ON SHUFFLE OF DOUBLE EISENSTEIN SERIES IN POSITIVE CHARACTERISTIC

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ABSTRACT. The study of the present paper is inspired by Gangl, Kaneko and Zagier's result of the connection with double zeta values and modular forms. We introduce double Eisenstein series $E_{r,s}$ in positive characteristic with double zeta values $\zeta_A(r,s)$ as their constant term and compute the t-expansions of the double Eisenstein series. Moreover, we derive the shuffle relations of double Eisenstein series which match the shuffle relations of double zeta values in [4].

1. INTRODUCTION

The multiple zeta values (abbreviated as MZV's) $\zeta(s_1, \cdots, s_r)$, where $s_1, \ldots, s_r$ are positive integers with $s_1 \geq 2$, are generalizations of zeta values. These numbers were first studied by Euler for the case of $r = 2$. He derived certain interesting relations between double zeta values, such as the sum formula $\sum_{n=2}^{k-1} \zeta(n, k-n) = \zeta(k)$. The MZV's have many connections with various arithmetic and geometric points of view, we refer the reader to [1, 12, 13]. Gangl-Kaneko-Zagier [5] introduced and studied double Eisenstein series $G_{r,s}$ with double zeta value $\zeta(r,s)$ as its constant term for $r \geq 3$ and $s \geq 2$. They showed that certain relations between double zeta values can be "lifted" to relations between corresponding double Eisenstein series.

Thakur [9] introduced the multizeta values $\zeta_A(s_1, \cdots, s_r)$ (abbreviated as MZV's too) in positive characteristic and [11] showed the existence of shuffle relations for multizeta values. These shuffle relations can be derived from explicit formulas (see [4]). Unlike the classical double shuffle relation, there is no other expression of the product of two single (Carlitz) zeta values than the shuffle relation mentioned above at date. In the recent work of Chang [3], he established an effective criterion for computing the dimension of the space generated by double zeta values and a fundamental period of the Carlitz module raised to the weight power in terms of special points in the tensor power of the Carlitz module. One views his work as an algebraic/arithmetic point of view.

In the present note the author also introduces double Eisenstein series $E_{r,s}$ in positive characteristic with double zeta values $\zeta_A(r,s)$ as their constant term. Moreover, the shuffle relations of double zeta values in [4] match the shuffle relations of the corresponding double Eisenstein series. We mention that the approach is from the idea of Gangl-Kaneko-Zagier, but the proofs are entire different. We use Goss polynomials to compute the t-expansions (an ananolgy to Fourier expansions) of the double Eisenstein series, and use partial fractional decomposition method to show the desired relation.

Let $\mathcal{DZ}_k$ denote the $\mathbb{F}_q(\theta)$-vector space ($\mathbb{Q}$-vector space in the classical case) spanned by double zeta values of weight k. It is a difficult problem in computing the dimension of $\mathcal{DZ}_k$. In the classical case, Gangl-Kaneko-Zagier [5] showed that the structure of the $\mathbb{Q}$-vector spaces $\mathcal{DZ}_k$ are well connected to the space of modular forms M_k of weight k for the full modular group $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$ by means of double Eisensetin series. With additional work, they bound the dimension of $\mathcal{DZ}_k$ in terms of the dimension of the space of cusp forms S_k of weight k for Γ_1. In a subsequent paper we shall explore the space $\mathcal{DZ}_k$ in connection with a suitable space of weight k Drinfeld modular forms for $\mathrm{GL}_2(\mathbb{F}_q[\theta])$.

2. Preliminaries

2.1. **Notation.** We adopt the notations below in the following sections.

$\mathbb{F}_q$:= a finite field with $q = p^m$ elements.
K := $\mathbb{F}_q(\theta)$, the rational function field in the variable θ.
∞ := the zero of $1/\theta$, the infinite place of K.
$|\ |_\infty$:= the nonarchimedean absolute value on K corresponding to ∞.
K_∞:= $\mathbb{F}_q((1/\theta))$, the completion of K with respect to $|\cdot|_\infty$.
$\mathbb{C}_\infty$:= the completion of a fixed algebraic closure of K_∞ with respect to $|\cdot|_\infty$.
A := $\mathbb{F}_q[\theta]$, the ring of polynomials in the variable θ over $\mathbb{F}_q$.
A_+ := the set of monic polynomials in A.
A_d := the set of polynomials in A of degree d.
A_{d+}:= $A_d \cap A_+$, the set of monic polynomials in A of degree d.
$[n]$:= $\theta^{q^n} - \theta$.
D_n := $\prod_{i=0}^{n-1} \theta^{q^n} - \theta^{q^i} = [n][n-1]^q \cdots [1]^{q^{n-1}}$.
L_n := $\prod_{i=1}^{n} \theta^{q^i} - \theta = [n][n-1]\cdots[1]$.
l_n := $(-1)^n L_n$.

2.2. **Shuffle relations of double zeta values.** For any tuple $(s_1,\cdots,s_r) \in \mathbb{N}^r$,

$$\zeta_A(s_1,\cdots,s_r) := \sum_{\substack{a_i \in A_+ \\ \deg_\theta a_1 > \cdots > \deg_\theta a_r}} \frac{1}{a_1^{s_1}\cdots a_r^{s_r}} \in K_\infty.$$

We mention that they are not only non-vanishing [10] but also transcendental over K [2]. Here r is called the depth and $w := s_1 + \cdots + s_r$ is called the weight of the presentation $\zeta_A(s_1,\cdots,s_r)$. Recall that for the case $r=2$, we have the following shuffle relation of double zeta values.

Theorem 2.3. [4] *For any $r,s\in\mathbb{N}$, we have*

$$\begin{aligned}\zeta_A(r)\zeta_A(s) &= \zeta_A(r,s)+\zeta_A(s,r)+\zeta_A(r+s)\\ &\quad+\sum_{\substack{i+j=r+s\\ q-1\mid j}}\left[(-1)^{r-1}\binom{j-1}{r-1}+(-1)^{s-1}\binom{j-1}{s-1}\right]\zeta_A(i,j).\end{aligned}$$

3. Eisenstein series and Double Eisenstein series

3.1. **Double Eisenstein series.** Let $\Omega:=\mathbb{C}_\infty-K_\infty=\{z\in\mathbb{C}_\infty\mid z\notin K_\infty\}$ be the Drinfeld upper half-plane, which has a natural structure as a connected admissible open subspace of the rigid analytic space $\mathbb{P}^1(\mathbb{C}_\infty)$ (see [7]).

Definition 3.2. Let $m,a\in A$ and $z\in\Omega$. We write "$mz+a\succ 0$" if $m\in A_+$ or $m=0$, $a\in A_+$. Suppose that $mz+a\succ 0$ and $nz+b\succ 0$. We write "$mz+a\succ nz+b$" if one of the following conditions holds: (i) $m\in A_+$, $n=0$ (ii) $m=n=0$, $a,b\in A_+$ with $\deg a>\deg b$. (iii) $m,n\in A_+$ with $\deg m>\deg n$.

Definition 3.3. For $r,s,k\in\mathbb{N}$, let E_k be the Eisenstein series of weight k defined on $z\in\Omega$:

$$E_k(z):=\sum_{\substack{mz+a\in Az+A\\ mz+a\succ 0}}\frac{1}{(mz+a)^k}.$$

The double Eisenstein series $E_{r,s}$ is defined by

$$E_{r,s}(z):=\sum_{\substack{mz+a,nz+b\in Az+A\\ mz+a\succ nz+b\succ 0}}\frac{1}{(mz+a)^r(nz+b)^s}.$$

Note that both $E_k(z)$ and $E_{r,s}(z)$ are rigid analytic functions on Ω [8].

Remark 3.3.1. The modular Eisenstein series in [8] are defined by

$$\mathcal{E}_k(z)=\sum_{mz+a\in Az+A}\frac{1}{(mz+a)^k}.$$

It can be checked that $\mathcal{E}_k(z)=0$ if $q-1\nmid k$ and $\mathcal{E}_k(z)=-E_k(z)$ if $q-1\mid k$. In the latter case $\mathcal{E}_k(z)$ is a Drinfeld modular form of weight k and type m (m is a class in $\mathbb{Z}/(q-1)\mathbb{Z}$) for $GL_2(A)$.

3.4. **t-expansions.** Let $(-\theta)^{\frac{1}{q-1}}$ be a fixed choice of $(q-1)$-st root of $-\theta$ and $\tilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty}(1 - \frac{\theta}{\theta^{q^i}})^{-1}$ be the fundamental period of the Carlitz module. Let $t(z)$ be the rigid analytic function on Ω given by

$$t(z) = \sum_{b \in \tilde{\pi}A} \frac{1}{(\tilde{\pi}z + b)}.$$

We have $t(z + a) = t(z)$ for any $a \in A$. The function $t(z)$ serves as a uniformizing parameter "at the infinity", so that every function $f(z)$ on Ω satisfying the property $f(z + a) = f(z)$ for any $a \in A$ can be written as the form $f(z) = \tilde{f}(t(z))$ with respect to t. This is analogues to $q(z) = \exp(2\pi\sqrt{-1}z)$ in the classical case.

Let $\exp_{\tilde{\pi}A}(z) := z \prod_{\lambda \in \tilde{\pi}A, \lambda \neq 0} (1 - \frac{z}{\lambda}) = \sum_{i \geq 0} \alpha_i z^{q^i}$ be the exponential function on $\mathbb{C}_\infty$ with respect to the lattice $\tilde{\pi}A$.

Proposition 3.5. [7] *There exists a sequence of polynomials $G_k(X) \in K[X]$ for $k \in \mathbb{N}$ satisfying the following properties:*
(i) For $z \in \Omega$,

$$\tilde{\pi}^k G_k(t(z)) = \sum_{b \in A} \frac{1}{(z + b)^k}.$$

(ii) $G_k(X) = X(G_{k-1} + \alpha_1 G_{k-q} + \cdots + \alpha_i G_{k-q^i})$, where $i = [\log_q k]$.
(iii) $G_k(0) = 0$ for all k.

Definition 3.6. We call $G_k(x)$ the k-th Goss polynomial.

On the domain Ω, $E_k(z)$ and $E_{r,s}(z)$ are invariant under the translations $z \mapsto z + b$ for $b \in A$. We can use Proposition 3.5 to derive the expansions of Eisenstein and double Eisenstein series with respect to t.

Proposition 3.7. *The t-expansions of $E_k(z)$ and $E_{r,s}(z)$ are given by*

$$E_k(z) = \zeta_A(k) + \sum_{m \in A_+} \tilde{\pi}^k G_k(t(mz))$$

and

$$\begin{aligned} E_{r,s}(z) = \zeta_A(r,s) &+ \sum_{m \in A_+} \tilde{\pi}^r \zeta_A(s) G_r(t(mz)) \\ &+ \sum_{\substack{m,n \in A_+ \\ \deg m > \deg n \geq 0}} \tilde{\pi}^{r+s} G_r(t(mz)) G_s(t(nz)). \end{aligned}$$

Proof.

$$\begin{aligned}E_k(z) &= \sum_{m=0, n\in A_+} \frac{1}{n^k} + \sum_{m\in A_+, n\in A} \frac{1}{(mz+n)^k} \\ &= \zeta_A(k) + \sum_{m\in A_+} \tilde{\pi}^k G_k(t(mz))\end{aligned}$$

$$\begin{aligned}E_{r,s}(z) &= \sum_{\substack{m=n=0, a,b\in A_+ \\ \deg a>\deg b}} \frac{1}{a^r b^s} + \sum_{m\in A_+, a\in A; n=0, b\in A_+} \frac{1}{(mz+a)^r b^s} + \sum_{\substack{m,n\in A_+, a,b\in A \\ \deg m>\deg n}} \frac{1}{(mz+a)^r (nz+b)^s} \\ &= \zeta_A(r,s) + \sum_{m\in A_+} \tilde{\pi}^r \zeta_A(s) G_r(t(mz)) + \sum_{\substack{m,n\in A_+ \\ \deg m>\deg n}} \tilde{\pi}^{r+s} G_r(t(mz)) G_s(t(nz)).\end{aligned}$$

□

4. Shuffle of Double Eisenstein series

In this section we will derive the shuffle relation of a product of two Eisenstein series.

Theorem 4.1. *On the domain Ω, we have*

$$E_r(z)E_s(z) = E_{r,s}(z)+E_{s,r}(z)+E_{r+s}(z)+\sum_{\substack{i+j=r+s \\ q-1\mid j}} \left[(-1)^{r-1}\binom{j-1}{r-1} + (-1)^{s-1}\binom{j-1}{s-1}\right] E_{i,j}(z).$$

Proof.

$$\begin{aligned}E_r(z)E_s(z) &= \zeta_A(r)\zeta_A(s) + \zeta_A(r)\sum_{n\in A_+} \tilde{\pi}^s G_s(t(nz)) \\ &\quad + \zeta_A(s)\sum_{m\in A_+} \tilde{\pi}^r G_r(t(mz)) + \sum_{m,n\in A_+} \tilde{\pi}^{r+s} G_r(t(mz)) G_s(t(nz)).\end{aligned}$$

Let $C_{r,s}^{i,j} := (-1)^{r-1}\binom{j-1}{r-1} + (-1)^{s-1}\binom{j-1}{s-1}$, then we can expand $E_{r,s}(z) + E_{s,r}(z) + E_{r+s}(z) + \sum_{\substack{i+j=r+s \\ q-1\mid j}} C_{r,s}^{i,j} E_{i,j}(z)$ as follows.

$$E_{r,s}(z) + E_{s,r}(z) + E_{r+s}(z) + \sum_{\substack{i+j=r+s \\ q-1\mid j}} C_{r,s}^{i,j} E_{i,j}(z)$$
$$= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4,$$

where

$$\Delta_1 = \zeta_A(r,s) + \zeta_A(s,r) + \zeta_A(r+s) + \sum_{\substack{i+j=r+s \\ q-1|j}} C^{i,j}_{r,s}\zeta_A(i,j),$$

$$\Delta_2 = \zeta_A(s)\sum_{m\in A_+} \tilde{\pi}^r G_r(t(mz)) + \zeta_A(r)\sum_{n\in A_+} \tilde{\pi}^s G_s(t(nz)),$$

$$\Delta_3 = \sum_{\substack{m,n\in A_+ \\ \deg n>\deg m\geq 0}} \tilde{\pi}^{r+s} G_r(t(nz))G_s(t(mz)) + \sum_{\substack{m,n\in A_+ \\ \deg n>\deg m\geq 0}} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(nz)),$$

$$\Delta_4 = \sum_{m\in A_+} \tilde{\pi}^{r+s} G_{r+s}(t(mz)) + \sum_{\substack{i+j=r+s \\ q-1|j}} C^{i,j}_{r,s}\left(\zeta_A(j)\sum_{m\in A_+}\tilde{\pi}^i G_i(t(mz)) + \sum_{\substack{m,n\in A_+ \\ \deg n>\deg m}} \tilde{\pi}^{r+s} G_i(t(mz))G_j(t(nz))\right).$$

By Theorem 2.3 we have $\zeta_A(r)\zeta_A(s) = \Delta_1$. So it suffices to show that

$$\sum_{l=0}^{\infty} \sum_{\substack{m,n\in A_+ \\ \deg m=\deg n=l}} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(nz)) = \Delta_4.$$

We will prove it by partial fraction decomposition. For any two distinct elements a, b in an integral domain R (we adopt R for A or $A[z]$ here), we have

$$\frac{1}{a^r b^s} = \sum_{i+j=r+s} \frac{1}{(a-b)^j}\left(\frac{(-1)^s\binom{j-1}{s-1}}{a^i} + \frac{(-1)^{j-r}\binom{j-1}{r-1}}{b^i}\right).$$

Consider

$$\begin{aligned}\sum_{l=0}^{\infty} \sum_{\substack{m,n\in A_+ \\ \deg m=\deg n=l}} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(nz)) &= \sum_{l=0}^{\infty} \sum_{\substack{m\in A_+ \\ \deg m=l}} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(mz)) \\ &+ \sum_{l=0}^{\infty} \sum_{\substack{m,n\in A_+ \\ \deg m=\deg n=l, m\neq n}} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(nz)).\end{aligned}$$

We rewrite the first part of the sum as

$$\begin{aligned}
\sum_{m\in A_+} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(mz)) &= \sum_{\substack{m\in A_+\\ a,b\in A}} \frac{1}{(mz+a)^r(mz+b)^s} \\
&= \sum_{\substack{m\in A_+\\ a=b\in A}} \frac{1}{(mz+a)^r(mz+b)^s} + \sum_{\substack{m\in A_+\\ a\neq b\in A}} \frac{1}{(mz+a)^r(mz+b)^s} \\
&= \sum_{m\in A_+} \tilde{\pi}^{r+s} G_{r+s}(t(mz)) \\
&+ \sum_{\substack{m\in A_+\\ a\neq b\in A}} \sum_{i+j=r+s} \frac{1}{(a-b)^j} \left(\frac{(-1)^s\binom{j-1}{s-1}}{(mz+a)^i} + \frac{(-1)^{j-r}\binom{j-1}{r-1}}{(mz+b)^i} \right) \\
&= \sum_{m\in A_+} \tilde{\pi}^{r+s} G_{r+s}(t(mz)) \\
&+ \sum_{i+j=r+s} \sum_{\substack{m\in A_+\\ a,f\in A, f\neq 0}} \frac{1}{f^j} \left(\frac{(-1)^s\binom{j-1}{s-1}}{(mz+a)^i} + \frac{(-1)^{j-r}\binom{j-1}{r-1}}{(mz+f+a)^i} \right).
\end{aligned}$$

For a fixed pair (i,j), we have that

$$\begin{aligned}
\sum_{\substack{m\in A_+\\ a,f\in A, f\neq 0}} \frac{1}{f^j} \frac{(-1)^s\binom{j-1}{s-1}}{(mz+a)^i} &= \sum_{\eta\in\mathbb{F}_q^*} \sum_{\substack{m\in A_+\\ a\in A, f\in A_+}} \frac{1}{(\eta f)^j} \frac{(-1)^s\binom{j-1}{s-1}}{(mz+a)^i} \\
&= \begin{cases} 0, & \text{if } q-1 \nmid j; \\ -\displaystyle\sum_{\substack{m\in A_+\\ a\in A, f\in A_+}} \frac{1}{f^j} \frac{(-1)^s\binom{j-1}{s-1}}{(mz+a)^i}, & \text{if } q-1 | j. \end{cases}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{\substack{m\in A_+\\ a,f\in A, f\neq 0}} \frac{1}{f^j} \frac{(-1)^{j-r}\binom{j-1}{r-1}}{(mz+f+a)^i} &= \sum_{\substack{m\in A_+\\ b,f\in A, f\neq 0}} \frac{1}{f^j} \frac{(-1)^{j-r}\binom{j-1}{r-1}}{(mz+b)^i} \\
&= \begin{cases} 0, & \text{if } q-1 \nmid j; \\ -\displaystyle\sum_{\substack{m\in A_+\\ b\in A, f\in A_+}} \frac{1}{f^j} \frac{(-1)^{j-r}\binom{j-1}{r-1}}{(mz+b)^i}, & \text{if } q-1 | j. \end{cases}
\end{aligned}$$

This implies that

$$\sum_{m\in A_+}\tilde{\pi}^{r+s}G_{r+s}(t(mz))+\sum_{i+j=r+s}\sum_{\substack{m\in A_+\\ a,f\in A,f\neq 0}}\frac{1}{f^j}\left(\frac{(-1)^s\binom{j-1}{s-1}}{(mz+a)^i}+\frac{(-1)^{j-r}\binom{j-1}{r-1}}{(mz+f+a)^i}\right)$$
$$=\sum_{m\in A_+}\tilde{\pi}^{r+s}G_{r+s}(t(mz))+\sum_{\substack{i+j=r+s\\ q-1|j}}\sum_{\substack{m\in A_+\\ b\in A,f\in A_+}}\frac{1}{f^j}\frac{(-1)^{s-1}\binom{j-1}{s-1}+(-1)^{j-r-1}\binom{j-1}{r-1})}{(mz+a)^i}$$
$$=\sum_{m\in A_+}\tilde{\pi}^{r+s}G_{r+s}(t(mz))+\sum_{\substack{i+j=r+s\\ q-1|j}}C^{i,j}_{r,s}\zeta_A(j)\sum_{m\in A_+}\tilde{\pi}^iG_i(t(mz)).$$

By using a similar argument, the second sum can be expressed as

$$\sum_{l=1}^{\infty}\sum_{\substack{m,n\in A_+,m\neq n\\ \deg m=\deg n=l}}\tilde{\pi}^{r+s}G_r(t(mz))G_s(t(nz))$$
$$=\sum_{l=1}^{\infty}\sum_{\substack{m\neq n\in A_+,a,b\in A\\ \deg m=\deg n=l}}\frac{1}{(mz+a)^r(mz+b)^s}$$
$$=\sum_{l=1}^{\infty}\sum_{\substack{m\neq n\in A_+,a,b\in A\\ \deg m=\deg n=l}}\sum_{i+j=r+s}\frac{1}{((m-n)z+(a-b))^j}\left(\frac{(-1)^s\binom{j-1}{s-1}}{(mz+a)^i}+\frac{(-1)^{j-r}\binom{j-1}{r-1}}{(nz+b)^i}\right)$$
$$=\sum_{l=1}^{\infty}\sum_{i+j=r+s}\sum_{\substack{m\neq n'\in A_+,a,b'\in A\\ \deg n'<\deg m=l}}C^{i,j}_{r,s}\frac{1}{(n'z+b')^j(mz+a)^i}$$
$$=\sum_{l=1}^{\infty}\sum_{\substack{i+j=r+s\\ q-1|j}}C^{i,j}_{r,s}\sum_{\substack{m,n'\in A_+\\ \deg n'<\deg m=l}}\tilde{\pi}^{r+s}G_i(t(mz))G_j(t(n'z)).$$

Combining the first sum with the second sum, we have

$$\begin{aligned}
&\sum_{l=0}^{\infty} \sum_{\substack{m,n\in A_+\\ \deg m=\deg n=l}} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(nz))\\
&= \sum_{l=0}^{\infty} \sum_{\substack{m\in A_+\\ \deg m=l}} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(mz))\\
&+ \sum_{l=0}^{\infty} \sum_{\substack{m,n\in A_+\\ \deg m=\deg n=l, m\neq n}} \tilde{\pi}^{r+s} G_r(t(mz))G_s(t(nz))\\
&= \sum_{m\in A_+} \tilde{\pi}^{r+s} G_{r+s}(t(mz)) + \sum_{\substack{i+j=r+s\\ q-1|j}} C_{r,s}^{i,j}\zeta_A(j) \sum_{m\in A_+} \tilde{\pi}^{i} G_i(t(mz))\\
&+ \sum_{l=1}^{\infty} \sum_{\substack{i+j=r+s\\ q-1|j}} C_{r,s}^{i,j} \sum_{\substack{m,n'\in A_+\\ \deg n'<\deg m=l}} \tilde{\pi}^{r+s} G_i(t(mz))G_j(t(n'z))\\
&= \Delta_4.
\end{aligned}$$

□

Remark 4.1.1. From Proposition 3.5 (iii), we see that the constant terms in the t-expansions of $E_r(z)E_s(z)$ and

$$E_{r,s}(z)+E_{s,r}(z)+E_{r+s}(z)+\sum_{\substack{i+j=r+s\\ q-1|j}} \left[(-1)^{r-1}\binom{j-1}{r-1}+(-1)^{s-1}\binom{j-1}{s-1}\right] E_{i,j}(z)$$

are $\zeta_A(r)\zeta_A(s)$ and

$$\zeta_A(r,s)+\zeta_A(s,r)+\zeta_A(r+s)+\sum_{\substack{i+j=r+s\\ q-1|j}} \left[(-1)^{r-1}\binom{j-1}{r-1}+(-1)^{s-1}\binom{j-1}{s-1}\right] \zeta_A(i,j)$$

respectively. It follows that the relation in Theorem 4.1 can be viewed as a “lifting” of the shuffle relation in Theorem 2.3.

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