BAKER-SCHMIDT THEOREM FOR HAUSDORFF DIMENSIONS IN FINITE CHARACTERISTIC

Abstract. In 1962 Sprindžuk proved Mahler's conjecture in both the real and complex cases. Baker gave a generalized result by using a modified version of Sprindžuk's method. Later, Baker and Schmidt derived the Hausdorff dimensions of sets which are defiend in terms of approximation by algebraic numbers of bounded degrees by using Baker's theorem. In this article we will prove two analogue theorems in the fields of formal power series over finite finite fields.

1. Introduction

For an algebraic number α, let $H(\alpha)$ denote the height of the irreducible polynomial of α with co-prime integral coefficients. For a real number ζ not algebraic of degree at most n, let

$$w_n(\zeta) = \limsup_{H\to\infty} \frac{-\log|\zeta-\alpha|}{\log H(\alpha)},$$

where α runs through all algebraic real numbers of degree less than n. A theorem due to Dirichlet says that $w_1(\theta) \geq 2$. Liouville's theorem implies that for any irrational real algebraic number ζ of degree $d(\zeta)$, $w_1(\zeta) \leq d(\zeta)$. Finally, a celebrated theorem proved by Roth [11] shows that if ζ is an irrational real algebraic number, $w_1(\zeta) = 2$.

For $\lambda \geq 1$, $\mathcal{K}(\lambda)$ is defined as the set of real numbers ζ for which there exist infinitely many rational numbers p/q such that $|\zeta - p/q| < 1/|q|^{2\lambda}$. According to Dirichlet's theorem, $\mathcal{K}(1)$ includes all irrational numbers. On the other hand, Roth's theorem implies that if $\lambda > 1$, then $\mathcal{K}(\lambda)$ contains no algebraic elements. Khintchine [7] proved that for $\lambda > 1$, $\mathcal{K}(\lambda)$ has (Lebesgue) measure zero. It is natural to measure the "size" of a set of measure zero in terms of its Hausdorff dimension. (The related definitions will be given in Section 3.) Due to Jarník [5], and Besicovitch [2], the Hausdorff dimension of $\mathcal{K}(\lambda)$ is $\frac{1}{\lambda}$ when $\lambda > 1$.

Wirsing tried to generalize Roth's theorem and considered the approximation by algebraic elements of bounded degree. He proved [21] that if ζ is an algebraic number, then $w_n(\zeta) \leq 2n$. However, a corollary to the celebrated Subspace Theorem of Schmidt [13] showed that the exponent $2n$ can be replaced by $n+1$.

For $n \in \mathbb{N}$ and $\lambda \geq 1$, we denote by $\mathcal{K}_n(\lambda)$ the set of all $\zeta \in \mathbb{R}$ such that, for any $\lambda' < \lambda$, there exist infinitely many algebraic numbers α with degree at most n satisfying

$$|\zeta - \alpha| < H(\alpha)^{-(n+1)\lambda'}.$$

We denote by $\mathcal{K}'_n(\lambda)$ the subset in $\mathcal{K}_n(\lambda)$ consisting of elements not belonging to $\mathcal{K}_n(\tau)$ for all $\tau > \lambda$. Sprindžuk's result [18, p.151] implies that for almost all real numbers ζ and any $n \in \mathbb{N}$, ζ belongs to $\mathcal{K}'_n(1)$. Baker and Schmidt [3] studied the Hausdorff dimensions of these sets and obtained the following result:

Theorem 1.1. (Baker-Schmidt) Let n be a positive integer. For $\lambda \geq 1$, the Hausdorff dimensions of $\mathcal{K}_n(\lambda)$ and $\mathcal{K}'_n(\lambda)$ equal $1/\lambda$.

In this article we will study Diophantine approximation in fields of power series over finite fields. Let $\mathbb{F}_q$ be a finite field with $q = p^s$ elements. We consider $A = \mathbb{F}_q[T]$, $K = \mathbb{F}_q(T)$

and $K_\infty = \mathbb{F}_q((1/T))$ as analogues of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. The nonarchimedean absolute value $|\cdot|$ comes from the infinity place $1/T$. For $\zeta \in K_\infty$ not algebraic of degree at most n, we define $w_n(\zeta)$ the same way as in the classical case. It is not difficult to see that Dirichlet's theorem still holds. Mahler [9] worked out an analogue in fields of power series to Liouville's theorem. However, an analog of Roth's theorem now fails. Mahler [9] gave a counterexample by constructing an algebraic element ζ with degree $d(\zeta) > 1$ such that $w_1(\zeta) = d(\zeta)$. For counterexamples of the approximation by algebraic elements of bounded degree, we refer the readers to [19, 20].

For $n \in \mathbb{N}$ and $\lambda \geq 1$, we define

$$\mathcal{K}_n(\lambda) = \{\zeta \in K_\infty \mid \text{for any } \lambda' < \lambda, \text{ there exist infinitely many } \alpha \in \overline{K} \text{ with } d(\alpha) \leq n \text{ satisfying } |\zeta - \alpha| < (H(\alpha))^{-(n+1)\lambda'}\},$$

and

$$\mathcal{K}'_n(\lambda) = \{\zeta \in \mathcal{K}_n(\lambda) \mid \zeta \notin \mathcal{K}_n(\lambda') \text{ for all } \lambda' > \lambda\}.$$

Despite the failures of Roth's theorem and Schmidt's subspace theorem, we have parallel results in the function field case by studying the measures and the Hausdorff dimensions of $\mathcal{K}_n(\lambda)$ and $\mathcal{K}'_n(\lambda)$. Guntermann [4] stated that one can deduce from Sprindžuk's result (c.f. [18, p.138]) that for almost all (in the sense of Haar measure) elements ζ in K_∞ and any $n \in \mathbb{N}$, ζ belongs to $\mathcal{K}'_n(1)$. The aim of the present paper is to derive the Hausdorff dimensions of $\mathcal{K}_n(\lambda)$ and $\mathcal{K}'_n(\lambda)$. We shall prove

Main Theorem 1. *For $n \in \mathbb{N}$ and $\lambda \geq 1$, the Hausdorff dimensions $\mathcal{K}_n(\lambda)$ and $\mathcal{K}'_n(\lambda)$ equal $1/\lambda$.*

The idea is inspired by Baker and Schmidt [3]. In Section 3.1 we consider a special family of closed balls covering $\mathcal{K}_n(\lambda)$ to obtain the upper bound $1/\lambda$. To give the lower bound, we need first prove

Main Theorem 2. *Let $\psi(H)$ be a positive monotonic decreasing function defined on positive integers such that $\sum \psi(H)$ converges. Then for almost all $\theta \in K_\infty$ and any $n \in \mathbb{N}$ there exist only finitely many polynomials $P(x) \in A[X]$ with degree n such that*

$$|P(\theta)| < (\psi(H(P)))^n,$$

where $H(P)$ denotes the height of P.

Then we use an analogue of Minkowski's geometry of numbers, which was proved by Mahler [8], and Main Theorem 2 to construct a regular system. (See Corollary 3.8 in Section 3.3.) Secondly we prove Proposition 3.10, which provides a way to estimate the Hausdorff dimensions of sets related to regular systems. Combining with Proposition 3.10 we can obtain the lower bound of the Hausdorff dimensions of $\mathcal{K}_n(\lambda)$ and $\mathcal{K}'_n(\lambda)$ in Section 3.4. One difficulty arises from the separability of an irreducible polynomial over $\mathbb{Q}$, which will be used to claim that the discriminant is not zero. Another one arises from the fact that we can assume that the leading coefficient of an irreducible integral polynomial is a positive integer in the classical case, which Baker uses to make a reduction and control the measures of the sets related to elements θ in Main Theorem 2. These two properties do not hold in K.

2. Main Theorem 2

2.1. **An analogue of Baker's theorem.** Main Theorem 2 is an analogue in the function field case of Baker's Theorem [1]. For any real number θ, let $||\theta||$ denotes the distance of θ from the nearest integer. Mahler conjectured that almost all real numbers θ have the following property. For any positive integer n and positive number ϵ, there are only finitely many positive integer b such that

$$\max_{j=1,\cdots,n} ||b\theta^j|| < b^{-(\frac{1}{n}+\epsilon)}.$$

This conjecture has been proved by Sprindžuk [15, 16]. Baker's theorem gives the following generalization of Sprindžuk's result.

Corollary 2.1. (*Baker*) *Let $\phi(b)$ be a positive monotone decreasing function defined on positive integers h such that $\sum b\phi(b)$ converges. Then for almost all real numbers θ and any positive integer n, there are only finitely many positive integers b such that*

$$\max_{j=1,\cdots,n} ||b\theta^j|| < b^{-\frac{1}{n}}\phi(b)^n.$$

2.2. **Key reduction.** The crucial part in the proof of Main Theorem 2 is to make a reduction to the approximation by irreducible polynomials with heights occurring in their leading coefficients. From now on we μ be the Haar measure on subsets in K_∞ such that $\mu(B(a, q^h)) = q^{h+1}$, where $B(a, q^h)$ denotes the closed ball of radius q^h centered at a.

Proposition 2.2. *For $n \in \mathbb{N}$, $\theta \in K_\infty$ and $\psi(H)$ a positive monotonic decreasing function defined on $\mathbb{N}$ such that $\sum \psi(H)$ converges, let $\mathcal{P}(n, \psi, \theta)$ denote the set of all polynomials $P(x)$ with degree n, integral coefficients and height H such that*

$$|P(\theta)| < (\psi(H))^n.$$

Let $\mathfrak{R}(n, \psi)$ denote the set of all $\theta \in K_\infty$ for which $\mathcal{P}(n, \psi, \theta)$ contains infinitely many elements. Further, let $\varphi(n, \psi)$ denote the more restricted set for which infinitely many polynomials in $\mathcal{P}$ are required (i) to be irreducible and (ii) to have $|a_n| \geq \max\{|a_0|, \cdots, |a_{n-1}|\}$, where $P(x) = a_0 + \cdots + a_n x^n$. Suppose that for every ψ, the sets $\varphi(m, \psi)$ $(m = 1, 2, \cdots, n)$ are of measure zero. Then $\mathfrak{R}(n, \psi)$ are also of measure zero for every ψ.

To prove Proposition 2.2 and Main Theorem 2, we need several lemmas in the next section. Then we will complete the proof in Section 2.5.

2.3. **Preliminary lemmas.**

Lemma 2.3. *Let n be a positive integer. Then there exists a constant $C(n) \geq 1$ such that for any polynomial $P(x) = a_n x^n + \cdots + a_0 \in A[x]$ of degree n,*

$$\max_{0\leq l\leq n} |P(T^l)| \geq C(n) \max_{0\leq i\leq n} |a_i|.$$

Proof. c.f.[18, p.120] □

Lemma 2.4. *Let $P(x)$ be a polynomial of degree n with distinct roots $\alpha_1, \cdots, \alpha_n$. Suppose that $|\theta - \alpha_1| \leq |\theta - \alpha_i|$ for $i = 2, \cdots, n$. Then*

$$|P(\theta)| \geq |P'(\alpha_1)||\theta - \alpha_1|$$

and

$$|P(\theta)||\alpha_1 - \alpha_2| \geq |P'(\alpha_1)||\theta - \alpha_1|^2.$$

If, further, $|\theta - \alpha_1| \leq |\alpha_1 - \alpha_i|$ for $i = 2, \cdots, n$, then

$$|P(\theta)| \leq |P'(\alpha_1)||\theta - \alpha_1|.$$

Proof. Suppose that the leading coefficient of $P(x)$ is a. By the hypothesis we have

$$|\alpha_1 - \alpha_i| \leq \max(|\theta - \alpha_1|, (|\theta - \alpha_i|) = |\theta - \alpha_i|$$

for $i = 2, \cdots, n$. It follows that

$$\begin{aligned} |\theta - \alpha_1||P'(\alpha_1)| &= |a||\theta - \alpha_1||\alpha_1 - \alpha_2| \cdots |\alpha_1 - \alpha_n| \\ &\leq |a||\theta - \alpha_1||\theta - \alpha_2| \cdots |\theta - \alpha_n| \\ &= |P(\theta)|. \end{aligned}$$

and

$$|\theta - \alpha_1|^2|P'(\alpha_1)| \leq |\theta - \alpha_1||\theta - \alpha_2||P'(\alpha_1)| \leq |\alpha_1 - \alpha_2||P(\theta)|.$$

If further, $|\theta - \alpha_1| \leq |\alpha_1 - \alpha_i|$, we have

$$|\theta - \alpha_i| \leq \max(|\theta - \alpha_1|, |\alpha_1 - \alpha_i|) = |\alpha_1 - \alpha_i|.$$

and hence

$$|P(\theta)| \leq |a||\theta - \alpha_1||\alpha_1 - \alpha_2| \cdots |\alpha_1 - \alpha_n| = |\theta - \alpha_1||P'(\alpha_1)|.$$

□

Lemma 2.5. *Let $d \geq 0$ be a fixed integer. Let $P_1(x), P_2(x)$ be polynomials over $A^{\frac{1}{p^d}} = \mathbb{F}_q[T^{\frac{1}{p^d}}]$ with $\deg P_1 = \deg P_2 = n \geq 2$. Suppose that the leading coefficients of P_1, P_2 are a, b respectively. Let $\alpha_1, \cdots, \alpha_n$ be roots of P_1 and $\beta_1, \cdots, \beta_n$ be roots of P_2. All of these roots are supposed distinct and to have absolute values at most k ($k \geq 1$). Suppose also that*

$$|\alpha_1 - \alpha_2| \leq |\alpha_1 - \alpha_i| \text{ and } |\beta_1 - \beta_2| \leq |\beta_1 - \beta_i|$$

for $i = 3, \cdots, n$. Let $u = |a^{n-1}P_1'(\alpha_1)(\alpha_1 - \alpha_2)^{-1}|$ and $v = |b^{n-1}P_2'(\beta_1)(\beta_1 - \beta_2)^{-1}|$. If $|\alpha_1 - \alpha_2|^2 < u^{-1}$ and $|\beta_1 - \beta_2|^2 < v^{-1}$, then

$$k^{2n^2}|\alpha_1 - \beta_1|^2 \max\{u, v\} \geq 1.$$

Proof. For any two polynomials $P_1, P_2 \in A^{\frac{1}{p^d}}[x]$, denote by $R(P_1, P_2)$ to be the resultant of P_1, P_2 and $D(P_i)$ to be the discriminant of P_i. Suppose that $k^{2n^2}|\alpha_1 - \beta_1|^2 \max\{u, v\} < 1$. Since $D(P_1) \in A^{\frac{1}{p^d}}$, it follows that

$$|D(P_1)| = |a^{2n-2}| \prod_{1 \leq i < j \leq n} |\alpha_i - \alpha_j|^2 \geq 1.$$

For any $j \geq 3$, we have

$$\begin{aligned} (u|\alpha_1 - \alpha_2||\alpha_2 - \alpha_j|)^2 &= (|a^{n-1}P_1'(\alpha_1)||\alpha_2 - \alpha_j|)^2 \\ &= |a^{n-1}\prod_{i=2}^{n}(\alpha_1 - \alpha_i)(\alpha_2 - \alpha_j)|^2 \\ &= \frac{D(P_1)|\alpha_2 - \alpha_j|^2}{\prod_{i=3}^{n}|\alpha_2 - \alpha_i|^2 \prod_{3 \leq i < i' \leq n}|\alpha_i - \alpha_{i'}|^2} \\ &\geq (\frac{1}{k})^{n^2 - 3n}, \end{aligned}$$

and this implies $k^{n^2-3n}(u|\alpha_1-\alpha_2||\alpha_2-\alpha_j|)^2 \geq 1$. By assumption that $|\alpha_1-\alpha_2| < u^{-1}$ and $|\alpha_1-\alpha_2| \leq |\alpha_1-\alpha_i|$, we have

$$(a)\ |\alpha_2-\alpha_j| \leq \max\{|\alpha_1-\alpha_2|, |\alpha_1-\alpha_j|\} = |\alpha_1-\alpha_j|,$$
$$(b)\ u^2|\alpha_1-\alpha_2|^2 < u.$$

We deduce that

$$k^{2n^2}u|\alpha_1-\alpha_j|^2 \geq k^{n^2-3n}u|\alpha_1-\alpha_j|^2 > 1 > k^{2n^2}u|\alpha_1-\beta_1|^2,$$

which implies $|\alpha_1-\beta_1| < |\alpha_1-\alpha_j|$ and hence $|\alpha_j-\beta_1| = |\alpha_1-\alpha_j|$. Similarly we have $|\alpha_1-\beta_1| < |\beta_1-\beta_j|$ and $|\beta_j-\alpha_1| = |\beta_1-\beta_j|$. Hence

$$(1)\ |a^n|\prod_{i'=1}^{n}|\beta_1-\alpha_{i'}| = u|\alpha_1-\beta_1||\alpha_2-\beta_1|,$$

and

$$(2)\ |b^n|\prod_{i'=1}^{n}|\alpha_1-\beta_{i'}| = v|\alpha_1-\beta_1||\beta_2-\alpha_1|.$$

Since $R(P_1,P_2) \in A^{\frac{1}{p^d}}$, by (1) and (2) it follows that

$$\begin{aligned}1 \leq R(P_1,P_2) &= (ab)^n \prod_{1\leq i,i'\leq n}|\alpha_i-\beta_{i'}| \\ &\leq k^{n^2}uv|\alpha_1-\beta_1||\alpha_2-\beta_1||\alpha_1-\beta_2||\alpha_2-\beta_2|.\end{aligned}$$

Note that $|\alpha_1-\beta_2| \leq \max\{|\alpha_1-\beta_1|, |\beta_1-\beta_2|\} \leq v^{-\frac{1}{2}}$ and similarly $|\beta_1-\alpha_2| \leq u^{-\frac{1}{2}}$. Moreover $|\alpha_2-\beta_2| \leq \max\{u^{-\frac{1}{2}}, v^{-\frac{1}{2}}\}$. We conclude that

$$\begin{aligned}u^{-1}v^{-1} &\leq k^{n^2}|\alpha_1-\beta_1||\alpha_2-\beta_1||\alpha_1-\beta_2||\alpha_2-\beta_2| \\ &\leq k^{n^2}|\alpha_1-\beta_1|\max\{u^{-\frac{1}{2}}, v^{-\frac{1}{2}}\}u^{-\frac{1}{2}}v^{-\frac{1}{2}},\end{aligned}$$

which implies

$$k^{2n^2}|\alpha_1-\beta_1|^2\max\{u,v\} \geq 1.$$

This contradicts the original assumption so the proof is complete. □

Lemma 2.6. *Let $\rho(n)$ be a positive monotonic decreasing function of the integral variable n such that $\sum\rho(n)$ converges. Then there exist a positive monotonic decreasing function $\sigma(n)$ such that $\sum\sigma(n)$ converges, $\sigma(n) \geq \rho(n)$ for all n, and for every positive integer r, $\frac{\sigma(n)}{\sigma(rn)} \leq 2r^2$ for $n \geq 2$.*

Proof. c.f. [1]. □

Lemma 2.7. *For each $H \in \mathbb{N}$ let $\mathcal{U}(H)$ denote a finite set of closed balls in K_∞. Let $\mathcal{V}(H)$ denote the subset of $\mathcal{U}(H)$ such that for each $I \in \mathcal{V}(H)$ there is $J \neq I$ in $\mathcal{U}(H)$ with*

$$\mu(I\cap J) \geq \frac{1}{2}\mu(I).$$

Let

$$V(H) = \bigcup_{I\in\mathcal{V}(H)} I \text{ and } v(H) = \bigcup_{I\in\mathcal{V}(H),J\neq I} I\cap J$$

Further, let W and w denote the set of points contained in infinitely many $V(H)$ and in infinitely many $v(H)$, respectively. Then if w has measure zero, so also has W.

Proof. Note that

$$W = \bigcap_{1 \le m < \infty} \bigcup_{H \ge m} V(H)$$

and

$$w = \bigcap_{1 \le m < \infty} \bigcup_{H \ge m} v(H).$$

If w has measure zero, then for any $\epsilon > 0$, there is a positive integer m such that

$$\mu(\bigcup_{m \le H \le n} v(H)) < \epsilon$$

for all $n \ge m$. Let $\{I_j^*\}$ be a subsystem of $\bigcup_{m \le H \le n} V(H)$ such that I_j^* are pairwise disjoint and the union of the I_j^* equals $\bigcup_{m \le H \le n} V(H)$. Then we take $J_{I_j^*} \in \mathcal{U}(H)$ with the property $\mu(I_j^* \cap J_{I_j^*}) \ge \frac{1}{2}\mu(I_j^*)$. Then

$$\begin{aligned}
\mu(\bigcup_{m \le H \le n} V(H)) &= \sum \mu(I_j^*) \\
&\le 2 \sum \mu(I_j^* \cap J_{I_j^*}) \\
&= 2\mu(\bigcup I_j^* \cap J_{I_j^*}) \\
&\le 2\mu(\bigcup_{m \le H \le n} v(H)) < 2\epsilon
\end{aligned}$$

for each $n \ge m$. This implies $\mu(W) = 0$. □

Lemma 2.8. *If S is a set in K_∞ of measure zero, then $S^{-1} = \{\alpha^{-1} \mid \alpha \in S,\ \alpha \ne 0\}$ also has measure zero.*

Proof. c.f.[18, p.70] □

2.4. Proof of Proposition 2.2.

Proof. Let $\mathcal{T}(n, \psi)$ be the set of real elements θ in $\mathfrak{R}(n, \psi)$ for which infinitely many polynomials in $\mathcal{P}$ are required (ii) to have $|a_n| \ge \max\{|a_0|, \cdots, |a_{n-1}|\}$. For $\theta \in \mathfrak{R}(n, \psi)$, by Lemma 2.3 we know there exists $l \in \{0, 1, \cdots, n\}$ such that there are infinitely many $P(x) \in \mathcal{P}(n, \psi, \theta)$ with $P(T^l) \ge C(n)H(P)$. Let S_l be the set of elements θ for which $P(x)$ satisfies this extra condition. Then it suffices to show that $\mu(S_l) = 0$ or equivalently, $\mu(S_l - T^l) = 0$ for $l = 0, \cdots, n$, where $S_l - T^l = \{\theta - T^l \mid \theta \in S_l\}$.

For $\zeta = \theta - T^l \in S_l - T^l$, we have

$$|P(\zeta + T^l)| = |P(\theta)| < \psi(H(P))^n$$

for all $P \in \mathcal{P}(n, \psi, \theta)$ and

$$H(P(x + T^l)) \le q^{ln} H(P) \le q^{n^2} H(P).$$

By Lemma 2.6 we can find a monotone decreasing function $\sigma(H)$ defined on positive integers H with $\sum \sigma(H)$ convergent and such that $\sigma(H) \ge \psi(H)$ for all H and $\sigma(H) \le 2q^{n^2}\sigma(q^{n^2}H)$

for all $H \geq 2$. It follows that

$$\begin{aligned}|P(\zeta+T^l)| < \psi(H(P))^n &\leq \sigma(H(P))^n \\ &\leq (2q^{n^2}\sigma(q^{n^2}H(P)))^n \\ &\leq (2q^{n^2}\sigma(H(P(x+T^l)))^n.\end{aligned}$$

Let $\phi = 2q^{n^2}\sigma$ and denote $P(x+T^l)$ by $\widetilde{P}(x)$. Then we have infinitely many $\widetilde{P}$ satisfying

$$|\widetilde{P}(\zeta)| < \phi(H(\widetilde{P}))^n,$$

where ϕ is a monotone decreasing function such that $\sum \phi$ converges and the constant coefficients of $\widetilde{P}$ is $P(T^l)$. So the absolute value of the constant coefficient of $\widetilde{P}$ has the property that

$$|P(T^l)| \geq C(n)H(P) \geq C(n)q^{-n^2}H(\widetilde{P}).$$

Let $\gamma = |T^{-n^2}| = q^{-n^2}$ and $Q(x) = x^n\widetilde{P}(\frac{1}{x}) = b_nx^n + \cdots + b_1x + b_0$. Denote by $l(Q)$ as the leading coefficient of Q. Then

$$|l(Q)| = |P(T^l)| \geq \gamma H(\widetilde{P}) = \gamma H(Q).$$

Now let $\widetilde{Q}(x) = Q(T^{n^2}x) = ((T^{n^2})^nb_n)x^n + \cdots + ((T^{n^2})b_1)x + b_0$. Then we have

$$|l(\widetilde{Q})| \geq |(T^{n^2})^ib_i|$$

for all $i = 0, \cdots, n-1$, and $H(\widetilde{Q}) \leq \gamma^{-n}H(\widetilde{P})$. If $|\zeta| \geq q^{-N}$, then

$$\begin{aligned}|\widetilde{Q}(\frac{\zeta^{-1}}{T^{n^2}})| = |Q(\zeta^{-1})| &= |\zeta^{-n}\widetilde{P}(\zeta)| \\ &< (q^N)^n\phi(H(\widetilde{P}))^n \\ &\leq (q^N)^n\phi(\gamma^nH(\widetilde{Q}))^n.\end{aligned}$$

We apply Lemma 2.6 again so that there exist $\sigma'(n) \geq \phi(n)$ satisfying the conditions in Lemma 2.2 and let $\phi'(n) = 2q^{nN}\gamma^{-2n^2}\sigma'(n)$. Then we deduce that if $|\zeta| \geq q^{-N}$, then we can find a function ϕ' such that

$$\frac{\zeta^{-1}}{T^{n^2}} \in \mathcal{T}(n, \phi')$$

Now let $E_1 = \{\zeta^{-1} \mid \zeta \in S_l - T^l\}$ and for $N \geq 0$, $E_{1,N} = \{\zeta^{-1} \in E_1 \mid |\zeta| \geq q^{-N}\}$. Then $E_1 = \bigcup_{N\geq 0} E_{1,N}$. By Lemma 2.8 it suffices to prove the claim that $\mu(\mathcal{T}(n, \phi')) = 0$ for any Φ'.

Now we prove the claim by mathematical induction. It is true for $n = 1$ since $\mathcal{T}(1, \phi') = \varphi(1, \phi')$. Assume that for any ϕ', $\mu(\mathcal{T}(m, \phi')) = 0$ for $m = 1, 2, \cdots, n-1$ and for any ϕ'. Hence by this assumption we have $\mu(\mathfrak{R}(m, \phi')) = 0$. For any $\theta \in \mathcal{T}(n, \phi')$, there are infinitely many integral polynomials $P(x)$ in $\mathcal{P}(n, \phi', \theta)$ satisfying condition (ii). Because the set of algebraic elements is of measure zero, we may assume that θ is transcendental over K. If there are infinitely many P also satisfying (i) then $\theta \in \varphi(n, \phi')$. We suppose therefore that all but finite P are reducible. Write $P = P_1 \cdots P_r$. Then we have

$$|P(\theta)| = |P_1(\theta)| \cdots |P_r(\theta)| < \phi'(H(P))^{\deg P_1 + \cdots + \deg P_r}.$$

So there is at least one P_i satisfying

$$|P_i(\theta)| < \phi'(H(P))^{\deg P_i} \leq \phi'(H(P_i))^{\deg P_i},$$

where the last inequality comes from the fact that

$$H(P) = H(P_1)\cdots H(P_r) \geq H(P_i)$$

for $i = 1, \cdots, r$. So we can find a positive integer $m < n$ to deduce that there exist infinitely many $Q \in \mathcal{P}(m, \phi', \theta)$. It follows that $\mathcal{T}(n, \phi') \subset \bigcup_{m=1}^{n-1} \mathfrak{R}(m, \phi')$ except a measure zero set. So by the induction hypothesis we complete our proof.

□

2.5. **Proof of Main Theorem 2.**

Proof. By Proposition 2.2 it suffices to show that $\mu(\varphi(n, \psi)) = 0$. For $\varphi(1, \psi)$, for every $\epsilon > 0$, there exists $H_0 \in q^{\mathbb{N}}$ such that

$$\sum_{H \geq H_0} \psi(H) < \epsilon.$$

Now we may assume $|\theta| \leq 1$. Consider $B(\frac{f}{g}, \frac{\psi(H)}{H})$ in K_∞ for $\frac{f}{g} \in K$ with $f, g \in A$ and $(f, g) = 1$. Then

$$\varphi(1, \psi) \subset \bigcup_{H \geq H_0} \bigcup_{f,g} B(\frac{f}{g}, \frac{\psi(H)}{H}).$$

Since

$$\mu(\bigcup_{f,g} B(\frac{f}{g}, \frac{\psi(H)}{H})) \leq q(q-1)H\psi(H)$$

and

$$\sum_{q^h+1}^{q^{h+1}} \psi(H) \geq q^h(q-1)\psi(q^{h+1}),$$

we have $\mu(\varphi(1, \psi)) \leq q \sum_{H \geq H_0} \psi(H) < q\epsilon$. Since ϵ is arbitrary, we complete the proof for $n = 1$.

Suppose that for any ψ, $\mu(\varphi(m, \psi)) = 0$ for $m = 1, 2, \cdots, n-1$. We shall prove $\mu(\varphi(n, \psi)) = 0$ for $n \geq 2$. Let $\mathfrak{z}(n, H)$ be the set of polynomials $P(x)$ with integral coefficients and degree n and height H satisfying (i) and (ii) in Proposition 2.2. For $P \in \mathfrak{z}(n, H)$, there exist a unique positive integer $d \leq n$ and a polynomial $Q(x) \in A^{\frac{1}{p^d}}[x]$ with no multiple roots such that $P(x) = Q(x)^{p^d}$. Denote by $\mathfrak{z}_d(n, H)$ as the subset for which the $P's$ are restricted to (iii) $P(x) = Q(x)^{p^d}$ so that $Q(x)$ has no multiple roots. Let $\varphi_d(n, \psi)$ be the subset for which infinitely many polynomials $P(x)$ belong to $P \in \mathfrak{z}_d(n, H)$. Since $\varphi(n, \psi) = \cup \varphi_d(n, \psi)$, it suffices to show that $\mu(\varphi_d(n, \psi)) = 0$. From now on we fix a positive integer $d \leq n$. If $n' = \dfrac{n}{p^d} = 1$, the proof is similar to the one of $\mu(\varphi(1, \psi)) = 0$. Hence we may assume $n' \geq 2$. Let $\alpha_1, \cdots, \alpha_{n'}$ denote the roots of Q. For each $j = 1, \cdots, n'$, put

$$\tau_j = \min_{i \neq j} |\alpha_i - \alpha_j|, \quad \nu_j = |Q'(\alpha_j)|^{-1}(\psi(H))^{n'} \quad \text{and} \quad \mu_j = \min\{\nu_j, (\tau_j \nu_j)^{\frac{1}{2}}\}$$

Consider $I_j(P) = B(\alpha_j, \mu_j) \cap K_\infty$, where $B(\alpha_j, \mu_j)$ denote closed balls in $\mathbb{C}_\infty$ ($I_j(P)$ may be empty.) We denote by $\mathcal{S}_j(H)$ the set of all $I_j(P)$ as P runs through the elements of $\mathfrak{z}(n, H)$ satisfying (iii). Then it follows from Lemma 2.4 that every element of $\varphi_d(n, \psi)$ is contained in infinitely many $\mathcal{S}_j(H)$ for some fixed j. We proceed to prove that the set of points contained in infinitely many $\mathcal{S}_1(H)$ has measure zero. (the proof for $j \neq 1$ is similar.) Now without loss of generality we suppose that the roots of Q are so ordered that $\tau_1 = |\alpha_1 - \alpha_2|$. We list some crucial sets and notation as follows.

$$u = |l(Q)^{n'-1}Q'(\alpha_1)(\alpha_1-\alpha_2)^{-1}|$$
$$\mathfrak{A}(n,H) = \{P \in \mathfrak{z}_d(n,H) \mid |\alpha_1-\alpha_2|^2 \geq u^{-1}\},$$
$$\mathfrak{B}(n,H) = \{P \in \mathfrak{z}_d(n,H) \mid |\alpha_1-\alpha_2|^2 < u^{-1}\},$$
$$\mathfrak{K}(H) = \bigcup_{P\in\mathfrak{A}(n,H)} I_1(P), \quad \mathfrak{L}(H) = \bigcup_{P\in\mathfrak{B}(n,H)} I_1(P),$$
$$\kappa = \bigcap_{m\geq 1}\bigcup_{H\geq m} \mathfrak{K}(H),$$
$$\iota = \bigcap_{m\geq 1}\bigcup_{H\geq m} \mathfrak{L}(H).$$

Since $\mathfrak{K}(H)\bigcup\mathfrak{L}(H) = \mathcal{S}_1(H)$, it suffices to prove that κ and ι have measure zero.

(i) $\mu(\kappa)$

Since $\psi(H)$ is decreasing and $\sum\psi(H)$ converges, we have

$$H\psi(H)\to 0 \text{ as } H\to\infty.$$

So we may assume that $\psi(H) < H^{-1}$ for all H. For each $P\in\mathfrak{A}(n,H)$, let

$$I(P) = B(\alpha_1, (\psi(H))^{-1}\mu_1)\cap K_\infty.$$

Then

$$I_1(P)\subset I(P) \text{ and } \mu(I_1(P)) \leq q\psi(H)\mu(I(P)).$$

Let $\mathcal{U}(H)$ denote the set of elements in $I(P)$ and $\mathcal{V}(H)$ the maximal subset of $\mathcal{U}(H)$ satisfying the property that for $I(P_1)\in\mathcal{V}(H)$, there is $I(P_2)\in\mathcal{U}(H)$ with $l(P_1)=l(P_2)$ such that

$$\mu(I(P_1)\cap I(P_2)) \geq \frac{1}{2}\mu(I(P_1)).$$

Let W and w be defined as in Lemma 2.7 respectively. To prove $\mu(\kappa)=0$ we only need to prove

(a) $\mu(w)=0$ and

(b) the set of elements contained in infinitely many $\mathfrak{K}(H)$ with those $I_1(P)$ excluded for which the corresponding $I(P)$ is in $\mathcal{V}(H)$, has measure zero.

(a) Observe that for each $\theta\in I(P)$ we have

$$|\theta-\alpha_1| \leq (\psi(H))^{-1}\nu_1 \leq H^{-n'+1}|Q(\alpha_1)|^{-1} \leq |(\alpha_1-\alpha_2)u|^{-1} \leq |\alpha_1-\alpha_2|.$$

So we can apply Lemma 2.4 and get

$$|P(\theta)| = |Q(\theta)|^{p^d} \leq (|Q'(\alpha_1)||\theta-\alpha_1|)^{p^d} = (\psi(H))^{(n'-1)p^d}.$$

If $\theta\in I(P_1)\cap I(P_2)$ with $P_1\neq P_2$ in $\mathfrak{A}(n,H)$, then the polynomial $R=P_1-P_2$ would satisfy

$$\deg R \leq (n'-1)p^d,$$
$$H(R)\leq H,$$
$$|R(\theta)| \leq (\psi(H))^{(n'-1)p^d}.$$

So we deduce that w is contained in

$$\bigcup_{m\leq n-1}\mathfrak{R}(m,\psi)$$

and then $\mu(w)=0$ by Proposition 2.2 and the induction hypothesis.

(b) If $I(P_1)$ and $I(P_2)$ are distinct elements in $\mathcal{U}(H)$ and $l(P_1)=l(P_2)$, then $I(P_1)\cap I(P_2)=\Phi$. Let S denote the set of elements contained in infinitely many $\mathfrak{K}(H)$ with those $I_1(P)$ excluded for which the corresponding $I(P)$ is in $\mathcal{V}(H)$. Since $|\theta-\alpha_1|\le|\alpha_1-\alpha_2|\le 1$, we have $I(P)\subset B(0,1)$. Moreover, there are $(q-1)H$ polynomials in A with absolute value H. It follows that

$$\begin{aligned}\sum_{h\ge 1, H=q^h}\mu(\bigcup_{I(P)\notin\mathcal{V}(H)}I_1(P)) &\le \sum_{h\ge 1,H=q^h}\sum_{a\in A,\ |a|=H}\ \sum_{I(P)\notin\mathcal{V}(H),\ l(P)=a} q\psi(H)\mu(I(P))\\ &\le \sum_{h\ge 1,H=q^h}\sum_{a\in A,\ |a|=H} q\psi(H)\mu(B(0,1))\\ &\le \sum_{h\ge 1,H=q^h}\mu(B(0,1))q(q-1)H\psi(H)<\infty.\end{aligned}$$

By measure theory, we conclude that

$$\mu(S)=\mu(\bigcap_{1\le m<\infty}\bigcup_{h\ge m}\bigcup_{I(P)\notin\mathcal{V}(H)}I_1(P))=0.$$

(ii) $\mu(\iota)$

Similarly we will claim that

$$\sum_H\sum_{P\in\mathfrak{B}(n,H)}\mu(I_1(P))<\infty.$$

For $H=q^h$ and $l\in\mathbb{Z}$, let

$$S(n,H,l)=\{P\in\mathfrak{B}(n,H)\mid q^{l-1}\le u<q^l\}.$$

Note that in the proof Lemma 2.4 we see that $u\ge u|\alpha_1-\alpha_2||\alpha_2-\alpha_j|\ge 1$, so we may assume $l\ge 1$. On the other hand, $u\le H^{\frac{n'-1}{p^d}}$, which implies $l\le\frac{h(n'-1)}{p^d}$. For $P_1,P_2\in S(n,H,l)$ with $I(P_1),I(P_2)\ne\Phi$, let α_1,β_1 be the roots of P_1,P_2 satisfying the conditions in Lemma 2.5 and u_1,u_2 be defined as in Lemma 2.5. Then we have

$$|\alpha_1-\beta_1|^2\max\{u_1,u_2\}\ge 1.$$

by Lemma 2.5 and the fact that all roots of P_1,P_2 have absolute value less than 1. Choose $\theta_i\in I(P_i)$. Then

$$\begin{aligned}|\theta_1-\alpha_1| &\le (\tau_1\nu_1)^{\frac12}\\ &= (u_1^{-1}H^{\frac{n'-1}{p^d}}(\psi(H)^{n'}))^{\frac12}\\ &\le (q^{-l+1}H^{\frac{n'(1-p^d)-1}{p^d}})^{\frac12}\\ &\le q^{-\frac{l}{2}}.\end{aligned}$$

Similarly we have

$$|\beta_1-\theta_2|\le q^{-\frac{l}{2}}.$$

Since $|\alpha_1-\beta_1|^2q^l>|\alpha_1-\beta_1|^2\max\{u_1,u_2\}\ge 1$, it follows that

$$|\theta_1-\theta_2|=|\alpha_1-\beta_1+\theta_1-\alpha_1+\beta_1-\theta_2|=|\alpha_1-\beta_1|>q^{\frac{-l}{2}}.$$

Combining with the fact that $|\theta_i|\le 1$, we obtain that there are at most $q^{\frac{l+1}{2}}$ polynomials $P\in S(n,H,l)$ for which $I_1(P)\ne\Phi$. We conclude that

$$\begin{aligned}\sum_{h\geq 1, H=q^h}\sum_{P\in\mathfrak{B}(n,H)}\mu(I_1(P)) &\leq \sum_{H}\sum_{l=1}^{\frac{h(n'-1)}{p^d}}\sum_{P\in S(n,H,l)}\mu(I_1(P))\\ &\leq \sum_{H}\sum_{l=1}^{\frac{h(n'-1)}{p^d}} q^{\frac{l+1}{2}}(q^{-l+1}H^{\frac{n'(1-p^d)-1}{p^d}})^{\frac{1}{2}}\\ &= q\sum_{H}\sum_{l=1}^{\frac{h(n'-1)}{p^d}} H^{\frac{n'(1-p^d)-1}{2p^d}}\\ &\leq q\sum_{h\geq 1}(q^{\frac{-h}{2p^d}})\frac{h(n'-1)}{p^d} < \infty.\end{aligned}$$

This completes the proof of $\mu(\iota) = 0$. □

3. Main theorem 1

In this section we will give the proof of Main theorem 1. First we recall the definitions of Hausdorff measures and Hausdorff dimensions. For any subset U of K_∞, let $|U|$ be the diameter of U.

Definition 3.1. Let S be a subset of K_∞. For any positive number d, let

$$\mathcal{H}_\delta^d(S) = \inf\sum_i |U_i|^d,$$

where $I = \{U_i\}$ runs through all collections of countable closed sets that cover S with $|U_i| \leq \delta$. The Hausdorff measure of S is defined by

$$\mathcal{H}^d(S) = \lim_{\delta\to 0}\mathcal{H}_\delta^d(S).$$

The Hausdorff dimension of S is

$$\dim_{\mathcal{H}}(S) = \inf\{d > 0 \mid \mathcal{H}^d(S) = 0\}.$$

Remark 3.2. We refer to [10] for the alternative definitions of Hausdorff measure and Hausdorff dimension.

First we give an upper bound of $\dim_{\mathcal{H}}(\mathcal{K}_n(\lambda))$ and $\dim_{\mathcal{H}}(\mathcal{K}'_n(\lambda))$ in Section 3.1. To obtain the lower bound of $\dim_{\mathcal{H}}(\mathcal{K}_n(\lambda))$ and $\dim_{\mathcal{H}}(\mathcal{K}'_n(\lambda))$, we use Main Theorem 2 to construct a regular system and apply it to Proposition 3.10. The consequence is stated in Section 3.4.

3.1. Upper bounds.

One can show that the following definition is equivalent to the above one.

$$\begin{aligned}\dim_{\mathcal{H}}(S) = \inf\{d > 0 \mid &\text{ for any } \delta > 0, \text{ there exists a collection of countable closed balls}\\ & I = \{B(a_i, r_i)\} \text{ that cover } S \text{ with } r_i \leq \delta \text{ and } \sum_i r_i^d < \infty\},\end{aligned}$$

where $r_i \in q^{\mathbb{Z}}$. We will obtain the upper bounds under this definition.

Proposition 3.3. *For any $n \in \mathbb{N}$ and $\lambda \geq 1$,*

$$\dim_{\mathcal{H}}(\mathcal{K}'_n(\lambda)) \leq \dim_{\mathcal{H}}(\mathcal{K}_n(\lambda)) \leq \frac{1}{\lambda}$$

Proof. It is clear that $\dim_{\mathcal{H}}(\mathcal{K}'_n(\lambda)) \leq \dim_{\mathcal{H}}(\mathcal{K}_n(\lambda))$. Given $0 < \lambda' < \lambda$, consider a family of closed balls $I(\alpha) = \{B(\alpha, (H(\alpha))^{-(n+1)\lambda'})\}$, where α runs through all algebraic elements with degree at most n. (Note that there are countable many such closed balls.) Since $|B(a,r)| = \max\{q^i \in q^{\mathbb{Z}} \mid q^i \leq r\}$, for any $\rho > \frac{1}{\lambda'}$,

$$\sum_{\alpha} |B(a,r)|^{\rho} \leq \sum_{\alpha} ((H(\alpha))^{-(n+1)\lambda'})^{\rho}.$$

Now for $q^h \in q^{\mathbb{N}}$, there are at most $q^{(n+1)(h+1)} - q^{(n+1)H}$ elements of α with $H(\alpha) = q^h$. It follows that

$$\begin{aligned}\sum_{\alpha} |I(\alpha)|^{\rho} &\leq \sum_{h=0}^{\infty} (q^{(n+1)(h+1)} - q^{(n+1)h})((q^h)^{-(n+1)\lambda'})^{\rho} \\ &\leq \sum_{h=0}^{\infty} (q^{n+1} - 1) q^{h(n+1)(1-\lambda'\rho)} < \infty\end{aligned}$$

Since every ζ in $\mathcal{K}_n(\lambda)$ lies in infinitely many $I(\alpha)$, it follows that for $\delta > 0$ the closed balls $I(\alpha)$ with $|I(\alpha)| \leq \delta$ form a closed coverings of $\mathcal{K}_n(\lambda)$. Thus $\dim_{\mathcal{H}} \mathcal{K}_n(\lambda) \leq \rho$. If we choose ρ arbitrarily close to $\dfrac{1}{\lambda'}$ and λ' arbitrarily close to λ, then we get

$$\dim_{\mathcal{H}} \mathcal{K}_n(\lambda) \leq \frac{1}{\lambda}.$$

□

3.2. **Regular systems.** Let Γ be a countable set of elements in K_∞ and N be a positive valued function on Γ. we call (Γ, N) a regular system if for any closed ball J , there is a positive number $k(J)$ such that, for all $k \geq k(J)$, there exist $\gamma_1, \cdots, \gamma_t \in \Gamma \bigcap J$ satisfying

$$N(\gamma_i) \leq k, \quad |\gamma_i - \gamma_j| \geq 1/k, \quad t \geq c_1 \mu(J) k,$$

where c_1 is a constant depending only on Γ and N.

Remark 3.4. The same property can be obtained for any finite union J of closed balls.

Definition 3.5.
1. For any regular system (Γ, N) and any positive valued function $f(x)$ defined for $x > 0$, we signify by $(\Gamma, N; f)$ the set of all elements ζ in K_∞ for which there exist infinitely many γ of Γ such that

$$|\zeta - \gamma| < f(N(\gamma)).$$

2. For any subset S of K_∞ and any positive valued function $g(x)$ defined for $x > 0$, we write $S \prec g$ if for every $\lambda > 0$, $\delta > 0$, S is covered by a countable set closed balls $I_1, I_2, \cdots$ with $|I_i| \leq \lambda$ and

$$\sum_{i=1}^{\infty} g(|I_i|) < \delta.$$

Remark 3.6. We list some properties which we will use in Proposition 3.10 and 3.11.
(a) $S_1 \prec g, S_2 \prec g \Rightarrow S_1 \bigcup S_2 \prec g$.
(b) $S \prec g$ for every countable set S provided that $g(x)$ tends to 0 with x.
(c) If $\dim_{\mathcal{H}}(S) = d$, then $S \prec x^\rho$ for $\rho > d$ and $S \not\prec x^\rho$ for $\rho < d$.

3.3. **On the distribution of algebraic elements.** Let J be a finite union of closed balls. For each positive integer n, let $R_J(n, H)$ denote the set of all ζ in J for which there exists a real algebraic element α with degree at most n and height at most H such that

$$|\zeta - \alpha| < \frac{(\log H)^{3n(n+1)}}{H^{n+1}}.$$

We prove

Proposition 3.7. *$\mu(R_J(n, H)) \to \mu(J)$ as $H \to \infty$.*

Proof. Let $D = \max\{|\gamma| \mid \gamma \in J\}$. For $H > 1$ with $H^{n+1} > D^{n(n-1)+1}$, let ζ be a transcendental element in J but not in $R_J(n, H)$. For any $c_1, c_2, c_3 > 0$ with $c_1 c_2 c_3^{n-1} > 1$, by Minkowski's linear form Theorem over function fields , there exists a polynomial

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

with integral coefficients and $\deg P(x) \leq n$ such that

$$\begin{aligned} &0 < |P(\zeta)| < c_1 \frac{(\log H)^{3n(n-1)}}{H^n}, \\ &0 < |P'(\zeta)| < c_2 H, \\ &|a_j| < c_3 \frac{H}{(\log H)^{3n}}. \end{aligned}$$

Furthermore, since

$$\min\{H^{n+1} \frac{H^{n+1}}{D^{n(n-1)+1}}\} > 1,$$

if we choose c_i to be such that

$$\begin{aligned} &c_1 \leq \frac{H^{n+1}}{(\log H)^{3n(n-1)}}, \\ &c_2 \leq \min\{1, \frac{1}{D}\}, \\ &c_3 \leq \min\{\frac{(\log H)^{3n}}{D^n}, (\log H)^{3n}\}, \end{aligned}$$

we can ensure that $H(P) \leq H$. Note that the numbers c_i can be chosen so that they only depend on n and J.

We now distinguish P into three cases:
1. $H(P) < \log H$. If α is the zero of P nearest to ζ, then

$$|\alpha - \zeta| \leq C|P(\zeta)|^{\frac{1}{n}} < C' \frac{(\log H)^{3n}}{H},$$

where C and C' are constants. Since the number of polynomials P with $H(P) < \log H$ is less than $(q \log H)^{n+1}$, it follows that

$$\zeta \in \bigcup B(\alpha, C' \frac{(\log H)^{3n}}{H}),$$

whose measure is less than $q^{n+2}C'\dfrac{(\log H)^{4n+1}}{H}$, which tends to 0 as H tends to infinity.

2. $H(P) \geq \log H$ and $|P'(\zeta)| \leq C''\dfrac{H}{(\log H)^{3n}}$ for some constant C''. Then $H(P) \leq \widetilde{C}\dfrac{H}{(\log H)^{3n}}$ for some constant $\widetilde{C}$, which implies

$$0 < |P(\zeta)| < c_1\frac{(\log H)^{3n(n-1)}}{H^n} \leq \widehat{C}H(P)^{-n}(\log H(P))^{-3n},$$

where $\widehat{C}$ is a constant obtained from $\widetilde{C}$ and c_1. Let $\phi(H)$ be a positive decreasing function defined by $\phi(H) = H^{-1}(\log H)^{-2}$. Now we let

$$\begin{aligned} S_J(n,H) = \{\zeta \in J \mid \exists \text{ a polynomial } P(x) \in \mathbb{F}_q[T][x] \text{ such that } \deg P \leq n \\ \text{and } H(P) \leq H \text{ and } 0 < |P(\zeta)| < (\phi(H))^n\}. \end{aligned}$$

Thus for H large enough, we have $\zeta \in S_J(n, H(P))$. It follows that ζ lies also in

$$\bigcup_{M \geq \log H} S_J(n,M),$$

a set whose measure tends to zero as H tends to infinity by Main Theorem 2.

3. $|P'(\zeta)| \geq C''\dfrac{H}{(\log H)^{3n}}$ for some constant C''. Suppose that $\alpha_1, \cdots, \alpha_n$ are zeros of P and α_1 is nearest to ζ. It follows that

$$|\zeta - \alpha_1| \leq \frac{|P(\zeta)|}{|P'(\zeta)|} \ll |P(\zeta)|H^{-1}(\log H)^{3n} \ll H^{-(n+1)(\log H)^{3n^2}}.$$

Hence if H is large, ζ lies in $R_J(n,H)$, which contradicts the assumption made at the beginning.

□

Corollary 3.8. *The set Γ of all real algebraic elements α with $\deg\alpha \leq n$, together with the function $N(\alpha) = \dfrac{H(\alpha)^{n+1}}{(\log H(\alpha))^{3n(n+1)}}$ is a regular system.*

Proof. Let $\varphi(x)$ be the function defined by $\varphi(H) = \frac{(\log H)^{3n}}{H}$. Proposition 3.7 implies

$$\mu(R_J(n,H)) \geq \frac{1}{2}\mu(J)$$

for sufficiently large H. For $k = \varphi(H)^{-(n+1)}$ with sufficiently large H, let $\{\gamma_1, \cdots, \gamma_t\}$ be a maximal subset of elements of Γ with $H(\gamma_i) \leq H$ and $|\gamma_i - \gamma_j| \geq \frac{1}{k}$ for all $i \neq j$, so that for any $\gamma \in \Gamma$ with $H(\gamma) \leq H$, there exists γ_i satisfying $|\gamma - \gamma_i| \leq \frac{1}{k}$. This implies

$$\bigcup_{\gamma} B(\gamma, k^{-1}) \subset \bigcup_{i=1}^{t} B(\gamma_i, k^{-1}).$$

Since $R_J(n,H) \subset \bigcup_{\gamma} B(\gamma, k^{-1})$ and $\mu(\bigcup_{\gamma} B(\gamma, k^{-1})) \leq qtk^{-1}$, we have $t \geq c_1 k\mu(J)$, where $c_1 = \dfrac{1}{2q}$.

□

Lemma 3.9. *Let* (Γ, N) *be a regular system and* $f(x)$ *be a positive valued function defined on* $x > 0$ *such that* $f(x)$ *decreases and* $N(\gamma)$ *goes to infinity as* $|\gamma|$ *grows. Then*

$$(\Gamma, N; f) \cup \Gamma = \{\zeta \in K_\infty \mid \exists \text{ arbitrary large } k \text{ and } \gamma \text{ with } N(\gamma) \leq k \text{ such that } |\zeta - \gamma| < f(k)\}.$$

Proof. Let S denote the the set on the right hand side. Since $f(x)$ decreases, it is clear that S is contained in $(\Gamma, N; f) \cup \Gamma$. On the other hand, Let $\{\gamma_i\}$ denote the set of all elements in Γ such that

$$|\zeta - \gamma_i| < f(N(\gamma_i)).$$

Since $N(\gamma_i) \to \infty$ as $|\gamma_i| \to \infty$, by taking $k_i = N(r_i)$ we see that

$$(\Gamma, N; f) \cup \Gamma \subset S.$$

□

Proposition 3.10. *Let* $f(x), g(x)$ *be positive functions defined for* $x > 0$ *such that*

(*a*) $f(x)$ *decreases and* $f(x) \leq \dfrac{1}{qx}$ *for large* x

(*b*) $g(x)$ *and* $\dfrac{x}{g(x)}$ *both increase and tend to zero with* x

(*c*) $xg(f(x))$ *tends to infinity as* x.

Then for any regular system (Γ, N), *we have* $(\Gamma, N; f) \not\prec g$. *In fact, for any regular systems* (Γ_i, N_i) $(i = 1, 2, \cdots)$, *we have*

$$\bigcap_{i=1}^{\infty} (\Gamma_i, N_i; f) \not\prec g.$$

Proof. Since Γ is countable, by Remark 3.6 it suffices to prove that

$$(\Gamma, N; f) \bigcup \Gamma \not\prec g.$$

Choose a number k_0 sufficiently large so that

$$f(x) \leq \frac{1}{qx}$$

for all $x \geq k_0$ and

$$q^3 \lambda_0 / g(\lambda_0) < c_1 = c_1(\Gamma, N),$$

where $\lambda_0 = f(k_0)$. This choice is possible since $f(x) \to 0$ as $x \to \infty$ and $x/g(x) \to 0$ as $x \to 0$. Now let $I(\lambda, g) = \{B(a_i, r_i)\}$ be any system of closed balls such that

$$r_i \leq \lambda < \lambda_0 \text{ and } \sum_{i=1}^{\infty} g(r_i) < \delta := 1.$$

We will prove that $I(\lambda, g)$ is not a covering of $(\Gamma, N; f) \bigcup \Gamma$, which implies $(\Gamma, N; f) \bigcup \Gamma \not\prec g$. We shall construct a sequence of increasing numbers $k_1, k_2, \cdots$ and a sequence of sets $J_0, J_1, \cdots$ each a finite union of closed balls, such that

$$k_j > 2k_{j-1} \quad \text{and} \quad J_j \subset J_{j-1},$$

which satisfies

(i) $J_j \cap I = \Phi$ for any $I \in I(\lambda, g)$ with $|I| > \lambda_j := f(k_j)$,

(ii) for every $\zeta \in J_j$ there exists $\gamma \in \Gamma$ such that $|\zeta - \gamma| < f(k_j)$ and $N(\gamma) \leq k_j$,

(iii) $\mu(J_j) > \frac{1}{q^2} c_1 \lambda_j k_j \mu(J_{j-1})$,

(iv) $k_j g(\lambda_j) > q^5 c_1^{-2} \mu(J_{j-1})^{-1}$.

Since $\{J_j\}$ are closed bounded sets in K_∞, we have $\bigcap_{j=0}^{\infty} J_j$ is non-empty. Moreover, by the construction and Lemma 3.9, it follows that $\bigcap_{j=0}^{\infty} J_j$ is contained in $(\Gamma, N; f) \bigcup \Gamma$ but disjoint from $I(\lambda, g)$. Thus $I(\lambda, g)$ cannot cover $(\Gamma, N; f) \bigcup \Gamma$ and this completes the proof.

Note that the only condition above that will be also required in the case $j = 0$ is (i) and this is plainly satisfied since $\lambda < \lambda_0$. So for J_0 we take the unit ball $B(a, 1)$. We assume that $k_0, \cdots, k_{j-1}$ and $J_0, \cdots, J_{j-1}$ have already been defined and we proceed to construct k_j and J_j. We take k_j sufficiently large so that

$$k_j \geq k(J_{j-1}),\ k_j > 2k_{j-1},\ (iv) \text{ holds and } c_1 k_j \mu(J_{j-1}) \geq q\nu(J_{j-1}),$$

where $\nu(S)$ denotes the number of closed balls in S and $k(J_{j-1})$ is referred to in the definition of (Γ, N) with $J = J_{j-1}$. Such k_j exists because $xg(f(x)) \to \infty$ as $x \to \infty$.

Now let $\gamma_1, \cdots, \gamma_t$ be the elements of Γ satisfying the conditions in the definition of (Γ, N) with $k = k_j$ and $J = J_{j-1}$, and let F be the set of balls $B(\gamma_i, \lambda_j)$ such that $r_i \neq r_j$ lie in a single closed ball of J_{j-1}. Since $t \geq c_1 k_j \mu(J_{j-1})$, we have

$$\nu(F) \geq t - \nu(J_{j-1}) \geq c_1 k_j \mu(J_{j-1}) - \nu(J_{j-1}) \geq \frac{q-1}{q} c_1 k_j \mu(J_{j-1}).$$

The fact that closed balls in F are disjoint follows from the fact that $\lambda_j = f(k_j) \leq \frac{1}{qk_j}$, and $|r_i - r_s| \geq \frac{1}{k_j}$. Furthermore, they are all contained in J_{j-1} and so, by (i), they are disjoint from any $I \in I(\lambda, g)$ with $|I| > \lambda_{j-1}$.

We write $F = F' + F''$, where F' consists of those closed balls in F that are contained in a closed ball of $I(\lambda, g)$ and F'' consists of all other closed balls in F. Let I be a closed ball in $I(\lambda, g)$. If I contains n closed balls in F, then at least $|I| \geq \lambda_j$. If $n > 1$, it follows that

$$n \leq qk_j |I|.$$

Thus, if $|I| \geq \lambda_j$, we have

$$n \leq qk_j |I| + 1.$$

Since $g(x)$ increases, one can show that

$$n \leq qk_j |I| + \frac{g(|I|)}{g(\lambda_j)}.$$

This implies $\nu(F') \leq \bigoplus^1 + \bigoplus^2$, where

$$\bigoplus^1 = \sum_I qk_j |I|,\ \bigoplus^2 = \sum_I \frac{g(|I|)}{g(\lambda_j)},$$

and the summations are taken over all closed balls I in $I(\lambda, g)$ with $|I| \leq \lambda_{j-1}$. Now, since $x/g(x)$ increases and $\sum_I g(|I|) \leq 1$, we have

$$\begin{aligned}\bigoplus^1 &= qk_j \sum_I \frac{|I|}{g(|I|)} g(|I|) \\ &\leq qk_j \frac{\lambda_{j-1}}{g(\lambda_{j-1})} \sum_I g(|I|) \\ &\leq qk_j \frac{\lambda_{j-1}}{g(\lambda_{j-1})}.\end{aligned}$$

When $j = 1$, it follows from the definitions of k_0 and J_0 that

$$\bigoplus^1 \leq qk_1 \frac{\lambda_0}{g(\lambda_0)} < \frac{1}{q^2} c_1 k_1 = \frac{1}{q^3} c_1 k_1 \mu(J_0)$$

When $j > 1$, by

$$(iii)\ \mu(J_{j-1}) > \frac{1}{q^2} c_1 \lambda_{j-1} k_{j-1} \mu(J_{j-2})$$

and

$$(iv)\ k_{j-1} g(\lambda_{j-1}) > q^5 c_1^{-2} \mu(J_{j-2})^{-1},$$

we have

$$\bigoplus^1 \leq \frac{1}{q^3} c_1 k_j \mu(J_{j-1}).$$

On the other hand, since we can choose $c_1 = c_1(\Gamma, N)$ under the restriction of $c_1 \leq 1$, we obtain from (iv) that

$$\bigoplus^2 \leq \frac{1}{g(\lambda_j)} \leq \frac{1}{q^5} c_1^2 k_j \mu(J_{j-1}) \leq \frac{1}{q^5} c_1 k_j \mu(J_{j-1}).$$

Hence we have

$$\nu(F') \leq \bigoplus^1 + \bigoplus^2 \leq \frac{q^2+1}{q^5} c_1 k_j \mu(J_{j-1}) \leq \frac{q^2+1}{q^4(q-1)} \nu(F);$$

whence

$$\nu(F'') \geq \frac{q^4(q-1) - q^2 - 1}{q^4(q-1)} \nu(F).$$

Now let J_j be the set of all closed balls in F'' if $\lambda_j \notin q^{\mathbb{Z}}$ and let J_j be the set

$$\{B(\gamma_i, \frac{\lambda_j}{q}) \mid B(\gamma_i, \lambda_j) \in F''\}$$

if $\lambda_j \in q^{\mathbb{Z}}$. It is clear that J_j is disjoint from any $I \in I(\lambda, g)$ with $|I| > \lambda_j$, and so (i) holds. Because closed balls in J_j are of the forms either $B(\gamma_i, \lambda_j)$ or $B(\gamma_i, \frac{\lambda_j}{q})$ (depending on whether or not λ_j belongs to $q^{\mathbb{Z}}$,) (ii) also holds. Since

$$\begin{aligned}\mu(J_j) \geq \lambda_j \nu(F'') &\geq \frac{q^4(q-1) - q^2 - 1}{q^4(q-1)} \nu(F) \lambda_j \\ &\geq \frac{q^4(q-1) - q^2 - 1}{q^5} c_1 \lambda_j k_j \mu(J_{j-1}), \\ &> \frac{1}{q^2} c_1 \lambda_j k_j \mu(J_{j-1})\end{aligned}$$

we see that (iii) is satisfied.

For the last part of the proposition we choose a sequence $\{l(n)\}, n = 1, 2, \cdots$ of positive integers such that for each positive integer i, $l(j) = i$ has infinitely many solutions j and we replace the condition (ii) by the condition that for each $\zeta \in J_j$ there is a $\gamma \in \Gamma_{l(j)}$ such that

$$|\zeta - \gamma| < f(k_j) \text{ and } N_{l(j)}(\gamma) \leq k_j.$$

□

3.4. **Lower bounds.**

Proposition 3.11. *If* $\lambda > 1$, *then*

$$\dim_{\mathcal{H}}(\mathcal{K}_n(\lambda)) \geq \dim_{\mathcal{H}}(\mathcal{K}'_n(\lambda)) \geq \frac{1}{\lambda}.$$

Proof. From Proposition 3.3 we have $\dim_{\mathcal{H}}(\mathcal{K}_n(\tau)) \leq \frac{1}{\tau} < \frac{1}{\lambda}$ for any $\tau > \lambda$. It follows that

$$\mathcal{K}_n(\lambda) \backslash \mathcal{K}'_n(\lambda) \prec x^{\frac{1}{\lambda}}.$$

Therefore by Remark 3.6 it suffices to show that $\mathcal{K}_n(\lambda) \not\prec x^{\frac{1}{\lambda}}$. Let

$$\begin{aligned}
&\Gamma = \text{the set of all real algebraic elements } \alpha \text{ with } \deg \alpha \leq n, \\
&N(\alpha) = \frac{H(\alpha)^{n+1}}{(\log H(\alpha))^{3n(n+1)}}, \\
&f(x) = \frac{\log x}{x^\lambda}, \\
&g(x) = x^{\frac{1}{\lambda}}.
\end{aligned}$$

Then (Γ, N) is a regular system and $(\Gamma, N; f) \not\prec g$ by Corollary 3.8 and Proposition 3.10. This implies that

$$\dim_{\mathcal{H}}(\Gamma, N; f) \geq \frac{1}{\lambda}.$$

On the other hand,

$$\begin{aligned}
f(N(\alpha)) &= H(\alpha)^{-(n+1)\lambda}(\log H(\alpha))^{3n(n+1)\lambda}(\log \frac{H(\alpha)^{n+1}}{(\log H(\alpha))^{3n(n+1)}}) \\
&\leq CH(\alpha)^{-(n+1)\lambda}(\log H(\alpha))^{3n(n+1)\lambda+1}
\end{aligned}$$

for some constant C. Note that for any $\lambda' < \lambda$, we have

$$CH(\alpha)^{-(n+1)(\lambda-\lambda')}(\log H(\alpha))^{3n(n+1)\lambda+1} \to 0 \text{ as } H(\alpha) \to \infty.$$

It follows that $f(N(\alpha)) < H(\alpha)^{-(n+1)\lambda'}$ for $H(\alpha)$ sufficiently large and hence

$$(\Gamma, N; f) \subset \mathcal{K}_n(\lambda).$$

This concludes the proof that $\dim_{\mathcal{H}}(\mathcal{K}_n(\lambda)) \geq \frac{1}{\lambda}$. □

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