$\frac{1}{2}\frac{d}{dt}\|\xi\|_0^2 \lesssim h^{-2}\|e\|_{0,\infty}^2\|\eta\|_0^2 + (1 + h^{-1}\|e\|_{0,\infty})\|\xi\|_0^2 + h^{-1}\|e\|_{0,\infty}^2\sum_{j=1}^N(\tilde{\eta}_{j+\frac{1}{2}})^2 + |I| + |J|, \tag{6}$$

*where $\tilde{\eta}$ is the numerical flux defined in (4) with $f(u_h)$ replaced by $\eta$, and*

$$I = \sum_{j=1}^N (\tilde{f}(u_h) - \hat{f}(u_h))[\xi]\big|_{j+\frac{1}{2}}, \tag{7}$$

$$J = -(\eta_t, \xi) + (f'(u)\eta, \xi_x) + \sum_{j=1}^N \left(f'(u)\tilde{\eta}[\xi]\right)\big|_{j+\frac{1}{2}}. \tag{8}$$

*Proof.* Since the exact solution also satisfies (2), we have for all $v \in V_h$

$$(e_t, v)_j - (f(u) - f(u_h), v_x)_j + (f(u) - \hat{f}(u_h))v^-\big|_{j+\frac{1}{2}} - (f(u) - \hat{f}(u_h))v^+\big|_{j-\frac{1}{2}} = 0. \tag{9}$$

Summing up over $j$ from 1 to $N$ and using the periodic boundary condition, we obtain

$$(e_t, v) = (f(u) - f(u_h), v_x) + \sum_{j=1}^N \Big((f(u) - \hat{f}(u_h))[v]\Big)\big|_{j+\frac{1}{2}},$$

where $(u,v)=\sum_{j=1}^N (u,v)_j$ and $[v]_{j+\frac12}=v^+_{j+\frac12}-v^-_{j+\frac12}$ denotes the jump of $v$ at the point $x_{j+\frac12}$. Now we take $v=\xi$ in the above identity to get

$$(\xi_t,\xi)=-(\eta_t,\xi)+(f(u)-f(u_h),\xi_x)+\sum_{j=1}^N\Big((f(u)-\tilde f(u_h))[\xi]\Big)\big|_{j+\frac12}+I. \tag{10}$$

By the Taylor expansion of $f$ about $u$, we have

$$\begin{aligned} f(u)-f(u_h)&=f'(u)\xi+f'(u)\eta-\frac12\bar f''_u(\xi+\eta)^2,\\ f(u)-\tilde f(u_h)&=f'(u)\tilde\xi+f'(u)\tilde\eta-\frac12\bar{\bar f}''_u(\tilde\xi+\tilde\eta)^2, \end{aligned} \tag{11}$$

where $\bar f''_u=f''(\alpha_1u+(1-\alpha_1)u_h)$ and $\bar{\bar f}''_u=f''(\alpha_2u+(1-\alpha_2)u_h)$ with $0\le\alpha_1,\alpha_2\le 1$. Then (10) can be rewritten as

$$(\xi_t,\xi)=I_1-\frac{I_2}{2}+I+J, \tag{12}$$

where

$$I_1=(f'(u)\xi,\xi_x)+\sum_{j=1}^N\Big(f'(u)\tilde\xi[\xi]\Big)\big|_{j+\frac12},\quad I_2=(\bar f''_u(\xi+\eta)^2,\xi_x)+\sum_{j=1}^N\Big(\bar{\bar f}''_u(\tilde\xi+\tilde\eta)^2[\xi]\Big)\big|_{j+\frac12}.$$

Let

$$\Omega_1=\{j\in\mathbb{Z}_N: f'_{j+\frac12}\ge 0\},\quad \Omega_2=\{j\in\mathbb{Z}_N: f'_{j+\frac12}<0\}, \tag{13}$$

where $f'_{j+\frac12}=f'(u(x_{j+\frac12},t))$. By a simple integration by parts, we have

$$\begin{aligned} I_1&=-\frac12(\partial_x f'(u),\xi^2)-\sum_{j=1}^N(f'(u)\{\xi\}[\xi])|_{j+\frac12}+\sum_{j\in\Omega_1}(f'(u)\xi^-[\xi])|_{j+\frac12}+\sum_{j\in\Omega_2}(f'(u)\xi^+[\xi])|_{j+\frac12}\\ &=-\frac12(\partial_x f'(u),\xi^2)-\sum_{j\in\Omega_1}(f'(u)[\xi]^2)|_{j+\frac12}+\sum_{j\in\Omega_2}(f'(u)[\xi]^2)|_{j+\frac12}\le-\frac12(\partial_x f'(u),\xi^2)\le C\|\xi\|_0^2. \end{aligned}$$

Here and in the following, $C$ is a positive constant independent of the mesh size $h$, and is not necessary to be the same at every occurrence. As for $I_2$, we have from Cauchy-Schwarz inequality

$$\begin{aligned} |I_2|&\le C\|e\|_{0,\infty}\left(\|e\|_0\|\xi_x\|_0+\|\xi\|^2_{\Gamma_h}+\sum_{j=1}^N\left|\tilde\eta_{j+\frac12}[\xi]_{j+\frac12}\right|\right)\\ &\lesssim h^{-1}\|e\|_{0,\infty}\left(\|e\|_0\|\xi\|_0+\|\xi\|_0^2\right)+\|e\|_{0,\infty}\sum_{j=1}^N\left|\tilde\eta_{j+\frac12}[\xi]_{j+\frac12}\right|\\ &\lesssim h^{-2}\|e\|^2_{0,\infty}\|\eta\|_0^2+(1+h^{-1}\|e\|_{0,\infty})\|\xi\|_0^2+h^{-1}\|e\|^2_{0,\infty}\sum_{j=1}^N(\tilde\eta_{j+\frac12})^2, \end{aligned}$$

where for any given $v$,

$$\|v\|_{\Gamma_h} = \left( \sum_{j=1}^{N} (v^-_{j+\frac{1}{2}})^2 + (v^+_{j+\frac{1}{2}})^2 \right)^{\frac{1}{2}},$$

and in the second and last steps, we have used the inverse inequality

$$\|v_x\|_0 \le Ch^{-1}\|v\|_0, \quad \|v\|_{\Gamma_h} \le Ch^{-\frac{1}{2}}\|v\|_0, \quad \|v\|_{0,\infty,\tau_j} \le Ch^{-\frac{1}{2}}\|v\|_{0,\tau_j}, \quad \forall v \in V_h.$$

Then (6) follows by substituting the estimates of $I_1$ and $I_2$ into (12). The proof is complete. ∎

**3. Analysis for problems with fixed wind direction.** In this section, we study the superconvergence of DG methods for problems with fixed wind direction. Without loss of generality, we consider the case $f'(u) \ge \delta > 0$. The same arguments can be applied to the case with $f'(u) \le -\delta < 0$.

To study the superconvergence properties of the DG solution, our analysis is along this line: We first construct a special interpolation function $u_I \in V_h$ of the exact solution such that the DG solution $u_h$ is super-close to $u_I$ under $L^2$ norm; and then we analyze the superconvergence behavior of the interpolation function $u_I$; finally, the superconvergence for the DG solution is reduced to the superconvergence of the interpolation function $u_I$ due to the super-closeness between $u_h$ and $u_I$. Therefore, our goal here is to design the special function $u_I$.

Note that

$$\|\xi(\cdot,t)\|_0^2 = \|\xi(\cdot,0)\|_0^2 + \int_0^t \frac{d}{dt}\|\xi\|_0^2 dt, \quad \forall t \in (0,T],$$

then the convergence rate of $\|\xi\|_0$ depends upon the term $\frac{d}{dt}\|\xi\|_0^2$. In light of (6) and the fact that $I$ in (6) vanishes for fixed wind directions, to achieve our superconvergence goal, $u_I$ should be specially designed such that $J$ in (6) is of high order. To this end, we begin with some preliminaries.

**3.1. Preliminaries.** First, we denote by $L_n$ and $L_{j,n}$ the traditional Legendre polynomials of degree $n$ in the intervals $[-1,1]$ and $\tau_j, j \in \mathbb{Z}_N$, respectively. For any $x \in \tau_j$, let

$$(14) \qquad \omega(x,t) = f'(u(x,t)), \quad \bar{\omega}_j(s,t) = \omega(x_j + \bar{h}_j s, t) = \omega(x,t), \quad s \in [-1,1].$$

Note that for any fixed $t$ and $j$, where $j \in \mathbb{Z}_N$, $\bar{\omega}_j \ge \delta > 0 \in L^1(\tau_0), \tau_0 = [-1,1]$, there exists a series of monic orthogonal polynomials $\{\bar{\phi}_{j,n}\}_{n=1}^{\infty}$ with respect to the weight function $\bar{\omega}_j$, i.e.,

$$(15) \qquad (\bar{\phi}_{j,n}, \bar{\phi}_{j,m})_{\bar{\omega}_j} := \int_{-1}^{1} \bar{\omega}_j \bar{\phi}_{j,n} \bar{\phi}_{j,m} ds = \gamma_n \delta_{mn},$$

where $\gamma_n = (\bar{\phi}_{j,n}, \bar{\phi}_{j,n})_{\bar{\omega}_j}$ and $\delta_{mn}$ is the Kronecker delta. Moreover, $\bar{\phi}_{j,n}$ can be constructed by the following three-term recurrence formula

$$(16) \qquad \bar{\phi}_{j,0} = 1, \; \bar{\phi}_{j,1} = s - \alpha_0, \quad \bar{\phi}_{j,n+1} = (s - \alpha_n)\bar{\phi}_{j,n} - \beta_n \bar{\phi}_{j,n-1}, n \ge 1,$$

where for all $n \ge 0, m \ge 1$,

$$(17) \quad \alpha_n = (s\bar{\phi}_{j,n}, \bar{\phi}_{j,n})_{\bar{\omega}_j} / (\bar{\phi}_{j,n}, \bar{\phi}_{j,n})_{\bar{\omega}_j}, \quad \beta_m = (\bar{\phi}_{j,m}, \bar{\phi}_{j,m})_{\bar{\omega}_j} / (\bar{\phi}_{j,m-1} \bar{\phi}_{j,m-1})_{\bar{\omega}_j}.$$

With the orthogonal polynomials $\{\bar{\phi}_{j,n}\}$ on $[-1,1]$, we can construct a set of orthogonal polynomials $\phi_{j,n}$ in each element $\tau_j$ as follows.

$$\phi_{j,n}(x,t) = \bar{\phi}_{j,n}(s,t) = \bar{\phi}_{j,n}\big(\frac{2(x-x_j)}{h_j},t\big), \quad x \in \tau_j.$$

Moreover, there holds the following orthogonal property

$$\int_{\tau_j} (\phi_{j,n}\phi_{j,m}\omega)(x,t)dx = \frac{h_j}{2}\int_{-1}^{1}(\bar{\phi}_{j,n}\bar{\phi}_{j,m}\bar{\omega}_j)(s,t)ds = \frac{h_j}{2}\gamma_n\delta_{mn}. \tag{18}$$

That is, $\{\phi_{j,n}\}$ are orthogonal polynomials with respect to the weight function $\omega = f'(u)$ in $\tau_j$.

LEMMA 2. *For any $j \in \mathbb{Z}_N$, let $\{\phi_{j,n}\}$ be a sequence of orthogonal polynomials with respect to the weight function $\omega = f'(u)$ in $\tau_j$, and $f'(u)$ be a sufficiently smooth function satisfying $|\partial_t^m f'(u)| \lesssim 1, m \le k+1$. Then*

$$\|\phi_{j,n}\|_{0,\tau_j} \lesssim h^{\frac{1}{2}}, \quad \|\phi_{j,n}\|_{0,\infty,\tau_j} \lesssim h^{-\frac{1}{2}}\|\phi_{j,n}\|_{0,\tau_j} \lesssim 1. \tag{19}$$

*Moreover, for all positive $m$, where $1 \le m \le k+1$, there holds*

$$\|\partial_t^m \phi_{j,n}\|_{0,\infty} \lesssim 1. \tag{20}$$

*Proof.* First, (19) can be easily verified by a direct calculation. To show (20), it is sufficient to prove

$$|\partial_t^m \bar{\phi}_{j,n}| \lesssim 1, \quad 1 \le m \le k+1. \tag{21}$$

We will prove (21) by induction. For any integer $m \ge 1$, we have, from (16) and the Newton-Leibniz formula of derivative,

$$\partial_t^m \bar{\phi}_{j,1} = -\partial_t^m \alpha_0 = -\partial_t^m \left(\frac{\int_{-1}^1 s\bar{\omega}_j ds}{\int_{-1}^1 \bar{\omega}_j ds}\right) \lesssim \frac{\sum_{r=0}^m \|\partial_t^r \bar{\omega}_j\|_{0,\infty}}{\left|\int_{-1}^1 \bar{\omega}_j ds\right|^{m+1}} \lesssim 1.$$

Then (21) holds for $n = 1$. Now we suppose (21) is valid for all $n$ and prove it also holds for $n+1$. Actually, recall the three-term recurrence formula of $\bar{\phi}_{j,n}$ in (16), we easily get

$$|\partial_t^m \bar{\phi}_{j,n+1}| \lesssim (|\partial_t^m \alpha_n| + |\partial_t^m \beta_n|)(|\partial_t^m \bar{\phi}_{j,n}| + |\partial_t^m \bar{\phi}_{j,n-1}|) \lesssim (|\partial_t^m \alpha_n| + |\partial_t^m \beta_n|).$$

Again, we use the Newton-Leibniz formula of derivative to obtain

$$(|\partial_t^m \alpha_n| + |\partial_t^m \beta_n|) \lesssim \frac{\sum_{r=0}^m |\partial_t^r \bar{\phi}_{j,n}|}{(\bar{\phi}_{j,n},\bar{\phi}_{j,n})_{\bar{\omega}_j}^{m+1}} + \frac{\sum_{r=0}^m (|\partial_t^r \bar{\phi}_{j,n}| + |\partial_t^r \bar{\phi}_{j,n-1}|)}{(\bar{\phi}_{j,n-1},\bar{\phi}_{j,n-1})_{\bar{\omega}_j}^{m+1}} \lesssim 1,$$

and thus,

$$|\partial_t^m \bar{\phi}_{j,n+1}| \lesssim 1.$$

Then (21) is also valid for $n+1$, and this finishes our proof. ∎

Let

$$H_h^1 = \{v: \ v|_{\tau_j} \in H^1(\tau_j), \ j \in \mathbb{Z}_N\},$$

and for all $\psi \in H_h^1$, we denote by $P_h^- \psi \in V_h$ the traditional Gauss-Radau projection of $\psi$, which is defined by

$$(P_h^- \psi, v)_j = (\psi, v)_j, \quad \forall v \in \mathbb{P}_{k-1}, \quad P_h^- \psi(x_{j+\frac{1}{2}}^-) = \psi_{j+\frac{1}{2}}^-, \quad j \in \mathbb{Z}_N. \tag{22}$$

Note that the Gauss-Radau projection $P_h^-$ is frequently used in the DG error analysis. Now we extend the definition of Gauss-Radau projection to a more general one. Given a positive function $\omega$, we define a projection $P_h : H_h^1 \to V_h$ by

$$\int_{\tau_j} \omega P_h \psi v dx = \int_{\tau_j} \omega \psi v dx, \quad \forall v \in \mathbb{P}_{k-1}, \quad P_h \psi(x_{j+\frac{1}{2}}^-) = \psi(x_{j+\frac{1}{2}}^-). \tag{23}$$

Note that $P_h$ is reduced to the traditional Gauss-Radau projection $P_h^-$ when $\omega = 1$. Furthermore, we have the following approximation properties for the projection $P_h$.

LEMMA 3. *Let $\omega = f'(u) \geq \delta > 0$. Then the projection $P_h$ in (23) is well-defined. Moreover, if $\psi \in W^{k+2,\infty}(\Omega)$ and $|\partial_x^n f'(u)| \lesssim 1, n \leq k$, there holds the following results.*

- *$P_h \psi$ is super-close to the Gauss-Radau projection $P_h^- \psi$, i.e.,*

$$\|\partial_t^m (P_h \psi - P_h^- \psi)\|_{0,\infty,\tau_j} \lesssim h^{k+2} \|\partial_t^m \psi\|_{k+1,\infty}, \quad m = 0, 1. \tag{24}$$

- *The function value approximation of $P_h \psi$ is superconvergent at the interior right Radau points $r_{j,m}, 1 \leq m \leq k+1$ (zeros of the right Radau polynomial $L_{j,k+1} - L_{j,k}$ except the point $x_{j+\frac{1}{2}}$), namely,*

$$(P_h \psi - \psi)(r_{j,m}) \lesssim h^{k+2} \|\psi\|_{k+2,\infty}, \quad m \in \mathbb{Z}_k. \tag{25}$$

- *The derivative value approximation of $P_h \psi$ is superconvergent at the interior left Radau points $l_{j,m}, m \in \mathbb{Z}_k$ (zeros of left Radau polynomial $L_{j,k+1} + L_{j,k}$ except the point $x = x_{j-\frac{1}{2}}$ ), i.e.,*

$$\partial_x (P_h \psi - \psi)(l_{j,m}) \lesssim h^{k+1} \|\psi\|_{k+2,\infty}, \quad m \in \mathbb{Z}_k. \tag{26}$$

- *The cell average of $P_h \psi$ in each element $\tau_j$ is superconvergent with an order of $2k+1$, i.e.,*

$$\left| \frac{1}{h_j} \int_{\tau_j} (\psi - P_h \psi)(x) dx \right| \lesssim h^{2k+1} \|\psi\|_{k+1,\tau_j}. \tag{27}$$

*Proof.* Since $P_h \psi \in V_h$, we express in each element $\tau_j$

$$P_h \psi \big|_{\tau_j} = \sum_{n=0}^{k} a_n \phi_{j,n}(x),$$

where $\{a_n\}_0^k$ are constants to be determined. Recall the definition of $P_h$ and the orthogonality of $\{\phi_{j,n}\}$ in (18), we easily obtain

$$a_n = \frac{\int_{\tau_j} \omega \psi \phi_{j,n} dx}{\int_{\tau_j} \omega \phi_{j,n} \phi_{j,n} dx} = \frac{2}{h_j \gamma_n} \int_{\tau_j} \omega \psi \phi_{j,n} dx, \quad n \leq k-1,$$

where $\gamma_n$ is the same as in (15). Moreover, noticing that the zeros of the orthogonal polynomials $\{\bar{\phi}_{j,n}\}$ are all real, simple, and lies in the interval $(-1,1)$ (see [22], Theorem 3.2), we have $\bar{\phi}_{j,n}(1)\neq 0$, or equivalently, $\phi_{j,n}(x^-_{j+\frac{1}{2}})\neq 0$, which yields, together with the second identity of (23),

$$a_k = \frac{1}{\phi_{j,k}(x^-_{j+\frac{1}{2}})}\left(\psi(x^-_{j+\frac{1}{2}}) - \sum_{n=0}^{k-1} a_n\phi_{j,n}(x^-_{j+\frac{1}{2}})\right) = \frac{1}{\bar{\phi}_{j,k}(1)}\left(\psi(x^-_{j+\frac{1}{2}}) - \sum_{n=0}^{k-1} a_n\bar{\phi}_{j,n}(1)\right).$$

Therefore, the projection $P_h\psi$ for any function $\psi\in H^1_h$ is uniquely determined.

Now we assume $P_h\psi - P_h^-\psi$ has the following expression in $\tau_j$

$$(P_h\psi - P_h^-\psi)(x,t) = \sum_{n=0}^{k} b_n\phi_{j,n}(x,t),\quad x\in\tau_j.$$

Recalling the definitions of $P_h$ and $P_h^-$ in (22) -(23), we have

$$b_n = \frac{\int_{\tau_j}\omega(P_h\psi - P_h^-\psi)\phi_{j,n}dx}{\int_{\tau_j}\omega\phi_{j,n}\phi_{j,n}dx} = \frac{2}{h_j\gamma_n}\int_{\tau_j}\omega(\psi - P_h^-\psi)\phi_{j,n}dx$$

for all $n\le k-1$, and

$$b_k = -\sum_{n=0}^{k-1} b_n\frac{\bar{\phi}_{j,n}(1)}{\bar{\phi}_{j,k}(1)}.$$

Noticing that $(\psi - P_h^-\psi)\perp\mathbb{P}_{k-1}$, there holds for all $n\le k-1$,

$$\begin{aligned}|b_n| &= \frac{2}{h_j\gamma_n}\left|\int_{\tau_j}(\omega-\omega(x_j))(\psi - P_h^-\psi)\phi_{j,n}dx\right|\\ &\lesssim \|\omega-\omega(x_j)\|_{0,\infty,\tau_j}\|\psi - P_h^-\psi\|_{0,\infty,\tau_j}\|\phi_{j,n}\|_{0,\infty,\tau_j}\lesssim h\|\omega\|_{1,\infty,\tau_j}\|\psi - P_h^-\psi\|_{0,\infty,\tau_j},\end{aligned}$$

where in the last step we have used (19). Consequently,

$$|b_k|\lesssim\sum_{n=0}^{k-1}|b_n|\lesssim h\|\omega\|_{1,\infty,\tau_j}\|\psi - P_h^-\psi\|_{0,\infty,\tau_j},$$

and thus,

$$\|P_h\psi - P_h^-\psi\|_{0,\infty,\tau_j}\lesssim\sum_{n=0}^{k}|b_n|\lesssim h\|\psi - P_h^-\psi\|_{0,\infty,\tau_j}. \tag{28}$$

Following the same line, we can prove

$$\|\partial_t^m(P_h\psi - P_h^-\psi)\|_{0,\infty,\tau_j}\lesssim h\|\partial_t^m(\psi - P_h^-\psi)\|_{0,\infty,\tau_j},\quad m\le k+1. \tag{29}$$

Then (24) follows from the standard approximation property of $P_h^-$ (see, e.g., [21]). Thanks to the super-closeness result (24), together with the superconvergence result for $P_h^-$ (see, [7])

$$\left|(\psi - P_h^-\psi)(r_{j,i})\right|\lesssim h^{k+2}\|\psi\|_{k+2,\infty},\quad \left|\partial_x(\psi - P_h^-\psi)(l_{j,i})\right|\lesssim h^{k+1}\|\psi\|_{k+2,\infty},$$

the desired results (25)-(26) follow immediately.

Now we prove (27). On the one hand, using the orthogonality of $\phi_{j,n}$, we get

$$\int_{\tau_j} \phi_{j,n} dx = \int_{\tau_j} \omega^{-1}\omega\phi_{j,n} dx = \int_{\tau_j} (\omega^{-1} - I_{n-1}\omega^{-1})\omega\phi_{j,n} dx,$$

where $I_n v \in \mathbb{P}_n$ is some interpolation function of $v$. Since $\omega = f'(u)$ is sufficiently smooth, i.e., $|\partial_x^n \omega| \lesssim 1$, we have

$$\|\partial_x^n \omega^{-1}\|_{0,\infty,\tau_j} \lesssim \frac{\|\omega\|_{n,\infty,\tau_j}}{\|\omega\|_{0,\infty,\tau_j}^{n+1}} \lesssim 1.$$

Then

$$\left|\int_{\tau_j} \phi_{j,n} dx\right| \lesssim h^{n+1}\|\partial_x^n \omega^{-1}\|_{0,\infty,\tau_j} \lesssim h^{n+1}.$$

On the other hand, by the orthogonality of $P_h^-$, that is, $(\psi - P_h^-\psi)\perp\mathbb{P}_{k-1}$, we have

$$\begin{aligned}|b_n| &= \left|\int_{\tau_j} \omega(\psi - P_h^-\psi)\phi_{j,n} dx\right| = \left|\int_{\tau_j} (\omega - I_{k-1-n}\omega)(\psi - P_h^-\psi)\phi_{j,n} dx\right| \\ &\lesssim h^{2k+1-n}\|\psi\|_{k+1,\tau_j}\|\partial_x^{k-n}\omega\|_{0,\tau_j} \lesssim h^{2k+1-n}\|\psi\|_{k+1,\tau_j}.\end{aligned}$$

Noticing that

$$\int_{\tau_j} (\psi - P_h\psi) dx = \int_{\tau_j} (P_h^-\psi - P_h\psi) dx = -\sum_{n=0}^{k} b_n \int_{\tau_j} \phi_{j,n} dx,$$

then

$$\left|\int_{\tau_j} (\psi - P_h\psi) dx\right| \lesssim h^{2k+2}\|\psi\|_{k+1,\tau_j},$$

which yields (27) directly. The proof is complete. □

To end this subsection, we introduce a special operator $\mathcal{L}$, which will be used in later analysis. For all $\psi \in H_h^1$, we define the operator $\mathcal{L} : H_h^1 \to V_h$ by

$$(f'(u)\mathcal{L}(\psi), v_x)_j = (\psi, v)_j, \quad \forall v \in \mathbb{P}_k(\tau_j) \setminus \mathbb{P}_0(\tau_j), \tag{30}$$

$$\mathcal{L}(\psi)(x_{j+\frac{1}{2}}^-) = 0, \quad \forall j \in \mathbb{Z}_N. \tag{31}$$

Noticing that $v_x \in \mathbb{P}_{k-1}(\tau_j)$ for all $v \in \mathbb{P}_k(\tau_j) \setminus \mathbb{P}_0(\tau_j)$, we have from the definition of $P_h$

$$(f'(u)P_h\psi, v_x)_j = (\omega P_h\psi, v_x)_j = (\omega\psi, v_x)_j, \quad v \in \mathbb{P}_k(\tau_j) \setminus \mathbb{P}_0(\tau_j).$$

Then the definition of $\mathcal{L}$ is similar to that of $P_h$ except the term in the right-hand side (with $\omega v_x$ replaced by $v$). Therefore, we can similarly prove that the operator $\mathcal{L}$ is well-defined.

**3.2. Construction of $u_I$ and its approximation properties.** We first construct a series of functions starting with $w_0 = u - P_h u$ and define

$$w_{i+1} = \mathcal{L}(\partial_t w_i), \quad 1 \le i \le k. \tag{32}$$

We have the following properties.

LEMMA 4. *Let $\omega = f'(u) \geq \delta > 0$ be a sufficiently smooth function satisfying*

$$|\partial_x^m f'(u)| \lesssim 1 \quad |\partial_t^m f'(u)| \lesssim 1, \quad m \leq k+1. \tag{33}$$

*In each element $\tau_j$, we express $w_i$'s in terms of orthogonal basis $\{\phi_{j,n}\}$ as*

$$w_i|_{\tau_j} = \sum_{n=0}^{k} c_{i,n}\phi_{j,n}.$$

*Then the coefficient $c_{i,n}$ satisfies*

$$|\partial_t^m c_{i,n}| \lesssim h^{\max(k+1+i,2k+1-n)} \|\partial_t^{m+i} u\|_{k+1,\infty}, \quad n \leq k, \quad m = 0, 1. \tag{34}$$

*Consequently, there hold for $m = 0, 1$,*

$$\|\partial_t^m w_i\|_{0,\infty,\tau_j} \lesssim h^{k+i+1} \|\partial_t^{m+i} u\|_{k+1,\infty}, \qquad \left| \frac{1}{h_j} \int_{\tau_j} \partial_t^m w_i dx \right| \lesssim h^{2k+1} \|\partial_t^{i+m} u\|_{k+1,\infty}. \tag{35}$$

*Proof.* We only prove (34)-(35) for $m = 0$ since the similar argument can be applied to $m = 1$. We first prove (34) by induction. As $v \in \mathbb{P}_k \setminus \mathbb{P}_0$, we have $v_x \in \mathbb{P}_{k-1}$. Then we choose $v_x = \phi_{j,n}, n = 0, \ldots, k-1$ in (30) to obtain

$$c_{i+1,n} = \frac{\int_{\tau_j} \partial_t w_i D_x^{-1} \phi_{j,n} dx}{\int_{\tau_j} \omega \phi_{j,n} \phi_{j,n} dx} = \frac{2}{h_j \gamma_n} \int_{\tau_j} \partial_t w_i D_x^{-1} \phi_{j,n} dx, \quad n \leq k-1, \tag{36}$$

$$c_{i+1,k} = -\sum_{n=0}^{k-1} c_{i+1,n} \frac{\bar{\phi}_{j,n}(1)}{\bar{\phi}_{j,k}(1)}, \tag{37}$$

where $\gamma_n$ is given by (15), and $D_x^{-1} v$ is a function defined by

$$D_x^{-1} v\big|_{\tau_j} = \int_{x_{j-\frac{1}{2}}}^{x} v(x) dx. \tag{38}$$

Since $D_x^{-1}\phi_{j,n} \in \mathbb{P}_{n+1}(\tau_j)$, we have, from the orthogonality of $P_h$,

$$\begin{aligned} c_{1,n} &= \frac{2}{h_j \gamma_n} \int_{\tau_j} \omega^{-1} \omega \partial_t (u - P_h u) D_x^{-1} \phi_{j,n} dx \\ &= \frac{2}{h_j \gamma_n} \int_{\tau_j} \left(\omega^{-1} - I_{k-2-n}\omega^{-1}\right) \omega \partial_t (u - P_h u) D_x^{-1} \phi_{j,n} dx. \end{aligned}$$

Here again $I_m \omega \in \mathbb{P}_m$ denotes the interpolation function of $\omega$. By (33) and standard approximation theory,

$$\|\omega^{-1} - I_{k-2-n}\omega^{-1}\|_{0,\infty,\tau_j} \lesssim h^{k-1-n} \|\partial_x^{k-1-n} \omega^{-1}\|_{0,\infty} \lesssim h^{k-1-n},$$

which yields, together with (19) and (24),

$$\begin{aligned} &|c_{1,n}| \lesssim h^{k-n-1} \|\partial_t (u - P_h u)\|_{0,\infty,\tau_j} \|D_x^{-1} \phi_{j,n}\|_{0,\infty,\tau_j} \\ &\lesssim h^{k-n} (\|\partial_t (u - P_h^- u)\|_{0,\infty} + |\partial_t (P_h u - P_h^- u)\|_{0,\infty}) \|\phi_{j,n}\|_{0,\infty} \lesssim h^{2k+1-n} \|\partial_t u\|_{k+1,\infty} \end{aligned}$$

for all $n \le k-1$, and thus,

$$|c_{1,k}| \lesssim \sum_{n=0}^{k-1} |c_{1,n}| \lesssim h^{k+2} \|\partial_t u\|_{k+1,\infty}.$$

Then (34) holds for $i = 1$. Now we suppose (34) holds for $i$ and prove that it is also valid for $i+1$. Actually, we have for all $n \le k-1$, by (36) and the induction hypothesis

$$\begin{aligned} |c_{i+1,n}| &= \frac{2}{h_j \gamma_n} \left| \sum_{r=0}^{n+1} \partial_t c_{i,r} \int_{\tau_j} \phi_{j,r} D_x^{-1} \phi_{j,n} dx + \sum_{r=n+2}^{k} \partial_t c_{i,r} \int_{\tau_j} \phi_{j,r} D_x^{-1} \phi_{j,n} dx \right| \\ &\lesssim h \sum_{r=0}^{n+1} |\partial_t c_{i,r}| + \sum_{r=n+2}^{k} h^{r-n} |\partial_t c_{i,r}| \lesssim h^{\max(k+i+2, 2k+1-n)} \|\partial_t^{i+1} u\|_{k+1,\infty}, \end{aligned}$$

where in the second step, we have used the orthogonality of $\phi_{j,r}$, which yields

$$\int_{\tau_j} \phi_{j,r} v dx = \int_{\tau_j} (\omega^{-1} - I_{r-n-2}\omega^{-1}) \omega \phi_{j,r} v dx \lesssim h^{r-n} \|v\|_{0,\infty,\tau_j}, \ \ \forall v \in \mathbb{P}_{n+1}, \ \ r \ge n+2.$$

Then

$$|c_{i+1,k}| \lesssim \sum_{n=0}^{k-1} |c_{i+1,n}| \lesssim h^{k+2+i} \|\partial_t^{i+1} u\|_{k+1,\infty}.$$

Consequently, (34) holds for $i+1$, and thus it is valid for all $n \le k$. Then the first inequality of (35) follows directly from (34). Noticing that $\|w_i\|_{0,\infty,\tau_j} \lesssim \sum_{n=0}^{k} |c_{i,n}|$ and

$$\begin{aligned} \left| \int_{\tau_j} w_i dx \right| &= \left| \sum_{n=0}^{k} c_{i,n} \int_{\tau_j} (\omega^{-1} - I_{n-1}\omega^{-1}) \omega \phi_{j,n} dx \right| \\ &\lesssim \sum_{n=0}^{k} h^{n+1} |c_{i,n}| \|\partial_x^n \omega^{-1}\|_{0,\infty} \lesssim h^{2k+2} \|\partial_t^i u\|_{k+1,\infty}, \end{aligned}$$

then (35) follows directly from (34). The proof is complete. ∎

With the help of functions $w_i, i \in \mathbb{Z}_k$, we are ready to construct the interpolation function $u_I$ as follows. Given any positive integer $l$, where $1 \le l \le k$, we define

$$u_I := u_I^l = P_h u - w^l, \quad w^l = \sum_{m=1}^{l} w_i. \tag{39}$$

In the error analysis of the upwind DG method for nonlinear hyperbolic equations, the standard method is to choose Gauss-Radau projection as the interpolation function. That is, $u_I = P_h^- u$. However, this way of choosing $u_I$ usually yields the optimal convergence rate by the standard approximation theory, or $(k+\frac{3}{2})$-th convergence rate by the superconvergence analysis technique in [20], which is far from the highest superconvergence rate $2k+1$. To achieve our superconvergence goal, we use the idea of correction function to construct $u_I$. The basic idea of the correction function is to design a special function $w \in V_h$ to correct the error between the DG solution $u_h$ and the Gauss Radau projection $P_h^- u$ (or $P_h u$). As we may observe in our later analysis, thanks to the correction function $w := w^l$ in (39), the convergence rate of the error $u_h - u_I$ can reach as high as $2k+1$. The idea is motivated from its successful applications to finite element methods (FEM) and finite volume methods (FVM) for elliptic equations (see, e.g. [8, 9]), and DG methods for linear equations (see, e.g., [7]). However, it is very different from the steady state problems using FVM or FEM due to the time dependent feature. It is also much so for the nonlinear equation from those of linear equations due to the effects of the nonlinear term.

We now study the approximation properties of $u_I$.

LEMMA 5. *Given any integer $l$, where $1 \le l \le k$, let $u \in W^{k+l+2,\infty}$ be the solution of (1), and the flux function $f(u)$ is sufficiently smooth such that $|D^{k+1}f(u)| \lesssim 1$. Let $u_I = u_I^l$ be the special interpolation function of the exact solution $u$ defined in (39). There hold the following properties.*

- *The numerical fluxes is exact, i.e.,*

$$\hat{u}_I(x_{j+\frac{1}{2}}) = u_I(x^-_{j+\frac{1}{2}}) = u(x_{j+\frac{1}{2}}), \quad j \in \mathbb{Z}_N. \tag{40}$$

- *The cell average is superconvergent with an order of $2k+1$ in each element $\tau_j$. To be more precise,*

$$\left| \frac{1}{h_j} \int_{\tau_j} (u - u_I)(x)dx \right| \lesssim h^{2k+1} \|u\|_{k+1+l,\infty}. \tag{41}$$

- *The function value approximation of $u_I$ is superconvergent at the interior right Radau points $r_{j,m}, m \in \mathbb{Z}_k$, namely,*

$$|(u - u_I)(r_{j,m})| \lesssim h^{k+2} \|u\|_{k+l+1,\infty}, \quad m \in \mathbb{Z}_k. \tag{42}$$

- *The derivative value approximation of $u_I$ is superconvergent at the interior left Radau points $l_{j,m}, m \in \mathbb{Z}_k$. That is*

$$|\partial_x(u - u_I)(l_{j,m})| \lesssim h^{k+1} \|u\|_{k+l+1,\infty}, \quad m \in \mathbb{Z}_k. \tag{43}$$

- *There holds for all $v \in V_h$*

$$|(f'(u)(u - u_I), v_x) - ((u - u_I)_t, v)| \lesssim h^{k+l+1} \|u\|_{k+l+2} \|v\|_0. \tag{44}$$

*Proof.* First, (40) follows directly from (39), (31)-(32), and (23). Second, for any positive integer $m \le k+1$, by the chain rule of derivative, the fact that $|D^{k+1}f(u)| \lesssim 1$, and the equation $u_t = -\partial_x f(u) = -f'(u)u_x$, we derive

$$|\partial_t^m u| \lesssim \|u\|_{m,\infty}, \quad |\partial_x^m f'(u)| + |\partial_t^m f'(u)| \lesssim \|u\|_{m,\infty},$$

which indicate that the condition (33) in Lemma 4 holds. Then (27) and (35) are valid and consequently (41) follows.

In light of (35), we have

$$\|w^l\|_{0,\infty,\tau_j} \lesssim \sum_{i=0}^{l} \|w_i\|_{0,\infty,\tau_j} \lesssim h^{k+2} \|\partial_t^l u\|_{k+1,\infty} \lesssim h^{k+2} \|u\|_{k+1+l,\infty},$$
$$\|w^l\|_{1,\infty,\tau_j} \lesssim h^{-1} \|w^l\|_{0,\infty,\tau_j} \lesssim h^{k+1} \|u\|_{k+1+l,\infty}.$$

Then (42)-(43) follow from (25)-(26) and the triangle inequality.

Now we are ready to prove (44). Recalling the definition of $P_h$ and $w_i$, there holds for all $v_1 \in \mathbb{P}_k \backslash \mathbb{P}_0$

$$
\begin{aligned}
&(f'(u)(u-u_I), \partial_x v_1) - ((u-u_I)_t, v_1) \\
= &-(\partial_t(u-P_h u), v_1) - \sum_{i=1}^{l}(\partial_t w_i, v_1) + \sum_{i=1}^{l}(f'(u)w_i, \partial_x v_1) = -(\partial_t w_l, v_1),
\end{aligned}
$$

and thus

$$
|(f'(u)(u-u_I), \partial_x v_1) - ((u-u_I)_t, v_1)| \lesssim \|\partial_t w_l\|_0 \|v_1\|_0 \lesssim h^{k+l+1}\|\partial_t u\|_{k+l+1,\infty}\|v_1\|_0.
$$

On the other hand, for all $\bar{v} = \bar{v}(x_j) = \bar{v}_j \in \mathbb{P}_0(\tau_j)$,

$$
\begin{aligned}
&|(f'(u)(u-u_I), \bar{v}_x)_j - ((u-u_I)_t, \bar{v})_j| = \left|\bar{v}_j \sum_{i=0}^{l}\int_{\tau_j} \partial_t w_i dx\right| + \left|\bar{v}_j \int_{\tau_j}(u_t - P_h u_t)dx\right| \\
&\lesssim h^{2k+2}|\bar{v}_j| \sum_{i=0}^{l} \|\partial_t^{i+1} u\|_{k+1,\infty} \lesssim h^{2k+\frac{3}{2}}\|\bar{v}\|_{0,\tau_j}\|u\|_{k+l+2,\infty},
\end{aligned}
$$

where in the second step, we have used (27) and (35), and the third step, we used the inverse inequality. Then summing up all $j$ from 1 to $N$ yields

$$
|(f'(u)(u-u_I), \bar{v}_x) - ((u-u_I)_t, \bar{v})| \lesssim h^{2k+1}\|\bar{v}\|_0\|u\|_{k+l+2,\infty}.
$$

Note that all $v \in \mathbb{P}_k$ can be decomposed into

$$
v = \bar{v} + v_1, \quad \bar{v} \in \mathbb{P}_0, \ v_1 \in \mathbb{P}_k \backslash \mathbb{P}_0.
$$

Then

$$
\begin{aligned}
|(f'(u)(u-u_I), v_x) - ((u-u_I)_t, v)| &\lesssim h^{k+l+1}(\|v_1\|_0 + \|\bar{v}\|_0)\|u\|_{k+l+2,\infty} \\
&\lesssim h^{k+l+1}\|v\|_0\|u\|_{k+l+2,\infty}.
\end{aligned}
$$

This finishes our proof. ∎

As a direct consequence of the conclusions in Theorem 1 and Lemma 5, we have the following energy inequality for problems with fixed wind directions.

COROLLARY 6. *Given any positive $l$, where $1 \leq l \leq k$, let $u \in W^{k+l+2,\infty}(\Omega)$ be the solution of (1) with the flux function $f(u)$ sufficiently smooth such that $|D^{k+1}f(u)| \lesssim 1$, and $u_h$ be the solution of (2). Suppose $u_I = u_I^l$ is the specially designed interpolation function defined in (39), and $J$ is defined in (8). Then*

$$
\tilde{u}_I(x_{j+\frac{1}{2}}) = u_I(x^-_{j+\frac{1}{2}}) = u(x_{j+\frac{1}{2}}), \ \ |J| = |(\eta_t, \xi) - (f'(u)\eta, \xi_x)| \lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}\|\xi\|_0.
$$

*Consequently,*

$$
\frac{1}{2}\frac{d}{dt}\|\xi\|_0^2 \lesssim h^{2(k+l+1)}\|u\|^2_{k+l+2,\infty} + (1 + h^{-1}\|e\|_{0,\infty})\|\xi\|_0^2 + h^{-2}\|e\|^2_{0,\infty}\|\eta\|_0^2. \tag{45}
$$

**3.3. Superconvergence.** We begin with the super-closeness between the DG solution $u_h$ and the specially designed interpolation function $u_I$ defined in (39). To this end, we need a priori error assumption

$$\|P_h^- u - u_h\|_0 \lesssim h^2. \tag{46}$$

We woulds like to point out that the assumption (46) is reasonable, and we will justify this a priori assumption for piecewise polynomial of degree $k$. Actually, this assumption is frequently used in the DG error analysis for nonlinear problems, see, e.g., [20].

With the assumption (46), we are ready to prove the superconvergence result for $u_h - u_I^l$.

THEOREM 7. *Given any positive $l$, where $1 \le l \le k$, let $u \in W^{k+l+2,\infty}(\Omega)$ be the solution of (1) with the flux function $f(u)$ sufficiently smooth such that $|D^{k+1}f(u)| \lesssim 1$, and $u_I = u_I^l$ be defined by (39). Suppose $u_h$ is the solution of (2) satisfying the error assumption (46). If the initial discretization is chosen such that*

$$\|u_h(\cdot,0) - u_I(\cdot,0)\|_0 \lesssim h^{k+l+1}\|u_0\|_{k+l+1}, \tag{47}$$

*then*

$$\|(u_I^l - u_h)(\cdot,t)\|_0 \lesssim h^{k+l+1} \sup_{\tau\in(0,t]} \|u(\cdot,\tau)\|_{k+l+2,\infty}. \tag{48}$$

*Proof.* By the a priori error assumption (46), we have

$$\|u_h - P_h^- u\|_{0,\infty} \lesssim h^{-\frac{1}{2}}\|u_h - P_h^- u\|_0 \lesssim h^{\frac{3}{2}},$$

which yields

$$\|e\|_{0,\infty} \le \|u - P_h^- u\|_{0,\infty} + \|P_h^- u - u_h\|_{0,\infty} \lesssim h^{\frac{3}{2}}. \tag{49}$$

Substituting the above inequality into (45), we immediately get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi\|_0^2 &\lesssim h^{2(k+l+1)}\|u\|_{k+l+2,\infty}^2 + \|\xi\|_0^2 + h\|\eta\|_0^2 \\ &\lesssim \|\xi\|_0^2 + h^{2(k+l+1)}\|u\|_{k+l+2,\infty}^2 + h^{2k+3}\|u\|_{k+l+2,\infty}^2,\end{aligned}$$

where in the last step, we have used (24) and (35), which yields

$$\|\eta\|_0 \le \|u - P_h^- u\|_0 + \|P_h u - P_h^- u\|_0 + \|w^l\|_0 \lesssim h^{k+1}\|u\|_{k+l+1,\infty}.$$

Especially, we choose $l = 1$ and use the Gronwall inequality and (47) to derive

$$\|(u_I^1 - u_h)(\cdot,t)\|_0 \lesssim h^{k+\frac{3}{2}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}, \quad \forall t \in (0,T],$$

and thus

$$\|(P_h u - u_h)(\cdot,t)\|_0 \lesssim \|(u_I^1 - u_h)(\cdot,t)\|_0 + \|w_1(\cdot,t)\|_0 \lesssim h^{k+\frac{3}{2}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

By the inverse inequality and (24), we have for all $t \in [0,T]$,

$$\|P_h^- u - u_h\|_{0,\infty} \lesssim \|P_h u - u_h\|_{0,\infty} + \|P_h u - P_h^- u\|_{0,\infty} \lesssim h^{k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

Consequently,

$$\|e\|_{0,\infty} \lesssim \|u - P_h^- u\|_{0,\infty} + \|P_h^- u - u_h\|_{0,\infty} \lesssim h^{k+1} \sup_{\tau\in[0,T]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

Again, we substitute the estimates of $\|e\|_{0,\infty}$ and $\|\eta\|_0$ into (45) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\xi\|_0^2 \lesssim \|\xi\|_0^2 + h^{2(k+l+1)}\|u\|_{k+l+2,\infty}^2 + h^{4k+2}\|u\|_{k+l+2,\infty}^2.$$

Thanks to the special initial discretization and the Gronwall inequality, the desired result (47) follows. The proof is complete. ∎

Due to the super-closeness between $u_I$ and $u_h$, together with the superconvergent approximation properties of $u_I$ in Lemma 5, we have the following superconvergence for the DG solution.

Theorem 8. *Suppose all the conditions of Theorem 7 are satisfied with $l = 1$. Then there hold the following superconvergence results.*

- *The DG solution is $k+2$-th order superconvergent to a particular projection of the exact solution, i.e.,*

(50)
$$e_p = \|(u_h - P_h^- u)(\cdot,t)\|_0 + \|(u_h - P_h u)(\cdot,t)\|_0 \lesssim h^{k+2} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

- *The function value approximation of the DG solution is $k+2$-th order superconvergent at all interior right Radau points $r_{j,m}, (j,m) \in \mathbb{Z}_N \times \mathbb{Z}_k$, i.e.,*

$$e_r := \left(\frac{1}{Nk}\sum_{j=1}^{N}\sum_{m=1}^{k}(u_h - u)^2(r_{j,m},t)\right)^{\frac{1}{2}} \lesssim h^{k+2} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}. \quad (51)$$

- *The derivative value approximation of DG solution is $k+1$-th order superconvergent at all interior left Radau points $l_{j,m}, (j,m) \in \mathbb{Z}_N \times \mathbb{Z}_k$, i.e.,*

(52)
$$e_l := \left(\frac{1}{Nk}\sum_{j=1}^{N}\sum_{m=1}^{k}\partial_x(u_h - u)^2(l_{j,m},t)\right)^{\frac{1}{2}} \lesssim h^{k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

*Moreover, if all the conditions of Theorem 7 hold with $l = k$, then*

- *The errors of numerical fluxes at nodes is superconvergent with an order of $2k+1$, i.e.,*

$$e_n := \left(\frac{1}{N}\sum_{j=1}^{N}(u - u_h)^2(x^-_{j+\frac{1}{2}},t)\right)^{\frac{1}{2}} \lesssim h^{2k+1} \sup_{\tau\in(0,t]} \|u(\cdot,\tau)\|_{2k+2,\infty}. \quad (53)$$

- *The error for the cell-average is superconvergent with an order of $2k+1$, i.e.,*

(54)
$$e_c := \left(\frac{1}{N}\sum_{j=1}^{N}\Big(\frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}(u - u_h)dx\Big)^2\right)^{\frac{1}{2}} \lesssim h^{2k+1} \sup_{\tau\in(0,t]} \|u(\cdot,\tau)\|_{2k+2,\infty}.$$

*Proof.* First, choosing $l = 1$ in (48) and using the estimate of $w_1$ in (35), we have for all $t \in (0, T]$

$$\|(u_h - P_h u)(\cdot,t)\|_0 \le \|(u_h - u_I^l)(\cdot,t)\|_0 + \|w_1(\cdot,t)\|_0 \lesssim h^{k+2} \sup_{\tau\in(0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

Then (50) follows directly from (24).

By (42) and the triangle inequality,

$$\begin{aligned} e_r &\lesssim \left( \frac{1}{Nk} \sum_{j=1}^{N} \sum_{m=1}^{k} (u - u_I^l)^2 (r_{j,m}, t) + (u_h - u_I^l)^2 (r_{j,m}, t) \right)^{\frac{1}{2}} \\ &\lesssim \|u_I^l - u_h\|_0 + h^{k+2} \|u\|_{k+2,\infty}. \end{aligned}$$

Here in the second step, we have used the inverse inequality $\|v\|_{0,\infty,\tau_j} \lesssim h^{-\frac{1}{2}} \|v\|_{0,\tau_j}$ for all $v \in V_h$. By the same argument and (43), we get

$$e_l \lesssim \|u_I^l - u_h\|_1 + h^{k+1} \|u\|_{k+2,\infty} \lesssim h^{-1} \|u_I^l - u_h\|_0 + h^{k+1} \|u\|_{k+2,\infty}.$$

By taking $l = 1$ in (48), the desired results (51)-(52) follows.

In light of (40), we easily obtain

$$e_n = \left( \frac{1}{N} \sum_{j=1}^{N} (u_I^l - u_h)^2 (x_{j+\frac{1}{2}}^-, t) \right)^{\frac{1}{2}} \lesssim \left( \frac{1}{N} \sum_{j=1}^{N} \|u_I^l - u_h\|_{0,\infty,\tau_j}^2 \right)^{\frac{1}{2}} \lesssim \|u_I^l - u_h\|_0.$$

On the other hand, it follows from (41) that

$$\begin{aligned} \frac{1}{h_j} \int_{\tau_j} (u - u_h)(x) dx &= \frac{1}{h_j} \int_{\tau_j} (u - u_I^l)(x) dx + \frac{1}{h_j} \int_{\tau_j} (u_I^l - u_h)(x) dx \\ &\lesssim h^{2k+1} \|u\|_{k+l+1,\infty} + h^{-\frac{1}{2}} \|u_I^l - u_h\|_{0,\tau_j}. \end{aligned}$$

Consequently,

$$e_c \lesssim h^{2k+1} \|u\|_{k+l+1,\infty} + \|u_I^l - u_h\|_0.$$

Then (53)-(54) follow from (48) by taking $l = k$. ∎

REMARK 9. *As demonstrated in (50), the DG solution is superconvergent with an order of $k+2$ towards the Gauss-Radau projection $P_h^- u$ of the exact solution $u$, which indicates that the a priori error assumption (46) is reasonable. More details of the verification of the a priori error assumption has been given in [20], so we skip this part.*

REMARK 10. *A nature way to guarantee the initial condition (47) with $l = 1$ is to choose $u_h(x, 0) = P_h^- u_0(x)$ or $P_h u_0(x)$. In other words, the way of choosing Guass-Radau projection as initial discretization is enough to assure the superconvergence rate $k + 2$. However, to achieve the highest superconvergence rate $2k + 1$, i.e. (47) is satisfied with $l = k$, we need a stronger method of initial discretization. In this case, we can take the correction initial discretization, that is, $u_h = u_I = P_h u - w^k$ at $t = 0$.*

**4. Analysis for problems with alternating wind directions.** The superconvergence analysis for problems with alternating wind directions is totally different from that for fixed wind directions, and the construction of $u_I$ for alternating wind directions is more complicated due to the existence of the sonic points. Moreover, the superconvergence results are also of great difference from those for fixed wind directions.

To construct the interpolation function $u_I$ for problems with alternating wind, we need to modify both the projection $P_h u$ and the correction function $w^l$ in (39).

**4.1. A special projection for problems with alternating wind directions.** Given a function $\psi$, we denote by $R_h\psi \in V_h$ and $P_h^+\psi \in V_h$ the traditional $L^2$ and Gauss-Radau projections of $\psi$, respectively. That is,

$$(R_h\psi, v_h) = (\psi, v_h), \quad \forall v_h \in \mathbb{P}_k(\tau_j),$$

and

$$(P_h^+\psi, v_h)_j = (\psi, v_h)_j, \quad \forall v_h \in \mathbb{P}_{k-1}(\tau_j), \quad P_h^+\psi(x^+_{j-\frac{1}{2}}) = \psi(x^+_{j-\frac{1}{2}}). \tag{55}$$

In addition, we also define the Gauss-Lobatto projection $Q_h\psi$ (for $k \geq 2$) as follows:

$$(Q_h\psi, v_h) = (\psi, v_h), \ \forall v_h \in \mathbb{P}_{k-2}(\tau_j), \ Q_h\psi(x^-_{j+\frac{1}{2}}) = \psi(x^-_{j+\frac{1}{2}}), \ Q_h\psi(x^+_{j-\frac{1}{2}}) = \psi(x^+_{j-\frac{1}{2}}).$$

While for $k = 1$, $Q_h\psi$ is the interpolant of $\psi$ satisfying

$$Q_h\psi(x^-_{j+\frac{1}{2}}) = \psi(x^-_{j+\frac{1}{2}}), \quad Q_h\psi(x^+_{j-\frac{1}{2}}) = \psi(x^+_{j-\frac{1}{2}}).$$

The standard approximation theory gives us, for $p \geq 1$,

$$\|\psi - P_h^+\psi\|_{0,p} + \|\psi - P_h^-\psi\|_{0,p} + \|\psi - R_h\psi\|_{0,p} + \|\psi - Q_h\psi\|_{0,p} \lesssim h^{k+1}|\psi|_{k+1,p}. \tag{56}$$

With the above projections, we define the projection $P_h u$ as follows.

$$P_h u = \begin{cases} R_h u, & \text{if } f'_{j+\frac{1}{2}} \leq 0, \ f'_{j-\frac{1}{2}} > 0, \\ P_h^+ u, & \text{if } f'_{j+\frac{1}{2}} \leq 0, \ f'_{j-\frac{1}{2}} \leq 0, \\ P_h^- u, & \text{if } f'_{j+\frac{1}{2}} > 0, \ f'_{j-\frac{1}{2}} > 0, \\ Q_h u, & \text{if } f'_{j+\frac{1}{2}} > 0, \ f'_{j-\frac{1}{2}} \leq 0. \end{cases} \tag{57}$$

We have the following superconvergent properties for $P_h u$ (see [5]).

LEMMA 11. *Let $u \in W^{k+2,\infty}$ and $P_h u \in V_h$ be defined by (57). Then the following approximation properties hold true.*

- *The numerical flux (4) for $P_h u$ is exact, i.e.,*

$$\tilde{P}_h u\big|_{j+\frac{1}{2}} = u_{j+\frac{1}{2}}. \tag{58}$$

- *The function value of $P_h u$ is superconvergent at the roots of the right Radau polynomial $L_{j,k+1} - L_{j,k}$ if $P_h u|_{\tau_j} = P_h^- u$; at the roots of the left Radau polynomial $L_{j,k+1} + L_{j,k}$ if $P_h u|_{\tau_j} = P_h^+ u$; at the roots of the Legendre polynomial $L_{j,k+1}$ if $P_h u|_{\tau_j} = R_h u$; and at the roots of the Lobatto polynomial $L_{j,k+1} - L_{j,k-1}$ if $P_h u|_{\tau_j} = Q_h u$. In other words,*

$$|(u - P_h u)(y_{j,i})| \lesssim h^{k+2}|u|_{k+2,\infty,\tau_j}, \tag{59}$$

*where*

$$y_{j,i} = \begin{cases} g_{j,i}, & \text{if } f'_{j+\frac{1}{2}} \le 0,\ f'_{j-\frac{1}{2}} > 0, \\ l_{j,i}, & \text{if } f'_{j+\frac{1}{2}} \le 0,\ f'_{j-\frac{1}{2}} \le 0, \\ r_{j,i}, & \text{if } f'_{j+\frac{1}{2}} > 0,\ f'_{j-\frac{1}{2}} > 0, \\ gl_{j,i}, & \text{if } f'_{j+\frac{1}{2}} > 0,\ f'_{j-\frac{1}{2}} \le 0 \end{cases}$$

*with* $g_{j,i}$, $l_{j,i}$, $r_{j,i}$ *and* $gl_{j,i}$, *for* $(j,i) \in \mathbb{Z}_N \times \mathbb{Z}_{k+1}$, *being the zeros of* $L_{j,k+1}$, $L_{j,k+1} + L_{j,k}$, $L_{j,k+1} - L_{j,k}$ *and* $L_{j,k+1} - L_{j,k-1}$, *respectively.*

- *The derivative value of* $P_h u$ *is superconvergent at the interior left Radau points* $l^*_{j,m}, m \in \mathbb{Z}_k$ *(the roots of* $L_{j,k+1} + L_{j,k}$ *except the point* $x = x_{j-\frac{1}{2}}$*) if* $P_h u|_{\tau_j} = P_h^- u$*; at the interior right Radau points* $r^*_{j,m}$ *(the roots of* $L_{j,k+1} - L_{j,k}$ *except the point* $x = x_{j+\frac{1}{2}}$*) if* $P_h u|_{\tau_j} = P_h^+ u$*; at the Gauss points* $g^*_{j,m}$ *of degree* $k$ *(i.e., the roots of* $L_{j,k}$*) if* $P_h u|_{\tau_j} = Q_h u$*; and at the* $k$ *interior Gauss-Lobatto points* $gl^*_{j,m}, m \in \mathbb{Z}_k$ *(i.e., the roots of* $L_{j,k+2} - L_{j,k}$ *except the two boundary points* $x = x_{j+\frac{1}{2}}, x_{j-\frac{1}{2}}$*) if* $P_h u|_{\tau_j} = R_h u$*. That is,*

$$|\partial_x (u - P_h u)(z_{j,m})| \lesssim h^{k+1} |u|_{k+2,\infty,\tau_j}, \tag{60}$$

*where*

$$z_{j,m} = \begin{cases} gl^*_{j,m}, & \text{if } f'_{j+\frac{1}{2}} \le 0,\ f'_{j-\frac{1}{2}} > 0, \\ r^*_{j,m}, & \text{if } f'_{j+\frac{1}{2}} \le 0,\ f'_{j-\frac{1}{2}} \le 0, \\ l^*_{j,m}, & \text{if } f'_{j+\frac{1}{2}} > 0,\ f'_{j-\frac{1}{2}} > 0, \\ g^*_{j,m}, & \text{if } f'_{j+\frac{1}{2}} > 0,\ f'_{j-\frac{1}{2}} \le 0. \end{cases}$$

We would like to point out that the above projection $P_h$ is very similar to that for the linear conservation laws with degenerate variable coefficients (see [5]). Since we assume no sources or sinks in the problem, $f'(u(x_{j+\frac{1}{2}}, t)$ does not change its sign in the time domain $(0, T]$ for any fixed point $x_{j+\frac{1}{2}}$, (otherwise the shock wave will appear). Therefore, $P_h u$ is continuous with respect to $t$. Actually, the projection $P_h u$ in (57) is exact the same as that in [5] for linear conservation laws $u_t + (\alpha u)_x$ by setting $\alpha(x) = f'(u)$.

**4.2. Construction of the correction functions.** Motivated from the correction functions for linear conservation laws with degenerate variable coefficients in [5], we define the correction function as follows. For any fixed $t$, let

$$|\bar{f}'_j| = \max_{x \in \tau_j} |f'(u(x,t))|, \quad j \in \mathbb{Z}_N, \tag{61}$$

and $w_1, w_2 \in V_h$ be functions satisfying

$$\begin{aligned} &\bar{f}'_j (w_1, v)_j = -\bar{h}_j (D_x^{-1}(u_t - P_h^- u_t), v)_j - ((f' - \bar{f}'_j)(u - P_h^- u), v)_j, \quad v \in \mathbb{P}_{k-1}, \\ &w_1(x^-_{j+\frac{1}{2}}) = 0, \quad \forall j \in \mathbb{Z}_N, \end{aligned} \tag{62}$$

and

$$\begin{aligned} &\bar{f}'_j (w_2, v)_j = -\bar{h}_j (D_x^{-1}(u_t - P_h^+ u_t), v)_j - ((f' - \bar{f}'_j)(u - P_h^+ u), v)_j, \quad v \in \mathbb{P}_{k-1}, \\ &w_2(x^+_{j-\frac{1}{2}}) = 0, \quad \forall j \in \mathbb{Z}_N. \end{aligned} \tag{63}$$

where $D_x^{-1}v$ for any function $v$ is defined in (38). With $w_1$ and $w_2$, we define the correction function $w$ in the whole domain as follows.

$$w = \begin{cases} 0, & f'_{j+\frac{1}{2}} \le 0, \quad f'_{j-\frac{1}{2}} > 0, \\ w_2, & f'_{j+\frac{1}{2}} \le 0, \quad f'_{j-\frac{1}{2}} \le 0, \\ w_1, & f'_{j+\frac{1}{2}} > 0, \quad f'_{j-\frac{1}{2}} > 0, \\ 0 & f'_{j+\frac{1}{2}} > 0, \quad f'_{j-\frac{1}{2}} \le 0. \end{cases} \tag{64}$$

On the other hand, note that the number of zeros of $f'(u)$ at any time $t \in [0, T]$ is the same as that for $t = 0$, we assume that the smooth function $f'(u)$ has only a finite number of zeros in $\Omega$ at $t = 0$. For simplicity, we suppose $f'(u_0)$ has only one zero $x = x_0$ on $\Omega$. At the zero point $x = x_0$, we assume there exists a positive integer $m$ such that

$$f'(u_0(x_0)) = \partial_x f'(u_0(x_0)) = \ldots = \partial_x^{m-1} f'(u_0(x_0)) = 0, \quad \partial_x^m f'(u_0(x_0)) \ne 0. \tag{65}$$

In other words, $x = x_0$ is the $m$ multiple root of $f'(u_0)$. Given a fixed $t \in [0, T]$, suppose the root of $f'(u(x,t))$ is located at $x = \bar{x}_0$. Let

$$m' = \min(m, k+3), \tag{66}$$

and $i_0, j_0$ be a positive integer such that

$$x_{i_0-\frac{1}{2}} - \bar{x}_0 \le h^{\frac{1}{m'}} \le x_{i_0+\frac{1}{2}} - \bar{x}_0, \qquad \bar{x}_0 - x_{j_0+\frac{1}{2}} \le h^{\frac{1}{m'}} \le \bar{x}_0 - x_{j_0-\frac{1}{2}}.$$

Note that if $\bar{x}_0 = 0$, we simply choose $j_0 = 0$. The correction function $w$ at time $t$ is defined by

$$\bar{w}_i(x,t)\big|_{\tau_j} = \begin{cases} 0, & \tau_j \subset \omega_m^{\perp} = [x_{j_0+\frac{1}{2}}, x_{i_0-\frac{1}{2}}], \\ w_i, & \tau_j \subset \omega_m = [0, 2\pi] \setminus \omega_m^{\perp}, \end{cases} \tag{67}$$

where $w_i, i \in \mathbb{Z}_2$ are defined by (62), (63), respectively. We observe that, from (67), the correction function equals to zero on the domain $|\bar{x}_0 - x| \le h^{\frac{1}{m'}}$, which indicates the correction is not necessary near the area of $\bar{x}_0$.

Now we define another correction function $\bar{w}$ as follows:

$$\bar{w} = \begin{cases} 0, & f'_{j+\frac{1}{2}} \le 0, \quad f'_{j-\frac{1}{2}} > 0, \\ \bar{w}_2, & f'_{j+\frac{1}{2}} \le 0, \quad f'_{j-\frac{1}{2}} \le 0, \\ \bar{w}_1, & f'_{j+\frac{1}{2}} > 0, \quad f'j - \frac{1}{2} > 0, \\ 0 & f'_{j+\frac{1}{2}} > 0, \quad f'_{j-\frac{1}{2}} \le 0. \end{cases} \tag{68}$$

As proved in [5], we have the following similar results (see Lemma 5 in [5]).

Lemma 12. *Let $u \in W^{k+3,\infty}$ be the solution of (1), and $P_h u$ be the projection of $u$ defined in (57). If $f'(u)$ is sufficiently smooth satisfying $|D^{k+3} f'(u)| \lesssim 1$, then*

$$\sum_{i=1}^{2} (\|\bar{w}_i\|_0 + \|\partial_t \bar{w}_i\|_0) \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty}, \tag{69}$$

$$\sum_{j=1}^{N} ((u - P_h u + \bar{w})_t, v)_j + (f'(u)(u - P_h u + \bar{w}), v_x)_j)$$

$$\lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v\|_0, \quad \forall v \in V_h. \tag{70}$$

*Here $m'$ is given in (66).*

**4.3. Superconvergence for the DG solution itself.** We first define the special interpolation function $u_I$ as

$$u_I = P_h u - \bar{w}, \tag{71}$$

where $\bar{w}$ is defined in (67). We have the following superconvergence result.

THEOREM 13. *Let $u \in W^{k+3,\infty}$ be the solution of (1) satisfying $|D^{k+3}f(u)| \lesssim 1$, and $u_I$ be defined in (71). Assume that $u_h$ is the solution of (2) satisfying the error assumption (46). Then*

$$\frac{1}{2}\frac{d}{dt}\|\xi\|_0^2 \lesssim h^{2(k+1+\frac{1}{2m'})}\|u\|_{k+3,\infty}^2 + (1+h^{-1}\|e\|_{0,\infty})\|\xi\|_0^2 + h^{-2}\|e\|_{0,\infty}^2\|\eta\|_{0,\infty}^2. \tag{72}$$

*Consequently, if the initial discretization is chosen as $u_h(x,0) = P_h u_0(x)$ with $P_h$ defined in (57). Then*

$$\|(u_h - P_h u)(\cdot,t)\|_0 \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}, \quad \forall t \in (0,T]. \tag{73}$$

*Proof.* To show (72), we need to need to estimate the term $I$ and $J$ in (6). Recalling the definitions of $J$ in (8) and $u_I$ in (71), we have

$$\tilde{u}_I(x_{j+\frac{1}{2}}) = \tilde{P}_h u(x_{j+\frac{1}{2}}) = u(x_{j+\frac{1}{2}}),$$

and thus

$$|J| = |((u - P_h u + \bar{w})_t, \xi) + (f'(u)(u - P_h u + \bar{w}), \xi_x)| \lesssim h^{k+1+\frac{1}{2m'}}\|u\|_{k+3,\infty}\|\xi\|_0.$$

Here $\tilde{u}_I$ is the numerical flux defined in (4), and in the last step, we have used (70).

We next estimate the term $I$ in (7). By (3) and (11), we have

$$\hat{f}(u_h) - f(u) = \begin{cases} \min_{u_h^- \le \mu \le u_h^+}(f'(u)(\mu - u) - \frac{\bar{f}_u''}{2}(u-\mu)^2), & \text{if } u_h^- < u_h^+, \\ \max_{u_h^+ \le \mu \le u_h^-}(f'(u)(\mu - u) - \frac{\bar{f}_u''}{2}(u-\mu)^2), & \text{if } u_h^- \ge u_h^+. \end{cases}$$

Since for any $\mu \in [\min(u_h^-, u_h^+), \max(u_h^-, u_h^+)]$,

$$|\bar{f}_u''(u-\mu)^2_{j+\frac{1}{2}}| \le \|f_u''\|_{0,\infty}(\|e\|_{0,\infty,\tau_j}^2 + \|e\|_{0,\infty,\tau_{j+1}}^2) = B_j,$$

we have

$$f'(u)(\mu - u)_{j+\frac{1}{2}} - B_j \le \left(f'(u)(\mu-u) - \frac{\bar{f}_u''}{2}(u-\mu)^2\right)_{j+\frac{1}{2}} \le f'(u)(\mu-u)_{j+\frac{1}{2}} + B_j.$$

Taking minimal or maximal in the above inequality for all $\mu \in [\min(u_h^-, u_h^+), \max(u_h^-, u_h^+)]$, we immediately get

$$f'(u)(\tilde{u}_h - u)_{j+\frac{1}{2}} - B_j \le (\hat{f}(u_h) - f(u))_{j+\frac{1}{2}} \le f'(u)(\tilde{u}_h - u)_{j+\frac{1}{2}} + B_j.$$

By Taylor expansion, there exists a $\bar{\bar{f}}_u'' = f''(\alpha_2 u + (1-\alpha_2)u_h)$ with $0 \le \alpha_2 \le 1$ such that

$$f(u) - \tilde{f}(u_h) = f'(u)(u - \tilde{u}_h) - \frac{1}{2}\bar{\bar{f}}_u''(u - \tilde{u}_h)^2. \tag{74}$$

Consequently,

$$\left|(\tilde{f}(u_h) - \hat{f}(u_h))_{j+\frac{1}{2}}\right| \lesssim \|f_u''\|_{0,\infty}(\|e\|^2_{0,\infty,\tau_j} + \|e\|^2_{0,\infty,\tau_{j+1}}). \tag{75}$$

Here we use the notation $\tau_{N+1} = \tau_1$. Consequently, substituting the above inequality into (7) and again using the Cauchy-Schwarz inequality and the inverse inequality, we get

$$\begin{aligned}|I| &\lesssim \|e\|_{0,\infty}\sum_{j=1}^{N}(\|e\|_{0,\infty,\tau_j} + \|e\|_{0,\infty,\tau_{j+1}})(\|\xi\|_{0,\infty,\tau_j} + \|\xi\|_{0,\infty,\tau_{j+1}}) \\ &\lesssim h^{-1}\|e\|^2_{0,\infty}\sum_{j=1}^{N}\|\eta\|^2_{0,\infty,\tau_j} + (\|e\|_{0,\infty} + h)\sum_{j=1}^{N}\|\xi\|^2_{0,\infty,\tau_j} \\ &\lesssim h^{-2}\|e\|^2_{0,\infty}\|\eta\|^2_{0,\infty} + (1 + h^{-1}\|e\|_{0,\infty})\|\xi\|^2_0.\end{aligned}$$

Substituting the estimates of $I$ and $J$ into (6) and using the fact $\|\cdot\|_0 \lesssim \|\cdot\|_{0,\infty}$, we obtain (72) immediately.

By (69), the triangle inequality and the inverse inequality,

$$\|\eta\|_{0,\infty} \le \|u - P_h u\|_{0,\infty} + h^{-\frac{1}{2}}\|w\|_0 \lesssim h^{k+\frac{1}{2}+\frac{1}{2m'}}\|u\|_{k+3,\infty}.$$

Plugging the above inequality into (72) and using (49) , we have,

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 \lesssim h^{2(k+1+\frac{1}{2m'})}\|u\|^2_{k+3,\infty} + \|\xi\|^2_0.$$

Therefore, if we choose the initial discretization $u_h(x, 0) = P_h u(x, 0)$, we immediately get

$$\|\xi(\cdot, t)\|_0 \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau\in[0,t]} \|u(\cdot, \tau)\|_{k+3,\infty}, \quad \forall t \in (0, T].$$

Then (73) follows from (69) and the triangle inequality. This finishes our proof. ◻

With the super-closeness result in (73), we have the following superconvergence result for the DG solution itself.

Theorem 14. *Suppose all the conditions of Theorem 13 hold. Then there hold the following superconvergence results.*

- *The DG solution is superconvergent to the particular projection $P_h u$ of the exact solution $u$, with an order of $k + 1 + \frac{1}{2m'}$, i.e.,*

$$e_p = \|u_h - P_h u\|_0 \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau\in[0,t]} \|u(\cdot, \tau)\|_{k+3,\infty}. \tag{76}$$

- *The cell average is the DG solution approximation is superconvergent. That is,*

$$e_c = \left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}(u_h - u)dx\right)^2\right)^{\frac{1}{2}} \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau\in[0,t]} \|u(\cdot, \tau)\|_{k+3,\infty}. \tag{77}$$

- *The function value approximation of the DG solution is superconvergent at a class of special point $y_{j,i}$, where $y_{j,i}$ is the same as in (59). To be more*

*precise,*
(78)
$$e_r := \left( \frac{1}{Nk} \sum_{j=1}^{N} \sum_{m=1}^{k} (u_h - u)^2 (y_{j,i}, t) \right)^{\frac{1}{2}} \lesssim h^{k+1+\frac{1}{2m'}} \sup_{\tau \in [0,t]} \|u(\cdot, \tau)\|_{k+3,\infty}.$$

- *The derivative value approximation of the DG solution is* $k + \frac{1}{2m'}$*-th order superconvergent at the point* $z_{j,i}$*, i.e.,*
(79)
$$e_l := \left( \frac{1}{Nk} \sum_{j=1}^{N} \sum_{m=1}^{k} \partial_x (u_h - u)^2 (z_{j,m}, t) \right)^{\frac{1}{2}} \lesssim h^{k+\frac{1}{2m'}} \sup_{\tau \in [0,t]} \|u(\cdot, \tau)\|_{k+3,\infty},$$
*where* $z_{j,i}$ *are the same as in* (60).

Here we skip the proof since it can be obtained by following the same argument as what we did for the problem with fixed wind directions.

**4.4. Superconvergence for the DG flux function** $f(u)$**.** This subsection is dedicated to the superconvergence of the DG flux function $f(u_h)$. As indicated by (11), the convergence rate of the error $f(u_h) - f(u)$ is mainly based on the linear part $f'(u)(u_h - u)$ as the nonlinear part is a high order term. Therefore, we next investigate the convergence order of the error $\|f'(u)(u - u_h)\|_0$. To this end, we define a weighted $L^2$-norm for all $v \in H_h^1$ associated with the function $f'(u)$ as follows:

$$|||v|||_f^2 = \sum_{j=1}^{N} \left( (f'_{j+\frac{1}{2}})^2 + (f'_{j-\frac{1}{2}})^2 \right) \int_{\tau_j} v^2(x) dx. \tag{80}$$

Note that the norm $|||v|||_f$ can be regarded as a discrete approximation to the $L^2$ norm of the function $f(u)v$.

We now first define the interpolation function $u_I$ by

$$u_I = P_h u - w \tag{81}$$

with $P_h u$ and $w$ given in (57) and (64), respectively, and then analyze the energy inequality of $u_h - u_I$ under the norm $||| \cdot |||_f$.

Theorem 15. *Let $u$ and $u_h$ be the solution of* (1) *and* (2)*, respectively. Assume that $u_I = P_h u - w \in V_h$ with $P_h u$ and $w$ defined in* (57) *and* (64) *respectively. Then*

$$\frac{d}{dt} |||\xi|||_f^2 \lesssim h^{2(k+2)} \|u\|_{k+3,\infty}^2 + |||\xi|||_f + h^{-1} \|e\|_{0,\infty}^2 \sum_{j=1}^{N} \omega_j^2 \|e\|_{0,\infty,\Lambda_j}^2, \tag{82}$$

*where*

$$\omega = f'(u), \quad \omega_j^2 = \omega^2(x_{j+\frac{1}{2}}) + \omega^2(x_{j-\frac{1}{2}}), \ \Lambda_j = \tau_j \cup \tau_{j-1} \cup \tau_{j+1}, \quad j \in \mathbb{Z}_N \tag{83}$$

*with* $\tau_{N+1} = \tau_1, \tau_0 = \tau_N$.

*Proof.* Multiply (9) by $\omega_j^2$ and sum up all $j$ from 1 to $N$, and then follow the same argument as what we have done in Theorem 1, we obtain

$$\sum_{j=1}^{N} \omega_j^2 (\xi_t, \xi)_j = J_1 - \frac{J_2}{2} + \bar{I} + \bar{J}, \tag{84}$$

where

$$J_1 = \sum_{j=1}^{N} \omega_j^2 \left( (\omega\xi, \xi_x)_j - \omega\tilde{\xi}\xi^-\big|_{j+\frac{1}{2}} + \omega\tilde{\xi}\xi^+\big|_{j-\frac{1}{2}} \right),$$

$$J_2 = \sum_{j=1}^{N} \omega_j^2 \left( (\bar{f}''_u e^2, \xi_x)_j - \bar{\bar{f}}''_u \tilde{e}^2\xi^-\big|_{j+\frac{1}{2}} + \bar{\bar{f}}''_u \tilde{e}^2\xi^+\big|_{j-\frac{1}{2}} \right),$$

$$\bar{J} = \sum_{j=1}^{N} \omega_j^2 \left( (\omega\eta, \xi_x)_j - (\eta_t, \xi)_j - \omega\tilde{\eta}\xi^-\big|_{j+\frac{1}{2}} + \omega\tilde{\eta}\xi^+\big|_{j-\frac{1}{2}} \right),$$

$$\bar{I} = \sum_{j=1}^{N} \omega_j^2 \left( (\tilde{f}(u_h) - \hat{f}(u_h))\xi^+\big|_{j-\frac{1}{2}} - (\tilde{f}(u_h) - \hat{f}(u_h))\xi^-\big|_{j+\frac{1}{2}} \right).$$

with $\bar{f}''_u$ and $\bar{\bar{f}}''_u$ the same as in (11).

Note that it has been proved in [5] (see Lemmas 3-4)

$$|||w|||_f + |||\partial_t w|||_f \lesssim h^{k+2}\|u\|_{k+3,\infty}, \quad \tilde{w}(x_{j+\frac{1}{2}}) = 0, \quad j \in \mathbb{Z}_N, \tag{85}$$

$$|A(u - P_h u + w, v)| \lesssim h^{k+2}\|u\|_{k+3,\infty}|||v|||_f, \quad \forall v \in V_h, \tag{86}$$

where

$$A(w, v) = \sum_{j=1}^{N} \left( (f'_{j+\frac{1}{2}})^2 + (f'_{j+\frac{1}{2}})^2 \right) ((w_t, v)_j - (f'(u)w, v_x)_j).$$

Then $\tilde{\eta}|_{j+\frac{1}{2}} = 0$ and thus

$$|\bar{J}| \lesssim h^{k+2}\|u\|_{k+3,\infty}|||\xi|||_f.$$

By integration by parts, we have

$$\begin{aligned} J_1 &= \sum_{j=1}^{N} \omega_j^2 \left( -(\frac{\partial_x\omega}{2}, \xi^2)_j + \frac{\omega}{2}\xi^-\xi^-|_{j+\frac{1}{2}} - \frac{\omega}{2}\xi^+\xi^+|_{j-\frac{1}{2}} - \omega\tilde{\xi}\xi^-_{j+\frac{1}{2}} + \omega\tilde{\xi}\xi^+_{j-\frac{1}{2}} \right) \\ &= -\frac{1}{2}\sum_{j=1}^{N} \omega_j^2(\partial_x\omega, \xi^2)_j + 2\sum_{j=1}^{N} \left( \omega^3(\tilde{\xi} - \{\xi\})[\xi] \right)\Big|_{j+\frac{1}{2}} + E_1 + E_2, \end{aligned}$$

where

$$E_1 = \sum_{j=1}^{N} \omega_{j-\frac{1}{2}}(\omega^2_{j+\frac{1}{2}} - \omega^2_{j-\frac{1}{2}})(\tilde{\xi} - \frac{\xi^+}{2})_{j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}},$$

$$E_2 = \sum_{j=1}^{N} \omega_{j+\frac{1}{2}}(\omega^2_{j+\frac{1}{2}} - \omega^2_{j-\frac{1}{2}})(\tilde{\xi} - \frac{\xi^-}{2})_{j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}}.$$

Due to the special choice of the numerical flux $\tilde{\xi}$ in (4),

$$2\sum_{j=1}^{N} \left( \omega^3(\tilde{\xi} - \{\xi\})[\xi] \right)\Big|_{j+\frac{1}{2}} = -\sum_{j\in\Omega_1} \omega^3[\xi]^2\big|_{j+\frac{1}{2}} + \sum_{j\in\Omega_2} \omega^3[\xi]^2\big|_{j+\frac{1}{2}} \le 0,$$

where $\Omega_1, \Omega_2$ are given in (13). On the other hand, by the inverse inequality and the Cauchy-Schwarz inequality,

$$|E_1| \leq \|\omega\|_{1,\infty} \sum_{j=1}^{N} h_j \left|\omega_{j-\frac{1}{2}}(\omega_{j+\frac{1}{2}} + \omega_{j-\frac{1}{2}})\right| \|\xi\|_{0,\infty,\tau_j}(\|\xi\|_{0,\infty,\tau_j} + \|\xi\|_{0,\infty,\tau_{j-1}})$$

$$\lesssim \sum_{j=1}^{N} \left|\omega_{j-\frac{1}{2}}(\omega_{j+\frac{1}{2}} + \omega_{j-\frac{1}{2}})\right| (\|\xi\|_{0,\tau_j}^2 + \|\xi\|_{0,\tau_j}\|\xi\|_{0,\tau_{j-1}}) \lesssim \sum_{j=1}^{N}(\omega_{j+\frac{1}{2}}^2 + \omega_{j-\frac{1}{2}}^2)\|\xi\|_{0,\tau_j}^2.$$

Following the same line, we can obtain

$$|E_2| \lesssim \sum_{j=1}^{N}(\omega_{j+\frac{1}{2}}^2 + \omega_{j-\frac{1}{2}}^2)\|\xi\|_{0,\tau_j}^2.$$

Then

$$|J_1| \leq \sum_{j=1}^{N} |\omega_j^2(\partial_x\omega, \xi^2)_j| + |E_1| + |E_2| \lesssim \sum_{j=1}^{N}(\omega_{j+\frac{1}{2}}^2 + \omega_{j-\frac{1}{2}}^2)\|\xi\|_{0,\tau_j}^2 = |||\xi|||_f^2.$$

By (75) and the inverse inequality,

$$|\bar{I}| \lesssim \|e\|_{0,\infty} \sum_{j=1}^{N} \omega_j^2 \|e\|_{0,\infty,\Lambda_j}\|\xi\|_{0,\infty,\tau_j} \lesssim h^{-1}\|e\|_{0,\infty}^2 \sum_{j=1}^{N} \omega_j^2 \|e\|_{0,\infty,\Lambda_j}^2 + |||\xi|||_f^2.$$

Similarly, we have from the Cauchy-Schwarz inequality and the inverse inequality

$$|J_2| \lesssim \|e\|_{0,\infty} \sum_{j=1}^{N} \omega_j^2 \left(\|e\|_{0,\infty,\tau_j}\|\xi_x\|_{0,1,\tau_j} + \|e\|_{0,\infty,\Lambda_j}\|\xi\|_{0,\infty,\tau_j}\right)$$

$$\lesssim h^{-1}\|e\|_{0,\infty}^2 \sum_{j=1}^{N} \omega_j^2 \|e\|_{0,\infty,\Lambda_j}^2 + |||\xi|||_f^2.$$

Substituting the estimates of $J_1, J_2, \bar{I}$ and $\bar{J}$ into (84) and using the Cauchy-Schwarz inequality, we get

$$\sum_{j=1}^{N} \omega_j^2 \frac{d}{dt}(\xi, \xi)_j \lesssim h^{2(k+2)}\|u\|_{k+3,\infty}^2 + |||\xi|||_f^2 + h^{-1}\|e\|_{0,\infty} \sum_{j=1}^{N} \omega_j^2 \|e\|_{0,\infty,\Lambda_j}^2.$$

Then the desired result (82) follows from the following chain rule of derivative

$$\frac{1}{2}\partial_t(f'(u))^2 = f'(u)\partial_t f'(u) = f'(u)f''(u)u_t = -(f'(u))^2 f''(u)u_x.$$

The proof is complete. ∎

With the energy estimate established in the above theorem, we are ready to present the superconvergence properties of the DG flux function.

THEOREM 16. *Let $u \in W^{k+3,\infty}$ be the solution of (1) satisfying $|D^{k+3}f(u)| \lesssim 1$, and $u_h$ be the solution of (2) satisfying the error assumption (46), and the initial discretization is chosen as $u_h(x, 0) = P_h u_0(x)$ with $P_h$ defined in (57). Denote*

$$\nu = \min(1, k - \frac{1}{2} + \frac{1}{2m'})$$

*with $m'$ the same as in (66). Then we have the following superconvergence results:*

- *The DG flux function $f(u_h)$ is superconvergent to the flux function $f(P_h u)$, with an order of $k+1+\nu$, i.e.,*

$$e_{f,p} = \|f(u_h) - f(P_h u)\|_0 \lesssim h^{k+1+\nu} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3}. \tag{87}$$

- *The cell average of the DG flux function approximation is superconvergent with an order of $k+1+\nu$, i.e.,*

(88)

$$e_{f,c} = \left(\frac{1}{N}\sum_{j=1}^{N}\Big(\frac{1}{h_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}}(f(u) - f(P_h u))dx\Big)^2\right)^{\frac12} \lesssim h^{k+1+\nu} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3}.$$

- *The function value approximation of the flux function is superconvergent at a class of special points $y_{j,i}$, i.e.,*

(89)

$$e_{f,r} := \left(\frac{1}{Nk}\sum_{j=1}^{N}\sum_{i=1}^{k}(f(u_h) - f(u))^2(y_{j,i},t)\right)^{\frac12} \lesssim h^{k+1+\nu} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3},$$

*where $y_{j,i}$ are the same as in (59).*

- *The derivative value approximation of DG solution is superconvergent at the points $z_{j,i}$, i.e.,*

(90)

$$e_{f,l} := \left(\frac{1}{Nk}\sum_{j=1}^{N}\sum_{i=1}^{k}(f'(u)\partial_x(u-u_h))^2(z_{j,m},t)\right)^{\frac12} h^{k+\nu} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3}.$$

*Here $z_{j,i}$ are the same as in (60).*

*Proof.* First, by (73) and (56), we have

$$\|e\|_{0,\infty} \le \|u - P_h u\|_{0,\infty} + h^{-\frac12}\|u_h - P_h u\|_0 \lesssim h^{k+\frac12+\frac{1}{2m'}}\|u\|_{k+3,\infty}.$$

Consequently,

$$\begin{aligned}
&h^{-1}\|e\|_{0,\infty}^2 \sum_{j=1}^{N}\omega_j^2\|e\|_{0,\infty,\Lambda_j}^2\\
\lesssim\ & h^{-1}\|e\|_{0,\infty}^2 \sum_{j=1}^{N}\omega_j^2(h^{-1}\|P_h u - u_h\|_{0,\Lambda_j}^2 + \|u - P_h u\|_{0,\infty,\Lambda_j})\\
\lesssim\ & h^{-2}\|e\|_{0,\infty}^2(\|P_h u - u_h\|_0^2 + \|u - P_h u\|_{0,\infty}^2) \lesssim h^{2(2k+\frac12+\frac{1}{2m'})} \sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty}^2.
\end{aligned}$$

Substituting the above inequality into (82), we get

$$\frac{d}{dt}|||\xi|||_f^2 \lesssim |||\xi|||_f^2 + h^{4k+1+\frac{1}{m'}} \sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+2}^2 + h^{2(k+2)}\|u\|_{k+3,\infty}^2.$$

In light of the first inequality of (85) and the special initial discretization, we have

$$\begin{aligned}
|||\xi(\cdot,t)|||_f &\lesssim |||\xi(\cdot,0)|||_f + h^{k+1+\min(1,k-\frac12+\frac{1}{2m'})} \sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty}\\
&\lesssim h^{k+1+\nu} \sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+3,\infty},
\end{aligned}$$

and thus,

$$|||(u_h - P_h u)(\cdot,t)|||_f \le |||\xi(\cdot,t)|||_f + |||w|||_f \lesssim h^{k+1+\nu} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3,\infty}.$$

By Taylor expansion with respect to the variable $u$, we have

$$f(u_h) - f(P_h u) = f'(u)(u_h - P_h u) - \frac{1}{2}\bar{f}''_u (u_h - u)^2 + \frac{1}{2}\bar{\bar{f}}''_u (P_h u - u)^2, \tag{91}$$

where $\bar{\bar{f}}''_u = f''(\alpha_1 u + (1-\alpha_1)P_h u)$ with $0 \le \alpha_1 \le 1$. Consequently, by (56)

$$\begin{aligned}\|f(u_h) - f(P_h u)\|_0 &\lesssim \left(\sum_{j=1}^N \int_{\tau_j} (f'(u))^2 (u_h - P_h u)^2 dx\right)^{\frac{1}{2}} + \|e\|_{0,\infty}^2 + \|u - P_h u\|_{0,\infty}^2 \\ &\lesssim |||u_h - P_h u|||_f + h\|u_h - P_h u\|_0 + h^{k+2} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3} \\ &\lesssim h^{k+1+\nu} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+3}.\end{aligned}$$

This finishes the proof of (87).

Now we prove (88). Recalling the definition of $P_h u$, we have for $P_h u|_{\tau_j} = R_h u, P_h^{\pm} u$,

$$\int_{\tau_j} f'(u)(P_h u - u)dx = \int_{\tau_j} (f'(u) - f'(u_{j+\frac{1}{2}}))(P_h u - u)dx.$$

If $P_h u|_{\tau_j} = Q_h u$, which indicates $f'_{j+\frac{1}{2}} f'_{j-\frac{1}{2}} \le 0$, then there exists at least one point $\bar{u}_j \in \tau_j$ such that $f'(\bar{u}_j) = 0$, and thus,

$$\int_{\tau_j} f'(u)(P_h u - u)dx = \int_{\tau_j} (f'(u) - f'(\bar{u}_j))(P_h u - u)dx.$$

Consequently,

$$\left|\int_{\tau_j} f'(u)(P_h u - u)dx\right| \lesssim h^{\frac{3}{2}} \|u - P_h u\|_{0,\tau_j},$$

which yields

$$\begin{aligned}\left|\int_{\tau_j} (f(P_h u) - f(u))dx\right| &= \left|\int_{\tau_j} f'(u)(P_h u - u) - \frac{1}{2}\bar{f}''_u (P_h u - u)^2 dx\right| \\ &\lesssim h^{\frac{3}{2}} \|u - P_h u\|_{0,\tau_j} + \|u - P_h u\|_{0,\tau_j}^2.\end{aligned}$$

Then

$$\begin{aligned}&\left(\frac{1}{N}\sum_{j=1}^N \Big(\frac{1}{h_j}\int_{\tau_j} (f(u) - f(P_h u))dx\Big)^2\right)^{\frac{1}{2}} \\ &\lesssim h\|u - P_h u\|_0 + h^{-\frac{1}{2}} \|u - P_h u\|_0 \|u - P_h u\|_{0,\infty} \lesssim h^{k+2} \|u\|_{k+2}.\end{aligned}$$

Noticing that

$$e_{f,c} \lesssim \left(\frac{1}{N}\sum_{j=1}^{N}\Big(\frac{1}{h_j}\int_{\tau_j}(f(u)-f(P_hu))dx\Big)^2\right)^{\frac12} + \left(\frac{1}{N}\sum_{j=1}^{N}\Big(\frac{1}{h_j}\int_{\tau_j}(f(u_h)-f(P_hu))dx\Big)^2\right)^{\frac12}$$
$$\lesssim \|f(u_h)-f(P_hu)\|_0 + h^{k+2}\|u\|_{k+2},$$

then (88) follows directly from (87).

By the triangle inequality, (91),and (59), there holds

$$e_{f,r} \le \left(\frac{1}{Nk}\sum_{j=1}^{N}\sum_{i=1}^{k}(f(u_h)-f(P_hu))^2(y_{j,i},t)\right)^{\frac12} + \left(\frac{1}{Nk}\sum_{j=1}^{N}\sum_{i=1}^{k}(f(u)-f(P_hu))^2(y_{j,i},t)\right)^{\frac12}$$
$$\lesssim \left(h\sum_{j=1}^{N}\sum_{i=1}^{k}(f'(u)(P_hu-u_h))^2(y_{j,i},t)\right)^{\frac12} + \|e\|_{0,\infty}^2 + \|u-P_hu\|_{0,\infty}^2 + h^{k+2}\|u\|_{k+2,\infty}$$
$$\lesssim |||P_hu-u_h\|_f + h^{k+2}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+2,\infty}.$$

Following the same argument, we can show

$$e_{f,l} \lesssim h^{-1}|||P_hu-u_h\|_f + h^{k+1}\sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{k+2,\infty}.$$

Then (89)-(90) follow. This finishes our proof. □

**5. Numerical experiments.** In this section, we use numerical examples to verify the theorems in Sections 3 and 4. If not otherwise stated, we use the classical fourth-order Runge-Kutta methods for time discretization with $\Delta t = 0.01h_{min}^3$ to reduce the time error.

**Example 1.** We solve the following model problem

$$\begin{aligned} &u_t + (\tfrac{u^2}{2})_x = g(x,t), && (x,t)\in[0,2\pi]\times(0,0.3],\\ &u(x,0) = \sin(x)+\tfrac32, && x\in[0,2\pi], \end{aligned} \tag{92}$$

with the periodic boundary condition $u(0,t) = u(2\pi,t)$. The source term $g(x,t)$ is specially chosen such that the exact solution is

$$u(x,t) = \sin(x+t) + \frac{3}{2}.$$

As we may observe in our analysis, the source term does not affect the superconvergence results for problems with fixed wind directions. In other words, the superconvergence results established in Theorem 8 still hold true for the hyperbolic equation $u_t + f(u)_x = g(x,t)$ with fixed wind directions.

The model problem is tested by using $\mathbb{P}_k$ polynomials with $k = 1,2,3$. We compute the numerical solution at $t = 0.3$. The initial discretization is given by the same way as in Remark 10. In Table 1, we list several errors between the numerical approximation and the exact solution in Theorem 8.

Table 1 demonstrates superconvergence rates of $(2k+1)$-th order for the numerical cell average and numerical flux ($e_c$ and $e_n$), $(k+2)$-th order for the numerical solution at the downwind-biased Radau points ($e_r$) and towards the $P_h^-$ projection of the exact

TABLE 1
*Various errors in Theorem 8 for $k = 1, 2, 3$.*

| | | $e_c$ | | $e_n$ | | $e_r$ | | $e_p$ | | $e_l$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $n$ | error | order | error | order | error | order | error | order | error | order |
| 1 | 40 | 1.51e-04 | – | 1.23e-04 | – | 1.31e-04 | – | 6.14e-05 | – | 5.04e-04 | – |
| | 80 | 1.95e-05 | 2.95 | 1.48e-05 | 3.06 | 1.56e-05 | 3.07 | 7.89e-06 | 2.96 | 1.42e-04 | 1.83 |
| | 160 | 2.49e-06 | 2.97 | 1.79e-06 | 3.05 | 1.88e-06 | 3.05 | 1.01e-06 | 2.97 | 3.83e-05 | 1.89 |
| | 320 | 3.15e-07 | 2.98 | 2.19e-07 | 3.03 | 2.31e-07 | 3.03 | 1.28e-07 | 2.98 | 9.99e-06 | 1.94 |
| 2 | 40 | 5.51e-07 | – | 4.46e-07 | – | 1.00e-06 | – | 5.98e-07 | – | 2.04e-05 | – |
| | 80 | 1.76e-08 | 4.97 | 1.44e-08 | 4.95 | 4.99e-08 | 4.33 | 3.68e-08 | 4.03 | 2.38e-06 | 3.10 |
| | 160 | 5.56e-10 | 4.98 | 4.57e-10 | 4.98 | 2.83e-09 | 4.14 | 2.313e-09 | 4.00 | 2.97e-07 | 3.00 |
| | 320 | 1.75e-11 | 4.99 | 1.44e-11 | 4.99 | 1.70e-10 | 4.05 | 1.44e-10 | 4.00 | 3.71e-08 | 3.00 |
| 3 | 40 | 2.58e-08 | – | 2.40e-08 | – | 2.46e-08 | – | 5.39e-09 | – | 5.73e-07 | – |
| | 80 | 3.08e-10 | 6.39 | 1.32e-10 | 7.51 | 4.16e-10 | 5.89 | 1.83e-10 | 4.88 | 3.01e-08 | 4.25 |
| | 160 | 4.43e-12 | 6.12 | 1.18e-12 | 6.81 | 9.99e-12 | 5.38 | 5.72e-12 | 5.00 | 1.59e-09 | 4.24 |
| | 320 | 3.38e-14 | 7.03 | 8.02e-15 | 7.20 | 2.35e-13 | 5.41 | 1.76e-13 | 5.02 | 8.33e-11 | 4.26 |

solution, and , $(k+1)$-th order for the derivative approximation of the numerical solution at the left Radau points ($e_l$). All convergent rates match our theoretical error bounds in Theorem 8.

**Example 2.** We solve the following problem

$$\begin{aligned} &u_t + (\tfrac{u^2}{2})_x = 0, && (x,t) \in [0, 2\pi] \times (0, 0.3], \\ &u(x,0) = \sin(x) + \tfrac{1}{2}, && x \in [0, 2\pi], \end{aligned} \tag{93}$$

with periodic boundary condition $u(0,t) = u(2\pi, t)$. Note that the wind direction is alternating, and at the zeros $x_0 = \frac{7\pi}{6}, \frac{11\pi}{6}$ of $f'(u_0) = u_0 = \sin(x_0) + \frac{1}{2}$,

$$\partial_x f'(u_0) = \cos(x_0) \neq 0,$$

which indicates that (65) holds with $m = 1$. Then

$$m' = \min(m, k+3) = 1.$$

This problem is different from Example 1 with fixed wind direction, where the source term may have effect on the superconvergence result for the DG solution including the flux function. As the exact solution is unknown, we adopt the Newton iteration method to obtain a "psudo-exact" solution of (93) in our numerical experiment.

Again, we compute the numerical approximation at $t = 0.3$. The initial discretization is given as $u_h = P_h u$, where $P_h u$ was given in (57). We use Godunov fluxes to test the example by using $\mathbb{P}_k$ polynomials with $k = 1, 2, 3$. In table 2, we list several errors of the flux function in Theorem 16.

Table 2 demonstrates superconvergence rates of $(k+2)$-th order for $e_{f,p}$ and $e_{f,r}$, which is consistent with the estimates in Theorem 16. As for the cell-average error $e_{f,c}$, we observe a convergence rate of $k+2$ for $k = 1$, and $k + \frac{5}{2}$ for $k = 2, 3$, which is $\frac{1}{2}$ order higher than our theoretical results for $k = 2, 3$.

Next, we proceed to test Theorem 14 and the results are given in Table 3, where we observe superconvergence rates of $(k+3/2)$-th order for $e_p$, $e_c$, and $e_r$. All convergent rates in Table 3 match our theoretical error bounds in Theorem 14 with $m' = 1$.

Table 2
*Various errors in Theorem 16 for $k = 1, 2, 3$.*

| | | $e_{f,p}$ | | $e_{f,c}$ | | $e_{f,r}$ | |
|---|---|---|---|---|---|---|---|
| $k$ | $n$ | error | order | error | order | error | order |
| 1 | 40 | 3.68e-04 | – | 2.96e-04 | – | 1.82e-04 | – |
| | 80 | 4.72e-05 | 2.96 | 3.97e-05 | 2.90 | 2.30e-05 | 2.98 |
| | 160 | 5.99e-06 | 2.98 | 5.11e-06 | 2.96 | 2.90e-06 | 2.99 |
| | 320 | 7.54e-07 | 2.99 | 6.46e-07 | 2.98 | 3.64e-07 | 2.99 |
| 2 | 40 | 1.18e-05 | – | 1.93e-06 | – | 8.81e-06 | – |
| | 80 | 7.77e-07 | 3.93 | 7.42e-08 | 4.70 | 6.17e-07 | 3.84 |
| | 160 | 4.95e-08 | 3.97 | 2.69e-09 | 4.79 | 4.06e-08 | 3.93 |
| | 320 | 3.11e-09 | 3.99 | 9.63e-11 | 4.80 | 2.59e-09 | 3.97 |
| 3 | 40 | 2.76e-07 | – | 2.64e-08 | – | 2.16e-07 | – |
| | 80 | 9.14e-09 | 4.92 | 4.28e-10 | 5.95 | 7.43e-09 | 4.86 |
| | 160 | 2.98e-10 | 4.94 | 7.84e-12 | 5.77 | 2.43e-10 | 4.93 |
| | 320 | 9.59e-12 | 4.96 | 1.64e-13 | 5.58 | 7.91e-12 | 4.94 |

Table 3
*Various errors in Theorem 14 for $k = 1, 2, 3$.*

| | | $e_p$ | | $e_c$ | | $e_r$ | |
|---|---|---|---|---|---|---|---|
| $k$ | $n$ | error | order | error | order | error | order |
| 1 | 20 | 2.98e-03 | - | 2.20e-03 | - | 3.36e-03 | - |
| | 40 | 5.41e-04 | 2.46 | 4.32e-04 | 2.35 | 5.95e-04 | 2.50 |
| | 80 | 9.91e-05 | 2.45 | 7.10e-05 | 2.61 | 1.07e-04 | 2.48 |
| | 160 | 1.79e-05 | 2.47 | 1.27e-05 | 2.48 | 1.80e-05 | 2.57 |
| | 320 | 3.23e-06 | 2.47 | 2.19e-06 | 2.54 | 3.37e-06 | 2.42 |
| 2 | 20 | 2.07e-04 | - | 3.90e-05 | - | 2.72e-04 | - |
| | 40 | 2.07e-05 | 3.32 | 2.40e-06 | 4.02 | 2.47e-05 | 3.46 |
| | 80 | 1.64e-06 | 3.66 | 1.44e-07 | 4.06 | 2.18e-06 | 3.50 |
| | 160 | 1.37e-07 | 3.58 | 1.01e-08 | 3.83 | 1.82e-07 | 3.58 |
| | 320 | 1.10e-08 | 3.64 | 8.26e-10 | 3.61 | 1.53e-08 | 3.57 |
| 3 | 20 | 7.64e-06 | - | 1.27e-06 | - | 1.38e-05 | - |
| | 40 | 75.09-07 | 3.91 | 6.69e-08 | 4.24 | 7.41e-07 | 4.22 |
| | 80 | 2.64e-08 | 4.27 | 3.04e-09 | 4.46 | 3.57e-08 | 4.37 |
| | 160 | 1.30e-09 | 4.34 | 1.33e-10 | 4.51 | 1.76e-09 | 4.35 |
| | 320 | 6.09e-11 | 4.42 | 5.91e-12 | 4.49 | 7.98e-11 | 4.46 |

**Example 3.** We solve the following problem

$$\begin{aligned} &u_t + (\sin(u))_x = 0, && (x,t) \in [0, 2\pi] \times (0, 0.1], \\ &u(x,0) = \sin(x), && x \in [0, 2\pi], \end{aligned} \tag{94}$$

with the periodic boundary condition $u(0,t) = u(2\pi, t)$. The example is tested by using $\mathbb{P}_k$ polynomials with $k = 1, 2, 3$. We compute the numerical solution at $t = 0.1$. In Table 4, we compute several errors between the numerical approximation and the exact solution in Theorem 8.

Table 4 demonstrates superconvergence rates of $(2k+1)$-th order for the numerical cell average and numerical flux ($e_c$ and $e_n$), $(k+2)$-th order for the numerical solution at the downwind-biased Radau points ($e_r$) and towards the $P_h^-$ projection of the

Table 4
*Various errors in Theorem 8 for $k = 1, 2, 3$.*

| | | $e_c$ | | $e_n$ | | $e_r$ | | $e_p$ | |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | $n$ | error | order | error | order | error | order | error | order |
| 1 | 20 | 9.81e-04 | - | 3.62e-04 | - | 1.54e-03 | - | 8.35e-04 | - |
| | 40 | 1.18e-04 | 3.05 | 7.05e-05 | 2.36 | 1.72e-04 | 3.15 | 1.02e-04 | 3.03 |
| | 80 | 1.98e-05 | 2.58 | 7.82e-06 | 3.17 | 3.00e-05 | 2.52 | 1.30e-05 | 2.97 |
| | 160 | 2.55e-06 | 2.96 | 8.94e-07 | 3.12 | 3.93e-06 | 2.93 | 1.62e-06 | 3.01 |
| | 320 | 3.23e-07 | 2.98 | 1.06e-07 | 3.07 | 4.85e-07 | 3.02 | 2.01e-07 | 3.01 |
| 2 | 20 | 1.73e-05 | - | 5.66e-06 | - | 6.21e-05 | - | 3.40e-05 | - |
| | 40 | 8.18e-07 | 4.40 | 5.29e-07 | 3.42 | 4.91e-06 | 3.66 | 2.51e-06 | 3.76 |
| | 80 | 2.80e-08 | 4.87 | 1.92e-08 | 4.78 | 2.61e-07 | 4.23 | 1.42e-07 | 4.15 |
| | 160 | 9.83e-10 | 4.83 | 4.71e-10 | 5.35 | 2.29e-08 | 3.52 | 8.92e-09 | 3.99 |
| | 320 | 3.50e-11 | 4.81 | 1.51e-11 | 4.96 | 1.57e-09 | 3.86 | 5.69e-10 | 3.97 |
| 3 | 20 | 6.75e-07 | - | 8.41e-07 | - | 1.83e-06 | - | 1.46e-06 | - |
| | 40 | 4.55e-09 | 7.21 | 5.80e-09 | 7.18 | 6.27e-08 | 4.87 | 4.38e-08 | 5.06 |
| | 80 | 2.95e-11 | 7.27 | 4.12e-11 | 7.14 | 2.08e-09 | 4.92 | 1.36e-09 | 5.00 |
| | 160 | 1.98e-13 | 7.22 | 2.93e-13 | 7.14 | 6.60e-11 | 4.98 | 4.24e-11 | 5.01 |
| | 320 | 1.36e-15 | 7.20 | 2.19e-15 | 7.06 | 2.07e-12 | 4.99 | 1.32e-12 | 5.00 |

exact solution. All convergent rates in Table 4 match our theoretical error bounds in Theorem 8.

**Example 4.** We solve the following problem

$$\begin{aligned} &u_t + (\sin(u))_x = 0, && (x,t) \in [0, 2\pi] \times (0, 0.1], \\ &u(x,0) = \sin(x) + 1, && x \in [0, 2\pi], \end{aligned} \tag{95}$$

with periodic boundary condition $u(0,t) = u(2\pi, t)$. We compute the numerical approximation at $t = 0.1$. We use Godunov fluxes test the example by using $\mathbb{P}_k$ polynomials with $k = 1, 2, 3$. In Table 5, we compute several errors of the flux function in Theorem 16. Note that $m' = 1$ in this case.

Table 5 demonstrates superconvergence rates of $(k+2)$-th order for $e_{f,p}$, $e_{f,c}$ and $e_{f,r}$. Next, we proceed to test Theorem 14 and the results are given in Table 6.

Table 6 demonstrates superconvergence rates of $(k+3/2)$-th order for $e_p$, $e_c$ and $e_r$. All convergent rates in Table 6 match our theoretical error bounds in Theorem 14.

**6. Conclusion.** In this work, we have studied superconvergence properties of the DG method for scalar nonlinear hyperbolic equations with fixed and alternating wind directions. For the problems with fixed wind direction, we established the same superconvergence results as those of the counterpart linear problems in [7]. That is, we provided a theoretic proof of the $2k+1$-th superconvergence rate of the DG solution at the downwind points and for the cell average, and $k+2$-th supercpnvergence rate of the DG solution at the right Radau points, and the derivative superconvergene orde $k+1$ at all interior left Radau points (see the conclusions in Theorem 8). As for the problems with alternating direction, we proved that the superconvergence rate $2k+1$ is no longer valid, and the superconvergence rate of the DG solution is dependent upon the specific properties of the flux function $f'(u)$, and the highest convergence order is $k + \frac{3}{2}$. We also presented the superconvergence properties of the DG flux

TABLE 5
*Various errors in Theorem 16 for $k = 1, 2, 3$.*

| | | $e_{f,p}$ | | $e_{f,c}$ | | $e_{f,r}$ | |
|---|---|---|---|---|---|---|---|
| $k$ | $n$ | error | order | error | order | error | order |
| 1 | 20 | 8.81e-04 | - | 6.84e-04 | - | 1.59e-03 | - |
| | 40 | 1.32e-04 | 2.74 | 8.07e-05 | 3.08 | 1.82e-04 | 3.13 |
| | 80 | 1.85e-05 | 2.83 | 9.86e-06 | 3.03 | 2.33e-05 | 2.97 |
| | 160 | 2.47e-06 | 2.90 | 1.25e-06 | 2.97 | 3.32e-06 | 2.81 |
| | 320 | 3.20e-07 | 2.95 | 1.64e-07 | 2.93 | 5.26e-07 | 2.66 |
| 2 | 20 | 5.84e-05 | - | 1.40e-05 | - | 5.22e-05 | - |
| | 40 | 4.92e-06 | 3.57 | 7.23e-07 | 4.27 | 4.08e-06 | 3.68 |
| | 80 | 3.24e-07 | 3.92 | 3.18e-08 | 4.51 | 2.79e-07 | 3.87 |
| | 160 | 1.97e-08 | 4.04 | 1.24e-09 | 4.68 | 1.72e-08 | 4.02 |
| | 320 | 1.23e-09 | 3.99 | 4.84e-11 | 4.68 | 1.09e-09 | 3.97 |
| 3 | 20 | 2.08e-06 | - | 3.13e-07 | - | 1.63e-06 | - |
| | 40 | 8.39e-08 | 4.63 | 7.11e-09 | 5.46 | 6.47e-08 | 4.66 |
| | 80 | 2.32e-09 | 5.17 | 1.31e-10 | 5.76 | 1.85e-09 | 5.13 |
| | 160 | 8.47e-11 | 4.78 | 3.55e-12 | 5.21 | 6.80e-11 | 4.77 |
| | 320 | 2.68e-12 | 4.98 | 9.27e-14 | 5.26 | 2.17e-12 | 4.97 |

TABLE 6
*Various errors in Theorem 14 for $k = 1, 2, 3$.*

| | | $e_p$ | | $e_c$ | | $e_r$ | |
|---|---|---|---|---|---|---|---|
| $k$ | $n$ | error | order | error | order | error | order |
| 1 | 20 | 1.29e-03 | - | 2.06e-03 | - | 2.99e-03 | - |
| | 40 | 2.44e-04 | 2.40 | 4.21e-04 | 2.29 | 5.99e-04 | 2.32 |
| | 80 | 4.69e-05 | 2.38 | 7.93e-05 | 2.41 | 1.11e-04 | 2.43 |
| | 160 | 8.75e-06 | 2.42 | 1.45e-05 | 2.46 | 1.98e-05 | 2.49 |
| | 320 | 1.62e-06 | 2.43 | 2.50e-06 | 2.53 | 3.59e-06 | 2.46 |
| 2 | 20 | 8.51e-05 | - | 8.91e-06 | - | 1.59e-04 | - |
| | 40 | 9.15e-06 | 3.22 | 6.50e-07 | 3.78 | 1.46e-05 | 3.44 |
| | 80 | 8.19e-07 | 3.48 | 4.35e-08 | 3.90 | 1.30e-06 | 3.49 |
| | 160 | 6.90e-08 | 3.57 | 2.90e-09 | 3.91 | 1.12e-07 | 3.55 |
| | 320 | 5.99e-09 | 3.53 | 2.87e-10 | 3.33 | 9.83e-09 | 3.50 |
| 3 | 20 | 2.98e-06 | - | 2.87e-07 | - | 4.21e-06 | - |
| | 40 | 1.51e-07 | 4.31 | 1.12e-08 | 4.68 | 2.09e-07 | 4.33 |
| | 80 | 7.63e-09 | 4.30 | 4.71e-10 | 4.57 | 1.04e-08 | 4.32 |
| | 160 | 3.92e-10 | 4.28 | 2.14e-11 | 4.46 | 5.26e-09 | 4.31 |
| | 320 | 1.89e-11 | 4.37 | 9.88e-13 | 4.44 | 2.50e-11 | 4.40 |

function and showed that the superconvergence rate of the error $f(u) - f(u_h)$ under the $L^2$ norm can arrive at $k+2$ in most of the case (see Theorem 16). As a by-product, superconvergence of the DG solution and the DG flux function were established at some special points (see Theorems 14-16, (51)-(52) and (89)- (90) ). Our ongoing work include the nonlinear problems with high order derivatives (e.g., KdV equations) and higher dimensional nonlinear hyperbolic equations.

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