Deformations and Rigidity of ℓ-adic Sheaves *

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Abstract

Let X be a smooth connected algebraic curve over an algebraically closed field, let S be a finite closed subset in X, and let $\mathcal{F}_0$ be a lisse ℓ-torsion sheaf on $X-S$. We study the deformation of $\mathcal{F}_0$. The universal deformation space is a formal scheme. Its generic fiber has a rigid analytic space structure. By studying this rigid analytic space, we prove a conjecture of Katz which says that if a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ is irreducible and physically rigid, then it is cohomologically rigid, under the extra condition that $\mathcal{F} \mod \ell$ is absolutely irreducible or that $\mathcal{F}$ has finite monodromy.

Key words: deformation of Galois representations, formal scheme, rigid analytic space.

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Introduction

In this paper, we work over an algebraically closed field k of characteristic p even though our results can be extended to non-algebraically closed fields. Let X be a smooth connected projective curve over k, let S be a finite closed subset of X, and let ℓ be a prime number distinct from p. For any $s \in S$, let η_s be the generic point of the strict henselization of X at s. A lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $X-S$ is called *physically rigid* if for any lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on $X-S$ with the property $\mathcal{F}|_{\eta_s} \cong \mathcal{G}|_{\eta_s}$ for any closed point s in S, we have $\mathcal{F} \cong \mathcal{G}$. The lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $X-S$ corresponds to a Galois representation

$$\rho : \mathrm{Gal}(\overline{K(X)}/K(X)) \to \mathrm{GL}(\overline{\mathbb{Q}}_\ell^r)$$

of the function field $K(X)$ unramified everywhere on $X-S$. $\mathcal{F}$ is physically rigid if and only if for any Galois representation ρ' of $\mathrm{Gal}(\overline{K(X)}/K(X))$ such that ρ' and ρ induce isomorphic Galois representations of local fields obtained by taking completions of $K(X)$ at places of $K(X)$, we have $\rho \cong \rho'$. To get a good notion of rigidity, we have to assume $X = \mathbb{P}^1$. Indeed, the abelian-pro-ℓ quotient of the étale fundamental group $\pi_1(X)$ of X is isomorphic to $\mathbb{Z}_\ell^{2g}$, where g is the genus of X. If $g \geq 1$, then there exists a character $\chi : \pi_1(X) \to \overline{\mathbb{Q}}_\ell^*$ such that χ^n are nontrivial for all n. So there exists a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}$ of rank 1 on X such that $\mathcal{L}^{\otimes n}$ are nontrivial. For any lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $X-S$, the lisse sheaf $\mathcal{G} = \mathcal{F} \otimes \mathcal{L}$ is not isomorphic to $\mathcal{F}$ since they have non-isomorphic determinant, but

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$\mathcal{F}|_{\eta_s} \cong \mathcal{G}|_{\eta_s}$ for all $s \in X$ since $\chi|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$ is unramfied and hence trivial. Thus $\mathcal{F}$ is not physically rigid. A lisse irreducible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $X - S$ is called *cohomologically rigid* if we have

$$\chi(X, j_*\mathcal{E}nd(\mathcal{F})) = 2,$$

where $j : X - S \hookrightarrow X$ is the canonical open immersion and $\chi(-,-)$ is the Euler characteristic. In [13, 5.0.2 and 1.1.2], Katz shows that for an irreducible lisse sheaf, cohomological rigidity implies physical rigidity, and conjectures that the converse is true. He proves the conjecture for complex local systems on X in the case where k is the complex field. In this paper, we prove the conjecture for any k of characteristic p under the extra condition that $\mathcal{F} \mod \ell$ is absolutely irreducible or that $\mathcal{F}$ has finite monodromy.

Katz's proof for the complex field case ([13, 1.1.2]) can be interpreted as a study of the moduli space of representations of the topological fundamental group of $X - S$. In [4, Theorem 4.10], Bloch and Esnault study deformations of locally free $\mathcal{O}_{X-S}$-modules provided with connections while keeping local (formal) data undeformed, and prove that the universal deformation space is algebraizable. Using this fact, they prove that physical rigidity and cohomological rigidity are equivalent for irreducible locally free $\mathcal{O}_{X-S}$-modules provided with connections. Our method is similar. We study deformations of lisse ℓ-torsion sheaves. The universal deformation space is a formal scheme, and its generic fiber is a rigid analytic space which can be used to produce families of $\overline{\mathbb{Q}}_\ell$-sheaves. By a counting argument on dimensions of rigid analytic spaces, we prove Katz's conjecture under the extra condition mentioned above.

Suppose k is an algebraic closure of the finite field $\mathbb{F}_q$, and X is obtained from an algebraic curve X_0 over $\mathbb{F}_q$ by base change. In [9], Deligne studied the counting of fixed points of the Frobenius map on the set of isomorphic classes of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on X, a problem which is studied by Drinfeld for the case where the sheaves have rank 2. In [9, 1.3-1.6], Deligne makes some speculation on the moduli space of $\overline{\mathbb{Q}}_\ell$-sheaves, which is not algebraizable. The construction in this paper suggests that a piece of the moduli space might have a rigid analytic space structure.

In the following, we take Λ to be either a finite extension E of $\mathbb{Q}_\ell$, or the integer ring $\mathcal{O}$ of such E, or the residue field of $\mathcal{O}$. Let $\mathfrak{m}$ be the maximal ideal of Λ, and let $\kappa = \Lambda/\mathfrak{m}$ be the residue field of Λ. Denote by $\mathcal{C}_\Lambda$ the category of Artinian local Λ-algebras with same residue field κ as Λ. Morphisms in $\mathcal{C}_\Lambda$ are homomorphisms of Λ-algebras. Using the fact that the maximal ideal of an Artinian local ring coincides with its nilpotent radical, one can check that morphisms in $\mathcal{C}_\Lambda$ are necessarily local homomorphisms, and they induce the identity homomorphism on the residue field. If A is an object in $\mathcal{C}_\Lambda$, we denote by $\mathfrak{m}_A$ the maximal ideal of A. Let η be the generic point of X, and let $\pi_1(X - S, \bar{\eta})$ be the étale fundamental group of $X - S$. Fix an embedding $\mathrm{Gal}(\bar{\eta}_s/\eta_s) \hookrightarrow \mathrm{Gal}(\bar{\eta}/\eta)$ for each $s \in S$ and let $\mathrm{Gal}(\bar{\eta}_s/\eta_s) \to \pi_1(X - S, \bar{\eta})$ be its composite with the canonical surjection $\mathrm{Gal}(\bar{\eta}/\eta) \to \pi_1(X - S, \bar{\eta})$. A homomorphism $\rho : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A^r)$ is called a representation if it is continuous. Here, if $\Lambda = \mathcal{O}$, then A is finite and we put the discrete topology on $\mathrm{GL}(A^r)$. If $\Lambda = E$, then A is a finite dimensional vector space over E, and we put the topology induced from the ℓ-adic topology on $\mathrm{GL}(A^r)$.

Suppose we are given a representation $\rho_\Lambda : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(\Lambda^r)$. Let $\rho_0 : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(\kappa^r)$ be the representation obtained from ρ_Λ by passing to residue field. We study deformations of ρ_0. Our treatment is similar to Mazur's theory of deformations of Galois representations ([15]) and Kisin's theory of framed deformations of Galois representations ([14]).

Throughout this paper, we assume that S is nonempty. Suppose we are given $P_{0,s} \in \mathrm{GL}(\kappa^r)$ for each $s \in S$. In application, we often take $P_{0,s}$ to be the identity matrix I for all $s \in S$. In this case, we denote the data $(\rho_0, (P_{0,s})_{s\in S})$ by $(\rho_0, (I)_{s\in S})$. For any $A \in \mathrm{ob}\,\mathcal{C}_\Lambda$, denote the composite

$$\pi_1(X - S, \bar{\eta}) \stackrel{\rho_\Lambda}{\to} \mathrm{GL}(\Lambda^r) \to \mathrm{GL}(A^r)$$

also by ρ_Λ. Define $F(A)$ to be the set of deformations $(\rho, (P_s)_{s\in S})$ of the data $(\rho_0, (P_{0,s})_{s\in S})$ with the prescribed local monodromy $\rho_\Lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$. More precisely, we define

$$\begin{aligned} F(A) = \{(\rho, (P_s)_{s\in S}) \quad | \quad & \rho : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A^r) \text{ is a representation, } P_s \in \mathrm{GL}(A^r), \\ & \rho \mod \mathfrak{m}_A = \rho_0, \quad P_s \mod \mathfrak{m}_A = P_{0,s}, \\ & P_s^{-1} \rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} P_s = \rho_\Lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \text{ for all } s \in S\}/\sim, \end{aligned}$$

where two tuples $(\rho^{(i)}, (P_s^{(i)})_{s\in S})$ $(i = 1, 2)$ are equivalent if there exists $P \in \mathrm{GL}(A^r)$ such that

$$(\rho^{(1)}, (P_s^{(1)})_{s\in S}) = (P^{-1}\rho^{(2)}P, (P^{-1}P_s^{(2)})_{s\in S}).$$

Note that the equation $P_s^{(1)} = P^{-1}P_s^{(2)}$ implies that $P \equiv I \mod \mathfrak{m}_A$ since we assume S is nonempty and $P_s^{(1)} \equiv P_s^{(2)} \mod \mathfrak{m}_A = P_{0,s}$. The column vectors of $P_{0,s}$ can be regarded as a basis, that is, a frame for κ^r, and P_s can be regarded as a lifting of this frame to a frame of A^r. Two tuples $(\rho^{(i)}, (P_s^{(i)})_{s\in S})$ $(i = 1, 2)$ are equivalent if any only if there exists an isomorphism of representations $P : A^r \to A^r$ from $\rho^{(1)}$ to $\rho^{(2)}$ which transforms the frame $P_s^{(1)}$ to the frame $P_s^{(2)}$ for each $s \in S$. For any morphism $A' \to A$ in $\mathcal{C}_\Lambda$, define $F(A') \to F(A)$ to be the map induced by $\mathrm{GL}(A'^r) \to \mathrm{GL}(A^r)$. We thus get a covariant functor $F : \mathcal{C}_\Lambda \to (\mathrm{Sets})$.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}_\Lambda$. Consider the map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

Using the fact that S is nonempty, it is straightforward to verify that this map is bijective if $A'' \to A$ is surjective. Proposition 0.1 below shows that $F(\kappa[\epsilon])$ is a finite dimensional vector space. So by the Schlessinger criteria [17, Theorem 2.11], the functor F is pro-representable.

Proposition 0.1. *Let $\kappa[\epsilon]$ be the ring of dual numbers. The κ-vector space $F(\kappa[\epsilon])$ is finite dimensional. Suppose furthermore that $X = \mathbb{P}^1$ and all elements in the set $\mathrm{End}_{\pi_1(X-S,\bar{\eta})}(\kappa^r)$ are scalar multiplications, where κ^r is considered as a $\pi_1(X - S, \bar{\eta})$-module through the representation ρ_0. (This condition holds if ρ_0 is absolutely irreducible by Schur's lemma). Then the functor F is smooth, and we have*

$$\dim F(\kappa[\epsilon]) = -\chi(X, j_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)) + \sum_{s\in S} \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_0)) - 1,$$

where $\mathrm{Ad}(\rho_0)$ is the κ-vector space of $r \times r$ matrices with entries in κ on which $\pi_1(X - S, \bar{\eta})$ and $\mathrm{Gal}(\bar{\eta}_s/\eta_s)$ act by the composition of ρ_0 with the adjoint representation of $\mathrm{GL}(\kappa^r)$, $\mathcal{F}_0$ is the lisse

κ-sheaf on $X-S$ corresponding to the representation $\rho_0 : \pi_1(X-S,\bar\eta) \to \mathrm{GL}(\kappa^r)$, $\mathcal{E}nd^{(0)}(\mathcal{F}_0)$ is the subsheaf of $\mathcal{E}nd(\mathcal{F}_0)$ formed by sections of trace 0, and $j : X-S \hookrightarrow X$ is the canonical open immersion.

Denote by $R(\rho_\Lambda)$ the universal deformation ring for the functor F. It is a complete noetherian local Λ-algebra with residue field κ. We have a homomorphism

$$\rho_{\mathrm{univ}} : \pi_1(X-S,\bar\eta) \to \mathrm{GL}(r, R(\rho_\Lambda))$$

with the property

$$\rho_{\mathrm{univ}} \mod \mathfrak{m}_{R(\rho_\Lambda)} = \rho_0,$$

and we have matrices $P_{\mathrm{univ},s} \in \mathrm{GL}(r, R(\rho_\Lambda))$ with the property

$$P_{\mathrm{univ},s} \mod \mathfrak{m}_{R(\rho_\Lambda)} = P_{0,s}, \quad P_{\mathrm{univ},s}^{-1}\rho_{\mathrm{univ}}|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}P_{\mathrm{univ},s} = \rho_\Lambda|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}$$

such that the homomorphism $\pi_1(X-S,\bar\eta) \to \mathrm{GL}(r, R(\rho_\Lambda)/\mathfrak{m}^i_{R(\rho_\Lambda)})$ induced by ρ_{univ} are continuous for all positive integers i, and for any element $(\rho,(P_s)_{s\in S})$ in $F(A)$, there exists a unique local Λ-algebra homomorphism $R(\rho_\Lambda) \to A$ which brings $(\rho_{\mathrm{univ}},(P_{\mathrm{univ},s})_{s\in S})$ to the equivalent class of $(\rho,(P_s)_{s\in S})$. More generally, we have the following.

Proposition 0.2. *Let A' be a local Artinian Λ-algebra so that its residue field $\kappa' = A'/\mathfrak{m}_{A'}$ is a finite extension of $\kappa = \Lambda/\mathfrak{m}$. Let $(\rho',(P'_s)_{s\in S})$ be an equivalent class of tuples, where $\rho' : \pi_1(X-S,\bar\eta) \to \mathrm{GL}(A'^r)$ is a representation and $P'_s \in \mathrm{GL}(A'^r)$ such that*

$$\rho' \mod \mathfrak{m}_{A'} = \rho_0, \quad P'_s \mod \mathfrak{m}_{A'} = P_{0,s}, \quad P_s^{-1}\rho|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}P_s = \rho_\Lambda|_{\mathrm{Gal}(\bar\eta_s/\eta_s)} \text{ for all } s \in S\},$$

and the equivalence relation is defined as before. Then there exists a unique local Λ-algebra homomorphism $R(\rho_\Lambda) \to A'$ which brings $(\rho_{\mathrm{univ}},(P_{\mathrm{univ},s})_{s\in S})$ to the equivalent class of $(\rho',(P'_s)_{s\in S})$.

Proof. Let A be the inverse image of κ under the projection $A' \to A'/\mathfrak{m}_{A'} = \kappa'$. Then A is a local ring with maximal ideal $\mathfrak{m}_{A'}$ and its residue field is isomorphic to κ. Moreover, A is complete since $\mathfrak{m}_{A'}$ is nilpotent. The vector space $\mathfrak{m}_{A'}/\mathfrak{m}^2_{A'}$ is finite dimensional over $\kappa' = A'/\mathfrak{m}_{A'}$ and hence finite dimensional over $\kappa \cong A/\mathfrak{m}_A$. Choose a basis $\{x_1,\ldots,x_n\}$ of $\mathfrak{m}_{A'}/\mathfrak{m}^2_{A'}$ over κ. Then we have an epimorphism

$$\kappa[t_1,\ldots,t_n] \to \oplus_{i\geq 0}\mathfrak{m}^i_{A'}/\mathfrak{m}^{i+1}_{A'}, \quad t_i \mapsto x_i.$$

It follows that $\oplus_{i\geq 0}\mathfrak{m}^i_{A'}/\mathfrak{m}^{i+1}_{A'}$ is Noetherian. By [1, Corollary 2.5], A is Noetherian. Since its maximal idea is nilpotent, A is Artinian. It follows that A is an object in $\mathcal{C}_\Lambda$. Since $\rho' \mod \mathfrak{m}_{A'} = \rho_0$, the image of ρ' lands in $\mathrm{GL}(A)$ and hence ρ' defines a representation $\rho : \pi_1(X-S,\bar\eta) \to \mathrm{GL}(A^r)$. Since $P'_s \mod \mathfrak{m}_{A'} = P_{0,s}$, P'_s defines an element $P_s \in \mathrm{GL}(A^r)$. The tuple $(\rho,(P_s)_{s\in S})$ then defines an element in $F(A)$. So there exists a unique Λ-algebra homomorphism $R(\rho_\Lambda) \to A$ which brings $(\rho_{\mathrm{univ}},(P_{\mathrm{univ},s})_{s\in S})$ to the equivalent class of $(\rho,(P_s)_{s\in S})$. The composite

$$R(\rho_\Lambda) \to A \hookrightarrow A'$$

brings $(\rho_{\mathrm{univ}},(P_{\mathrm{univ},s})_{s\in S})$ to the equivalent class of $(\rho',(P'_s)_{s\in S})$. Since the residue field of $R(\rho_\Lambda)$ is isomorphic to κ, any local Λ-algebra homomorphism $R(\rho_\Lambda) \to A'$ factors through the subring A of A'.

So any local Λ-algebra homomorphism $R(\rho_\Lambda) \to A'$ which brings $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ to the equivalent class of $(\rho', (P'_s)_{s\in S})$ gives rise to a homomorphism $R(\rho_\Lambda) \to A$ which brings $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ to the equivalent class of $(\rho, (P_s)_{s\in S})$, and hence is unique. $\square$

Let E be a finite extension of $\mathbb{Q}_\ell$, let $\mathcal{O}$ be its integer ring, and let κ be its residue field. Suppose $\mathcal{F}_E$ is a lisse E-sheaf on $X - S$ of rank r. Choose a torsion free lisse $\mathcal{O}$-sheaf $\mathcal{F}_\mathcal{O}$ such that $\mathcal{F}_E \cong \mathcal{F}_\mathcal{O} \otimes_\mathcal{O} E$. Let $\mathcal{F}_0 = \mathcal{F}_\mathcal{O} \otimes_\mathcal{O} \kappa$, let $\rho_\mathcal{O} : \pi_1(X - S, \bar\eta) \to \mathrm{GL}(\mathcal{O}^r)$ be the representation corresponding to the sheaf $\mathcal{F}_\mathcal{O}$, and let $\rho_E : \pi_1(X - S, \bar\eta) \to \mathrm{GL}(E^r)$ and $\rho_0 : \pi_1(X - S, \bar\eta) \to \mathrm{GL}(\kappa^r)$ be the representations obtained from $\rho_\mathcal{O}$ by passing to the fraction field and the residue field of $\mathcal{O}$, respectively. Note that ρ_E and ρ_0 are also the representations corresponding to the E-sheaf $\mathcal{F}_E$ and the κ-sheaf $\mathcal{F}_0$, respectively. Take $\Lambda = \mathcal{O}$. Consider the universal deformation ring $R(\rho_\mathcal{O})$ of the functor $F : \mathcal{C}_\mathcal{O} \to (\mathrm{Sets})$ for the data $(\rho_0, (I)_{s\in S})$, where $P_{0,s} = I$ for all $s \in S$. As a local $\mathcal{O}$-algebra, it is isomorphic to a quotient of $\mathcal{O}[[y_1, \ldots, y_n]]$ for some n.

Let's recall a construction of Berthelot which associates a rigid analytic space to any noetherian adic formal scheme $\mathfrak{X}$ over $\mathrm{Spf}\,\mathcal{O}$ whose reduction $\mathfrak{X}_{\mathrm{red}}$ is a scheme of finite type over κ. First consider the case where $\mathfrak{X} = \mathrm{Spf}\,R$ is affine, where R is a complete adic noetherian $\mathcal{O}$-algebra such that the largest ideal of definition J of R contains $\mathfrak{m}_\mathcal{O} R$, and R/J is a finitely generated κ-algebra. One can show ([2, Lemma 1.2]) that R is a quotient of the ring $\mathcal{O}\{x_1, \ldots, x_n\}[[y_1, \ldots, y_m]]$. Recall that $\mathcal{O}\{x_1, \ldots, x_n\}$ is the ring of power series $\sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$ with the property $a_{i_1\ldots i_n} \in \mathcal{O}$ and $a_{i_1\ldots i_n} \to 0$ as $i_1 + \cdots + i_n \to \infty$. We define the rigid analytic space $\mathfrak{X}^{\mathrm{rig}}$ to be $E(0,1)^n \times D(0,1)^m$ for $R = \mathcal{O}\{x_1, \ldots, x_n\}[[y_1, \ldots, y_m]]$, where $E(0,1) = \mathrm{Sp}\,E\{x\}$ is the closed unit disc over E, and $D(0,1) = \bigcup_{i=1}^\infty \mathrm{Sp}\,E\{r_i^{-1}x\}$ is the open unit disc. Here $E\{x\} = E \otimes_\mathcal{O} \mathcal{O}\{x\}$, $\{r_i\}$ is an increasing sequence of positive real numbers with limit 1 such that for each r_i, an integral power of r_i is equal to the norm of an element in E, and $E\{r^{-1}x\}$ is the ring of power series $\sum_{i\geq 0} a_i x^i$ with the property $a_i \in E$ and $a_i r^i \to 0$ as $i \to \infty$. In general, if R is the quotient of $\mathcal{O}\{x_1, \ldots, x_n\}[[y_1, \ldots, y_m]]$ by an ideal generated by $g_1, \ldots, g_k$, we define $\mathfrak{X}^{\mathrm{rig}}$ to be the closed analytic subvariety $g_1 = \ldots = g_k = 0$ of $E(0,1)^n \times D(0,1)^m$. One extends this construction to a formal scheme $\mathfrak{X}$ over $\mathcal{O}$ by gluing the rigid analytic spaces constructed from an affine open covering of $\mathfrak{X}$. The rigid analytic space $\mathfrak{X}^{\mathrm{rig}}$ can be thought as the generic fiber of the formal scheme $\mathfrak{X}$ over $\mathrm{Spf}\,\mathcal{O}$. The construction $\mathfrak{X} \to \mathfrak{X}^{\mathrm{rig}}$ defines a functor from the category of noetherian adic formal schemes $\mathfrak{X}$ over $\mathrm{Spf}\,\mathcal{O}$ whose reduction $\mathfrak{X}_{\mathrm{red}}$ are schemes locally of finite type over $\mathrm{Spec}\,\kappa$ to the category of rigid analytic spaces over E. This functor commutes with fiber products. We refer the reader to [2, §1] and [12, §7] for details of Berthelot's construction.

Let $\mathfrak{F} = \mathrm{Spf}\,R(\rho_\mathcal{O})$ be the formal scheme associated to the universal deformation ring $R(\rho_\mathcal{O})$ of the functor $F : \mathcal{C}_\mathcal{O} \to (\mathrm{Sets})$ for the data $(\rho_0, (I)_{s\in S})$, and let $\mathfrak{F}^{\mathrm{rig}}$ be the associated rigid analytic space. By [12, 7.1.10], there is a one-to-one correspondence between the set of point in $\mathfrak{F}^{\mathrm{rig}}$ and the set of equivalent classes of local homomorphisms $R(\rho_\mathcal{O}) \to \mathcal{O}'$ of $\mathcal{O}$-algebras, where $\mathcal{O}'$ is the integer ring of a finite extension E' of E, and two such homomorphisms $R(\rho_\mathcal{O}) \to \mathcal{O}'$ and $R(\rho_\mathcal{O}) \to \mathcal{O}''$ are equivalent if there exists a commutative diagram

$$\begin{array}{ccc} R(\rho_\mathcal{O}) & \to & \mathcal{O}' \\ \downarrow & & \downarrow \\ \mathcal{O}'' & \to & \mathcal{O}''' \end{array}$$

such that $\mathcal{O}'''$ is the integer ring of a finite extension E''' of E containing both the fraction fields E' and E'' of $\mathcal{O}'$ and $\mathcal{O}''$ respectively. Applying the universal property of $R(\rho_{\mathcal{O}})$ to the tuples $(\rho_{\mathcal{O}}, (I)_{s\in S})$ mod $\mathfrak{m}_{\mathcal{O}}^i$ for all i, we get a unique $\mathcal{O}$-algebra homomorphism

$$\varphi_0 : R(\rho_{\mathcal{O}}) \to \mathcal{O}$$

which brings the universal representation $\rho_{\mathrm{univ}} : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(r, R(\rho_{\mathcal{O}}))$ to $\rho_{\mathcal{O}}$, and brings $P_{\mathrm{univ},s}$ to I for all $s \in S$. The homomorphism φ_0 defines a point t_0 in $\mathfrak{F}^{\mathrm{rig}}$. Let t be a point in $\mathfrak{F}^{\mathrm{rig}}$ corresponding to a local homomorphism

$$\varphi_t : R(\rho_{\mathcal{O}}) \to \mathcal{O}'$$

of $\mathcal{O}$-algebras. Let $(\rho_t, (P_{t,s})_{s\in S})$ be the tuple obtained by pushing forward the universal tuple $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ through the homomorphism φ_t. Note that $\rho_t : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(\mathcal{O}'^r)$ is a representation, $P_{t,s} \in \mathrm{GL}(\mathcal{O}'^r)$, and

$$\begin{aligned}
&\rho_{t_0} = \rho_{\mathcal{O}}, \quad P_{t_0,s} \mod \mathfrak{m}_{\mathcal{O}} = I, \\
&\rho_t \mod \mathfrak{m}_{\mathcal{O}'} = \rho_0, \quad P_{t,s} \mod \mathfrak{m}_{\mathcal{O}'} = I, \\
&P_{t,s}^{-1} \rho_t|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} P_{t,s} = \rho_{\mathcal{O}}|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)},
\end{aligned}$$

Suppose ρ_E is physically rigid. Then the third line in the above equations implies that ρ_t is isomorphic to ρ_E as $\overline{\mathbb{Q}}_\ell$-representations, that is, after enlarging the field E', there exists $P \in \mathrm{GL}(E'^r)$ such that $P^{-1}\rho_t P = \rho_E$. We conjecture that for those t close to t_0, we can choose P so that $P \in \mathrm{GL}(\mathcal{O}^r_{\overline{\mathbb{Q}}_\ell})$ and $P \equiv I \mod \mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}$, where $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$ is the integer ring of $\overline{\mathbb{Q}}_\ell$. More precisely, we should have the following conjecture.

Conjecture 0.3. *Notation as above. Suppose that $X = \mathbb{P}^1$, that $\mathrm{End}(\mathcal{F}_E)$ consists of scalar multiplications, and that $\mathcal{F}_E$ is physically rigid. Then there exists an admissible neighborhood V of the point t_0 in $\mathfrak{F}^{\mathrm{rig}}$ such that for any $t \in V$, there exists $P \in \mathrm{GL}(\mathcal{O}^r_{\overline{\mathbb{Q}}_\ell})$ such that $P \equiv I \mod \mathfrak{m}_{\mathcal{O}_{\mathbb{Q}_\ell}}$ and $P^{-1}\rho_t P = \rho_{\mathcal{O}}$.*

Remark 0.4. Note that under the assumption of Conjecture 0.3, P is uniquely determined up to scalar. Indeed, provided E^r and E'^r with the $\pi_1(X - S, \bar{\eta})$-module structure via the representation ρ_E. By [11, Lemma 1.1], we have

$$\mathrm{End}_{\pi_1(X-S,\bar{\eta})}(E'^r) \cong \mathrm{End}_{\pi_1(X-S,\bar{\eta})}(E^r) \otimes_E E'.$$

As $\mathrm{End}_{\pi_1(X-S,\bar{\eta})}(E^r) \cong \mathrm{End}(\mathcal{F}_E)$ consists of scalar multiplications, the same is true for $\mathrm{End}_{\pi_1(X-S,\bar{\eta})}(E'^r)$. If P and P' are two matrices in $\mathrm{GL}(E'^r)$ such that $P^{-1}\rho_t P = P'^{-1}\rho_t P' = \rho_E$, then we have

$$\rho_E P^{-1} P' = P^{-1} P' \rho_E.$$

So $P^{-1}P'$ lies in $\mathrm{End}_{\pi_1(X-S,\bar{\eta})}(E'^r)$, and hence is a scalar matrix.

In §3, we prove the following proposition:

Proposition 0.5. *Conjecture 0.3 holds under either one of the following conditions:*
(i) $\mathrm{End}(\mathcal{F}_0)$ *consists of scalar multiplications.*
(ii) $\mathcal{F}_E$ *has finite monodromy, that is,* $\mathrm{im}(\rho_E)$ *is finite.*

Let $G : \mathcal{C}_{\mathcal{O}} \to (\mathrm{Sets})$ be the functor defined by

$$G(A) = \{(P_s)_{s\in S} | P_s \in \mathrm{Aut}_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}(A^r),\ P_s \equiv I \mod \mathfrak{m}_A\}/\sim,$$

where A^r is provided with the $\mathrm{Gal}(\bar{\eta}_s/\eta_s)$-action via $\rho_{\mathcal{O}}$, and two tuples $(P_s^{(i)})_{s\in S}$ $(i = 1, 2)$ are equivalent if there exists an invertible scalar $(r \times r)$-matrix $P = uI$ for some unit u in A such that $P_s^{(1)} = P^{-1}P_s^{(2)}$ for all $s \in S$. Using Schlessinger's criteria, one can verify that functor G is pro-representable. One can describe the universal deformation ring of G as follows: Let H_s be the wild inertia subgroup of $\mathrm{Gal}(\bar{\eta}_s/\eta_s)$. Then the (tame) quotient group $\mathrm{Gal}(\bar{\eta}_s/\eta_s)/H_s$ is topologically generated by one element, say by the image of $g_s \in \mathrm{Gal}(\bar{\eta}_s/\eta_s)$. Since H_s is a pro-p-group, its image under $\rho_{\mathcal{O}}$ is finite. Choose $h_{s,1}, \ldots, h_{s,n_s} \in H_s$ so that

$$\mathrm{im}\,\rho_{\mathcal{O}} = \{\rho_{\mathcal{O}}(h_{s,1}), \ldots, \rho_{\mathcal{O}}(h_{s,n_s})\}.$$

Set

$$A_s = \rho_{\mathcal{O}}(g_s), \quad A_{s,k} = \rho_{\mathcal{O}}(h_{s,k})\ (k = 1, \ldots, n_s).$$

Then we have $P_s \in \mathrm{Aut}_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}(A^r)$ if and only if

$$P_s A_s = A_s P_s, \quad P_s A_{s,k} = A_{s,k} P_s.$$

Let I be the homogeneous ideal of $\mathcal{O}[t_{s,ij}]_{s\in S, 1\le i,j\le r}$ generated by the entries of the matrices

$$(t_{s,ij})A_s - A_s(t_{s,ij}), \quad (t_{s,ij})A_{s,k} - A_{s,k}(t_{s,ij}) \quad (s \in S,\ 1 \le k \le n_s).$$

It defines a closed subscheme X of the projective space $\mathbb{P}^{|S|r^2-1} = \mathrm{Proj}\,\mathcal{O}[t_{s,ij}]_{s\in S,1\le i,j\le r}$. Let $\mathfrak{p}$ be the kernel of the homomorphism

$$T = (\mathcal{O}[t_{s,ij}]_{s\in S,1\le i,j\le r})/I \to \kappa[t], \quad t_{s,ij} \mapsto \delta_{ij}t.$$

Then $\mathfrak{p}$ is a homogeneous prime ideal of T (resp. a Zariski closed point x of $X = \mathrm{Proj}\,T$). The universal deformation ring $R(G)$ for the functor G is isomorphic to the completion $T^{\wedge}_{(\mathfrak{p})}$ (resp. $\hat{\mathcal{O}}_{X,x}$) of the homogeneous localization $T_{(\mathfrak{p})}$ (resp. $\mathcal{O}_{X,x}$). Note that for any local $\mathcal{O}$-algebra A and any equivalent class of tuples $(P'_s)_{s\in S}$, where $P'_s \in \mathrm{Aut}_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}(A'^r)$ and $P_s \equiv I \mod \mathfrak{m}_{A'}$ and the equivalence relation is defined as before, there exists a unique local Λ-algebra homomorphism $R(G) \to A'$ which brings the universal tuple $((t_{s,ij})_{s\in S})$ to (P'_s).

The rigid analytic space $\mathfrak{G}^{\mathrm{rig}}$ associated to the formal scheme $\mathrm{Spf}\,R(G)$ is a group object, and its points can be identified with the set

$$\{(P_s)_{s\in S} | P_s \in \mathrm{Aut}_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}(\mathcal{O}^r_{\overline{\mathbb{Q}}_\ell}),\ P_s \equiv I \mod \mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}\}/\sim,$$

where $\mathcal{O}^r_{\overline{\mathbb{Q}}_\ell}$ is provided with the $\mathrm{Gal}(\bar{\eta}_s/\eta_s)$-action via $\rho_{\mathcal{O}}$, and two tuples $(P_s^{(i)})_{s\in S}$ $(i = 1, 2)$ are equivalent if $P_s^{(1)} = u^{-1}P_s^{(2)}$ for all $s \in S$ for some units u in $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$. Note that

$$\dim \mathfrak{G}^{\mathrm{rig}} = \sum_{s\in S} \dim \mathrm{End}_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}(E^r) - 1,$$

where E^r is provided with the $\mathrm{Gal}(\bar{\eta}_s/\eta_s)$-action via ρ_E.

Let $F : \mathcal{C}_{\mathcal{O}} \to (\mathrm{Sets})$ be the functor introduced before for the data $(\rho_{\mathcal{O}}, (I)_{s\in S})$. We have a morphism of functors $G \to F$ defined by

$$G(A) \to F(A), \quad (P_s)_{s\in S} \mapsto (\rho_{\mathcal{O}}, (P_s)_{s\in S}).$$

Lemma 0.6. *Let $\mathfrak{F}$ (resp. $\mathfrak{G}$) be the formal scheme associated to the universal deformation ring for the functor F (resp. G), let $\mathfrak{F}^{\mathrm{rig}}$ (resp. $\mathfrak{G}^{\mathrm{rig}}$) be the associated rigid analytic space, and let $f : \mathfrak{G}^{\mathrm{rig}} \to \mathfrak{F}^{\mathrm{rig}}$ be the morphism on rigid analytic spaces induced by the morphism of functors $G \to F$. Suppose that $X = \mathbb{P}^1$, that $\mathrm{End}(\mathcal{F}_E)$ consists of scalar multiplications, and that $\mathcal{F}_E$ is physically rigid. If Conjecture 0.3 is true, then $f : f^{-1}(V) \to V$ is surjective in the sense that every point in V is the image of a point in $\mathfrak{G}^{\mathrm{rig}}$.*

Proof. Let t be a point in V, and let $\varphi_t : R(\rho_{\mathcal{O}}) \to \mathcal{O}'$ be the corresponding local homomorphism of $\mathcal{O}$-algebras ([12, 7.1.10]), where $\mathcal{O}'$ is the integer ring of a finite extension E' of E. Let $(\rho_t, (P_{t,s})_{s\in S})$ be the tuple obtained by pushing forward the universal tuple $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ through the homomorphism φ_t. By our assumption, there exists $P \in \mathrm{GL}(\mathcal{O}^r_{\overline{\mathbb{Q}}_\ell})$ such that $P \equiv I \mod \mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}$ and $P^{-1}\rho_t P = \rho_{\mathcal{O}}$. By enlarging E', we may assume $P \in \mathrm{GL}(\mathcal{O}'^r)$. Then for each i, the tuple $(\rho_t, (P_{t,s})_{s\in S}) \mod \mathfrak{m}^i_{\mathcal{O}'}$ is equivalent to the tuple $(\rho_{\mathcal{O}}, (P^{-1}P_{t,s})_{s\in S}) \mod \mathfrak{m}^i_{\mathcal{O}'}$. The family of tuples $(P^{-1}P_{t,s})_{s\in S} \mod \mathfrak{m}^i_{\mathcal{O}'}$ defines a family of local $\mathcal{O}$-algebra homomorphisms $R(G) \to \mathcal{O}'/\mathfrak{m}^i_{\mathcal{O}'}$, where $R(G)$ is the universal deformation ring for G. This family is compatible and defines a local homomorphism $R(G) \to \mathcal{O}'$ of $\mathcal{O}$-algebras. It corresponds to a point in $\mathfrak{G}^{\mathrm{rig}}$ that is mapped to the point t of $\mathfrak{F}^{\mathrm{rig}}$. □

Lemma 0.7. *Let $f : X \to Y$ be a separated morphism of rigid analytic spaces over a nonarchimedean field k with non-trivial valuation. Suppose X can be covered by countably many k-affinoid subspaces X_n $(n = 1, 2, ...)$. If f is surjective on the underlying sets of points, then* $\dim Y \leq \dim X$.

Proof. The following proof is due to Junyi Xie. Making a base change to the completion of the algebraic closure of k, we may assume k is a complete algebraically closed field. We may reduced to the case where $Y = \mathrm{Sp}\, A$ for a strictly k-affinoid algebra A (in the language of Berkovich [3]). Then by definition, $\dim Y$ is the Krull dimension of A. By the Noether normalization theory ([5, Corollary 6.1.2/2]), there exists a finite monomorphism $T_d \to A$ for some Tate algebra $T_d = k\{t_1, \ldots, t_d\}$ with $d = \dim Y$. Note that the induced morphism $Y \to \mathrm{Sp}\, T_d$ is surjective on the underlying set of points. So we can reduce to the case where $A = T_d$ and $Y = E(0, 1)^d$ is the unit polydisc of dimension d.

Let $\mathcal{X}$ (resp. $\mathcal{X}_n$, resp. $\mathcal{Y}$) be the Berkovich space associated to X(resp. X_n, resp. Y). Denote the morphism $\mathcal{X} \to \mathcal{Y}$ corresponding to $f : X \to Y$ also by f. For any real multiple $\underline{r} = (r_1, \ldots, r_d)$ with $0 < r_i < 1$ and $r_i \in |k^*|$, and any rigid point $\underline{a} = (a_1, \ldots, a_d)$ in $E(0, 1)^d$, where $a_i \in k$ and $|a_i| \leq 1$, consider the polydisc

$$E(\underline{a}, \underline{r}) = \{(x_1, \ldots, x_d) \in k^d : |x_i - a_i| \leq r_i\}.$$

We have $E(\underline{a}, \underline{r}) \subset E(0, 1)^d$. We define the associated Gauss norm $|\cdot|_{E(\underline{a},\underline{r})}$ on T_d by

$$|f|_{E(\underline{a},\underline{r})} = \max\{|f(x)| : x \in E(a, r)\}$$

for any $f \in T_d$. If

$$f = \sum_{i_1,\ldots,i_d} a_{i_1\ldots i_d}(t_1 - a_1)^{i_1} \cdots (t_d - a_d)^{i_d}$$

is the Taylor expansion of f at a, then we have

$$|f|_{E(\underline{a},\underline{r})} = \max_{i_1,\ldots,i_d} |a_{i_1\ldots i_d}| r_1^{i_1} \cdots r_d^{i_d}.$$

The Gauss norms $|\cdot|_{E(\underline{a},\underline{r})}$ are points in $\mathcal{Y}$. Let $\mathcal{S}$ be the subset of $\mathcal{Y}$ consisting of all Gauss norms associated to all polydiscs $E(\underline{a},\underline{r})$. Note that $\mathcal{S}$ is dense in Y. Indeed, as the radius $\underline{r} = (r_1, \ldots, r_d)$ approaches to 0, the Gauss norm $|\cdot|_{E(\underline{a},\underline{r})}$ approaches to the rigid point $\underline{a}$, and it is known that the set of all rigid points is dense in $\mathcal{Y}$ ([3, 2.1.15]). Moreover, for any $y \in \mathcal{S}$, we have $s(\mathcal{H}(y)/k) = d$, where $\mathcal{H}(y)$ is the field defined in [3, 1.2.2 (i)], and $s(\mathcal{H}(y)/k) = \mathrm{tr.deg}(\widetilde{\mathcal{H}(y)}/\tilde{k})$ is defined in [3, 9.1]. We claim that $f(\mathcal{X}) \cap \mathcal{S}$ in nonempty. Otherwise, $f(\mathcal{X}_n)$ is disjoint from $\mathcal{S}$ for each n, that is, $\mathcal{S} \subset \mathcal{Y} - f(\mathcal{X}_n)$. Hence $\mathcal{Y} - f(\mathcal{X}_n)$ is dense in $\mathcal{Y}$. Since $\mathcal{X}_n$ is affiniod, it is compact ([3, 1.2.1]). So $f(\mathcal{X}_n)$ is a compact subset in the Hausdorff space $\mathcal{Y}$, and hence it is a closed subset. It follows that $\mathcal{Y} - f(\mathcal{X}_n)$ is a dense open subset of $\mathcal{Y}$. By [3, 2.1.15], the subset of rigid points $(\mathcal{Y} - f(\mathcal{X}_n)) \cap Y$ is open dense in Y. Here we provide Y with the topology induced from the Berkovich space $\mathcal{Y}$. But this topology on Y is induced by a complete metric. In fact, it is unit polydisc in k^n provided with the metric given by the valuation of k. By the Baire category theorem ([16, 9.1]), the set $\cap_{n=1}^{\infty} (\mathcal{Y} - f(\mathcal{X}_n)) \cap Y = (\mathcal{Y} - f(\mathcal{X})) \cap Y$ is dense in Y. In particular, it is nonempty. This contradicts to the assumption that $f : X \to Y$ is surjective. So $f(\mathcal{X}) \cap \mathcal{S}$ is nonemtpy. Take $x \in \mathcal{X}$ such that $f(x) \in \mathcal{S}$. Then by [3, 9.1.3], we have

$$\dim X \geq d(\mathcal{H}(x)/k) \geq s(\mathcal{H}(x)/k) \geq s(\mathcal{H}(f(x))/k) = \dim Y.$$

□

Proposition 0.8. *Let $\varphi_t : R(\rho_{\mathcal{O}}) \to \mathcal{O}$ be an $\mathcal{O}$-algebra homomorphism, let $\mathfrak{p}_t = \ker \varphi_t$, and let $(\rho_t, (P_{t,s})_{s\in S})$ be the tuple obtained by pushing forward the universal tuple $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ through the homomorphism φ_t.*

(i) The completion $R(\rho_{\mathcal{O}})^{\wedge}_{\mathfrak{p}_t}$ of the local ring $R(\rho_{\mathcal{O}})_{\mathfrak{p}_t}$ is canonically isomorphic to the universal deformation ring $R(\rho_t \otimes_{\mathcal{O}} E)$ for the functor $F : \mathcal{C}_E \to$ (Sets) defined by

$$\begin{aligned} F(A) = \{(\rho, (P_s)_{s\in S}) \quad | \quad & \rho : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A^r) \text{ is a representation, } P_s \in \mathrm{GL}(A^r), \\ & \rho \mod \mathfrak{m}_A = \rho_t, \quad P_s \mod \mathfrak{m}_A = P_{t,s}, \\ & P_s^{-1} \rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} P_s = \rho_t|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \text{ for all } s \in S\}/\sim, \end{aligned}$$

for any local Artinian E algebra $A \in \mathrm{ob}\,\mathcal{C}_E$ with residue field E.

(ii) Let t be the point in $\mathfrak{F}^{\mathrm{rig}}$ corresponding to φ_t. We have $\widehat{\mathcal{O}}_{\mathfrak{F}^{\mathrm{rig}},t} \cong R(\rho_{\mathcal{O}})^{\wedge}_{\mathfrak{p}_t}$.

We will prove this proposition in §4. We are now ready to prove the main result of this paper:

Theorem 0.9. *Suppose that $X = \mathbb{P}^1$, that $\mathcal{F}_E$ is physically rigid, and that one of the following condition holds:*

(i) $\mathrm{End}(\mathcal{F}_0)$ consists of scalar multiplications.

(ii) $\mathrm{End}(\mathcal{F}_E)$ consists of scalar multiplications and $\mathcal{F}_E$ has finite monodromy.

Then $\chi(X, j_*\mathcal{E}nd(\mathcal{F}_E)) = 2$.

Proof. To prove this theorem, we may assume S is nonempty. Indeed, replacing S by $S \cup \{x\}$ for any closed point x in $X - S$ has no effect on the theorem. Let $\varphi_0 : R(\rho_{\mathcal{O}}) \to \mathcal{O}$ be the local homomorphism of $\mathcal{O}$-algebras corresponding to the point t_0 in $\mathfrak{F}^{\mathrm{rig}}$, and let $\mathfrak{p}_0 = \ker \varphi_0$. By Proposition 0.8, we have

$$R(\rho_{\mathcal{O}})^{\wedge}_{\mathfrak{p}_0} \cong \widehat{\mathcal{O}}_{\mathfrak{F}^{\mathrm{rig}}, t_0}.$$

The condition that $\mathrm{End}(\mathcal{F}_0)$ consists of scalar multiplications implies that $\mathrm{End}(\mathcal{F}_E)$ consist of scalar multiplications. For a proof of this fact, see Lemma 3.1 (iii). By Proposition 0.1 applied to the case $\Lambda = E$ and Proposition 0.8, $R(\rho_{\mathcal{O}})^{\wedge}_{\mathfrak{p}_0} \cong R(\rho_E)$ is a formally smooth E-algebra, and we have

$$\dim R(\rho_{\mathcal{O}})^{\wedge}_{\mathfrak{p}_0} = -\chi(X, j_* \mathcal{E}nd^{(0)}(\mathcal{F}_E)) + \sum_{s \in S} \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_E)) - 1.$$

So we have

$$\dim \widehat{\mathcal{O}}_{\mathfrak{F}^{\mathrm{rig}}, t_0} = -\chi(X, j_* \mathcal{E}nd^{(0)}(\mathcal{F}_E)) + \sum_{s \in S} \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_E)) - 1.$$

On the other hand, by Proposition 0.5, Conjecture 0.3 holds under the assumption (i) or (ii). So by Lemmas 0.6 and 0.7, we have

$$\dim \widehat{\mathcal{O}}_{\mathfrak{F}^{\mathrm{rig}}, t_0} \leq \dim V \leq \dim f^{-1}(V) \leq \dim \mathfrak{G}^{\mathrm{rig}} = \sum_{s \in S} \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_E)) - 1.$$

Comparing the above expressions for $\dim \widehat{\mathcal{O}}_{\mathfrak{F}^{\mathrm{rig}}, t_0}$, we get

$$\chi(X, j_* \mathcal{E}nd^{(0)}(\mathcal{F}_E)) \geq 0.$$

As $\mathcal{E}nd(\mathcal{F}_E) \cong \mathcal{E}nd^{(0)}(\mathcal{F}_E) \oplus E$, we have

$$\chi(X, j_* \mathcal{E}nd(\mathcal{F}_E)) = \chi(X, j_* \mathcal{E}nd^{(0)}(\mathcal{F}_E)) + \chi(X, E) \geq 2,$$

Here we use the fact that $\chi(X, E) = 2$ since $X = \mathbb{P}^1$. By our assumption, the space $H^0(X, j_* \mathcal{E}nd(\mathcal{F}_E)) \cong \mathrm{End}(\mathcal{F}_E)$ consists of scalar multiplications and hence has dimension 1. The pairing

$$\mathcal{E}nd(\mathcal{F}_E) \times \mathcal{E}nd(\mathcal{F}_E) \to E, \quad (\phi, \psi) \mapsto \mathrm{Tr}(\psi \circ \phi)$$

defines a self-duality on $\mathcal{E}nd(\mathcal{F}_E)$. By the Poincaré duality ([8, 1.3 and 2.2]), we have a perfect pairing

$$H^2(X, j_* \mathcal{E}nd(\mathcal{F}_E)) \times H^0(X, j_* \mathcal{E}nd(\mathcal{F}_E)(1)) \to E.$$

It follows that $H^2(X, j_* \mathcal{E}nd(\mathcal{F}_E))$ also has dimension 1. So

$$\begin{aligned} \chi(X, j_* \mathcal{E}nd(\mathcal{F}_E)) &= \sum_{i=0}^{2} (-1)^i \dim H^i(X, j_* \mathcal{E}nd(\mathcal{F}_E)) \\ &= 2 - \dim H^1(X, j_* \mathcal{E}nd(\mathcal{F}_E)) \\ &\leq 2. \end{aligned}$$

Compared with the previous opposite inequality, we get $\chi(X, j_* \mathcal{E}nd(\mathcal{F}_E)) = 2$. This proves that $\mathcal{F}_E$ is cohomologically rigid. □

In Lemma 3.1 (iii), we prove that the condition that $\mathrm{End}(\mathcal{F}_0)$ consists of scalar multiplications implies that $\mathrm{End}(\mathcal{F}_E)$ consist of scalar multiplications. If Conjecture 0.3 is true, then Theorem 0.9 holds under the weaker condition that $\mathrm{End}(\mathcal{F}_E)$ consists of scalar multiplications.

The paper is organized as follows. In §1, we introduced a family of functors of framed deformations of representations of $\pi_1(X-S,\bar{\eta})$. These functors are pro-representable and we calculate the dimensions of their Zariski tangent spaces. In §2, we study the obstruction to lifting a deformation. Using results in §1 and §2, we prove Proposition 0.1 at the end of §2. We prove Proposition 0.5 in §3, and Lemma 0.8 in §4.

1 Calculation of dimensions of tangent spaces

In this section, we suppose that S is nonempty and that we are given a representation $\rho_\Lambda : \pi_1(X-S,\bar{\eta}) \to \mathrm{GL}(\Lambda^r)$ and a frame $P_{0,s} \in \mathrm{GL}(\kappa^r)$ for each $s \in S$. Let $\lambda : \pi_1(X-S,\bar{\eta}) \to \Lambda^*$ be the determinant of ρ_Λ and let $\rho_0 : \pi_1(X-S,\bar{\eta}) \to \mathrm{GL}(\kappa^r)$ be $\rho_\Lambda \mod \mathfrak{m}_\Lambda$. For any $A \in \mathrm{ob}\,\mathcal{C}_\Lambda$, denote the composite

$$\pi_1(X-S,\bar{\eta}) \xrightarrow{\lambda} \Lambda^* \to A^*$$

also by λ. Define

$$\begin{aligned} D_S^\Box(A) &= \{(\rho,(P_s)_{s\in S}) \,|\, \rho : \pi_1(X-S,\bar{\eta}) \to \mathrm{GL}(A^r) \text{ is a representation, } P_s \in \mathrm{GL}(A^r), \\ &\quad\ \rho \mod \mathfrak{m}_A = \rho_0,\ P_s \mod \mathfrak{m}_A = P_{s,0}\}/\sim, \\ D_S^{\Box,\lambda}(A) &= \{(\rho,(P_s)_{s\in S}) \in D_S^\Box(A) \,|\, \det(\rho) = \lambda\}, \end{aligned}$$

where two tuples $(\rho^{(i)},(P_s^{(i)})_{s\in S})$ $(i=1,2)$ are equivalent if there exists $P \in \mathrm{GL}(A^r)$ such that

$$(\rho^{(1)},(P_s^{(1)})_{s\in S}) = (P^{-1}\rho^{(2)}P,(P^{-1}P_s^{(2)})_{s\in S}).$$

For any $s \in S$, define

$$\begin{aligned} D_s^\Box(A) &= \{\rho \,|\, \rho : \mathrm{Gal}(\bar{\eta}_s/\eta_s) \to \mathrm{GL}(A^r) \text{ is a representation, } \rho \mod \mathfrak{m}_A = \rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}\}, \\ D_s^{\Box,\lambda}(A) &= \{\rho \in D_s^\Box(A) \,|\, \det(\rho) = \lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}\}. \end{aligned}$$

For any morphism $A' \to A$ in $\mathcal{C}_\Lambda$, define $D(A') \to D(A)$ to be the map induced by $\mathrm{GL}(A'^r) \to \mathrm{GL}(A^r)$ for $D = D_S^\Box, D_S^{\Box,\lambda}, D_s^\Box, D_s^{\Box,\lambda}$. Then $D_S^\Box, D_S^{\Box,\lambda}, D_s^\Box, D_s^{\Box,\lambda}$ are functors from the category $\mathcal{C}_\Lambda$ to the category of sets. We have canonical morphisms of functors

$$D_S^\Box \to \prod_{s\in S} D_s^\Box, \quad D_S^{\Box,\lambda} \to \prod_{s\in S} D_s^{\Box,\lambda}$$

given by

$$(\rho,(P_s)_{s\in S}) \to (P_s^{-1}\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P_s)_{s\in S}.$$

Finally, let F (resp. F^λ) be the subfunctor of $D_S^\Box$ (resp. $D_S^{\Box,\lambda}$) defined by

$$\begin{aligned} F(A) &= \{(\rho,(P_s)_{s\in S}) \in D_S^\Box(A) | P_s^{-1}\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P_s = \rho_\Lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}\}, \\ (\text{resp. } F^\lambda(A) &= \{(\rho,(P_s)_{s\in S}) \in D_S^{\Box,\lambda}(A) | P_s^{-1}\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P_s = \rho_\Lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}\}). \end{aligned}$$

Note that F (resp. F^λ) can be thought as the fiber of the morphism of functors

$$D_S^\square \to \prod_{s\in S} D_s^\square \quad (\text{resp. } D_S^{\square,\lambda} \to \prod_{s\in S} D_s^{\square,\lambda})$$

over $(\rho_\Lambda|_{\mathrm{Gal}(\bar\eta_s/\eta_s)})_{s\in S}$. For any morphism $A'\to A$ and $A''\to A$ in $\mathcal{C}_\Lambda$, one can verify that the canonical map

$$D(A'\times_A A'')\to D(A')\times_{D(A)} D(A'')$$

is bijective if $A''\to A$ is surjective for each of the functors $D = D_S^\square, D_S^{\square,\lambda}, D_s^\square, D_s^{\square,\lambda}, F, F^\lambda$. Proposition 1.1 below shows that $D(\kappa[\epsilon])$ is finite dimensional. So the functors $D_S^\square, D_S^{\square,\lambda}, D_s^\square, D_s^{\square,\lambda}, F, F^\lambda$ are pro-representable by Schlessinger's criteria [17, Theorem 2.11].

Proposition 1.1. *Let $\kappa[\epsilon]$ be the ring of dual numbers. We have*

$$\begin{aligned}
\dim D_S^\square(\kappa[\epsilon]) &= -\chi(\pi_1(X-S,\bar\eta),\mathrm{Ad}(\rho_0)) + |S|r^2\\
&= -\chi(X-S,\mathcal{E}nd(\mathcal{F}_0)) + |S|r^2,\\
\dim D_S^{\square,\lambda}(\kappa[\epsilon]) &= -\chi(\pi_1(X-S,\bar\eta),\mathrm{Ad}^{(0)}(\rho_0)) + |S|r^2 - 1\\
&= -\chi(X-S,\mathcal{E}nd^{(0)}(\mathcal{F}_0)) + |S|r^2 - 1,\\
\dim D_s^\square(\kappa[\epsilon]) &= -\chi(\mathrm{Gal}(\bar\eta_s/\eta_s),\mathrm{Ad}(\rho_0)) + r^2,\\
\dim D_s^{\square,\lambda}(\kappa[\epsilon]) &= -\chi(\mathrm{Gal}(\bar\eta_s/\eta_s),\mathrm{Ad}^{(0)}(\rho_0)) + r^2 - 1,
\end{aligned}$$

where $\mathrm{Ad}(\rho_0)$ is the κ-vector space of $r\times r$ matrices with entries in κ on which $\pi_1(X-S,\bar\eta)$ and $\mathrm{Gal}(\bar\eta_s/\eta_s)$ act by the composition of ρ_0 with the adjoint representation of $\mathrm{GL}(\kappa^r)$, $\mathrm{Ad}^{(0)}(\rho_0)$ is the subspace of $\mathrm{Ad}(\rho_0)$ consisting of matrices of trace 0, $\mathcal{F}_0$ is the lisse κ-sheaf on $X-S$ corresponding to the representation $\rho_0:\pi_1(X-S,\bar\eta)\to\mathrm{GL}(\kappa^r)$, and $\mathcal{E}nd^{(0)}(\mathcal{F}_0)$ is the subsheaf of $\mathcal{E}nd(\mathcal{F}_0)$ formed by sections of trace 0.

Proof. Let's calculate the dimensions of $D_S^{\square,\lambda}(\kappa[\epsilon])$. Fix $s_0\in S$. Any element in $D_S^{\square,\lambda}(\kappa[\epsilon])$ is equivalent to an element $(\rho,(P_s)_{s\in S})$ with the property $P_{s_0}=P_{0,s_0}$. Two elements $(\rho^{(i)},(P_s^{(i)})_{s\in S})$ $(i=1,2)$ in $D_S^{\square,\lambda}(F[\epsilon])$ with the property $P_{s_0}^{(i)}=P_{0,s_0}$ are equivalent if any only if $(\rho^{(1)},(P_s^{(1)})_{s\in S}) = (\rho^{(2)},(P_s^{(2)})_{s\in S})$. Let $(\rho,(P_s)_{s\in S})$ be an element in $D_S^{\square,\lambda}(F[\epsilon])$ with $P_{s_0}=P_{0,s_0}$. We can write

$$\rho(g)=\rho_0(g)+\epsilon M(g)\rho_0(g),\quad P_s = P_{0,s}+\epsilon Q_s P_{0,s}$$

for some $(r\times r)$-matrices $M(g)$ and Q_s with entries in κ. That ρ is a homomorphism is equivalent to saying the map

$$\pi_1(X-S,\bar\eta)\to\mathrm{End}^{(0)}(\kappa^r),\quad g\mapsto M(g)$$

is a 1-cocycle for $\mathrm{Ad}(\rho_0)$. That $\det(\rho)=\lambda$ is equivalent to saying $\mathrm{Tr}(M(g))=0$ for all $g\in\pi_1(X-S,\bar\eta)$. We have $Q_{s_0}=0$, and there is no restriction for Q_s if $s\in S-\{s_0\}$. So

$$\dim D_S^{\square,\lambda}(F[\epsilon]) = \dim Z^1(\pi_1(X-S,\bar\eta),\mathrm{Ad}^{(0)}(\rho_0)) + (|S|-1)r^2,$$

where $Z^1(\pi_1(X-S,\bar\eta),\mathrm{Ad}^{(0)}(\rho_0))$ is the group of 1-cocycles. Let $B^1(\pi_1(X-S,\bar\eta),\mathrm{Ad}^{(0)}(\rho_0))$ be the group of 1-coboundaries. Its elements are of the form

$$\pi_1(X-S,\bar\eta)\to\mathrm{End}^{(0)}(\kappa^r),\quad g\mapsto\rho_0(g)A\rho_0(g)^{-1}-A$$

for some $(r \times r)$-matrix A of trace 0 with entries in κ. A 1-coboundary of the above form is 0 if any only if $\rho_0(g)A\rho_0(g)^{-1} - A = 0$ for all $g \in \pi_1(X - S, \bar{\eta})$, that is,

$$A \in \mathrm{End}^{(0)}_{\pi_1(X-S,\bar{\eta})}(\kappa^r) \cong H^0(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)).$$

So we have exact sequences

$$0 \to H^0(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) \to \mathrm{End}^{(0)}(\kappa^r) \to B^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) \to 0,$$
$$0 \to B^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) \to Z^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) \to H^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) \to 0.$$

It follows that

$$\begin{aligned}
& \dim D_S^{\Box,\lambda}(\kappa[\epsilon]) \\
= \;& \dim Z^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) + (|S| - 1)r^2 \\
= \;& \dim H^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) + \dim B^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) + (|S| - 1)r^2 \\
= \;& \dim H^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) + \dim \mathrm{End}^{(0)}(\kappa^r) - \dim H^0(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) + (|S| - 1)r^2 \\
= \;& -\chi(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) + |S|r^2 - 1,
\end{aligned}$$

where for the last equality, we use the fact that $H^2(\pi_1(X - S, \eta), \mathrm{Ad}^{(0)}(\rho_0)) = 0$ ([11, Lemma 2.1]). On the other hand, we have $H^2(X - S, \mathcal{E}nd^{(0)}(\mathcal{F}_0)) = 0$ since $X - S$ is an affine curve. We have $H^1(\pi_1(X-S, \eta), \mathrm{Ad}^{(0)}(\rho_0)) \cong H^1(X-S, \mathcal{E}nd^{(0)}(\mathcal{F}_0))$ by [11, Lemma 1.6]. It follows from the definition that $H^0(\pi_1(X - S, \eta), \mathrm{Ad}^{(0)}(\rho_0)) \cong H^0(X - S, \mathcal{E}nd^{(0)}(\mathcal{F}_0))$. So we have

$$\chi(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}^{(0)}(\rho_0)) = \chi(X - S, \mathcal{E}nd^{(0)}(\mathcal{F}_0)).$$

We leave it to the read to calculate $\dim D_S^{\Box}(\kappa[\epsilon])$, $\dim D_s^{\Box}(\kappa[\epsilon])$, $\dim D_s^{\Box,\lambda}(\kappa[\epsilon])$. $\Box$

2 Obstruction theory

Let $A' \to A$ be an epimorphism in the category $\mathcal{C}_\Lambda$ such that its kernel $\mathfrak{a}$ has the property $\mathfrak{m}_{A'}\mathfrak{a} = 0$. We can regard $\mathfrak{a}$ as a vector space over $\kappa \cong A'/\mathfrak{m}_{A'}$. Let $\rho : \pi_1(X-S, \bar{\eta}) \to \mathrm{GL}(A^r)$ be a representation such that $\rho \mod \mathfrak{m}_A = \rho_0$. Fix a set theoretic continuous lifting $\gamma : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A'^r)$ of ρ. Consider the map

$$\begin{aligned}
c : \pi_1(X - S, \bar{\eta}) \times \pi_1(X - S, \bar{\eta}) &\to \mathrm{End}(\kappa^r) \otimes_\kappa \mathfrak{a} \cong \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}, \\
c(g_1, g_2) &= \gamma(g_1 g_2)\gamma(g_2)^{-1}\gamma(g_1)^{-1} - I.
\end{aligned}$$

One can show that c is a 2-cocycle. By [11, Lemma 2.1], c must be a 2-coboundary. Choose a continuous map

$$\delta : \pi_1(X - S, \bar{\eta}) \to \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}$$

such that $c = d(\delta\gamma^{-1})$. Then $\rho' = \gamma + \delta : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A'^r)$ is a representation lifting ρ. We conclude that ρ can always be lifted to a representation $\rho' : \pi_1(X-S, \bar{\eta}) \to \mathrm{GL}(A'^r)$. Similarly, for any $s \in S$, one can prove that any representation $\mathrm{Gal}(\bar{\eta}_s/\eta_s) \to \mathrm{GL}(A^r)$ can be lifted to a representation $\mathrm{Gal}(\bar{\eta}_s/\eta_s) \to \mathrm{GL}(A'^r)$. This proves the functor $D_s^{\Box}$ is smooth.

Let $(\rho, (P_s)_{s\in S})$ be an element in $D_S^\square(A)$, let $\rho_s = P_s^{-1}\rho|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}P_s$, and let $\rho'_s : \mathrm{Gal}(\bar\eta_s/\eta) \to \mathrm{GL}(A'^r)$ be a representation so that

$$\rho'_s \mod \mathfrak{a} = \rho_s$$

for all $s \in S$. Choose a lifting $P'_s \in \mathrm{GL}(A'^r)$ for P_s. Then $P'_s\rho'_sP'^{-1}_s$ is a lifting of $P_s\rho_sP_s^{-1} = \rho|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}$. Now $\rho'|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}$ is also a lifting of $\rho|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}$. As in the proof of [11, Lemma 1.7], the continuous map $\delta_s : \mathrm{Gal}(\bar\eta_s/\eta_s) \to \mathrm{Ad}(\rho_0)\otimes_\kappa \mathfrak{a}$ defined by

$$\rho'(g) = P'_s\rho'_s(g)P'^{-1}_s + \delta_s(g)P'_s\rho'_s(g)P'^{-1}_s \quad (g \in \mathrm{Gal}(\bar\eta_s/\eta_s))$$

is a 1-cocycle. Let $[\delta_s]$ be the cohomology class of δ_s in $H^1(\mathrm{Gal}(\bar\eta_s/\eta_s), \mathrm{Ad}(\rho_0)\otimes_\kappa \mathfrak{a})$ and let c be the image of $([\delta_s])_{s\in S}$ in the cokernel of the canonical homomorphism

$$H^1(\pi_1(X-S,\bar\eta), \mathrm{Ad}(\rho_0)\otimes_\kappa \mathfrak{a}) \to \bigoplus_{s\in S} H^1(\mathrm{Gal}(\bar\eta_s/\eta_s), \mathrm{Ad}(\rho_0)\otimes_\kappa \mathfrak{a}).$$

By [11, Lemmas 1.6, 1.8], this cokernel can be considered as a subspace of $H^2(X, j_*\mathcal{E}nd(\mathcal{F}_0))\otimes_\kappa \mathfrak{a}$, where $\mathcal{F}_0$ is the lisse κ-sheaf on $X-S$ corresponding to the representation ρ_0. So we can also regard c as an element of in $H^2(X, j_*\mathcal{E}nd(\mathcal{F}_0))\otimes_\kappa \mathfrak{a}$. We call c the *obstruction class to lifting* $(\rho, (P_s)_{s\in S})$ *with the prescribed local data* $(\rho'_s)_{s\in S}$. For simplicity, in the sequel we call c the obstruction class to lifting ρ. In Lemma 2.1 below, we will show that c is independent of the choice of ρ' and P'_s. Note that we have

$$\begin{aligned}\det(\rho'(g)) &= \det\Big((I+\delta_s(g))P'_s\rho'_s(g)P'^{-1}_s\Big)\\ &= \big(1+\mathrm{Tr}(\delta_s(g))\big)\det(\rho'_s(g))\\ &= \det(\rho'_s(g)) + \mathrm{Tr}(\delta_s(g))\det(\rho'_s(g)).\end{aligned}$$

It follows that the obstruction class to lifting $\det(\rho)$ is the image of the obstruction class to lifting ρ under the homomorphism

$$H^2(X, j_*\mathcal{E}nd(\mathcal{F}_0))\otimes_\kappa \mathfrak{a} \to H^2(X,\kappa)\otimes_\kappa \mathfrak{a}$$

induced by

$$\mathrm{Tr} : \mathcal{E}nd(\mathcal{F}_0) \to \kappa.$$

Lemma 2.1. *Suppose S is nonempty. Let $A' \to A$ be an epimorphism in the category $\mathcal{C}_\Lambda$ such that its kernel $\mathfrak{a}$ has the property $\mathfrak{m}_{A'}\mathfrak{a} = 0$. Let $(\rho, (P_s)_{s\in S})$ be an element in $D_S^\square(A)$, let $\rho_s = P_s^{-1}\rho|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}P_s$, and let $\rho'_s : \mathrm{Gal}(\bar\eta_s/\eta) \to \mathrm{GL}(A'^r)$ be a representation such that*

$$\rho'_s \mod \mathfrak{a} = \rho_s.$$

Choose a representation $\rho' : \pi_1(X-S,\bar\eta) \to \mathrm{GL}(A'^r)$ and matrices $P'_s \in \mathrm{GL}(A'^r)$ such that

$$\rho' \mod \mathfrak{a} = \rho, \quad P'_s \mod \mathfrak{a} = P_s$$

for all $s \in S$. Define the obstruction class c to lifting $(\rho, (P_s)_{s\in S})$ with the prescribed local data $(\rho'_s)_{s\in S}$ as above.

(i) c is independent of the choices of ρ' and P'_s, and c vanishes if and only if $(\rho, (P_s)_{s\in S})$ can be lifted to a tuple $(\rho'', (P''_s)_{s\in S})$ such that $\rho'' : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A'^r)$ is a representation lifting ρ, $P''_s \in \mathrm{GL}(A'^r)$ lift P_s and $P''^{-1}_s \rho''|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} P''_s = \rho'_s$ for all $s \in S$.

(ii) The obstruction class to lifting $\det(\rho)$ is the image of c under the homomorphism

$$H^2(X, j_*\mathcal{E}nd(\mathcal{F}_0)) \otimes_F \mathfrak{a} \to H^2(X, \kappa) \otimes_\kappa \mathfrak{a}$$

induced by $\mathrm{Tr} : \mathcal{E}nd(\mathcal{F}_0) \to \kappa$.

Proof. We have already shown (ii). Let us prove (i). Let $\rho'' : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A'^r)$ be another lifting of ρ, and define 1-cocycles

$$\delta_s, \theta_s : \mathrm{Gal}(\bar{\eta}_s/\eta_s) \to \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}$$

by

$$\begin{aligned} \rho'(g) &= P'_s \rho'_s(g) P'^{-1}_s + \delta_s(g) P'_s \rho'_s(g) P'^{-1}_s, \\ \rho''(g) &= P'_s \rho'_s(g) P'^{-1}_s + \theta_s(g) P'_s \rho'_s(g) P'^{-1}_s \end{aligned}$$

for all $g \in \mathrm{Gal}(\bar{\eta}_s/\eta_s)$. Since ρ' and ρ'' are liftings of ρ, the continuous map

$$\psi : \pi_1(X - S, \bar{\eta}) \to \mathrm{Ad}(\rho_0) \otimes_F \mathfrak{a}$$

defined by

$$\rho''(g) = \rho'(g) + \psi(g)\rho'(g) \quad (g \in \pi_1(X - S, \bar{\eta}))$$

is a 1-cocycle for the group $\pi_1(X - S, \bar{\eta})$ as shown in the proof of [11, Lemma 1.7]. For any $g \in \mathrm{Gal}(\bar{\eta}_s/\eta_s)$, we have

$$\begin{aligned} (\theta_s(g) - \delta_s(g)) P'_s \rho'_s(g) P'^{-1}_s &= \rho''(g) - \rho'(g) \\ &= \psi(g)\rho'(g) \\ &= \psi(g)(P'_s \rho'_s(g) P'^{-1}_s + \delta_s(g) P'_s \rho'_s(g) P'^{-1}_s) \\ &= \psi(g) P'_s \rho'_s(g) P'^{-1}_s, \end{aligned}$$

where the last equality follows from the fact that $\mathfrak{a}^2 = 0$. It follows that

$$\theta_s(g) - \delta_s(g) = \psi(g)$$

for all $g \in \mathrm{Gal}(\bar{\eta}_s/\eta_s)$. Hence the cohomology class $[\theta_s] - [\delta_s]$ is the image of the cohomology class $[\psi]$ under the canonical homomorphism

$$H^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}) \to H^1(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}).$$

It follows that $([\theta_s])_{s\in S}$ and $([\delta_s])_{s\in S}$ define the same element in the cokernel of the canonical homomorphism

$$H^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}) \to \bigoplus_{s\in S} H^1(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}).$$

So the obstruction class to lifting ρ is independent of the choice of the lifting ρ' of ρ.

Choose another lifting $\tilde{P}'_s \in \mathrm{GL}(A'^r)$ of P_s for each $s \in S$. As representations of $\mathrm{Gal}(\bar{\eta}_s/\eta_s)$, $\tilde{P}'_s\rho'_s\tilde{P}'^{-1}_s$ and $P'_s\rho'_sP'^{-1}_s$ are strictly equivalent relative to $A' \to A$. (Let G be a group. Recall that two representations $\rho^{(1)}, \rho^{(2)} : G \to \mathrm{GL}(A'^r)$ are called *strictly equivalent* relative to $A' \to A$ if there exists $P \in \mathrm{GL}(A'^r)$ with the property $P \equiv I \mod \mathfrak{a}$ such that $P^{-1}\rho^{(1)}(g)P = \rho^{(2)}(g)$ for all $g \in G$.) Define

$$\delta''_s : \mathrm{Gal}(\bar{\eta}_s/\eta_s) \to \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}$$

by

$$\tilde{P}'_s\rho'_s(g)\tilde{P}'^{-1}_s = P'_s\rho'_s(g)P'^{-1}_s + \delta''_s(g)P'_s\rho'_s(g)P'^{-1}_s$$

for all $g \in \mathrm{Gal}(\bar{\eta}_s/\eta_s)$. Then δ''_s is a 1-coboundary by the proof [11, Lemma 1.7]. Define 1-cocycles

$$\delta_s, \tilde{\delta}_s : \mathrm{Gal}(\bar{\eta}_s/\eta_s) \to \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}$$

by

$$\begin{aligned} \rho'(g) &= P'_s\rho'_s(g)P'^{-1}_s + \delta_s(g)P'_s\rho'_s(g)P'^{-1}_s, \\ \rho'(g) &= \tilde{P}'_s\rho'_s(g)\tilde{P}'^{-1}_s + \tilde{\delta}_s(g)\tilde{P}'_s\rho'_s(g)\tilde{P}'^{-1}_s \end{aligned}$$

for all $g \in \mathrm{Gal}(\bar{\eta}_s/\eta_s)$. Then we have

$$\begin{aligned} \rho'(g) &= \tilde{P}'_s\rho'_s(g)\tilde{P}'^{-1}_s + \tilde{\delta}_s(g)\tilde{P}'_s\rho'_s(g)\tilde{P}'^{-1}_s \\ &= (P'_s\rho'_s(g)P'^{-1}_s + \delta''_s(g)P'_s\rho'_s(g)P'^{-1}_s) + \tilde{\delta}_s(g)(P'_s\rho'_s(g)P'^{-1}_s + \delta''_s(g)P'_s\rho'_s(g)P'^{-1}_s) \\ &= P'_s\rho'_s(g)P'^{-1}_s + (\delta''_s(g) + \tilde{\delta}_s(g))P'_s\rho'_s(g)P'^{-1}_s. \end{aligned}$$

It follows that

$$\delta_s = \delta''_s + \tilde{\delta}_s$$

and hence δ_s and $\tilde{\delta}_s$ differ by a 1-coboundary. So the obstruction class to lifting ρ is independent of the choice of liftings P'_s of P_s.

Suppose the tuple $(\rho, (P_s)_{s\in S})$ can be lifted to a tuple $(\rho'', (P''_s)_{s\in S}) \in D^\square_S(A')$ such that $\rho'' : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A'^r)$ is a representation and

$$P''^{-1}_s\rho''|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P''_s = \rho'_s.$$

By the above discussion, to define the obstruction class to lifting ρ, we can use ρ'' instead of ρ', and use P''_s instead of P'_s. The 1-cocyle δ_s defined at the beginning then vanishes.

Conversely, suppose the obstruction class to lifting ρ vanishes. Then we can find a 1-cocycle $\psi : \pi_1(X - S, \bar{\eta}) \to \mathrm{Ad}(\rho_0) \otimes_\kappa \mathfrak{a}$ such that $\psi|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} + \delta_s$ are 1-coboundaries for all $s \in S$. Set

$$\rho'' = \rho' + \psi\rho'.$$

Then ρ'' is a representation and a lifting of ρ. Moreover, for any $g \in \mathrm{Gal}(\bar{\eta}_s/\eta_s)$, we have

$$\begin{aligned} \rho''(g) &= \rho'(g) + \psi(g)\rho'(g) \\ &= P'_s\rho'_s(g)P'^{-1}_s + \delta_s(g)P'_s\rho'_s(g)P'^{-1}_s + \psi(g)(P'_s\rho'_s(g)P'^{-1}_s + \delta_s(g)P'_s\rho'_s(g)P'^{-1}_s) \\ &= P'_s\rho'_s(g)P'^{-1}_s + (\psi(g) + \delta_s(g))P'_s\rho'_s(g)P'^{-1}_s. \end{aligned}$$

Since $\psi|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} + \delta_s$ is a 1-coboundary for each $s \in S$, $\rho''|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$ must be strictly equivalent to $P'_s \rho'_s(g) P'^{-1}_s$ relative to $A' \to A$ by [11, Lemma 1.7]. Choose $Q'_s \in \mathrm{GL}(A'^r)$ such that $Q'_s \equiv I \mod \mathfrak{a}$ and that $\rho''|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} = Q'_s P'_s \rho'_s(g) P'^{-1}_s Q'^{-1}_s$. Then $(\rho'', (Q'_s P'_s)_{s\in S})$ is a lifting of $(\rho, (P_s)_{s\in S})$. □

Lemma 2.2. *Let $A' \to A$ be an epimorphism in $\mathcal{C}_\Lambda$ such that its kernel $\mathfrak{a}$ has the property $\mathfrak{m}_{A'}\mathfrak{a} = 0$. Let $(\rho, (P_s)_{s\in S})$ be an element in $D_S^{\square}(A)$, let $\rho_s = P_s^{-1}\rho_s|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P_s$, and let $\rho'_s : \mathrm{Gal}(\bar{\eta}_s/\eta) \to \mathrm{GL}(A'^r)$ be representations such that*

$$\rho'_s \mod \mathfrak{a} = \rho_s$$

for all $s \in S$. Suppose all elements in $\mathrm{End}_{\pi_1(X-S,\bar{\eta})}(\kappa^r)$ are scalar multiplications and suppose $\det(\rho)$ can be lifted to a representation $\lambda' : \pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A'^r)$ with the property $\lambda'|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} = \det \rho'_s$. Then there exists a tuple $(\rho', (P'_s)_{s\in S})$ in $D_S^{\square}(A')$ lifting $(\rho, (P_s)_{s\in S})$ such that $P'^{-1}_s \rho'|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} P'_s = \rho'_s$ for all $s \in S$.

Proof. As in the proof of Theorem 0.9, the pairing

$$\mathcal{E}nd(\mathcal{F}_0) \times \mathcal{E}nd(\mathcal{F}_0) \to F, \quad (\phi, \psi) \mapsto \mathrm{Tr}(\psi \circ \phi)$$

induces a perfect pairing

$$H^2(X, j_*\mathcal{E}nd(\mathcal{F}_0)) \times H^0(X, j_*\mathcal{E}nd(\mathcal{F}_0)(1)) \to \kappa.$$

If all elements in $\mathrm{End}_{\pi_1(X-S,\bar{\eta})}(\kappa^r)$ are scalar multiplications, then we have

$$\kappa \cong \mathrm{End}(\mathcal{F}_0) \cong H^0(X, j_*\mathcal{E}nd(\mathcal{F}_0)).$$

So the morphism

$$\kappa \to \mathcal{E}nd(\mathcal{F}_0), \quad a \mapsto a\mathrm{Id}$$

induces an isomorphism

$$H^0(X, \kappa) \cong H^0(X, j_*\mathcal{E}nd(\mathcal{F}_0)).$$

This implies that $\mathrm{Tr} : \mathcal{E}nd(\mathcal{F}_0) \to \kappa$ induces an isomorphism

$$H^2(X, j_*\mathcal{E}nd(\mathcal{F}_0)) \xrightarrow{\cong} H^2(X, \kappa).$$

By Lemma 2.1 (ii), this last isomorphism maps the obstruction class to lifting ρ to the obstruction class to lifting $\det(\rho)$. By our assumption, there is no obstruction to lifting $\det(\rho)$. It follows from Lemma 2.1 (i) that there is no obstruction to lifting $(\rho, (P_s)_{s\in S})$ with the prescribed local data (ρ'_s). □

Theorem 2.3. *Suppose $X = \mathbb{P}^1$ and all elements in the set $\mathrm{End}_{\pi_1(X-S,\bar{\eta})}(\kappa^r)$ are scalar multiplications. Then the morphism of functors*

$$D_S^{\square,\lambda} \to \prod_{s\in S} D_s^{\square,\lambda}, \quad (\rho, (P_s)_{s\in S}) \mapsto (P_s^{-1}\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P_s)_{s\in S}$$

is smooth.

Proof. Keep the notation in Lemma 2.2, and suppose furthermore that $X = \mathbb{P}^1$, $\det(\rho) = \lambda$ and $\det(\rho'_s) = \lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$ so that $(\rho, (P_s)_{s\in S})$ is an element of $D_S^{\square,\lambda}(A)$ and $(\rho'_s)_{s\in S}$ is an element in $\prod_{s\in S} D_s^{\square,\lambda}(A')$. There is no obstruction to lifting λ. By Lemma 2.2, we can lift $(\rho, (P_s)_{s\in S})$ to $(\rho', (P'_s)_{s\in S})$ such that $P_s'^{-1}\rho'|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P'_s = \rho'_s$. We have

$$\det(\rho')|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} = \det(\rho'_s) = \lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}.$$

In particular, $\lambda^{-1}\det(\rho')$ is unramified at $s \in S$. It is also unramified outside S. As $\mathbb{P}^1$ is simply connected, we must have $\lambda^{-1}\det(\rho') = 1$. Hence $(\rho', (P'_s)_{s\in S})$ is an element in $D_S^{\square,\lambda}(A')$. This proves the morphism $D_S^{\square,\lambda} \to \prod_{s\in S} D_s^{\square,\lambda}$ is smooth. □

We are now ready to prove Proposition 0.1.

Proof of Proposition 0.1. First note that in the case $X = \mathbb{P}^1$, the functors F and F^λ coincide. Indeed, for any $(\rho, (P_s)_{s\in S})$ in $F(A)$, we have $P_s^{-1}\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}P_s = \rho_\Lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$. It follows that $\det(\rho)|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} = \lambda|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$. As in the proof of Theorem 2.3, this implies that $\det(\rho) = \lambda$ by the fact that $\mathbb{P}^1$ is simply connected. So $(\rho, (P_s)_{s\in S}) \in F^\lambda(A)$. By Theorem 2.3, the canonical morphism $D_S^{\square,\lambda} \to \prod_{s\in S} D_s^{\square,\lambda}$ is smooth. As a fiber of this morphism, the functor F is smooth. Combined with Proposition 1.1, we have

$$\begin{aligned}
&\dim F(\kappa[\epsilon]) \\
=\ &\dim D_S^{\square,\lambda}(\kappa[\epsilon]) - \sum_{s\in S} \dim D_s^{\square,\lambda}(\kappa[\epsilon]) \\
=\ &-\chi(X - S, \mathcal{E}nd^{(0)}(\mathcal{F}_0)) + |S|r^2 - 1 - \sum_{s\in S}\Big(-\chi(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)) + r^2 - 1\Big) \\
=\ &-\chi(X - S, \mathcal{E}nd^{(0)}(\mathcal{F}_0)) + \sum_{s\in S}\chi(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)) + |S| - 1.
\end{aligned}$$

Let Δ be the mapping cone of the canonical morphism $j_*\mathcal{E}nd^{(0)}(\mathcal{F}_0) \to Rj_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)$. Note that $\mathcal{H}^i(\Delta) = 0$ for $i \neq 1$, $\mathcal{H}^1(\Delta)$ is a skyscraper sheaf supported on S, and

$$H^1(X, \Delta) = \bigoplus_{s\in S} H^1(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)).$$

We thus have

$$\begin{aligned}
&\chi(X - S, \mathcal{E}nd^{(0)}(\mathcal{F}_0)) \\
=\ &\chi(X, Rj_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)) \\
=\ &\chi(X, j_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)) + \chi(X, \Delta) \\
=\ &\chi(X, j_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)) - \sum_{s\in S}\dim H^1(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)) \\
=\ &\chi(X, j_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)) + \sum_{s\in S}\Big(\chi(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)) - \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0))\Big).
\end{aligned}$$

Here we use the fact that $H^i(\mathrm{Gal}(\bar{\eta}_s/\eta_s), -) = 0$ for $i \geq 2$. So we have

$$\begin{aligned} & -\chi(X - S, \mathcal{E}nd^{(0)}(\mathcal{F}_0)) + \sum_{s\in S} \chi(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)) \\ = \quad & -\chi(X, j_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)) + \sum_{s\in S} \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)) \end{aligned}$$

Therefore

$$\dim F(\kappa[\epsilon]) = -\chi(X, j_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)) + \sum_{s\in S} \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)) + |S| - 1.$$

As $\mathrm{Ad}(\rho_0) \cong \mathrm{Ad}^{(0)}(\rho_0) \oplus \kappa$, we have

$$\dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_0)) = \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}^{(0)}(\rho_0)) + 1.$$

It follows that

$$\dim F(\kappa[\epsilon]) = -\chi(X, j_*\mathcal{E}nd^{(0)}(\mathcal{F}_0)) + \sum_{s\in S} \dim H^0(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_0)) - 1.$$

□

3 Proof of Proposition 0.5

Lemma 3.1. *Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be a lisse torsion free $\mathcal{O}$-sheaf on $X - S$, let $\mathcal{F}_i = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^{i+1}$ (resp. $\mathcal{F}'_i = \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^{i+1}$) be the locally free lisse sheaf of $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}^{i+1}$-modules on $X - S$ corresponding to $\mathcal{F}$ (resp. $\mathcal{F}'$), and let $\mathcal{F}_E = \mathcal{F} \otimes_{\mathcal{O}} E$ (resp. $\mathcal{F}'_E = \mathcal{F}' \otimes_{\mathcal{O}} E$) be the lisse E-sheaf corresponding to $\mathcal{F}$ (resp. $\mathcal{F}'$). Fix a uniformizer π of $\mathcal{O}$.*

(i) We have a canonical exact sequence

$$0 \to \mathrm{Hom}(\mathcal{F}, \mathcal{F}') \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}_{\mathcal{O}} \to \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}'_0) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}')_\pi \to 0,$$

where $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}')_\pi$ is the kernel of the endomorphism on $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}')$ defined by multiplication by π.

(ii) Suppose $\mathrm{End}(\mathcal{F}_0)$ consists of scalar multiplications, and suppose $\mathcal{F}_E \cong \mathcal{F}'_E$ and $\mathcal{F}_0 \cong \mathcal{F}'_0$. Then we have an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}_{\mathcal{O}} \cong \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}'_0).$$

(iii) Suppose $\mathrm{End}(\mathcal{F}_0)$ consists of scalar multiplications. Then $\mathrm{End}(\mathcal{F}_E)$ consists of scalar multiplications.

Proof.

(i) Let $\mathcal{G}$ be a lisse torsion free $\mathcal{O}$-sheaf on $X - S$, and let $\mathcal{G}_i = \mathcal{G} \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^{i+1}$. Then we have an exact sequence

$$0 \to \mathcal{G}_{i-1} \stackrel{[\pi]}{\to} \mathcal{G}_i \to \mathcal{G}_0 \to 0$$

for each $i \geq 1$, where $[\pi] : \mathcal{G}_{i-1} \to \mathcal{G}_i$ is induced by the morphism $\pi : \mathcal{G}_i \to \mathcal{G}_i$ defined by multiplication by π and note that $\mathcal{G}_i/\ker \pi \cong \mathcal{G}_{i-1}$ since $\mathcal{G}_i$ is a locally free lisse sheaf of $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}^{i+1}$-modules. This gives rise to an exact sequence

$$H^0(X - S, \mathcal{G}_{i-1}) \stackrel{[\pi]}{\to} H^0(X - S, \mathcal{G}_i) \to H^0(X - S, \mathcal{G}_0) \to H^1(X - S, \mathcal{G}_{i-1}) \stackrel{[\pi]}{\to} H^1(X - S, \mathcal{G}_i).$$

Taking $\varprojlim_i$, we get an exact sequence

$$H^0(X-S,\mathcal{G}) \xrightarrow{\pi} H^0(X-S,\mathcal{G}) \to H^0(X-S,\mathcal{G}_0) \to H^1(X-S,\mathcal{G}) \xrightarrow{\pi} H^1(X-S,\mathcal{G}).$$

It induces an exact sequence

$$0 \to H^0(X-S,\mathcal{G})/\pi H^0(X-S,\mathcal{G}) \to H^0(X-S,\mathcal{G}_0) \to H^1(X-S,\mathcal{G})_\pi \to 0,$$

where $H^1(X-S,\mathcal{G})_\pi$ is the kernel of multiplication by π on $H^1(X-S,\mathcal{G})$.

Let's apply the above result to $\mathcal{G} = \mathcal{H}om(\mathcal{F},\mathcal{F}')$. We have $\mathcal{E}xt^q(\mathcal{F},\mathcal{F}')=0$ for $q\geq 1$ since $\mathcal{F}$ and $\mathcal{F}'$ are lisse. So the spectral sequence

$$H^p(X-S,\mathcal{E}xt^q(\mathcal{F},\mathcal{F}')) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F},\mathcal{F}')$$

degenerates. This gives

$$H^0(X-S,\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F},\mathcal{F}'), \quad H^1(X-S,\mathcal{G}) \cong \mathrm{Ext}^1(\mathcal{F},\mathcal{F}').$$

Our assertion follows.

(ii) Note that $\mathrm{Hom}(\mathcal{F},\mathcal{F}')$ is a finitely generated torsion free $\mathcal{O}$-module. So we have

$$\dim_E(\mathrm{Hom}(\mathcal{F},\mathcal{F}')\otimes_{\mathcal{O}} E) = \dim_{\mathcal{O}/\mathfrak{m}_{\mathcal{O}}}(\mathrm{Hom}(\mathcal{F},\mathcal{F}')\otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}_{\mathcal{O}}).$$

We have

$$\mathrm{Hom}(\mathcal{F}_E,\mathcal{F}'_E) \cong \mathrm{Hom}(\mathcal{F},\mathcal{F}')\otimes_{\mathcal{O}} E,$$

and by (i), we have

$$\dim_{\mathcal{O}/\mathfrak{m}_{\mathcal{O}}}(\mathrm{Hom}(\mathcal{F},\mathcal{F}')\otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}_{\mathcal{O}}) \leq \dim_{\mathcal{O}/\mathfrak{m}_{\mathcal{O}}}\mathrm{Hom}(\mathcal{F}_0,\mathcal{F}'_0).$$

So we have

$$\dim_E\mathrm{Hom}(\mathcal{F}_E,\mathcal{F}'_E) \leq \dim_{\mathcal{O}/\mathfrak{m}_{\mathcal{O}}}\mathrm{Hom}(\mathcal{F}_0,\mathcal{F}'_0).$$

If $\mathrm{End}(\mathcal{F}_0)$ consists of scalar multiplications, then since $\mathcal{F}_0\cong\mathcal{F}'_0$, we have $\dim_{\mathcal{O}/\mathfrak{m}_{\mathcal{O}}}\mathrm{Hom}(\mathcal{F}_0,\mathcal{F}'_0)=1$. So $\dim_E\mathrm{Hom}(\mathcal{F}_E,\mathcal{F}'_E)\leq 1$. Note that $\dim_E\mathrm{End}(\mathcal{F}_E)\geq 1$ since scalar multiplications form a one dimensional subspace of $\mathrm{End}(\mathcal{F}_E)$. As $\mathcal{F}_E\cong\mathcal{F}'_E$, we must have $\dim_E\mathrm{Hom}(\mathcal{F}_E,\mathcal{F}'_E)\geq 1$. We thus get $\dim_E\mathrm{Hom}(\mathcal{F}_E,\mathcal{F}'_E)=1$. Then monomorphism $\mathrm{Hom}(\mathcal{F},\mathcal{F}')\otimes_{\mathcal{O}}\mathcal{O}/\mathfrak{m}_{\mathcal{O}}\to\mathrm{Hom}(\mathcal{F}_0,\mathcal{F}'_0)$ in (i) between one dimensional vector spaces is necessarily an isomorphism.

(iii) We have seen in the proof of (ii) that $\dim_E\mathrm{End}(\mathcal{F}_E)=1$. So $\mathrm{End}(\mathcal{F}_E)$ coincides with the one dimensional subspace consisting of scalar multiplications. □

Proof of Proposition 0.5 (i). Let's prove that we can take $V=\mathcal{F}^{\mathrm{rig}}$. By [12, 7.1.10], a point in $\mathcal{F}^{\mathrm{rig}}$ corresponds to a local $\mathcal{O}$-algebra homomorphism $R(\rho_{\mathcal{O}})\to\mathcal{O}'$, where $\mathcal{O}'$ is the integer ring of a finite extension E' of E. Let $(\rho',(P'_s)_{s\in S})$ be the tuple obtained by pushing-forward the universal tuple $(\rho_{\mathrm{univ}},(P_{\mathrm{univ},s})_{s\in S})$ through this homomorphism. Then $\rho':\pi_1(X-S,\bar\eta)\to\mathrm{GL}(\mathcal{O}'^r)$ is a continuous homomorphism, $P'_s\in\mathrm{GL}(\mathcal{O}'^r)$, and

$$\rho' \mod \mathfrak{m}_{\mathcal{O}'}=\rho_0, \quad P'^{-1}_s\rho'|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}P'_s=\rho_{\mathcal{O}}|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}$$

for all $s \in S$. In particular, we have $\rho'|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_E|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$. Since ρ_E is physically rigid, by taking E' sufficiently large, we have $\rho' \cong \rho_E$ as E'-representations of $\pi_1(X-S, \bar{\eta})$. Let $\mathcal{F}'$ (resp. $\mathcal{F}$) be the lisse $\mathcal{O}'$-sheaf on $X - S$ associated to the representation ρ' (resp. $\rho_{\mathcal{O}}$). Then we have $\mathcal{F}'_{E'} \cong \mathcal{F}_{E'}$ and $\mathcal{F}'_0 \cong \mathcal{F}_0$. Applying Lemma 3.1 (ii) (with $\mathcal{O}$ replaced by $\mathcal{O}'$), we see that the canonical map

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') \to \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}'_0)$$

is surjective. So the isomorphism $\mathcal{F}'_0 \cong \mathcal{F}_0$ in $\mathrm{Hom}(\mathcal{F}_0, \mathcal{F}'_0)$ can be lifted $\mathrm{Hom}(\mathcal{F}, \mathcal{F}')$, which is necessarily an isomorphism. (A homomorphism $\mathcal{F}' \to \mathcal{F}$ is an isomorphism if and only if it induces an isomorphism $\mathcal{F}'_0 \stackrel{\cong}{\to} \mathcal{F}_0$.) Hence there exists $P \in \mathrm{GL}(\mathcal{O}'^r)$ such that $P \equiv I \mod \mathfrak{m}_{\mathcal{O}'}$ and $P^{-1}\rho' P = \rho_{\mathcal{O}}$. □

The following is a theorem of Maranda ([7, Theorem 30.14]). We give a proof for completeness.

Lemma 3.2. *Let H be finite group and let $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$ be the integer ring of $\overline{\mathbb{Q}}_\ell$. There exists an integer n such that for any free $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$-modules M and N of finite ranks with H-action, and any isomorphism $f_0 : M/\ell^{n+1}M \to N/\ell^{n+1}N$ of $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}[H]$-modules, there exists an isomorphism $f : M \to N$ of $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}[H]$-modules such that f and f_0 induce the same isomorphism $M/\ell M \to N/\ell N$.*

Proof. Write $|H| = m\ell^n$, where m is relatively prime to ℓ. We can lift $f_0 : M/\ell^{n+1}M \to N/\ell^{n+1}N$ to a homomorphism $f : M \to N$ of $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$-modules since M is a free $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$-module. For any $g \in H$ and $x \in M$, we have $(gfg^{-1} - f)(x) \in \ell^{n+1}N$. Define $F_g \in \mathrm{Hom}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}(M, N)$ by

$$gfg^{-1} - f = \ell^{n+1}F_g.$$

Then we have

$$F_{g_1g_2} = g_1F_{g_2}g_1^{-1} + F_{g_1}.$$

So $F : H \to \mathrm{Hom}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}(M, N)$ is a 1-cocyle, where $\mathrm{Hom}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}(M, N)$ is provided with the H-action

$$H \times \mathrm{Hom}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}(M, N) \to \mathrm{Hom}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}(M, N), \quad (g, \phi) \mapsto g\phi g^{-1}.$$

By [18, Corollary 1 in VIII §2], $H^1(H, \mathrm{Hom}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}(M, N))$ is annihilated by $|H| = m\ell^n$. Since m is a unit in $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$, $\ell^n F$ must be a 1-coboundary. Choose $\delta \in \mathrm{Hom}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}}(M, N)$ such that

$$\ell^n F_g = g\delta g^{-1} - \delta$$

for all $g \in G$. We then have

$$g(f - \ell\delta)g^{-1} = f - \ell\delta$$

for all $g \in H$. So $f - \ell\delta : M \to N$ is homomorphism of $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}[H]$-modules. Modulo ℓ, it coincides with the isomorphism f_0. So $f - \ell\delta : M \to N$ is an isomorphism. □

Proof of Proposition 0.5 (ii). Let $H = \mathrm{im}(\rho_E)$, and let n be the integer satisfying the conclusion of Lemma 3.2. It is known that $R(\rho_{\mathcal{O}}) \cong \mathcal{O}[[t_1, \ldots, t_d]]/I$ for some d and some ideal I of $\mathcal{O}[[t_1, \ldots, t_d]]$, and $\mathfrak{F}^{\mathrm{rig}}$ is defined to be the zero set of the ideal I in the open unit polydisc $D(0,1)^d$. Consider the analytic subdomain V of $\mathfrak{F}^{\mathrm{rig}}$ defined by

$$|t_i| \leq |\ell^{n+1}| \quad (i = 1, \ldots, n+1).$$

For any $t \in V$, let $\varphi_t : R(\rho_{\mathcal{O}}) \to \mathcal{O}_{\overline{\mathbb{Q}}_\ell}$ be the corresponding local homomorphism of $\mathcal{O}$-algebras. We then have $\varphi_t \equiv \varphi_0 \mod \ell^{n+1}\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$, where $\varphi_0 : R(\rho_{\mathcal{O}}) \to \mathcal{O}$ is the homomorphism corresponding to the point t_0 in $\mathfrak{F}^{\mathrm{rig}}$. Let $(\rho_t, (P_{t,s})_{t\in S})$ be the tuple obtained by pushing-forward the universal tuple $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ through φ_t. Then we have

$$\rho_t \equiv \rho_{\mathcal{O}} \mod \ell^{n+1}\mathcal{O}_{\overline{\mathbb{Q}}_\ell}.$$

Moreover, we have $P_{t,s}^{-1}\rho_t|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}P_{t,s} = \rho_{\mathcal{O}}|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}$ for all $s \in S$. So $\rho_t|_{\mathrm{Gal}(\bar\eta_s)} \cong \rho_{\mathcal{O}}|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}$. Since ρ_E is physically rigid, we have $\rho_t \cong \rho_E$. In particular, the representation $\rho_t : \pi_1(X - S, \bar\eta) \to \mathrm{GL}(\mathcal{O}_{\overline{\mathbb{Q}}_\ell})$ factors through the quotient $H = \mathrm{im}(\rho_E)$ of $\pi_1(X - S, \bar\eta)$, and ρ_t is also a representation of H. By Lemma 3.2, the identity endomorphism of

$$\rho_t \equiv \rho_{\mathcal{O}} \mod \mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_\ell}} = \rho_0$$

can be lifted to an isomorphism $\rho_t \cong \rho_{\mathcal{O}}$ of $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$-representations of H, that is, there exists $P \in \mathrm{GL}(\mathcal{O}_{\overline{\mathbb{Q}}_\ell})$ such that $P^{-1}\rho_t P = \rho_{\mathcal{O}}$. □

4 Proof of Proposition 0.8

The second statement follows from [12, Lemma 7.1.9]. Let's prove the first statement. The argument is the same as B. Conrad's lecture note [6, §7], and we include it for completeness. Since $R(\rho_{\mathcal{O}})$ is a noetherian ring, so is $R(\rho_{\mathcal{O}})^\wedge_{\mathfrak{p}_t}$. Since $\varphi_t : R(\rho_{\mathcal{O}}) \to \mathcal{O}$ is an $\mathcal{O}$-algebra homomorphism, the prime ideal $\mathfrak{p}_t = \ker\varphi_t$ lies in the generic fiber of $\operatorname{Spec} R(\rho_{\mathcal{O}}) \to \operatorname{Spec}\mathcal{O}$. Let $\mathfrak{m}_t$ be the kernel of the homomorphism

$$\varphi_t \otimes \mathrm{id}_E : R(\rho_{\mathcal{O}}) \otimes_{\mathcal{O}} E \to E.$$

Then $\mathfrak{m}_t$ is a maximal ideal of $R(\rho_{\mathcal{O}})\otimes_{\mathcal{O}}E$ with residue field E. Since $\operatorname{Spec}(R(\rho_{\mathcal{O}})\otimes_{\mathcal{O}}E) \to \operatorname{Spec} R(\rho_{\mathcal{O}})$ is an open immersion, we have

$$R(\rho_{\mathcal{O}})_{\mathfrak{p}_t} \cong (R(\rho_{\mathcal{O}}) \otimes_{\mathcal{O}} E)_{\mathfrak{m}_t}.$$

It follows that $R(\rho_{\mathcal{O}})^\wedge_{\mathfrak{p}_t}$ is a complete local noetherian E-algebra with residue field E.

Let A be an Artinian local E-algebra with residue field E and let $(\rho, (P_s)_{s\in S})$ be a tuple such that $\rho : \pi_1(X - S, \bar\eta) \to \mathrm{GL}(A^r)$ is a representation, $P_s \in \mathrm{GL}(A^r)$ and

$$\rho \mod \mathfrak{m}_A = \rho_t, \quad P_s \mod \mathfrak{m}_A = P_{t,s}, \quad P_s^{-1}\rho|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}P_s = \rho_t|_{\mathrm{Gal}(\bar\eta_s/\eta_s)}.$$

Note that the maximal ideal $\mathfrak{m}_A$ of A consists of nilpotent elements of A and we have $A = E \oplus \mathfrak{m}_A$. For any finite family of element $x_1, \ldots, x_n$ in $\mathfrak{m}_A$, the subring $\mathcal{O}[x_1, \ldots, x_n]$ of A is finitely generated as an $\mathcal{O}$-module. Note that $\mathfrak{m}_A$ is finite dimensional as a E-vector space. Choose a basis $\{e_1, \ldots, e_m\}$ of $\mathfrak{m}_A$ over E. The set of matrices in $\mathrm{GL}(A^r)$ with entries lying in $\mathcal{O}[e_1, \ldots, e_m]$ is open. So there exists an open subgroup G of $\pi_1(X - S, \bar\eta)$ such that the entries of $\rho(g) \in \mathrm{GL}(A^r)$ lie in $\mathcal{O}[e_1, \ldots, e_m]$ for all $g \in G$. The family of left cosets $\pi_1(X - S, \bar\eta)/G$ is finite. Choose finitely many elements $g_1, \ldots, g_k$ in $\pi_1(X - S, \bar\eta)$ so that any left coset is of the form g_iG for some i. We have $\rho_t(g) \in \mathrm{GL}(\mathcal{O}^r)$ for any $g \in \pi_1(X - S, \bar\eta)$. Since $\rho \mod \mathfrak{m}_A = \rho_t$, the entries of the matrices $\rho(g_1), \ldots, \rho(g_k)$ lie in the subset $\mathcal{O} \oplus \mathfrak{m}_A$ of A. Since $P_s \mod \mathfrak{m}_A = P_{t,s}$ and $P_{t,s} \in \mathrm{GL}(\mathcal{O}^r)$ for all $s \in S$, the entries of

P_s also lie in this subset. We can choose a finite subset $\{x_1, \ldots, x_n\}$ of $\mathfrak{m}_A$ containing $e_1, \ldots, e_m$ so that the entries of the finite family of matrices $\rho(g_1), \ldots, \rho(g_k), P_s(s \in S)$ all lie in $\mathcal{O}[x_1, \ldots, x_n]$. Set $A_{\mathcal{O}} = \mathcal{O}[x_1, \ldots, x_n]$. Then ρ induces a continuous homomorphism $\pi_1(X - S, \bar{\eta}) \to \mathrm{GL}(A_{\mathcal{O}}^r)$ which we still denote by ρ.

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. Note that $A_{\mathcal{O}}$ is a complete noetherian local ring and its maximal ideal consists of elements of the form $\sum_{i_1,\ldots,i_n} a_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$ with $a_{0\ldots 0} \in \mathfrak{m}$ and $a_{i_1\ldots i_n} \in \mathcal{O}$. The $\mathcal{O}$-algebra $A_{\mathcal{O}}$ is finitely generated as an $\mathcal{O}$-module, and its topology coincides with the $\mathfrak{m}$-adic topology. For each i, $A_{\mathcal{O}}/\mathfrak{m}^i A_{\mathcal{O}}$ is an Artinian local $\mathcal{O}$-algebra with residue field κ, and modulo $\mathfrak{m}^i A_{\mathcal{O}}$, the tuple $(\rho, (P_s)_{s\in S})$ is an object in $F(A_{\mathcal{O}}/\mathfrak{m}^i A_{\mathcal{O}})$. So there exists a unique $\mathcal{O}$-algebra homomorphism $R(\rho_{\mathcal{O}}) \to A_{\mathcal{O}}/\mathfrak{m}_{\mathcal{O}}^i A_{\mathcal{O}}$ which brings the universal data $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ to $(\rho, (P_s)_{s\in S}) \mod \mathfrak{m}^i A_{\mathcal{O}}$. These homomorphisms form a compatible family for various i. So we have a local $\mathcal{O}$-algebra homomorphism $\psi : R(\rho_{\mathcal{O}}) \to A_{\mathcal{O}}$ which brings $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ to $(\rho, (P_s)_{s\in S})$. Moreover, such a local $\mathcal{O}$-algebra homomorphism is unique. Since $(\rho, (P_s)_{s\in S})$ lies over $(\rho_t, (P_{t,s})_{s\in S})$, the following diagram commutes:

$$\begin{array}{ccc} R(\rho_{\mathcal{O}}) & \stackrel{\psi}{\to} & A_{\mathcal{O}} \\ {\scriptstyle \varphi_t} \downarrow & \swarrow & \\ \mathcal{O} & & \end{array}$$

where the oblique arrow is the local homomorphism

$$A_{\mathcal{O}} = \mathcal{O}[x_1, \ldots, x_n] \to \mathcal{O}, \quad f(x_1, \ldots, x_n) \mapsto f(0, \ldots, 0).$$

It induces a commutative diagram

$$\begin{array}{ccc} R(\rho_{\mathcal{O}}) \otimes_{\mathcal{O}} E & \stackrel{\psi\otimes \mathrm{id}_E}{\to} & A = E \oplus \mathfrak{m}_A \\ {\scriptstyle \varphi_t \otimes \mathrm{id}_E} \downarrow & \swarrow & \\ E. & & \end{array}$$

It follows that the maximal idea of A lies above $\mathfrak{m}_t = \ker(\varphi_t \otimes \mathrm{id}_E)$. Passing to localization and completion, the homomorphism $\psi \otimes \mathrm{id}_E$ induces a local E-algebra homomorphism

$$R(\rho_{\mathcal{O}})^{\wedge}_{\mathfrak{p}_t} \cong (R(\rho_{\mathcal{O}}) \otimes_{\mathcal{O}} E)^{\wedge}_{\mathfrak{m}_t} \to A$$

which brings the universal data $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ to $(\rho, (P_s)_{s\in S})$.

Suppose we enlarge the set $\{x_1, \ldots, x_n\}$ to $\{x_1, \ldots, x_n, x_{n+1}, \ldots x_{n+m}\}$, and let $\tilde{A}_{\mathcal{O}} = \mathcal{O}[x_1, \ldots, x_{n+m}]$. Repeating the above argument, we get a unique local $\mathcal{O}$-algebra homomorphism $\tilde{\psi} : R(\rho_{\mathcal{O}}) \to \tilde{A}_{\mathcal{O}}$ which brings $(\rho_{\mathrm{univ}}, (P_{\mathrm{univ},s})_{s\in S})$ to $(\rho, (P_s)_{s\in S})$. By the uniqueness, the following diagram commutes:

$$\begin{array}{ccc} R(\rho_{\mathcal{O}}) & & \\ {\scriptstyle \psi} \downarrow & \searrow {\scriptstyle \tilde{\psi}} & \\ A_{\mathcal{O}} & \hookrightarrow & \tilde{A}_{\mathcal{O}}. \end{array}$$

This shows that the homomorphism $R(\rho_{\mathcal{O}})^{\wedge}_{\mathfrak{p}_t} \to A$ constructed above does not change if we enlarge the set $\{x_1, \ldots, x_n\}$.

Suppose we have a local E-algebra homomorphism

$$R(\rho_{\mathcal{O}})^{\wedge}_{\mathfrak{p}_t} \to A$$

which brings $(\rho_{\text{univ}}, (P_{\text{univ},s})_{s\in S})$ to $(\rho, (P_s)_{s\in S})$, and let's prove it must coincide with the one constructed above. It induces an $\mathcal{O}$-algebra homomorphism

$$\theta : R(\rho_{\mathcal{O}}) \to A = E \oplus \mathfrak{m}_A$$

which brings $(\rho_{\text{univ}}, (P_{\text{univ},s})_{s\in S})$ to $(\rho, (P_s)_{s\in S})$ such that the maximal ideal $\mathfrak{m}_A$ lies above the ideal $\mathfrak{p}_t$. Let $a_1, \ldots, a_n \in \mathfrak{p}_t$ be a family of generators for the prime ideal $\mathfrak{p}_t$. We have $\theta(a_i) \in \mathfrak{m}_A$. Choose a large N such that $\theta(a_i)^N = 0$ for all i. Since the $\mathcal{O}$-algebra structure homomorphism $\mathcal{O} \to R(\rho_{\mathcal{O}})$ is a section of the homomorphism $\varphi_t : R(\rho_{\mathcal{O}}) \to \mathcal{O}$, we have a decomposition

$$R(\rho_{\mathcal{O}}) = \mathcal{O} \oplus \mathfrak{p}_t.$$

Using this decomposition, we can write any element $a \in R$ in the form

$$a = \sum_{0\leq i_1,\ldots,i_n\leq N-1} c_{i_1\ldots i_m} a_1^{i_1} \cdots a_n^{i_n} + \sum_{i=1}^{n} \delta_i a_i^N$$

with $c_{i_1\ldots i_n} \in \mathcal{O}$ and $\delta_i \in R(\rho_{\mathcal{O}})$. We then have

$$\theta(a) = \sum_{0\leq i_1,\ldots,i_n\leq N-1} c_{i_1\ldots i_n} \theta(a_1)^{i_1} \cdots \theta(a_n)^{i_n}.$$

Let $x_i = \theta(a_i)$. Then $R(\rho_{\mathcal{O}}) \to A$ factors through an $\mathcal{O}$-algebra homomorphism $R(\rho_{\mathcal{O}}) \to A_{\mathcal{O}} = \mathcal{O}[x_1, \ldots, x_n]$ which which brings $(\rho_{\text{univ}}, (P_{\text{univ},s})_{s\in S})$ to $(\rho, (P_s)_{s\in S})$. By our previous discussion, this homomorphism must coincide with ψ. Our assertion follows.

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