ANDERSON-THAKUR POLYNOMIALS AND MULTIZETA VALUES IN POSITIVE CHARACTERISTIC

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Abstract. Multizeta values in positive characteristic were first introduced and studied by Thakur. He and Lara Rodríguez [9] discovered and conjectured certain zeta-like families. Kuan, Lin and Yu stated more conjectures about zeta-like multizeta values in [7]. In the present paper we study and give the transparent formula for certain Anderson-Thakur polynomials. This enables us to confirm the conjectured zeta-like families.

1. Introduction

The study of arithmetic of zeta values begins by Euler's famous evaluations: for $m \in \mathbb{N}$,

$$\zeta(2m) = \frac{-B_{2m}\left(2\pi\sqrt{-1}\right)^{2m}}{2(2m)!},$$

where $B_{2m} \in \mathbb{Q}$ are Bernoulli numbers. Euler's formula implies that $\zeta(n)/(2\pi\sqrt{-1})^n$ is rational if and only if n is even. The multiple zeta values $\zeta(s_1, \cdots, s_r)$, where $s_1, \dots, s_r$ are positive integers with $s_1 \geq 2$, are generalizations of zeta values. These numbers were first studied by Euler for the case of $r = 2$. Although there exist simple relations between zeta and multiple zeta values, such as $\zeta(2,1) = \zeta(3)$, sorting out all relations among these multiple zeta values is a much involved problem. Here r is called the depth and $w := \sum_{i=1}^{r} s_i$ is called the weight of $\zeta(s_1, \dots, s_r)$. We call $\zeta(s_1, \dots, s_r)$ *Eulerian* if the ratio $\zeta(s_1, \dots, s_r)/\left(2\pi\sqrt{-1}\right)^w$ is rational.

Carlitz introduced and derived an analogue of Euler's formula for what we now called Carlitz zeta values $\zeta_A(n)$ for $n \geq 1$. Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ and $K = \mathbb{F}_q(\theta)$ be its quotient field. Let t be a variable independent of θ. Let $\mathbf{C}$ be the Carlitz module and $\widetilde{\pi}$ is a fundamental period of $\mathbf{C}$. The Carlitz exponential function is defined by $\exp_{\mathbf{C}}(z) = \sum_{n\geq 0} \dfrac{z^{q^n}}{D_n}$, where $D_n = \prod_{i=0}^{n-1}(\theta^{q^n} - \theta^{q^i})$. We denote by $\Gamma_{n+1} \in A$ the Carlitz factorials and $BC(n) \in K$ by the Bernoulli-Carlitz numbers. Carlitz showed that

$$\zeta_A(n) := \sum_{a\in A+} \frac{1}{a^n} = \frac{BC(n)}{\Gamma_{n+1}}\widetilde{\pi}^n$$

if $q-1|n$. Carlitz's result implies that $\zeta_A(n)/\tilde{\pi}^n$ is rational in K if and only if $q-1|n$.

Anderson and Thakur [1] related the interesting value $\zeta_A(n)$ to a special integral point Z_n in $\mathbf{C}^{\otimes n}(A)$ via the logarithm map of $\mathbf{C}^{\otimes n}$, where $\mathbf{C}^{\otimes n}$ denotes the n-th tensor power of the Carlitz module (viewed as a Carlitz-Tate t-motive). As a consequence, one has that $\zeta_A(n)/\tilde{\pi}^n$ is rational if and only if Z_n is an $\mathbb{F}_q[t]$-torsion point, and this condition is equivalent to n being divisible by $q-1$. In [1] a key role is played by a sequence of distinguished polynomials $H_n \in A[t]$, now called the Anderson-Thakur polynomials. On the other hand, Yu [13] also showed that the transcendence of $\zeta_A(n)/\tilde{\pi}^n$ over K is equivalent to Z_n being non-torsion on $\mathbf{C}^{\otimes n}(A)$, whence deriving that $\zeta_A(n)/\tilde{\pi}^n$ is algebraic over K if and only if $\zeta_A(n)/\tilde{\pi}^n$ is rational in K.

In the last decade, Thakur [10, 11] initiated the study of multizeta values $\zeta_A(s_1, \cdots, s_r)$, where $s_1, \ldots, s_r$ are positive integers. He and his co-workers discovered interesting relations among some multizeta values. Call $\zeta_A(s_1, \ldots, s_r)$ *Eulerian* (*zeta-like* resp.) if the ratio $\zeta_A(s_1, \cdots, s_r)/\tilde{\pi}^w$ ($\zeta_A(s_1, \ldots, s_r)/\zeta_A(w)$ resp.) is rational in K. A basic question in this respect is to find all Eulerian/zeta-like multizeta values. In [9], Lara Rodriguez and Thakur gave particularly precise formulas for certain families of Eulerian/zeta-like multizeta values and conjectured other ones. Their conjectures are supported by numerical data from continued fraction computations. On the other hand, Chang [3] also proved the subtle fact that these ratios $\zeta_A(s_1, \cdots, s_r)/\tilde{\pi}^w$, $\zeta_A(s_1, \ldots, s_r)/\zeta_A(w)$ are either rational or transcendental over K.

In an effort to understand relations among multizeta values, Chang, Papanikolas and Yu [4] established an effective criterion for Eulerian/zeta-like multizeta values by constructing an abelian t-module E' defined over A and relating the values $\zeta_A(s_1, \cdots, s_r)$, $\zeta_A(w)$ to specific integral points $\mathbf{v}_{\mathfrak{s}}$, $\mathbf{u}_{\mathfrak{s}}$ on $E'(A)$. They proved that $\zeta_A(s_1, \cdots, s_r)$ is Eulerian (zeta-like) if and only if $\mathbf{v}_{\mathfrak{s}}$ is an $\mathbb{F}_q[t]$- torsion point (respectively, $\mathbf{u}_{\mathfrak{s}}$ and $\mathbf{v}_{\mathfrak{s}}$ have an $\mathbb{F}_q[t]$-linear relation inside $E'(A)$). The integral points $\mathbf{v}_{\mathfrak{s}}$, $\mathbf{u}_{\mathfrak{s}}$ are constructed using the Anderson-Thakur polynomials. Their theory connects possible $\mathbb{F}_q(\theta)$-linear relation of $\zeta_A(s_1, \cdots, s_r)$ and $\zeta_A(w)$ explicitly with the possible $\mathbb{F}_q[t]$-linear relation among $\mathbf{v}_{\mathfrak{s}}$ and $\mathbf{u}_{\mathfrak{s}}$ inside $E'(A)$.

Just recently, Kuan-Lin [7] implemented algorithms basing on the criterion of Chang-Papanikolas-Yu. They have collected more extensive data on zeta-like and Eulerian multizeta values over the polynomial rings $\mathbb{F}_q[\theta]$. Particularly in [4, 9], a conjectured rule is spelled out to specify all Eulerian multizeta values. Lists given in [7] suggest more families of zeta-like multizeta values of arbitrary depth. These families are not covered by [9]. It is observed that there should be only a few zeta-like families in higher depth, because of the conjectured "splicing" condition (cf. [9]). Finding all zeta-like multizeta values is now in sight.

Inspired by this development we study Anderson-Thakur polynomials in more details in this paper, for the purpose of deriving exact rational ratio between $\zeta_A(s_1, \cdots, s_r)$ and $\zeta_A(w)$ whenever such a ratio exists. In particular, we are able to verify : (1) Conjecture 4.6 of [9], (2) Conjecture 5 of [7], (3) the conjectured list of all Eulerian multizeta values given in [4], Section 6.2, are indeed Eulerian. The strategy for proving zeta-like property for given multizeta values is to handle recurrence relations among Anderson-Thakur polynomials H_n. In view of the fact that these H_n are polynomials in both θ and t over $\mathbb{F}_q$, we use Lucas Theorem to establish q-th power recurrence when n has particular q-adic "shape". Combining with the obvious linear recurrence relating H_n to H_{n-q^i}, we eventually arrive at more transparent formulas for H_n.

The contents of this paper are arranged as follows. In Section 2, we set up preliminaries and introduce the conjectured families of zeta-like multizeta values given in [7] and [9], which we will prove later. In Section 3, we use generalized Lucas Theorem [6, p.75-76] to study Anderson-Thakur polynomials. Then in Section 4 we apply Chang-Papanikolas-Yu's theorem [4, Theorem 2.5.2] to verify that all previously conjectured families of zeta-like multizeta values are indeed zeta-like with exact formulas given in Theorem 4.4. At the end of this paper we provide 'recursive' relations for two very special families of multizeta values and derive that they are Eulerian (Theorem 5.1, 5.2) in Section 5.

2. Preliminaries for Multizeta values

2.1. **Notation.** We adopt the notation below in the following chapters.

$\mathbb{F}_q$ = a finite field with $q = p^l$ elements.
K = $\mathbb{F}_q(\theta)$, the rational function field in the variable θ.
∞ = zero of $1/\theta$, the infinite place of K.
$|\ |$ = the nonarchimedean absolute value on K corresponding to ∞.
K_∞ = $\mathbb{F}_q((1/\theta))$, the completion of K with respect to the absolute value $|\cdot|$.
A = $\mathbb{F}_q[\theta]$, the ring of polynomials in the variable θ.
A_+ = the set of monic polynomials in A.
A_d = the set of polynomials in A of degree d.
A_{d+} = $A_d \cap A_+$, the set of monic polynomials in A of degree d.
$[n] = \theta^{q^n} - \theta$.
$D_n = \prod_{i=0}^{n-1} \theta^{q^n} - \theta^{q^i} = [n][n-1]^q \cdots [1]^{q^{n-1}}$.
$L_n = \prod_{i=1}^{n} \theta^{q^i} - \theta = [n][n-1]\cdots[1]$.
$l_n = (-1)^n L_n$.
t = a variable independent of θ.

2.2. **Multizeta values.** For $s \in \mathbb{N}$ and $d \in \mathbb{Z}_{\geq 0}$, put

$$S_d(s) = \sum_{a \in A_{d+}} \frac{1}{a^s} \in K.$$

For $s \in \mathbb{N}$ the Carlitz-Goss zeta values are defined by

$$\zeta_A(s) = \sum_{d=0}^{\infty} S_d(s) = \sum_{a \in A_+} \frac{1}{a^s} \in K_\infty.$$

For a given tuple $(s_1, \cdots, s_r) \in \mathbb{N}^r$, the Thakur multizeta values of depth r and weight $w = \sum s_i$ are defined by

$$\zeta_A(s_1, \cdots, s_r) = \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum_{\substack{a_i \in A_+ \\ \deg a_1 > \cdots > \deg a_r \geq 0}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}}.$$

2.3. **Bernoulli-Carlitz numbers $BC(n)$.** For a non-negative integer n, we express n as

$$n = \sum_{i=0}^{\infty} n_i q^i \quad (0 \leq n_i \leq q-1,\ n_i = 0 \text{ for } i \gg 0),$$

and we recall the definition of the arithmetic Γ-function $\Gamma_{n+1} := \prod_{i=0}^{\infty} D_i^{n_i} \in A$. We denote by $(-\theta)^{\frac{1}{q-1}}$ a fixed $(q-1)$-th root of $-\theta$. Let $\mathbf{C}$ be the Carlitz module and $\tilde{\pi} = (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} (1 - \frac{\theta}{\theta^{q^i}})^{-1}$ be a fundamental period of $\mathbf{C}$. The Carlitz exponential function is defined by $\exp_{\mathbf{C}}(z) = \sum_{n \geq 0} \frac{z^{q^n}}{D_n}$. The Bernoulli-Carlitz numbers $BC(n) \in K$ are defined by

$\frac{z}{\exp_{\mathbf{C}}(z)} = \sum_{n\geq 0} \frac{BC(n)}{\Gamma_{n+1}} z^n.$

When n is 'even' i.e., $q-1|n$, Carlitz derived an analogue of Euler's formula as follows:

Lemma 2.4. (*Carlitz* [2])

(*a*) *For* $n \geq 1$, $\zeta_A(n) = \frac{BC(n)}{\Gamma_{n+1}} \widetilde{\pi}^n$ *if* $q-1|n$.

(*b*) *For* $0 \leq i \leq n$, $BC(q^n - q^i) = \frac{(-1)^{n-i}\Gamma_{q^n-q^i+1}}{L_{n-i}^{q^i}}$.

Combining (*a*), (*b*) we get $\zeta_A(q^n-1) = \frac{(-1)^n}{L_n} \widetilde{\pi}^{q^n-1}$.

2.5. **Anderson-Thakur polynomials.** First we define polynomials $G_i \in \mathbb{F}_q[t,\theta]$ for $i \in \mathbb{Z}_{\geq 0}$. Put $G_0 = 1$. For $i \in \mathbb{N}$, let

$$G_i = \prod_{j=1}^{i} (t^{q^i} - \theta^{q^j}).$$

For $n = 0, 1, 2, \ldots$, we define the sequence of Anderson-Thakur polynomials $H_n \in A[t]$ by the generating function identity

$$\left(1 - \sum_{i=0}^{\infty} \frac{G_i}{D_i|_{\theta=t}} x^{q^i}\right)^{-1} = \sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}|_{\theta=t}} x^n.$$

We note that for $0 \leq n \leq q-1$ we have $H_n = 1$. For any infinite vector $\underline{a} = (a_0, a_1, a_2, \cdots)$ with integers $a_i \geq 0$ and $a_j = 0$ for $j \gg 0$, put $m(\underline{a})$:= last index i such that $a_i \neq 0$. We define $C_{\underline{a}} := \frac{(a_0 + \cdots + a_{m(\underline{a})})!}{a_0! \cdots a_{m(\underline{a})}!}$. For $n \in \mathbb{N}$, a q power weighted partition of n is an infinite vector $\underline{a}$ satisfying $n = \sum_{i=0}^{\infty} a_i q^i$. We have the following lemma giving two ways for explicitly writing Anderson–Thakur polynomials:

Lemma 2.6.

(*a*) *For* $n \in \mathbb{Z}_{\geq 0}$, *let* $S_n = \{\underline{a} \mid n = \sum a_i q^i,\ C_{\underline{a}} \not\equiv 0 \mod p\}$ *denote the set of all possible* q *power weighted partition of* n *with nonzero* $C_{\underline{a}} \mod p$. *Then*

$$\frac{H_n}{\Gamma_{n+1}|_{\theta=t}} = \sum_{\underline{a} \in S_n} C_{\underline{a}} \prod_{i=0}^{\infty} \left(\frac{G_i}{D_i|_{\theta=t}}\right)^{a_i}.$$

(*b*) *For* $n \in \mathbb{N}$,

$$\frac{H_n}{\Gamma_{n+1}|_{\theta=t}} = \sum_{i=0}^{[log_q n]} \frac{G_i}{D_i|_{\theta=t}} \frac{H_{n-q^i}}{\Gamma_{n-q^i+1}|_{\theta=t}}.$$

We will discuss more details about Anderson-Thakur polynomials in Section 3.

2.7. **Revisiting Lucas Theorem.** To compute $C_{\underline{a}} \mod p$, a useful tool is a generalization of the Lucas Theorem.

Theorem 2.8. (*Dickson* [6]) *For any infinite vector* $\underline{a}$, $C_{\underline{a}} \not\equiv 0 \mod p$ *if and only if there is no carrying in computing the sum* $a_0+\cdots+a_{m(\underline{a})}$ *in terms of base* p *expansion. Furthermore, if* $\sum a_i = \sum_{j=0}^{m} n_j p^j$, $a_i = \sum_{j=0}^{m} n_{i,j} p^j$, *with* $0 \leq n_j, n_{i,j} \leq p-1$ *and* $n_j = \sum_{i=0}^{m(\underline{a})} n_{i,j}$. *Then* $C_{\underline{a}} \equiv \prod C_{\underline{n_j}}$ *in* $\mathbb{F}_p$, *where* $\underline{n_j} = (n_{0,j}, \cdots, n_{m(\underline{a}),j}, 0, 0, \cdots)$.

Proof. See [6, p.75-76]. □

By Theorem 2.8 we see that $C_{\underline{a}} \mod p$ can be computed as digits in base p expansion separately. So we try to descend H_n via the maps below. For simplicity we view $C_{\underline{a}}$ as elements in $\mathbb{F}_p$.

Definition 2.9. For any infinite vector $\underline{a}$ with $C_{\underline{a}} \neq 0$, let $\underline{\tilde{a}} = (\tilde{a_0}, \tilde{a_1}, \cdots)$, where $a_i \equiv \tilde{a_i} \mod q$ with $0 \leq \tilde{a_i} \leq q-1$. We define the following 'reduction map' of vectors.

$$r(\underline{a}) := (\frac{a_0 - \tilde{a_0}}{q}, \frac{a_1 - \tilde{a_1}}{q}, \cdots).$$

By Theorem 2.8 we see that $C_{\underline{a}} = C_{\underline{\tilde{a}}} C_{r(\underline{a})}$.

2.10. **Binomial series to the Carlitz module.** For $k \in \mathbb{Z}_{\geq 0}$, let $\Psi_k(x)$ be the polynomials in $K[x]$ defined by

$$\exp_{\mathbf{C}}(x \log_{\mathbf{C}}(u)) = \sum_{k \geq 0} \Psi_k(x) u^{q^k}.$$

Here $\log_{\mathbf{C}}(z)$ is the Carlitz logarithm defined by

$$\log_{\mathbf{C}}(z) = \sum_{n \geq 0} \frac{z^{q^n}}{l_n}.$$

Then $\Psi_k(x)$ can be expressed as follows:

Proposition 2.11. (*Anderson-Thakur* [1])

$$\Psi_k(x) = \sum_{i=0}^{k} \frac{\prod_{j=1}^{i} (\theta^{q^i} - \theta^{q^{k+j}})}{D_i} (\frac{x}{(-1)^k L_k})^{q^i}.$$

Moreover, $\Psi_k(a) = 0$ *for all* $a \in \mathbb{F}_q[\theta]$ *with* $\deg_\theta a < k$ *and* $\Psi_k(\theta^k) = 1$.

This result is another key tool in the proof of Theorem 3.3. For our purpose, we replace θ by t in Anderson-Thakur's result so that $\Psi_k|_{\theta=t}(x) \in \mathbb{F}_q(t)[x]$.

2.12. **Conjectures on Eulerian/Zeta-like Multizeta Values.** There are families of zeta-like multizeta values of arbitrary depth, for instance, in [9], they showed that for any q, $\zeta_A(1, q-1, (q-1)q, \cdots, (q-1)q^n)$ is zeta-like by giving the ratio of it to $\zeta_A(q^{n+1})$. There are certainly more families of zeta-like multizeta values of arbitrary depth, the following conjecture is given in [9]:

Conjecture 2.13.

(*a*) *For any* $q, n \geq 1$ *and* $r \geq 2$,

$$\zeta_A(q^n-1,(q-1)q^n,\cdots,(q-1)q^{n+r-2}) = \frac{[n+r-2][n+r-3]\cdots[n]}{[1]^{q^{n+r-2}}[2]^{q^{n+r-3}}\cdots[r-1]^{q^n}}\zeta_A(q^{n+r-1}-1).$$

(*b*) *For any* $q, n \geq 0$,

$$\zeta_A(1,q^2-1,(q-1)q^2,\cdots,(q-1)q^{n+1}) = \frac{[n+2]-1}{l_1[n+2]}\frac{1}{l_1^{(q-1)q^n}l_2^{(q-1)q^{n-1}}\cdots l_{n-1}^{(q-1)q^2}l_n^{q^2}}\zeta_A(q^{n+2}).$$

(*c*) *For* $q > 2$, $n \geq 0$ *and* $r \geq 2$,

$$\zeta_A((q-1)q^n-1,(q-1)q^{n+1},\cdots,(q-1)q^{n+r-1}) = \frac{(-1)^{r+1}[n+r-1][n+r-2]\cdots[n+1]}{[1]^{(q^{r-1}-1)q^n}[2]^{(q^{r-2}-1)q^n}\cdots[r-1]^{(q-1)q^n}}\zeta_A(q^{n+r}-q^n-1).$$

Remark 2.13.1. In [9] Conjecture 2.13 is proved in the depth 2 case. We refer to [9] for more details, in particular [9, Theorem 3.1], where many depth 2 zeta-like multizeta values are given with precise ratio to $\zeta_A(w)$.

According to Chang-Papanikolas-Yu criterion for zeta-like multizeta values in [4], Kuan-Lin [7] wrote an algorithm and tested multizeta values with bounded weights and depths by computer. From their output data, they gave another more extensive conjecture about zeta-like families of arbitrary depth and also specific depth 3 zeta-like multizeta values.

Conjecture 2.14. (*Kuan-Lin-Yu*) *Suppose that* $q > 2$. *Then we have the following families of zeta-like multizeta values:*

(*a*) *For* $q = p^l > 2$, $1 \leq p^m \leq q$, $n > 0$ *and* $r \geq 2$, *consider* $N_i \in \mathbb{Z}_{\geq 0}$ *for* $0 \leq i \leq n-1$ *such that* $1 \leq \sum N_i \leq q-1$. *If* $(q-1)(q^n - \sum N_i q^i) \leq p^m(q-1)q^{n-1}$, *then*

$$\zeta_A(q^n - \sum N_i q^i, p^m(q-1)q^{n-1},\cdots,p^m(q-1)q^{n+r-3})$$

is zeta-like. In particular

$$\zeta_A(1,p^m(q-1),p^m q(q-1),\cdots,p^m q^{r-2}(q-1))$$

is zeta-like.

(*b*) *In the case of depth* $r = 3$,

$$\zeta_A(1,q(q-1),q^3-q^2+q-1) = \frac{[3]-1}{[3][2][1]^{q^2-q+1}}\zeta_A(q^3).$$

Remark 2.14.1. When $n \geq 1$ in Conjecture 2.13 (c), it corresponds to a special case of Conjecture 2.14 (a) by taking $p^m = q$, $N_0 = N_{n-1} = 1$ and $N_i = 0$ for $0 < i < n-1$. When $n = 0$ in Conjecture 2.13 (c), it corresponds to the case when $n = 1$ and $N_0 = 2$ in Conjecture 2.14 (a).

Note that when the weight w is 'even', the statement that $\zeta_A(s_1,\cdots,s_r)$ is zeta-like is equivalent to that it is Eulerian. In Section 4 we will prove non-Eulerian part of Conjecture 2.13 and Conjecture 2.14. The Eulerian part of Conjecture 2.13 will be treated in Section 5

3. Investigation into Anderson-Thakur polynomials

In general Anderson-Thakur polynomials H_n are complicated to investigate. However, for index n having very special q-adic expansion, we can give a nicer and simpler formula for such H_n. For example, to prove Conjecture 2.14 (b), we need to compute the corresponding Anderson-Thakur polynomials H_0, H_{q^2-q-1}, $H_{q^3-q^2+q-2}$ and H_{q^3-1}. It is known that $H_0=1$. On the other hand, it can be directly proved that $H_{q^2-q-1}=\Gamma_{q^2-q}|_{\theta=t}$, $H_{q^3-1}=\Gamma_{q^3}|_{\theta=t}$ and $H_{q^2-q}=\Gamma_{q^2-q+1}|_{\theta=t}\dfrac{(t-\theta^q)^{q-1}}{L_1^{q-1}|_{\theta=t}}$ (These are special cases of Theorem 3.3). Recall that S_m is the set of all q power weighted partition $\underline{a}$ of m with $C_{\underline{a}}\neq 0$ in $\mathbb{F}_p$. Furthermore, for given $\underline{b}$ with $0\le b_i\le q-1$, let $S_{m,\underline{b}}$ be the subset of S_m collecting $\underline{a}$ satisfying $a_i\equiv b_i \mod q$ for all i. By Lemma 2.6 and to use the reduction maps $r(\underline{a})$, we can compute $H_{q^3-q^2+q-2}$.

Proposition 3.1.

$$H_{q^3-q^2+q-2}=-[2]^{q-2}|_{\theta=t}\left[(t-\theta^q)^{q^2-q+1}+[1]^{q^2-1}|_{\theta=t}(t-\theta^q)\right].$$

Proof. For any q power weighted partition $(a_0,a_1,a_2,0,0,\cdots)$ of q^3-q^2+q-2, we see that $a_0\equiv q-2=p-2+(p-1)p+\cdots+(p-1)p^{l-1} \mod q$. Let $a_i\equiv a_{i,0}+a_{i,1}p+\cdots+a_{i,l-1}p^{l-1} \mod q$, where $i=1,2$ and $0\le a_{i,j}\le p-1$ for $j\ge 0$. It follows that if $C_{\underline{a}}\neq 0$, then $a_{1,0}+a_{2,0}\le 1$ and $a_{1,j}+a_{2,j}=0$ for $j>0$. This implies $\tilde{\underline{a}}=(q-2,0,0,\cdots)$ or $(q-2,1,0,\cdots)$ or $(q-2,0,1,0,\cdots)$ and hence $S_{q^3-q^2+q-2}$ is the disjoint union of $S_{q^3-q^2+q-2,(q-2,0,0,\cdots)}$, $S_{q^3-q^2+q-2,(q-2,1,0,\cdots)}$ and $S_{q^3-q^2+q-2,(q-2,0,1,\cdots)}$. The reduction map $\underline{a}\mapsto r(\underline{a})$ induces bijections

$$\begin{aligned}
r: S_{q^3-q^2+q-2,(q-2,0,0,\cdots)} &\to S_{q^2-q},\\
r: S_{q^3-q^2+q-2,(q-2,1,0,\cdots)} &\to S_{q^2-q-1},\\
r: S_{q^3-q^2+q-2,(q-2,0,1,\cdots)} &\to S_{q^2-2q}.
\end{aligned}$$

Moveover, we see that $C_{\underline{a}}=C_{r(\underline{a})}$ if $\tilde{\underline{a}}=(q-2,0,0,\cdots)$ and $C_{\underline{a}}=-C_{r(\underline{a})}$ if $\tilde{\underline{a}}=(q-2,1,0,\cdots)$ or $(q-2,0,1,\cdots)$. We obtain from Lemma 2.6 that

$$\begin{aligned}
\frac{H_{q^3-q^2+q-2}}{\Gamma_{q^3-q^2+q-1}|_{\theta=t}} &= \sum_{\tilde{\underline{a}}}\sum_{\underline{a}\in S_{q^3-q^2+q-2,\tilde{\underline{a}}}} C_{\underline{a}}\Big(\frac{G_0}{D_0|_{\theta=t}}\Big)^{a_0}\prod_{i\ge 1}\Big(\frac{G_i}{D_i|_{\theta=t}}\Big)^{a_i}\\
&= \sum_{\tilde{\underline{a}}}\sum_{\underline{a}\in S_{q^3-q^2+q-2,\tilde{\underline{a}}}} C_{\tilde{\underline{a}}}C_{r(\underline{a})}\Big(\frac{G_0}{D_0|_{\theta=t}}\Big)^{q\tilde{\tilde{a}}_0+\tilde{a}_0}\prod_{i\ge 1}\Big(\frac{G_i}{D_i|_{\theta=t}}\Big)^{q\tilde{\tilde{a}}_i+\tilde{a}_i}\\
&= \Big(\frac{H_{q^2-q}}{\Gamma_{q^2-q+1}|_{\theta=t}}\Big)^q-\frac{G_1}{D_1|_{\theta=t}}\Big(\frac{H_{q^2-q-1}}{\Gamma_{q^2-q}|_{\theta=t}}\Big)^q-\frac{G_2}{D_2|_{\theta=t}}\Big(\frac{H_{q^2-2q}}{\Gamma_{q^2-2q+1}|_{\theta=t}}\Big)^q
\end{aligned}$$

On the other hand, by Lemma 2.6 (b) for $n=q^2-q$, we have

$$\frac{H_{q^2-2q}}{\Gamma_{q^2-2q+1}|_{\theta=t}}=\frac{D_1|_{\theta=t}}{G_1}\left[\frac{H_{q^2-q}}{\Gamma_{q^2-q+1}|_{\theta=t}}-\frac{H_{q^2-q-1}}{\Gamma_{q^2-q}|_{\theta=t}}\right].$$

It follows that $\frac{H_{q^3-q^2+q-2}}{\Gamma_{q^3-q^2+q-1}|_{\theta=t}} = \frac{-(t-\theta^q)^{q^2-q+1}}{[2][1]^{q(q-1)}|_{\theta=t}} + \frac{-(t-\theta^q)[1]^{q-1}|_{\theta=t}}{[2]|_{\theta=t}}$. By definition of Γ-function, we can easily derive that $\Gamma_{q^3-q^2+q-1} = D_2^{q-1}$ and the result follows.

□

3.2. **Formula for Anderson-Thakur polynomials H_m with $m = q^n - \sum N_i q^i$.** For $n \in \mathbb{N}$, consider a tuple $(N_0, \cdots, N_{n-1})$ with $N_i \in \mathbb{Z}_{\geq 0}$ satisfying $0 \leq \sum_{i=0}^{n-1} N_i \leq q-1$. This implies $q^n - \sum_{i=0}^{n-1} N_i q^i - 1 \geq 0$. We have the following formula for these special polynomials:

Theorem 3.3. *Let $n \in \mathbb{N}$ and $N_i \in \mathbb{Z}_{\geq 0}$ satisfying $0 \leq \sum_{i=0}^{n-1} N_i \leq q-1$. Then*

$$\frac{H_{q^n-\sum N_i q^i-1}}{\Gamma_{q^n-\sum N_i q^i}|_{\theta=t}} = \frac{(-1)^{\sum_{i=0}^{n-2}(n-1-i)N_i q^i}}{\prod_{i=0}^{n-2} L_{n-1-i}^{N_i q^i}|_{\theta=t}} \prod_{v=1}^{n-1} (t-\theta^{q^v})^{\sum_{j=0}^{n-1-v} N_j q^j}. \tag{3.3.1}$$

The key idea is using Lemma 2.6 and Theorem 2.8 to descend Anderson-Thakur polynomials from H_n to suitable H_m with $m < n$.

Proof. We will prove by induction on n and N_i. For $n = 1$ it is clear that $\frac{H_{q-N_0-1}}{\Gamma_{q-N_0}|_{\theta=t}} = 1$ satisfying (3.3.1) for any $0 \leq N_0 \leq q-1$. For $M \geq 2$, suppose that the statement holds for $H_{q^n-\sum_{i=0}^{n-1} N_i q^i-1}$ with $1 \leq n < M$. Our goal is to prove the formula (3.3.1) holds for $H_{q^M-\sum_{i=0}^{M-1} N_i q^i-1}$. For the case $N_0 = 0$, let $\underline{a} = (a_0, \cdots, a_{M-1}, 0, 0, \cdots)$ be a q power weighted partition of $q^M - \sum_{i=0}^{M-1} N_i q^i - 1$ with $C_{\underline{a}} \neq 0$. Then $a_0 \equiv q-1 = (p-1) + \cdots + (p-1)p^{l-1} \mod q$. By Theorem 2.8, it forces $a_i \equiv 0 \mod q$ for $i \geq 1$. Then by considering $\tilde{\underline{a}} = (q-1, 0, 0, \cdots)$, we have that $r(\underline{a})$ is a q power weighted partition of $q^{M-1} - \sum_{i=0}^{M-2} N_i' q^i - 1$, where $N_i' = N_{i+1}$. Conversely, if $\underline{a}' \in S_{q^{M-1}-\sum_{i=0}^{M-2} N_i' q^i-1}$, let $\underline{a} = (qa_0' + q - 1, qa_1', \cdots, qa_{M-1}', 0, 0, \cdots)$. We put $N_0 = 0$ and $N_i = N_{i-1}'$ for $i \geq 1$. Then $\underline{a} \in S_{q^M-\sum_{i=0}^{M-1} N_i q^i-1}$ and $r(\underline{a}) = \underline{a}'$. Therefore, $r : S_{q^M-\sum_{i=0}^{M-1} N_i q^i-1} \to S_{q^{M-1}-\sum_{i=0}^{M-2} N_i' q^i-1}$ is bijective. Moreover, $C_{\underline{a}} = C_{r(\underline{a})}$. It follows that

$$\begin{aligned}
\frac{H_{q^M-\sum_{i=0}^{M-1} N_i q^i-1}}{\Gamma_{q^M-\sum_{i=0}^{M-1} N_i q^i}|_{\theta=t}} &= \sum_{\underline{a} \in S_{q^M-\sum_{i=0}^{M-1} N_i q^i-1}} C_{\underline{a}} \left(\frac{G_0}{D_0|_{\theta=t}}\right)^{a_0} \prod_{\nu \geq 1} \left(\frac{G_\nu}{D_\nu|_{\theta=t}}\right)^{a_\nu} \\
&= \sum_{\underline{a}' \in S_{q^{M-1}-\sum_{i=0}^{M-2} N_i' q^i-1}} C_{\underline{a}'} \left(\frac{G_0}{D_0|_{\theta=t}}\right)^{qa_0'+q-1} \prod_{\nu \geq 1} \left(\frac{G_\nu}{D_\nu|_{\theta=t}}\right)^{qa_\nu'} \\
&= \left(\frac{H_{q^{M-1}-\sum_{i=0}^{M-2} N_i' q^i-1}}{\Gamma_{q^{M-1}-\sum_{i=0}^{M-2} N_i' q^i}|_{\theta=t}}\right)^q \\
&= \left(\frac{(-1)^{\sum_{i=0}^{M-3}(M-2-i)N_i' q^i}}{\prod_{i=0}^{M-3} L_{M-2-i}^{N_i' q^i}|_{\theta=t}} \prod_{i=1}^{M-2} (t-\theta^{q^i})^{\sum_{j=0}^{M-2-i} N_j' q^j}\right)^q \\
&= \frac{(-1)^{\sum_{i=0}^{M-3}(M-2-i)N_{i+1} q^{i+1}}}{\prod_{i=0}^{M-3} L_{M-2-i}^{N_{i+1} q^{i+1}}|_{\theta=t}} \prod_{i=1}^{M-2} (t-\theta^{q^i})^{\sum_{j=0}^{M-2-i} N_{j+1} q^{j+1}} \\
&= \frac{(-1)^{\sum_{i=1}^{M-2}(M-1-i)N_i q^i}}{\prod_{i=1}^{M-2} L_{M-1-i}^{N_i q^i}|_{\theta=t}} \prod_{i=1}^{M-1} (t-\theta^{q^i})^{\sum_{j=0}^{M-1-i} N_j q^j}.
\end{aligned}$$

Note that the last step follows from the induction hypothesis and $N_0 = 0$.

Now we assume that $N_0 \geq 1$. By Lemma 2.6 (b) we have

$$(3.3.2)\quad \frac{H_{q^M-\sum_{i=1}^{M-1}N_iq^i-(N_0-1)-1}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i+1}|_{\theta=t}} = \frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i-1}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i}|_{\theta=t}} + \sum_{j=1}^{M-1}\frac{G_j}{D_j|_{\theta=t}}\frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i-q^j}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i-q^j+1}|_{\theta=t}}.$$

Note that $q^M-\sum_{i=0}^{M-1}N_jq^i-q^j$ and $q^M-\sum_{i=0}^{M-1}N_iq^i$ are congruent to $q-N_0 \mod q$. If we consider all possible vectors $\underline{\tilde{a}}=(q-N_0,\tilde{a_1},\cdots,\tilde{a_{M-1}},0,0,\cdots)$ with $0\leq\tilde{a}_i\leq q-1$ and $C_{\underline{\tilde{a}}}\neq 0$, then $\sum_{j=0}^{\infty}\tilde{a}_j=\sum_{j=0}^{M-1}\tilde{a}_j\leq q-1$. This implies $q^M-\sum_{i=0}^{M-1}N_iq^i-q^j-\sum_{i\geq 0}\tilde{a}_iq^i\geq 0$ and $q^M-\sum_{i=0}^{M-1}N_iq^i-\sum_{i\geq 0}\tilde{a}_iq^i\geq 0$. Hence for a given $\underline{\tilde{a}}$, The reduction map $\underline{a}\mapsto r(\underline{a})$ induces bijections

$$r: S_{q^M-\sum_{i=0}^{M-1}N_iq^i-q^j,\underline{\tilde{a}}}\to S_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-q^{j-1}-1}$$
$$r: S_{q^M-\sum_{i=0}^{M-1}N_iq^i,\underline{\tilde{a}}}\to S_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-1}.$$

By Lemma 2.6 (a) we have

$$\begin{aligned}\frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i-q^j}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i-q^j+1}|_{\theta=t}} &= \sum_{\underline{\tilde{a}}}\sum_{\underline{a}\in S_{q^M-\sum_{i=0}^{M-1}N_iq^i-q^j,\underline{\tilde{a}}}} C_{\underline{a}}\Big(\frac{G_0}{D_0|_{\theta=t}}\Big)^{a_0}\prod_{v\geq 1}\Big(\frac{G_v}{D_v|_{\theta=t}}\Big)^{a_v}\\ &= \sum_{\underline{\tilde{a}}}\sum_{\underline{a'}\in S_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-q^{j-1}-1}} C_{\underline{\tilde{a}}}C_{\underline{a'}}\Big(\frac{G_0}{D_0|_{\theta=t}}\Big)^{qa'_0+\tilde{a_0}}\prod_{v\geq 1}\Big(\frac{G_v}{D_v|_{\theta=t}}\Big)^{qa'_v+\tilde{a_v}}\\ &= \sum_{\underline{\tilde{a}}}C_{\underline{\tilde{a}}}\prod_{v\geq 0}\Big(\frac{G_v}{D_v|_{\theta=t}}\Big)^{\tilde{a_v}}\Big(\frac{H_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-q^{j-1}-1}}{\Gamma_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-q^{j-1}}|_{\theta=t}}\Big)^q\end{aligned}$$

Similarly,

$$\frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i+1}|_{\theta=t}} = \sum_{\underline{\tilde{a}}}C_{\underline{\tilde{a}}}\prod_{v\geq 0}\Big(\frac{G_v}{D_v|_{\theta=t}}\Big)^{\tilde{a_v}}\Big(\frac{H_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-1}}{\Gamma_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}}|_{\theta=t}}\Big)^q.$$

Since $\sum_{i=1}^{M-1}N_i+\tilde{a}_i\leq q-1-N_0+N_0-1=q-2$, by the induction hypothesis on $M-1<M$, one can show that

$$\frac{H_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-q^{j-1}-1}}{\Gamma_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-q^{j-1}}|_{\theta=t}} = \frac{H_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}-1}}{\Gamma_{q^{M-1}-\sum_{i=1}^{M-1}(N_i+\tilde{a_i})q^{i-1}}|_{\theta=t}}\frac{(-1)^{(M-2-(j-1))q^{j-1}}}{L^{q^{j-1}}_{M-2-(j-1)}|_{\theta=t}}\prod_{v=1}^{M-2-(j-1)}(t-\theta^{q^v})^{q^{j-1}}.$$

It follows that

$$\frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i-q^j}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i-q^j+1}|_{\theta=t}} = \frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i+1}|_{\theta=t}}\frac{(-1)^{M-j-1}}{L^{q^j}_{M-j-1}|_{\theta=t}}\prod_{v=1}^{M-j-1}(t-\theta^{q^v})^{q^j}.$$

By (3.3.2), we have

$$\frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i-1}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i}|_{\theta=t}} = \frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i+1}|_{\theta=t}}\Big(1-\sum_{j=1}^{M-1}\frac{G_j}{D_j|_{\theta=t}}\frac{(-1)^{M-j-1}}{L^{q^j}_{M-j-1}|_{\theta=t}}\prod_{v=1}^{M-j-1}(t-\theta^{q^v})^{q^j}\Big).$$

Now we begin to prove that the formula (3.3.1) holds for $q^M-\sum_{i=0}^{M-1}N_iq^i-1$. Let

$$f(\theta):=1-\sum_{j=0}^{M-1}\frac{G_j}{D_j|_{\theta=t}}\frac{(-1)^{M-j-1}}{L_{M-j-1}^{q^j}|_{\theta=t}}\prod_{v=1}^{M-j-1}(t-\theta^{q^v})^{q^j}.$$

Let U denotes the collection of all subsets of $\{1,2,\cdots,M-1\}$. For $I=\{i_1,\cdots,i_m\}\in U$, we put $\theta^I=\theta^{q^{i_1}+\cdots+q^{i_m}}$, and $|I|=m$, the number of elements in I. Then for $i=0,\cdots,M-1$,

$$G_j\prod_{v=1}^{M-j-1}(t-\theta^{q^v})^{q^j}=\prod_{v=1}^{M-1}(t^{q^j}-\theta^{q^v})=\sum_{I\in U}\theta^I(-1)^{|I|}(t^{q^j})^{M-1-|I|}.$$

Since for distinct I_1,I_2, $\theta^{I_1}\neq\theta^{I_2}$, we have

$$f(\theta)=1-\sum_{I\in U}(\sum_{j=0}^{M-1}\frac{(t^{q^j})^{M-1-|I|}}{(-1)^{M-j-1}D_jL_{M-j-1}^{q^j}|_{\theta=t}})\theta^I(-1)^{|I|}.$$

Observe that

$$\frac{1}{(-1)^{M-j-1}L_{M-j-1}^{q^j}|_{\theta=t}}=\frac{\prod_{v=1}^{j}(t^{q^j}-t^{q^{v+M-1}})}{(-1)^{M-1}L_{M-1}^{q^j}|_{\theta=t}}.$$

It follows that

$$f(\theta)=1-\sum_{I\in U}(\sum_{j=0}^{M-1}\frac{\prod_{v=1}^{i}(t^{q^j}-t^{q^{v+M-1}})(t^{q^j})^{M-1-|I|}}{(-1)^{M-1}D_jL_{M-1}^{q^j}|_{\theta=t}})\theta^I(-1)^{|I|}=1-\sum_{I\in U}\Psi_{M-1}|_{\theta=t}(t^{M-1-|I|})\theta^I(-1)^{|I|}.$$

By Proposition 2.11, $\Psi_{M-1}|_{\theta=t}(t^{M-1-|I|})=0$ if $|I|>0$, and $\Psi_{M-1}|_{\theta=t}(t^{M-1-|I|})=1$ if I is the empty set. In the later condition $\theta^I(-1)^{|I|}=1$ and hence $f(\theta)=1-1=0$. This implies $1-\sum_{j=1}^{M-1}\frac{G_j}{D_j|_{\theta=t}}\frac{(-1)^{M-j-1}}{L_{M-j-1}^{q^j}|_{\theta=t}}\prod_{v=1}^{M-j-1}(t-\theta^{q^v})^{q^j}=\frac{(-1)^{M-1}}{L_{M-1}|_{\theta=t}}\prod_{v=1}^{M-1}(t-\theta^{q^v})$ and we deduce that

$$\begin{aligned}\frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i-1}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i}|_{\theta=t}}&=\frac{H_{q^M-\sum_{i=0}^{M-1}N_iq^i}}{\Gamma_{q^M-\sum_{i=0}^{M-1}N_iq^i+1}|_{\theta=t}}\frac{(-1)^{M-1}}{L_{M-1}|_{\theta=t}}\prod_{v=1}^{M-1}(t-\theta^{q^v})\\&=\frac{(-1)^{\sum_{i=0}^{M-2}(M-1-i)N_iq^i}}{\prod_{i=0}^{M-2}L_{M-1-i}^{N_iq^i}|_{\theta=t}}\prod_{v=1}^{M-1}(t-\theta^{q^v})^{\sum_{j=0}^{M-1-v}N_jq^j}.\end{aligned}$$

$\square$

4. Main result on Zeta-like Multizeta values

In this section we will prove Conjecture 2.13 (b) with $q>2$ and Conjecture 2.14.

4.1. **Frobenius twisting.** We fix the following automorphism of the field of Laurent series over $\mathbb{C}_\infty$, which is referred to as *Frobenius twisting*:

$$\begin{array}{rcl}\mathbb{C}_\infty(\!(t)\!)&\to&\mathbb{C}_\infty(\!(t)\!),\\ f:=\sum_i a_it^i&\mapsto&f^{(-1)}:=\sum_i a_i^{\frac{1}{q}}t^i.\end{array}$$

In [4], the following criterion is proved for deciding zeta-like multizeta values in terms of Anderson-Thakur polynomials.

Theorem 4.2. (*Chang-Papanikolas-Yu* [4, Theorem 2.5.2, 4.4.2]) *Given a tuple* $(s_1, s_2, \cdots, s_r) \in \mathbb{N}^r$*, then* $\zeta_A(s_1, \cdots, s_r)$ *is zeta-like if and only if there exist* $\delta_1, \cdots, \delta_r \in \overline{K}[t]$ *and* $a, b \in \mathbb{F}_q[t]$ *with* $a \neq 0$ *such that*

$$
\begin{aligned}
\delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1-1}^{(-1)}(t-\theta)^w + bH_{w-1}^{(-1)}(t-\theta)^w;\\
\delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r} + \delta_3^{(-1)} H_{s_2-1}^{(-1)}(t-\theta)^{s_2+\cdots+s_r};\\
&\vdots\\
\delta_{r-1} &= \delta_{r-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} + \delta_r^{(-1)} H_{s_{r-1}-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r};\\
\delta_r &= \delta_r^{(-1)}(t-\theta)^{s_r} + aH_{s_r-1}^{(-1)}(t-\theta)^{s_r},
\end{aligned}
\tag{4.2.1}
$$

where $H_{s_1-1}, \cdots, H_{s_r-1}, H_{w-1}$ *are Anderson-Thakur polynomials.*
If $q-1$ *does not divide* $w := \sum s_i$*, then we have*

$$
a(\theta)\Gamma_{s_1}\cdots\Gamma_{s_r}\zeta_A(s_1, \cdots, s_r) + b(\theta)\Gamma_w\zeta_A(w) = 0
$$

Remark 4.2.2. If $(\delta_1, \cdots, \delta_r, a, b)$ is a solution of (4.2.1), then for any nonzero $f \in \mathbb{F}_q[t]$, $(f\delta_1, \cdots, f\delta_r, fa, fb)$ is also a solution of (4.2.1) since the set of elements in $\overline{K}[t]$ fixed by the Frobenius twisting is $\mathbb{F}_q[t]$.

According to this theorem, our strategy for proving given multizeta values to be zeta-like is to actually solve system of Equations 4.2.1 by finding $\delta_1, \cdots, \delta_r \in \overline{K}[t]$ and $a, b \in \mathbb{F}_q[t]$. Since we are interested in tuples $(s_1, \cdots, s_r)$ with s_i of very special q-adic "shape ", solutions (δ_r, a) can be given immediately. Then an inductive procedure is used to go from a solution $(\delta_j', \cdots, \delta_r', a')$ of a subsystem of (4.2.1) with $r-j+1$ equations to a solution $(\delta_{j-1}, \cdots, \delta_r, a)$ of a subsystem of (4.2.1) with $r-j+2$ equations. The precise statement is incorporated in the following proposition.

Proposition 4.3. *Fix* $1 \leq p^M \leq q$*. For any* $r \geq 2$ *and* $m \in \mathbb{Z}_{\geq 0}$*, let* $s_i = p^M(q-1)q^{m+i-2}$ *for* $i = 2, \cdots r$*. Let* $(\delta_2, \cdots, \delta_r, a)$ *be defined as follows:*

$$
\begin{aligned}
f_r &:= [2]^{p^M q^{r+m-3}} \cdots [r-1]^{p^M q^m}\Gamma_{p^M q^{m+r-2}(q-1)};\\
f_i &:= \frac{-f_{i+1}}{[r-i+1]^{p^M q^{m+i-2}}}\Gamma_{p^M q^{m+i-2}(q-1)} \text{ for } j \leq i < r;\\
\delta_i &:= f_i|_{\theta=t}[(t-\theta)\cdots(t-\theta^{\frac{1}{q^{r-i}}})]^{p^M q^{r+m-1}};\\
a &:= -\left[[1]^{p^M q^{r+m-2}}[2]^{p^M q^{r+m-3}}\cdots[r-1]^{p^M q^m}\right]|_{\theta=t}.
\end{aligned}
\tag{4.3.1}
$$

Then for any j *with* $2 \leq j < r$*, the system of equations*

$$
\begin{aligned}
\delta_j &= \delta_j^{(-1)}(t-\theta)^{s_j+\cdots+s_r} + \delta_{j+1}^{(-1)} H_{s_j-1}^{(-1)}(t-\theta)^{s_j+\cdots+s_r};\\
&\vdots\\
\delta_{r-1} &= \delta_{r-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} + \delta_r^{(-1)} H_{s_{r-1}-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r};\\
\delta_r &= \delta_r^{(-1)}(t-\theta)^{s_r} + aH_{s_r-1}^{(-1)}(t-\theta)^{s_r}.
\end{aligned}
\tag{4.3.2}
$$

can be solved explicitly with $(\delta_j, \cdots, \delta_r, a)$ *given by* (4.3.1).

Remark 4.3.3. It follows from the recursive definition of f_i that

$$
f_2 = (-1)^r\Gamma_{p^M(q-1)q^m}\cdots\Gamma_{p^M(q-1)q^{m+r-2}}.
$$

Proof. By using similar arguments in the proof of Theorem 3.3, one can show that

$$H_{p^M q^{m+i-2}(q-1)-1} = \Gamma_{p^M q^{m+i-2}(q-1)}|_{\theta=t}$$

for $1 \le p^M \le q$. We obtain that $(\delta_j, \cdots, \delta_r, a)$ defined in (4.3.1) is a solution of the system of equations (4.3.2) by straight forward manipulation. □

Now we begin to prove Conjecture 2.13 (b) and Conjecture 2.14 (a).

Theorem 4.4. *Suppose that $q > 2$. Then we have the following zeta-like families.*

$$(a)\ \zeta_A(1, q^2-1, (q-1)q^2, \cdots, (q-1)q^{n+1}) = \frac{(-1)^{n+1}([n+2]-1)}{[1][n+2]} \frac{1}{[1]^{q^{n+1}} \cdots [n]^{q^2}} \zeta_A(q^{n+2}).$$

(b) For $n \ge 1$, $r \ge 2$ and $1 \le p^m \le q$, consider $N_i \in \mathbb{Z}_{\ge 0}$ for $0 \le i \le n-1$ such that $1 \le \sum N_i \le q-1$. If $(q-1)(q^n - \sum N_i q^i) \le p^m (q-1) q^{n-1}$, then

$$\zeta_A(q^n - \sum N_i q^i, p^m(q-1)q^{n-1}, \cdots, p^m(q-1)q^{n+r-3})$$

is zeta-like. In particular, if $q-1$ does not divide $q^n - \sum N_i q^i$, then we have

$$\begin{aligned} &\zeta_A(q^n - \sum N_i q^i, p^m(q-1)q^{n-1}, \cdots, p^m(q-1)q^{n+r-3}) \\ &= \frac{(-1)^{(r-1)(1+\sum N_i)} L_{n+r-2}^{N_0 q^0} \cdots L_{r-1}^{(N_{n-1}-(q-p^m))q^{n-1}}}{[1]^{p^m q^{n+r-3}} [2]^{p^m q^{n+r-4}} \cdots [r-1]^{p^m q^{n-1}} L_{n-1}^{N_0 q^0} \cdots L_1^{N_{n-2} q^{n-2}}} \zeta_A(p^m q^{n+r-2} - p^m q^{n-1} + q^n - \sum N_i q^i). \end{aligned}$$

Remark 4.4.1. The zeta-like part of Theorem 3.1 (1) in [9] is a special case of Theorem 4.4 (b) by taking $r = 2$. Also, the zeta-like part of Theorem 3.2 in [9] is a special case of Theorem 4.4 (b) by taking $n = 0, N_i = 0$.

Proof. (a) Let $w = q^{n+2}$. Let $s_1 = 1$, $s_2 = q^2 - 1$ and $s_i = (q-1)q^{i-1}$ for $3 \le i \le n+2$. If $n+2 \ge 3$, we choose $f_i \in A$, $\delta_i \in \overline{K}[t]$ and $a \in \mathbb{F}_q[t]$ as same as in Proposition 4.3 when $3 \le i \le n+2$. Then δ_i and a satisfy the subsystem of equations (4.3.1) for $j = 3$. If $n+2 = 2$, we define $a = -[1]^q|_{\theta=t}$. Now we let

$$\begin{aligned} &\delta_2 = (-1)^n [n+1]^q|_{\theta=t} \Gamma_w|_{\theta=t} [(t-\theta)\cdots(t-\theta^{\frac{1}{q^n}})]^{q^{n+2}} (t-\theta^q) \in A[t], \\ &f = \{-[n+2] L_1 L_2 \cdots L_{n+1}\}|_{\theta=t}, \\ &b = (-1)^n ([n+2]-1)|_{\theta=t} [n+1]^q|_{\theta=t}. \end{aligned}$$

We further put

$$\begin{aligned} &B = (-1)^{n+1} [n+1]^q|_{\theta=t} \Gamma_w|_{\theta=t}, \\ &F_0 = B(t-\theta)^w,\ F_1 = B(t-\theta)^w (t-\theta^{\frac{w}{q}}), \\ &F_i = F_{i-1}^{(-1)} (t-\theta)^w \quad \text{for } i = 2, \cdots n+1. \end{aligned}$$

and let $\delta_1 = \sum_{j=0}^{n+1} F_j$. By this recursive formula we can see that $\delta_2 = -F_{n+1}$. Note that by Theorem 3.3, $H_{s_1-1} = 1, H_{s_2-1} = H_{q^2-2} = (\theta^q - t) D_1^{q-2}|_{\theta=t}$ and $H_{w-1} = \Gamma_w|_{\theta=t} = (\prod_{i=1}^{n+1} D_i^{q-1})|_{\theta=t}$. It follows from direct algebraic manipulation that the tuple $(\delta_1, \delta_2, f\delta_3, \cdots, f\delta_{n+2}, fa, b)$ satisfies the system of equations (4.2.1). Since $q-1$ does not divide $\sum s_i$, by Theorem 4.2

we obtain that

$$\begin{aligned}\zeta_A(1, q^2-1, q^2(q-1), \cdots, q^{n+1}(q-1)) &= \frac{-b(\theta)\Gamma_w}{f(\theta)a(\theta)\Gamma_1\Gamma_{q^2-1}\Gamma_{(q-1)q^2}\cdots\Gamma_{(q-1)q^{n+1}}}\zeta_A(w)\\ &= \frac{(-1)^{n+1}([n+2]-1)}{[n+2]L_1[1]^{q^{n+1}}[2]^{q^n}\cdots[n]^{q^2}}\zeta_A(w).\end{aligned}$$

(b) Let $s_1 = q^n - \sum_{i=0}^{n-1} N_i q^i$, $s_i = p^m(q-1)q^{n+i-3}$ for $2 \le i \le r$ and $w = \sum s_i = p^m q^{n+r-2} - p^m q^{n-1} + q^n - \sum N_i q^i$. Let $(\delta_2, \delta_3, \cdots, \delta_r, a)$ be chosen in Proposition 4.3. Note that the condition $(q-1)s_1 \le s_2$ is equivalent to $N_{n-1} \ge q - p^m$. Therefore we can rewrite the weight w as $w = q^{n+r-1} - \sum_{i=0}^{n+r-2} N'_i q^i$, where

$$\begin{aligned}&N'_{n+r-2} = q - p^m, N'_j = 0 \text{ for } n \le j \le n+r-3,\\ &N'_{n-1} = N_{n-1} - (q-p^m), N'_j = N_j \text{ for } 0 \le j \le n-2.\end{aligned}$$

For $b \in \mathbb{F}_q[t]$, we put

$$\begin{aligned}&B_1 = \Gamma_{q^n - \sum N_i q^i}|_{\theta=t} \frac{(-1)^{\sum_{i=0}^{n-2}(n-1-i)N_i q^i}}{\prod_{i=0}^{n-2} L_{n-1-i}^{N_i q^i}|_{\theta=t}},\ B_2 = \Gamma_{q^{n+r-1} - \sum_{i=0}^{n+r-2} N'_i q^i}|_{\theta=t} \frac{(-1)^{\sum_{i=0}^{n-1}(n+r-2-i)N'_i q^i}}{\prod_{i=0}^{n-1} L_{n+r-2-i}^{N'_i q^i}|_{\theta=t}},\\ &b = -f_2|_{\theta=t} B_1,\ f_1 = B_2,\\ &F_0 = bH_{w-1}^{(-1)}(t-\theta)^w,\ F_i = F_{i-1}^{(-1)}(t-\theta)^w \quad \text{for } i = 1, \cdots r-1,\\ &\delta_1 = \sum_{j=0}^{r-2} F_j \in \overline{K}[t].\end{aligned}$$

Moreover, these N'_i satisfy the condition

$$0 \le \sum_{i=0}^{n+r-2} N'_i = \sum_{i=0}^{n-1} N_i \le q-1.$$

By Theorem 3.3 and the straight forward manipulation, we have

$$H_{w-1}^{(-r)} = \frac{B_2}{B_1} H_{s_1-1}^{(-1)} \prod_{i=1}^{r-1} (t - \theta^{\frac{1}{q^i}})^{\sum_{j=0}^{n-1} N'_j q^j}.$$

It follows that $F_{r-1} + f_1 \delta_2^{(-1)} H_{s_1-1}^{(-1)}(t-\theta)^w = 0$ and then

$$\delta_1^{(-1)}(t-\theta)^w + f_1\delta_2^{(-1)} H_{s_1-1}^{(-1)}(t-\theta)^w + bH_{w-1}^{(-1)}(t-\theta)^w = \delta_1.$$

Combining with Proposition 4.3, we see that the tuple $(\delta_1, f_1\delta_2, \cdots, f_1\delta_r, f_1 a, b)$ satisfies the system of equations (4.2.1).

If $q-1$ does not divide $\sum s_i$, we can apply f_2 in Remark 4.3.3 and get

$$\begin{aligned}
&\zeta_A(q^n-\sum N_iq^i, p^m(q-1)q^{n-1},\cdots,p^m(q-1)q^{n+r-3})\\
&=\frac{-b(\theta)\Gamma_{q^{n+r-1}-\sum N_i'q^i}}{f_1(\theta)a(\theta)\Gamma_{q^n-\sum N_iq^i}\Gamma_{p^m(q-1)q^{n-1}}\cdots\Gamma_{p^m(q-1)q^{n+r-3}}}\zeta_A(q^{n+r-1}-\sum N_i'q^i)\\
&=\frac{(-1)^{(r-1)(1+\sum N_i)}L_{n+r-2}^{N_0q^0}\cdots L_{r-1}^{(N_{n-1}-(q-p^m))q^{n-1}}}{[1]^{p^mq^{n+r-3}}[2]^{p^mq^{n+r-4}}\cdots[r-1]^{p^mq^{n-1}}L_{n-1}^{N_0q^0}\cdots L_1^{N_{n-2}q^{n-2}}}\zeta_A(q^{n+r-1}-\sum N_i'q^i).
\end{aligned}$$

□

At the end of this section we prove Conjecture 2.14 (b).

Theorem 4.5. *For any $q>2$, $\zeta_A(1,q(q-1),q^3-q^2+q-1)$ is zeta-like. Furthermore,*

$$\zeta_A(1,q(q-1),q^3-q^2+q-1)=\frac{[3]-1}{[3][2][1]^{q^2-q+1}}\zeta_A(q^3).$$

Proof. Let

$$\begin{aligned}
&a=\{\Gamma_{q^3}[3]\}|_{\theta=t},\ \ b=\{[1]^{q-3}[2]^{q-2}(-[3]+1)\}|_{\theta=t},\\
&\delta_1=\frac{a[2]^{q-2}[1]^{q-3}|_{\theta=t}}{[3]|_{\theta=t}}(t-\theta)^{q^3}\left[(t-\theta^q)(t-\theta^{\frac{1}{q}})^{q^3}-\theta^{q^2}+t+1\right],\\
&\delta_2=\frac{-a[2]^{q-2}[1]^{q-3}|_{\theta=t}}{[3]|_{\theta=t}}(t-\theta)^{q^3}(t-\theta^{\frac{1}{q}})^{q^3}(t-\theta^q),\\
&\delta_3=\frac{a[2]^{q-2}|_{\theta=t}}{[1]|_{\theta=t}}(t-\theta)^{q^3}(t-\theta^q).
\end{aligned}$$

Then by Proposition 3.1 and Theorem 3.3, one can show that $(\delta_1,\delta_2,\delta_3,a,b)$ satisfies (4.2.1). Since $q>2$, the ratio of $\zeta_A(1,q(q-1),q^3-q^2+q-1)$ to $\zeta_A(q^3)$ is

$$\frac{-b(\theta)\Gamma_{q^3}}{a(\theta)\Gamma_1\Gamma_{q^2-q}\Gamma_{q^3-q^2+q-1}}=\frac{[3]-1}{[3][2][1]^{q^2-q+1}}.$$

□

5. Main result on Eulerian multizeta values

In this section we will present two families of Eulerian multizeta values mentioned in Conjecture 2.13 (a) and Conjecture 2.13 (b) for $q=2$. As a consequence, this confirms that all the multizeta values conjectured to be Eulerian in [4], Section 6.2, are indeed Eulerian.

Theorem 5.1. *For any positive integer $r>1$ and n, we have*

$$\begin{aligned}
&\zeta_A(q^n-1,(q-1)q^n,\cdots,(q-1)q^{n+r-2})\\
&=\zeta_A(q^n-1)\zeta_A((q-1),\cdots,(q-1)q^{r-2})^{q^n}-\zeta_A(q^{n+1}-1,(q-1)q^{n+1},\cdots,(q-1)q^{n+r-2}).
\end{aligned}$$

We obtained that

$$\zeta_A(q^n-1,(q-1)q^n,\cdots,(q-1)q^{n+r-2})=\frac{[n+r-2][n+r-3]\cdots[n]}{[1]^{q^{n+r-2}}[2]^{q^{n+r-3}}\cdots[r-1]^{q^n}}\zeta_A(q^{n+r-1}-1).$$

Theorem 5.2. *Let r be a positive integer greater than 1. If $q=2$, then we have*

$$\zeta_A(1)\zeta_A(1,2,\cdots,2^{r-1})=\zeta_A(1,3,2^2,\cdots,2^{r-1})+\zeta_A(1,1,2,\cdots,2^{r-2})^2.$$

Furthermore, we have

$$\zeta_A(1,3,2^2,\cdots,2^{r-1})=\frac{[r]-1}{L_1[r]}\frac{1}{L_1^{2^{r-2}}L_2^{2^{r-3}}\cdots L_{r-3}^{2^2}L_{r-2}^{2^2}}\zeta_A(2^r).$$

Aside from Carlitz's evaluations in Lemma 2.4, the key point of the proof is the relations among the power sums $S_d(m)$.

Lemma 5.3. *For any $d\geq 1$, one has*

$$S_d(q^n-1)S_d((q-1)q^n)=S_d(q^{n+1}-1)-S_d((q-1)q^n)\sum_{d'<d}S_{d'}(q^n-1)$$

Proof. See [11, pp.2332]. □

Proof of Theorem 5.1 By Lemma 5.3, we have

$$\begin{aligned}
&\sum_{\substack{d_1>d_2>\cdots>d_{r-1}\geq 0\\ d=d_1}} S_d(q^n-1)S_{d_1}((q-1)q^n)\cdots S_{d_{r-1}}((q-1)q^{n+r-2})\\
=&\sum_{d_1>d_2>\cdots>d_{r-1}\geq 0} S_{d_1}(q^{n+1}-1)S_{d_2}((q-1)q^{n+1})\cdots S_{d_{r-1}}((q-1)q^{n+r-2})\\
-&\sum_{\substack{d_1>d_2>\cdots>d_{r-1}\geq 0\\ d'<d_1}} S_{d_1}((q-1)q^n)S_{d'}(q^n-1)S_{d_2}((q-1)q^{n+1})\cdots S_{d_{r-1}}((q-1)q^{n+r-2})
\end{aligned}$$

It follows that

$$\begin{aligned}
&\zeta_A(q^n-1)\zeta_A((q-1)q^n,\cdots,(q-1)q^{n+r-2})\\
=&\sum_{\substack{d_1>d_2>\cdots>d_{r-1}\geq 0\\ d>d_1}}+\sum_{\substack{d_1>d_2>\cdots>d_{r-1}\geq 0\\ d=d_1}}+\sum_{\substack{d_1>d_2>\cdots>d_{r-1}\geq 0\\ d<d_1}} S_d(q^n-1)S_{d_1}((q-1)q^n)\cdots S_{d_{r-1}}((q-1)q^{n+r-2})\\
=&\sum_{\substack{d_1>d_2>\cdots>d_{r-1}\geq 0\\ d>d_1}} S_d(q^n-1)S_{d_1}((q-1)q^n)\cdots S_{d_{r-1}}((q-1)q^{n+r-2})\\
+&\sum_{d_1>d_2>\cdots>d_{r-1}\geq 0} S_{d_1}(q^{n+1}-1)S_{d_2}((q-1)q^{n+1})\cdots S_{d_{r-1}}((q-1)q^{n+r-2})\\
=&\,\zeta_A(q^n-1,(q-1)q^n,\cdots,(q-1)q^{n+r-2})+\zeta_A(q^{n+1}-1,(q-1)q^{n+1},\cdots,(q-1)q^{n+r-2})
\end{aligned}$$

We will prove the second part by mathematical induction on $r>1$. In fact we also prove that

$$\zeta_A(q^n-1,(q-1)q^n,\cdots,(q-1)q^{n+r-2})=\frac{(-1)^{n+r-1}[n+r-2]\cdots[n+1][n]}{[1]^{q^{n+r-2}}\cdots[r-2]^{q^{n+1}}[r-1]^{q^n}L_{n+r-1}}\tilde{\pi}^{q^{n+r-1}-1}.$$

When $r=2$, by Lemma 2.4, we have

$$\zeta_A(q^n-1,(q-1)q^n)=\zeta_A(q^n-1)\zeta_A((q-1)q^n)-\zeta_A(q^{n+1}-1)=\frac{[n]}{[1]^{q^n}}\zeta_A(q^{n+1}-1).$$

Assume that the statement holds for any n with depth $<r$, then by the above recursive formula and induction hypothesis we have

$$\begin{aligned}
&\zeta_A(q^n-1,(q-1)q^n,\cdots,(q-1)q^{n+r-2})\\
&=\zeta_A(q^n-1)\zeta_A((q-1),\cdots,(q-1)q^{r-2})^{q^n}-\zeta_A(q^{n+1}-1,(q-1)q^{n+1},\cdots,(q-1)q^{n+r-2}).\\
&=\frac{(-1)^n}{L_n}\tilde{\pi}^{q^n-1}\Big(\frac{(-1)^{r-1}[r-2]\cdots[1]}{[1]^{q^{r-2}}\cdots[r-2]^qL_{r-1}}\tilde{\pi}^{q^{r-1}-1}\Big)^{q^n}-\frac{(-1)^{n+r-1}[n+r-2]\cdots[n+1]}{[1]^{q^{n+r-2}}\cdots[r-2]^{q^{n+1}}L_{n+r-1}}\tilde{\pi}^{q^{n+r-1}-1}\\
&=\frac{(-1)^{n+r-1}[n+r-2]\cdots[n+1][n]}{[1]^{q^{n+r-2}}\cdots[r-2]^{q^{n+1}]}[r-1]^{q^n}L_{n+r-1}}\tilde{\pi}^{q^{n+r-1}-1}\\
&=\frac{[n+r-2][n+r-3]\cdots[n]}{[1]^{q^{n+r-2}}[2]^{q^{n+r-3}}\cdots[r-1]^{q^n}}\zeta_A(q^{n+r-1}-1).
\end{aligned}$$

□

Similarly, by Carlitz's evaluations and relations on power sums we can prove Theorem 5.2.

Proof of Theorem 5.2 Observe that

$$\begin{aligned}
\zeta_A(1,1,2,\cdots,2^{r-1}) &= \sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d>d_1}} S_d(1)S_{d_1}(1)\cdots S_{d_r}(2^{r-1})\\
&= \sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d_2<d<d_1}} S_d(1)S_{d_1}(1)\cdots S_{d_r}(2^{r-1}).
\end{aligned}$$

It follows from Lemma 5.3 that

$$\begin{aligned}
&\zeta_A(1)\zeta_A(1,2,\cdots,2^{r-1})\\
&=\sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d>d_1}}+\sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d=d_1}}+\sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d_2<d<d_1}}+\sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d=d_2}}+\sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d<d_2}} S_d(1)S_{d_1}(1)\cdots S_{d_r}(2^{r-1})\\
&=\sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d=d_1}} S_{d_1}(2)S_{d_2}(2)\cdots S_{d_r}(2^{r-1})+\sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d=d_2}} S_{d_1}(1)[S_{d_2}(3)+S_{d_2}(2,1)]\cdots S_{d_r}(2^{r-1})\\
&+\sum_{\substack{d_1>d_2>\cdots>d_r\geq 0\\ d<d_2}} S_d(1)S_{d_1}(1)\cdots S_{d_r}(2^{r-1})\\
&=\zeta_A(1,1,2,\cdots,2^{r-2})^2+\zeta_A(1,3,2^2,\cdots,2^{r-1}).
\end{aligned}$$

By the above formula and Lemma 2.4, Theorem 5.1 and [9, Theorem3.2], we have

$$\zeta_A(1,3,2^2,\cdots,2^{r-1})=\frac{1}{L_1^{2^{r-2}}L_2^{2^{r-3}}\cdots L_{r-3}^{2^2}L_{r-2}^{2^2}}\frac{[r]-1}{L_1[r]}\zeta_A(2^r).$$

□

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References

[1] Greg W. Anderson; Dinesh S. Thakur, Tensor powers of the Carlitz module and zeta values, *Ann. of Math.*, (2) 132 (1990), no. 1, 159–191.

[2] Leonard Carlitz, On certain functions connected with polynomials in a Galois field. *Duke Math. J.*, 1 (1935), no. 2, 137–168.

[3] Chieh-Yu Chang, Linear independence of monomials of multizeta values in positive characteristic, *Compos. Math.*, 150 (2014), no. 11, 1789–1808.

[4] Chieh-Yu Chang; Matthew A. Papanikolas; Jing Yu, An effective criterion for Eulerian multizeta values in positive characteristic, *arXiv:1411.0124*.

[5] Huei-Jeng Chen, On Shuffle of Double Zeta Values over $\mathbb{F}_q[t]$, Journal of Number Theory, 148 (2015), 153–163.

[6] Leonard Eugene Dickson, The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group. *Ann. of Math.*, 11 (1896/97), no. 1-6, 65–120.

[7] Yen-Liang Kuan; Yi-Hsuan Lin, Criterion for deciding zeta-like multizeta values in positive characteristic. *To appear in Experimental Mathematica.*

[8] Jose Alejandro Lara Rodríguez, Relations between multizeta values in characteristic p, *Journal of Number Theory*, 2011, 131 (11), 2081–2099.

[9] Jose Alejandro Lara Rodríguez; Dinesh S. Thakur, Zeta-like multizeta values for $\mathbb{F}_q[t]$, *Indian J. Pure Appl. Math.*, 45 (5), 785–798 (2014).

[10] Dinesh S. Thakur, Function field Arithmetic, *World Scientific Publishing Co. Inc. River Edge*, NJ, 2004.

[11] Dinesh S. Thakur, Relations between multizeta values for $F_q[t]$, *International Mathematics Research Notices. IMRN*, 2009, (12), 2318–2346.

[12] Dinesh S. Thakur, Multizeta in function field arithmetic , *To appear in proceedings of Banff workshop to be published by EMS.*

[13] Jing Yu, Transcendence and special zeta values in characteristic p, *Ann. of Math.*, (2) 134 (1991), no. 1, 1–23.