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(d, d')-elliptic curves of genus two

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Abstract. We study stable curves of arithmetic genus 2 which admit two morphisms of finite degree d, resp. d', onto smooth elliptic curves, with particular attention to the case d prime.

1. Introduction

In this paper we consider stable curves of arithmetic genus two which admit a (d, d')-elliptic configurations, namely two morphisms of finite degree d, resp. d', onto smooth elliptic curves D and E:

$$\begin{array}{ccc} & C & \\ {}^{f}\swarrow_{d:1} & & {}_{d':1}\searrow^{g} \\ E & & D \end{array} .$$

A curve of genus two is called a (d, d')-elliptic curve if it admits a (d, d')-elliptic configuration. When $d=2$ and $d'=3$ we use the terminology bi-tri-elliptic curve.

The study of genus two smooth curves with a degree d morphism onto an elliptic curve, i.e. genus two d-elliptic curves, goes back to the 19th century, where the attention was on the analysis of elliptic integrals (cf. the last chapter of [Krazer]).

More recently, Frey and Kani revived the subject in [FK91]; then in [FK09], [Kan97], [Kani03], [Kani14] and [Kani16] the arithmetic properties of d-elliptic curves of genus 2 were studied in detail, also providing existence results.

Our starting point for the study of (d, d')-elliptic is a classical construction of the Jacobian of a d-elliptic curve of genus two described by Frey and Kani in [FK91]. Since stable (d, d')-elliptic curves of arithmetic genus two are automatically of compact type, i.e., they have compact Jacobian (Corollary 3.4), in §2 we recall the Frey-Kani construction, noting that it extends to curves of compact type.

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In §3 we study (d,d')-elliptic curves, with particular attention to the case d prime. In Theorem 3.14 we give a classification of such curves and in §3.3 we show that for every pair of integers $d,d'>1$ there exists a smooth (d,d')-elliptic curve of genus two.

The original motivation for this article was the study of bi-tri-elliptic configurations, which parametrise certain strata in the boundary of the moduli space of stable Godeaux surfaces (see [FPR17]). Thus we describe the geometry of bi-tri-elliptic configurations in a little more detail in the last section.

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2. *d*-elliptic curves of genus two

Here we recall and slightly refine some results from [FK91, §1], where the focus is on smooth curves and on the case d odd (see below).

2.1. Set-up and preliminaries

We work over an algebraically closed field $\mathbb{K}$ whose characteristic does not divide the degree d of the finite morphisms that we consider. Throughout all this section C is a stable curve of genus two and $J=J(C)$ is the Jacobian of C.

Definition 2.1. Let $d\geq 2$ be an integer. We say that C is *d-elliptic* if there exists a finite degree d morphism $f\colon C\to E$ such that E is a smooth curve of genus 1 and f does not factor through an étale cover of E; we call f a *d-elliptic map*. Sometimes, a d-elliptic map is called an “elliptic subcover” and the curve C is said to have an “elliptic differential” (cf. [Kan97]); our choice of terminology is due to the fact that we wish to emphasize the degree d of the map. For $d=2,3$, the curve C is also called *bi-elliptic*, resp. *tri-elliptic*.

An isomorphism of d-elliptic curves $f_i\colon C_i\to E_i$, $i=1,2$, is a pair of isomorphisms $\varphi\colon C_1\to C_2$ and $\bar\varphi\colon E_1\to E_2$ such that $f_2\circ\varphi=\bar\varphi\circ f_1$.

For an abelian variety A we denote by $A[d]$ its subgroup of d-torsion points. If A is principally polarised then there is a non-degenerate alternating pairing $e_d\colon A[d]\times$

$A[d]\to\mu_d$ (where μ_d denotes the d-th roots of unity) called Weil pairing (or Riemann form [Mum74, Chapter IV, §20]).

If A' is an abelian variety, then we call a group homomorphism $\alpha\colon A[d]\to A'[d]$ *anti-symplectic* if for every $P,Q\in A[d]$ one has:

$$e_d(\alpha(P),\alpha(Q))=e_d(P,Q)^{-1},$$

or, equivalently, if the graph of α is an isotropic subgroup of $(A\times A')[d]$.

2.2. The Frey-Kani construction

Now assume that C is a stable genus two curve of compact type, i.e., it is either smooth or the union of two elliptic curves intersecting in one point. Notice that the Jacobian $J=J(C)$ is a principally polarised abelian surface.

Let $f\colon C\to E$ be a d-elliptic map on C. The pull back map $f^*\colon E\to J$ is injective, hence the norm map $f_*\colon J\to E$ has connected kernel E'. We denote by $h\colon E\times E'\to J$ the map induced by f^* and by the inclusion $E'\hookrightarrow J$. Since the composition $f_*f^*\colon E\to E$ is multiplication by d, the abelian subvarieties E' and f^*E of J intersect in $E[d]$ and we have a tower of isogenies

$$E\times E' \xrightarrow[d^2:1]{h} J \xrightarrow[d^2:1]{h'} E\times E' \ , \tag{1}$$

whose composition is multiplication by d and h' is determined by this property. Composing the Abel-Jacobi map $C\hookrightarrow J$ with the projection to E' we get a second d-elliptic map $f'\colon C\to E'$, which we call the *complementary* d-elliptic map. Composing h with the inclusions $E,E'\hookrightarrow J$ one sees that $h'=(f_*,f'_*)$.

The construction that follows, which we call the Frey-Kani construction, has been described in [FK91, §1] for smooth curves, but the proof works verbatim for stable curves of compact type. Therefore one has:

Proposition 2.2. *Let C be a stable d-elliptic curve of genus two of compact type, let $f\colon C\to E$ and $f'\colon C\to E'$ be complementary d-elliptic maps and let $h\colon E\times E'\to J=J(C)$ be as in (1). Then:*

(i) *there exists an anti-symplectic isomorphism $\alpha\colon E[d]\to E'[d]$ such that* $\ker h$ *is the graph H_α of α;*

(ii) *the principal polarization on J pulls back to $d(E\times\{0\}+\{0\}\times E')$.*

Notice that if $d=2$, then any isomorphism α as in Proposition 2.2 is anti-symplectic. More generally, for a prime d the number of anti-symplectic isomorphisms $E[d]\to E'[d]$ is equal to $d(d^2-1)$ (cf. [FK91]).

The above proposition has a converse (see [FK91]):

Proposition 2.3. *Let E, E' be elliptic curves and let $\alpha\colon E[d]\to E'[d]$ be an anti-symplectic isomorphism. Denote by H_α the graph of α; set $A:=(E\times E')/H_\alpha$ and denote by $h\colon E\times E'\to A$ the quotient map.*

Then

(i) *$d(E\times\{0\}+\{0\}\times E')$ descends to a principal polarization Θ on A;*

(ii) *let C be a theta-divisor on A; then C is a stable curve of genus two of compact type and the maps $f\colon C\to E$ and $f'\colon C\to E'$ induced by the natural maps $A\to E$ and $A\to E'$ are complementary d-elliptic maps;*

(iii) *if d is odd, then there is precisely one symmetric Theta-divisor on A that is linearly equivalent to $d(E\times\{0\}+\{0\}\times E')$.*

2.3. Special geometry for small d

The question of under what conditions the polarisation coming from the Frey-Kani construction is reducible has been answered by Kani in [Kan97, Theorem 3].

Here we are interested mainly in the case $d=2$; below we spell out Kani's result in this case.

Lemma 2.4. *Let A be constructed as in Proposition 2.3 for $d=2$. Then the principal polarization Θ of A is reducible if and only if there exists an isomorphism $\psi\colon E'\to E$ such that the map $E\times E'\to E\times E$ defined by $(x,y)\mapsto(x,\psi(y))$ maps H_α to the subgroup $\Delta[2]=\{(\eta,\eta)|\eta\in E[2]\}$.*

Moreover, up to isomorphism the bi-elliptic map f is given by the composition

$$C=E\times\{0\}\cup\{0\}\times E\hookrightarrow J(C)=E\times E\xrightarrow{+}E,$$

that is, it is the identity on each component of C; the complementary map f' is the identity on one component and multiplication by -1 on the other.

Proof. The first part follows from [Kan97, Theorem 3] in the special case $d=2$. Moreover, by ibid. we have the following commutative diagram:

$$\begin{array}{ccc} E\times E' & \xrightarrow{(\mathrm{id},\psi)} & E\times E \\ \downarrow h & & \downarrow q \\ A & \longrightarrow & E\times E \end{array}, \tag{2}$$

where h is the quotient map and $q(x,y)=(x+y,x-y)$.

To conclude the proof, assume now $E=E'$ and α is the identity. The map $q\colon E\times E\to E\times E$ defined by $q(x,y)=(x+y,x-y)$ has kernel $H_\alpha=\Delta[2]$, hence A is isomorphic to $E\times E$. Let $C=E\times\{0\}+\{0\}\times E$; then $q^*C=\Delta+\Delta^-$, where Δ is the diagonal and Δ^- is the antidiagonal. Since $(q^*C)^2=8$ by the pull-back formula and $q^*C(E\times\{0\}+\{0\}\times E)=4$, the divisors q^*C and $2(E\times\{0\}+\{0\}\times E)$ are algebraically equivalent by the Index Theorem, hence C is the principal polarisation of Proposition 2.3. (More precisely, since q^*C and $2(E\times\{0\}+\{0\}\times E)$ restrict to the same divisor on $E\times\{0\}$ and $\{0\}\times E$ they are actually linearly equivalent.) The final part of the statement follows. □

We close this section with an alternative description of bi-elliptic curves of genus two of compact type, which basically stems from the fact that a double cover is the quotient by an involution.

Lemma 2.5. *Let C be a genus two stable curve of compact type, let $f\colon C\to E$ and $f'\colon C\to E'$ be complementary bi-elliptic maps and let σ, resp. σ', be the involution induced by f, resp. f'. Then the group $\langle\sigma,\sigma'\rangle$ is isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2$ and $\tau:=\sigma\sigma'$ is the hyperelliptic involution.*

Proof. Let J be the Jacobian of C. The involution σ on J is induced by the involution $(x,y)\mapsto(x,-y)$ of $E\times E'$ (cf. (1)) and, similarly, σ' is induced by $(x,y)\mapsto(-x,y)$. So $\tau=\sigma\sigma'$ acts as multiplication by -1 on J and therefore, if C is smooth, is the hyperelliptic involution. If C is reducible, then τ is multiplication by -1 on both components of C. Since τ is in the center of $\mathrm{Aut}(C)$, σ and σ' commute and $\langle\sigma,\sigma'\rangle$ has order 4. □

It is well known that a normal abelian cover $X\to Y$, with X normal and Y smooth projective and simply connected, can be reconstructed from its branch data, i.e. from a certain decomposition of the branch divisor (cf. [Par91, §2], [AP12, § 1.2]). We explain this in the case at hand.

Choose points P_1,P_2 and distinct points $Q_1,Q_2,Q_3\in\mathbb{P}^1$ that are also distinct from P_1 and P_2 (P_1 and P_2 are allowed to coincide). Let $\pi\colon C\to\mathbb{P}^1$ be the bidouble cover branched on $D_1=P_2$, $D_2=P_1$ and $D_3=Q_1+Q_2+Q_3$, denote by G the Galois group of π and by $\sigma\in G$ (resp. σ' and τ) the involution the fixes the preimage of D_1 (resp. D_2, D_3). Assume first that $P_1\neq P_2$; in this case C is a smooth curve of genus two and for $i=1,2$ the quotients $E=C/\sigma$ and $E'=C/\sigma'$ are smooth curves of genus 1. The involution τ has 6 fixed points and therefore is the hyperelliptic involution.

If $P_1=P_2$, then C has a node over $P_1=P_2$ and the normalization is the bidouble cover of $\mathbb{P}^1$ with branch divisors $D_1=D_2=0$ and $D_3=P_1+Q_1+Q_2+Q_3$. So C is reducible and has two components, both isomorphic to the double cover of $\mathbb{P}^1$ branched on $P_1+Q_1+Q_2+Q_3$.

This construction is related to the construction given in Proposition 2.3 as follows.

Let $\pi\colon C\to\mathbb{P}^1$ be as above, with C smooth, and take the preimage of P_1 as the origin $0\in E$ and the preimage of P_2 as the origin $0'\in E'$. Denote by A_1, A_2, A_3 (resp. by B_1, B_2, B_3) the preimages of Q_1, Q_2, Q_3 in E (resp. in E'). Then the nonzero elements of $E[2]$ (resp. $E'[2]$) are $\eta_i{:=}A_i-0$ (resp. $\eta'_i{:=}B_i-0'$), $i=1,2,3$); we define $\alpha\colon E[2]\to E'[2]$ as the isomorphism that maps $\eta_i\to\eta'_i$.

We claim that the bi-elliptic structure on C is obtained via the Frey-Kani construction with the above choice of α, i.e., the kernel of the pull-back map $h'^*\colon E\times E'\to J{:=}J(C)$ is the graph H_α of α.

Indeed, since the kernel Γ of the pull-back map has order 4, it is enough to show that H_α is in contained in Γ. In addition one has $f^*A_i=f'^*B_i$ for $i=1,2,3$, hence we only need to show that f^*0 and f'^*0' are linearly equivalent. The divisor f^*0 is the ramification divisor of f', hence $f^*0\equiv K_C$; the same argument shows that $f'^*0'\equiv K_C$ and we are done.

The case C reducible is obvious.

Remark 2.6. For tri-elliptic curves one can apply the general theory of triple covers [Mir85] to deduce the following result [FPR17, Lemma 2.8]: a stable curve C of genus two admits a tri-elliptic map $C\to E$ such that C embeds into the symmetric square of E as a tri-section of the Albanese map $S^2E\to E$.

Note, however, that a tri-elliptic map $C\to E$ cannot be a cyclic cover, since by the Hurwitz formula it would be ramified over precisely one point and this is impossible, for instance, by [Par91, Proposition 2.1] (more generally, in [Kani03] it is proven that the Galois group of a map $C\to E$ of degree $d>2$ is trivial). So there is no elementary description of C just in terms of the ramification divisor.

Finally, notice that the genus two curve C and the degree d map $f\colon C\to E$ can be constructed as the fibre product of two covers of $\mathbb{P}^1$ (cf. the discussion of [FK09, Section 2.2]).

3. (d, d')-elliptic curves of genus two

We consider stable curves of genus two admitting two distinct maps to elliptic curves.

3.1. (d, d')-elliptic curves and configurations

Definition 3.1. Let C be a stable curve of genus two. A (d, d')-elliptic configuration (C, f, g) is a diagram

(3)
$$\begin{array}{ccccc} & & C & & \\ & \overset{f}{\swarrow} {\scriptstyle d:1} & & {\scriptstyle d':1} \overset{g}{\searrow} & \\ E & & & & D \end{array},$$

where f is a d-elliptic map and g is a d'-elliptic map such that there is no isomorphism $\psi\colon E\to D$ such that $g=\psi\circ f$. We refer to C as to a (d, d')-elliptic curve (of genus two).

An isomorphism of (d, d')-elliptic configurations is an isomorphism of diagrams like (3).

Lemma 3.2. *Let C be a d-elliptic stable curve of genus two, and let $f\colon C\to E$ be the d-elliptic map. Then C is one of the following*:

(i) *a smooth curve of genus* 2

(ii) *the union of two elliptic curves equipped with isogenies of degrees d_1, d_2 onto E, where $d=d_1+d_2$*

(iii) *a curve with one node, such that the induced map $\psi\colon C^\nu\to E$ is a degree d isogeny (C^ν being the normalization); C is obtained from C^ν by gluing the origin to a point P that generates* $\ker\psi$.

In case (iii) the d-elliptic structure on C is unique and for $d'\neq d$ there is no d'-elliptic structures on C.

Proof. Let $f\colon C\to E$ be the d-elliptic map.

The curve C cannot have rational components, nor more than one singular point, since it has a finite map onto an elliptic curve, hence the only possibilities for C are as in *(i)*, *(ii)* and *(iii)*.

To prove the last statement, assume by contradiction that there exists another d'-elliptic structure $g\colon C\to D$, let $C^\nu\to C$ be the normalization map and denote by $\psi_1\colon C^\nu\to E$ (resp. $\psi_2\colon C^\nu\to D$) the map of degree d (resp. d') induced by f (resp. g). With a suitable choice of the origins in E and D we can assume that ψ_1 and ψ_2 are isogenies.

Since ψ_1 and ψ_2 factor through $C^\nu\to C$, P belongs to $\ker\psi_1\cap\ker\psi_2$ and for $i=1,2$ ψ_i factors through the étale covers $C^\nu/\langle P\rangle\to E$ and $C^\nu/\langle P\rangle\to D$. It follows that f and g also factor through $C^\nu/\langle P\rangle\to E$ and $C^\nu/\langle P\rangle\to D$ hence, by the definition of d-elliptic curve, it follows that $C^\nu/\langle P\rangle\to E$ and $C^\nu/\langle P\rangle\to D$ are isomorphisms, hence $d'=d$ and the two d-elliptic structures differ by an isomorphism $E\to D$, a contradiction. $\square$

Remark 3.3. The above lemma can be seen as a special case of the analysis of moduli functors of normalized genus two covers of elliptic curves and their compactification given in [FK09].

As an immediate consequence of Lemma 3.2 we obtain the following

Corollary 3.4. *If C is a (d,d')-elliptic stable curve of genus two, then it is of compact type.*

Remark 3.5. By the Frey-Kani construction given in §2, if C is of compact type and has a d-elliptic map $f\colon C \to E$ then, if we denote $f'\colon C \to E'$ the complementary d-elliptic map, (C, f, f') is a (d,d)-elliptic configuration. We refer to this as to the *trivial (d,d)-elliptic configuration.*

3.2. Existence of (d, d')-elliptic curves

Frey and Kani in [FK09, Section 6.1] developed a method to construct d-elliptic curves via isogenies $E \to E'$, obtaining a genus two curve C such that $J(C) \cong E \times E'$ (see also [Frey95]).

Here we use an analogous construction to produce a (d,d')-elliptic curve. We start by giving the definition of twisting number, which turns out to be useful also in the analysis of stable Godeaux surfaces given in [FPR17].

Assume we are given a (d,d')-elliptic configuration as in (3), which is non-trivial in the sense of Remark 3.5. Then both elliptic maps factor, up to isomorphism, through the Abel-Jacobi map of C and are thus uniquely determined by the subgroups $\ker f_*$ and $\ker g_*$.

Definition 3.6. For a given (d,d')-elliptic configuration (C,f,g) we denote $\overline{F} = \ker g_*$ and $\overline{E}' = \ker f_*$ and we define the *twisting number* of (C,f,g) as

$$m = m(C,f,g) := \overline{F}\,\overline{E}' = \deg(\overline{F} \times \overline{E}' \longrightarrow J(C))$$

Even if the above definition is symmetric with respect to f and g we view it via the Frey-Kani construction applied to the d-elliptic map given by f, i.e., we extend diagram (1) to the following commutative diagram

$$
\begin{array}{ccccccc}
F & \overset{h_F}{\dashrightarrow} & \overline{F} & \xrightarrow{m:1} & & & E \\
{\scriptstyle(\varphi,\varphi')}\downarrow & & \downarrow & & & & \\
E\times E' & \xrightarrow{h} & J(C) & \xrightarrow{h'} & E \times E' & \longrightarrow & E \\
 & & \downarrow{\scriptstyle g_*} & & & & \\
 & & D & & & &
\end{array}
\tag{4}
$$

where $\overline{F}\Theta = d'$ (Θ denoting the principal polarization), F is the connected component of $h^{-1}\overline{F}$ containing the origin and φ, φ' are the isogenies induced by the two projections of $E \times E'$.

Remark 3.7. A genus two curve of compact type has a d-elliptic structure if and only if its Jacobian J contains a connected 1-dimensional subgroup $\overline{E}'$ such that $\overline{E}'\Theta = d$.

Therefore a d-elliptic curve C has a (d,d')-elliptic structure if and only if J contains a second connected 1-dimensional subgroup $\overline{F}$ such that $\overline{F}\Theta = d'$ and $\overline{F} \neq \overline{E}'$. So, except in the case of a trivial (d,d)-structure (cf. Remark 3.5), the Jacobian of a (d,d')-elliptic curve of genus two contains at least three, hence infinitely many, connected 1-dimensional subgroups. In particular the curve C has infinitely many elliptic structures, and the curves E and E' are isogenous. This is a classical theorem of Bolza and Picard (see e.g. [Krazer]).

Since the map $h' \circ h$ is the multiplication by d, diagram (4) yields the following equalities

$$m = m(C,f,g) := \overline{F}\overline{E}' = \overline{F} \ker f_* = \deg(\overline{F} \longrightarrow E) = d^2 \frac{\deg \varphi}{\deg h_F}. \tag{5}$$

Remark 3.8. One has $m > 0$, by the definition of (d,d')-elliptic curve.

Denote by $\overline{E}$ the kernel of $f'_* \colon J \to E'$, where f' is the complementary map of f. Then by the Frey-Kani construction we have $\Theta = \frac{\overline{E}+\overline{E}'}{d}$, hence $dd' = d\overline{F}\Theta = m + \overline{F}\overline{E}$. It follows that $m \leq dd'$, with equality holding if and only if $\overline{F}\overline{E} = 0$, namely if we are in the trivial case $g = f'$.

Moreover, we have

$$d(\deg(\varphi) + \deg(\varphi')) = \deg(h_F)d'.$$

We first provide three examples that fit into this general pattern and then prove that when d is a prime these cover all possibilities for non trivial elliptic configurations.

Example 3.9. Let d, d' be integers. Let F be an elliptic curve and let $\varphi \colon F \to E$ and $\varphi' \colon F \to E'$ be isogenies such that:

– $\ker \varphi \cap \ker \varphi' = \{0\}$, hence $(\varphi, \varphi') \colon F \to E \times E'$ is injective;

– $\deg \varphi + \deg \varphi' = dd'$, and d and $\deg \varphi$ are coprime.

We abuse notation and denote again by F the image of (φ,φ'). The subgroup $H := F[d] \subset (E \times E')[d]$ satisfies $H \cap E = H \cap E' = \{0\}$, since $EF = \deg \varphi'$ and $E'F = \deg \varphi$ are coprime to d. Hence H is the graph of an isomorphism $E[d] \to E'[d]$. The polarization $d(E \times \{0\} + \{0\} \times E')$ restricts on F to a divisor of degree $d^2 d'$, which therefore is a pull back via the map $F \to F$ defined by multiplication by d. By the functorial properties of the Weil pairing (see statement (1) of [Mum74, Chapter IV, §23, p.228]) it follows that $F[d]$ is an isotropic subspace of $(E \times E')[d]$.

Let $A{=}(E{\times}E')/H$ and let Θ be the principal polarization of A (see Proposition 2.3). Denote by $\overline{F}$ be the image of F in A: then we have $d^2\overline{F}\Theta{=}dF(E\times\{0\}+\{0\}\times E'){=}d^2d'$, namely $\overline{F}\Theta{=}d'$. By Remark 3.7 we obtain a (d,d')-elliptic configuration with twisting number $m{=}d^2\frac{\deg\varphi}{\deg h_F}{=}\deg\varphi$ (cf. (5)).

Remark 3.10. The above example is closely connected with the construction of covers induced by isogenies (see [FK09]).

To see the connection, let $\varphi^t\colon E\to F$ denote the dual isogeny and set $\widetilde{\varphi}{:=}\varphi'\circ\varphi^t\colon E\to E'$. Fix an integer $z{\in}\mathbb{Z}$ such that $z\deg\varphi{\equiv}1 \mod d$ (which exists by our hypothesis). Then

$$F[d]=\mathrm{Graph}(z\widetilde{\varphi}|_{E[d]}),$$

and so the anti-symplectic isomorphism is induced by the isogeny $z\widetilde{\varphi}$. It thus follows from [FK09, Proposition 6.2] (see also [DF08]) that $J(C){\cong}E{\times}E'$ as abelian surfaces (but not as principally polarized abelian varieties). The existence, structure, and moduli of such Jacobians was studied in detail in [Kani16].

In the next two examples, we will focus on the case where d is a prime number.

Example 3.11. Let d,d' be integers and assume that d is a prime. Let F be an elliptic curve and let $\varphi\colon F\to E$ and $\varphi'\colon F\to E'$ be isogenies such that:

– $\ker\varphi{\cap}\ker\varphi'{=}\{0\}$;
– $\deg\varphi{+}\deg\varphi'{=}d'$;
– $F[d]{\not\subset}\ker\varphi$ and $F[d]{\not\subset}\ker\varphi'$.

Under the above conditions, it is possible to find an antisymplectic isomorphism $\alpha\colon E[d]\to E'[d]$ such that $H_\alpha{\cap}F$ has order d, where H_α is the graph of α. This follows because by our assumptions there exists $0{\neq}v{\in}F[d]$ such that $v{\notin}\ker\varphi\cup\ker\varphi'$. Moreover, since the Weil pairing of a product is given by the product of the Weil pairings (see statement (2) of [Mum74, Chapter IV, §23, p.228]), the annihilator W of v in $(E{\times}E')[d]$ does not contain $E[d]{\times}\{0\}$ nor $\{0\}{\times}E'[d]$. The linear subspace W is three dimensional, hence $\mathbb{P}(W)$ is a projective plane over $\mathbb{F}_d$. Now consider in $\mathbb{P}(W)$ the pencil $\mathcal{F}$ of lines through $[v]$: since $\mathcal{F}$ consists of $d{+}1$ lines, if $d{>}2$ there is at least a line $l{\in}\mathcal{F}$ that does not intersect the lines $r{:=}\mathbb{P}(E[d]{\times}\{0\})$ and $s{:=}\mathbb{P}(\{0\}{\times}E'[d])$ and distinct from $t{:=}\mathbb{P}(F[d])$. The subspace of $(E{\times}E')[d]$ corresponding to l is the graph of an isomorphism $\alpha\colon E[d]\to E'[d]$ and is also isotropic, hence α is anti-symplectic. For $d{=}2$, any isomorphism α is antisymplectic, hence it is enough to find a line in $\mathbb{P}((E{\times}E')[2])$ that contains $[v]$, which is distinct from t and does not intersect r and s. An elementary geometrical argument shows that there exist two lines with this property.

Therefore, we can consider $A{:=}(E{\times}E')/H_\alpha$ and the principal polarization Θ of A (cf. Proposition 2.3). Again, we denote by $\overline{F}$ the image of F in A, obtaining

$d^2\overline{F}\Theta{=}dF(d(E\times\{0\}+\{0\}\times E')){=}d^2d'$, namely $\overline{F}\Theta{=}d'$, i.e. by Remark 3.7 we get a (d, d')-elliptic configuration. In this case by (5) we have $m{=}d^2\frac{\deg\varphi}{\deg h_F}{=}d\deg\varphi$.

Example 3.12. Let d, d' be integers such that d is a prime and d' is divisible by d. Write $d'{=}d\delta$ and let F be an elliptic curve with $\varphi\colon F\to E$ and $\varphi'\colon F\to E'$ isogenies such that:

– $\ker\varphi\cap\ker\varphi'=\{0\}$;

– $\deg\varphi+\deg\varphi'=\delta$;

We look for an antisymplectic isomorphism $\alpha\colon E[d]\to E'[d]$ such that $H_\alpha\cap F{=}\{0\}$, H_α being the graph of α.

To see that such α exists we argue as follows. As in Example 3.11, we identify 2-dimensional subspaces of $(E\times E')[d]$ with lines in $\mathbb{P}^3(\mathbb{F}_d){:=}\mathbb{P}((E\times E')[d])$. We have seen in Example 3.11 that there are $d{+}1$ isotropic lines through any point, hence there exist $(d{+}1)(d^2{+}1)$ isotropic lines.

The isotropic lines meeting a given line are $d(d{+}1){+}1$ or $(d{+}1)^2$, according to whether the line is isotropic or not. Set $r{:=}\mathbb{P}(E[d]\times\{0\})$, $s{:=}\mathbb{P}(\{0\}\times E'[d])$ and $t{=}\mathbb{P}(F[d])$; note that r and s are not isotropic. Hence there are at most $3(d{+}1)^2$ isotropic lines meeting $r\cup s\cup t$. However, all the lines joining a point of r and a point of s are isotropic hence, by subtracting these lines (that we had counted twice) we get the better upper estimate $2(d{+}1)^2$ for the number of isotropic lines meeting $r\cup s\cup t$. For $d\geq 3$ this shows the existence of the isotropic subspace H_α that we are looking for, since $(d{+}1)(d^2{+}1){-}2(d{+}1)^2{=}(d{+}1)(d^2{-}2d{-}1){>}0$.

For $d{=}2$, we need only find a line that is disjoint from $r\cup t\cup s$. We observe that $\mathbb{P}^3(\mathbb{F}_2)$ contains 35 lines, that the lines intersecting a given line are 19, that the lines meeting two given skew lines are 9 and the lines meeting three mutually skew lines are 3.

If $\deg\varphi$ and $\deg\varphi'$ are odd, then the three lines r, s and t are mutually skew: then the set of lines meeting at least one of these consists of $3{\cdot}19{-}3{\cdot}9{+}3{=}33$ lines, hence there are 2 possibilities for H_α.

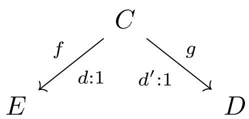
Now assume that both $\deg\varphi$ and $\deg\varphi'$ are even. In this case, t meets both r and s. The number of lines intersecting $r\cup t$ is equal to $7{+}7{-}3{=}11$, since there are 7 lines in plane spanned by r and t, there are 7 lines passing through $r\cap t$, and 3 lines common to these two sets. An analogous argument shows that there are $3{+}3{-}1{=}5$ lines meeting t, r and s. So the number of lines intersecting $r\cup t\cup s$ is equal to $3{\cdot}19{-}9{-}2{\cdot}11{+}5{=}31$, so there are 4 possibilities for H_α.

Finally we consider the case where $\deg\varphi$ is even and $\deg\varphi'$ is odd. There are two possibilities: either $r{=}t$ or r and t are coplanar but distinct. In the former case, the number of lines meeting $r\cup t\cup s{=}r\cup s$ is equal to $2{\cdot}19{-}9{=}29$, so there are 6 possibilities for H_α. In the latter case, the number of lines meeting $r\cup t\cup s$ is equal to $3{\cdot}19{-}2{\cdot}9{-}11{+}5{=}33$, so there are 2 possibilities for H_α.

Taking $A{:=}(E\times E')/H_\alpha$ and Θ the principal polarization of A (cf. Proposition 2.3) and denoting by $\overline{F}$ the image of F in A, we get $d^2\overline{F}\Theta{=}d^2F(d(E\times\{0\}+\{0\}\times E'))=d^2d'$, namely $\overline{F}\Theta{=}d'$, i.e. by Remark 3.7 we get a (d,d')-elliptic configuration. In this case (5) yields $m{=}d^2\frac{\deg\varphi}{\deg h_F}{=}d^2\deg\varphi$.

The above examples yield the following existence result.

Theorem 3.13. *Let $d,d'{>}1$ and $0{<}m{<}dd'$ be integers and let F be an elliptic curve. There exists a stable genus two curve C and a non trivial (d,d')-elliptic configuration with twisting number m*



such that E and D are isogenous to F in the following cases:

(a) *d and m are coprime*

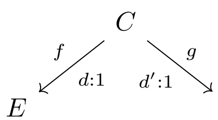
(b) *d is a prime number.*

Proof. Case (a) can be obtained as in Example 3.9: it suffices to take Γ, Γ' finite subgroups of F of orders m and $dd'-m$, respectively, such that $\Gamma\cap\Gamma'{=}\{0\}$ and let $\varphi\colon F{\to}E{:=}F/\Gamma$ and $\varphi'\colon F{\to}E'{:=}F/\Gamma'$ be the quotient maps.

Assume now that d is a prime. Since we have already proven existence in case (a), it is enough to consider the case when $m{=}td$ is divisible by d. We choose Γ, Γ' finite cyclic subgroups of F of orders t, $d'-t$ respectively, such that $\Gamma\cap\Gamma'{=}\{0\}$, we let $\varphi\colon F{\to}E{:=}F/\Gamma$ and $\varphi'\colon F{\to}E'{:=}F/\Gamma$ be the quotient maps and we use the construction of Example 3.11. $\square$

Conversely, we have the following

Theorem 3.14. *Let d be a prime and let d' be a positive integer. Let C be a stable curve of genus two and let (C,f,g) be a non-trivial (cf. Remark 3.5) (d,d')-elliptic configuration*



Denote by $\overline{E}'$ (resp. $\overline{F}$) the kernel of $f_*\colon J{=}J(C){\to}E$ (resp. $g_*\colon J{\to}D$) and let $m{=}\overline{E}'\overline{F}$ be the twisting number as in (5). Then

(i) *the (d,d')-elliptic configuration arises as in Example 3.9, or 3.11, or 3.12, with $1{\le}m{\le}dd'-1$;*

(ii) *the case of Example* 3.12 *can occur only if* d *divides* d' *and* d^2 *divides* m;
(iii) *the case of Example* 3.9 *occurs if and only if* m *is not divisible by* d.

Proof. By Remark 3.8 we have $1 \le m \le dd'$, and $m = dd'$ holds only in the trivial case $g = f'$. Therefore, by our assumptions, it is $1 \le m \le dd' - 1$.

We use freely the notation of §3.2 and diagram 4 and we denote by $\varphi\colon F \to E$ and $\varphi'\colon F \to E'$ the isogenies induced by the two projections of $E \times E'$. Note that $\ker \varphi \cap \ker \varphi' = \{0\}$ by construction. The pull-back $h^* \overline{F} \subset E \times E'$ is algebraically equivalent to νF for some integer $\nu \in \{1, d, d^2\}$ (one has $d^2 = \nu |H \cap F|$). We have $d^2 m = d^2 \overline{F}\,\overline{E}' = \nu F(d^2(\{0\} \times E'))$, i.e., $m = \nu F(\{0\} \times E') = \nu \deg \varphi$. In the same way, one obtains $dd' - m = \nu F(E \times \{0\}) = \nu \deg \varphi'$. In particular, $\nu = 1$ if m is not divisible by d.

Consider the case $\nu = 1$, i.e., $H = F[d]$. In this case, the map $E \times E' \to J(C)$ induces a degree d^2 isogeny $F \to \overline{F} \cong F$, the degree of φ is equal to m and the degree of φ' is equal to $dd' - m$. Since H, being a graph, intersects $E \times \{0\}$ and $\{0\} \times E'$ only in 0, it follows that m, which is equal to the order of $(\{0\} \times E') \cap F$, is prime to d, and the same is true for $\deg \varphi' = dd' - m$. So, C is constructed as in Example 3.9.

Next, assume that $\nu = d$, i.e. $H \cap F$ has order d. In this case, one has $m = d \deg \varphi$ and $\deg \varphi + \deg \varphi' = d'$. Since $H \cap F$ has order d and H is a graph, it follows that $F[d] \not\subset \ker \varphi$ and $F[d] \not\subset \ker \varphi'$, hence C is constructed as in Example 3.11.

Finally, consider the case $\nu = d^2$. In this case, one has $m = d^2 \deg \varphi$ and $d' = d(\deg \varphi + \deg \varphi')$, hence C is constructed as in Example 3.12. □

3.3. Existence of smooth (d, d')-elliptic curves

First of all, let us recall that an irreducible (d, d')-elliptic curve is smooth by Corollary 3.4.

By Lemma 2.4, for $d = 2$ a necessary condition for the irreducibility of the genus two curve C constructed as in Proposition 2.3 is that the curves E and E' are isomorphic, hence if E does not have complex multiplication then the constructions of Examples 3.9, 3.11 and 3.12 yield examples of smooth $(2, d')$-elliptic curves of genus two for every $d' > 2$.

In general, it is not clear whether the constructions of Examples 3.9, 3.11 and 3.12 give rise to irreducible, hence smooth, curves. We are able to settle this point at least in a special case:

Proposition 3.15. *Let* $d \ge 2, d' \ge 3$ *be integers*; *let* E *be an elliptic curve without complex multiplication*, $\xi \in E$ *an element of order* $r := dd' - 1$, *and* $\varphi'\colon E \to E' := E/\langle \xi \rangle$ *the quotient map.*

Then the (d, d')*-elliptic genus two curve constructed as in Example 3.9 with* $F{=}E$, $\varphi{=}\mathrm{Id}_E$ *and* φ' *as above is smooth.*

As an immediate consequence we obtain:

Corollary 3.16. *For every pair of integers* $d, d'{>}1$ *there exists a smooth* (d, d')*-elliptic curve of genus two with twisting number* $m{=}1$.

Proof. For $d{=}d'{=}2$ the claim follows by Lemma 2.4, for instance by using the construction of Example 3.9, and by Proposition 3.15 in the remaining cases. □

Remark 3.17. The above results are strictly related with the theory developed by Kani in [Kani16]. Arguing as in Remark 3.10, we have $J(C){\cong}E{\times}E'$ and therefore the problem of finding a smooth (d, d')-elliptic curve of genus two becomes the problem of finding a smooth genus two curves lying on $E{\times}E'$.

By the irreducibility criterion (cf. [Kani16, Proposition 6]), such a curve does exist if and only if the refined Humbert invariant never takes the value 1 (see [Kani14] for the definition and main properties), and in the situation of Proposition 3.15 one can deduce that this is the case (one can compute the refined Humbert invariant via [Kani16, Proposition 29]).

Before giving the proof of Proposition 3.15 we recall a well known fact, for which we give a proof due to the lack of a suitable reference.

Lemma 3.18. *Let* E *be an elliptic curve without complex multiplication.*

Then the connected 1*-dimensional subgroups of* $E{\times}E$ *distinct from* $E{\times}\{0\}$ *and* $\{0\}{\times}E$ *are of the form* $\{(ax, bx) \mid x{\in}E\}$, *with* a, b *coprime integers.*

Proof. Let G be such a subgroup, and denote by $\psi_i\colon G{\to}E$, $i{=}1, 2$ the isogenies induced by the two projections. Note that $\ker\psi_1{\cap}\ker\psi_2{=}\{0\}$. If G is isomorphic to E, then the ψ_i are multiplication maps and G is of the form $\{(ax, bx) \mid x{\in}E\}$ for some pair of coprime integers a, b. So assume that G and E are not isomorphic and consider an isogeny $\chi\colon E{\to}G$. Since χ is not a multiplication map, there exists an integer k and elements $u, v{\in}E[k]$ such that $\chi(u){=}0$ and $\chi(v){=}v'{\neq}0$. Now consider the maps $\mu_i{:=}\psi_i{\circ}\chi\colon E{\to}E$, which are multiplication maps by integer t_i, $i{=}1, 2$. Both t_1 and t_2 are divisible by k, since for $i{=}1, 2$ we have $\mu_i(u){=}0$, hence $\psi_i(v'){=}\mu_i(v){=}0$ and so $v'{=}0$, a contradiction. □

Proof of Proposition 3.15. Denote by Ξ the product polarization on $E{\times}E'$. Set $H{:=}\{(\eta, \varphi'(\eta)) | \eta{\in}E[d]\}$ and let $h\colon E{\times}E'{\to}A{:=}(E{\times}E')/H$ be the quotient map.

We argue by contradiction, so assume that the principal polarization of A induced by $d\Xi$ is reducible and denote it by $C{=}C_1{+}C_2$, where C_1 and C_2 are smooth elliptic curves meeting transversally in a unique point. Up to a translation we may assume that the singular point of C is the origin of A. Let $\widetilde{C}_i$ be the connected component of the preimage of C_i containing the origin of $E\times E'$, $i{=}1,2$, so that h^*C_i is numerically equivalent to $\nu_i\widetilde{C}_i$ for a positive integer ν_i. One has

$$d^2=\nu_i|H\cap\widetilde{C}_i| \quad \text{and} \quad \frac{d}{\nu_i}=\widetilde{C}_i\Xi\in\mathbb{Z}. \tag{6}$$

By Lemma 3.18 the connected 1-dimensional subgroups of $E\times E'$ distinct from $E\times\{0\}$ and $\{0\}\times E'$ are of the form $D_{a,b}{:=}\{(ax,b\varphi'(x))\mid x{\in}E\}$ with a,b coprime integers.

Since $D_{a,b}{=}D_{-a,-b}$, we may always assume $a{\geq}0$.

Notice that the kernel of the induced map $E{\to}D_{a,b}$ is the cyclic subgroup of $\langle\xi\rangle$ of order $\delta{:=}g.c.d.(a,r)$. Using this observation one computes:

$$D_{a,b}(\{0\}\times E')=\frac{a^2}{\delta},\quad D_{a,b}(E\times\{0\})=\frac{b^2r}{\delta},\quad D_{a,b}D_{1,1}=(b-a)^2\frac{r}{\delta}. \tag{7}$$

For $i{=}1,2$, let $a_i,b_i{\in}\mathbb{Z}$ be such that $\widetilde{C}_i{=}D_{a_i,b_i}$, with $a_i{\geq}0$; set $\delta_i{=}g.c.d.(a_i,r)$. We will now derive a contradiction using intersection numbers.

Step 1. We have $a_i{>}0$. Indeed if $a_i{=}0$ we have $\widetilde{C}_i\Xi{=}1$ and $|H{\cap}\widetilde{C}_i|{=}1$, so (6) gives $d{=}\nu_i$ and $\nu_i{=}d^2$, against our assumptions.

Step 2. We show $(a_i,b_i){\neq}(1,1)$. Indeed, assuming $\widetilde{C}_i{=}D_{1,1}$ (6) gives

$$\frac{d}{\nu_i}=\widetilde{C}_i\Xi=D_{1,1}\Xi=1+r=dd',$$

which is impossible since $d'{>}1$. In particular, since a_i and b_i are coprime, we have $a_i{\neq}b_i$.

Step 3. From the above steps we derive two inequalities and a divisibility property which will lead to a contradiction.

First of all we have, for $i{=}1,2$,

$$d|\nu_i(a_i-b_i). \tag{8}$$

Indeed, since $D_{1,1}{\cap}\widetilde{C}_i$ is a subgroup containing $H{\cap}\widetilde{C}_i$ we have that $\widetilde{C}_iD_{1,1}{=}(b_i-a_i)^2\frac{r}{\delta_i}$ is divisible by $\frac{d^2}{\nu_i}$, hence $(b_i-a_i)^2$ is divisible by $\frac{d^2}{\nu_i}$, since $\frac{r}{\delta_i}$ is an integer prime to d. So $\nu_i^2(b_i-a_i)^2$ is divisible by d^2, and therefore $\nu_i(a_i-b_i)$ is divisible by d.

Secondly, by (7) we have $d{=}(\nu_1\widetilde{C}_1{+}\nu_2\widetilde{C}_2)(\{0\}\times E'){=}\nu_1a_1\frac{a_1}{\delta_1}{+}\nu_2a_2\frac{a_2}{\delta_2}$ and $d{=}(\nu_1\widetilde{C}_1{+}\nu_2\widetilde{C}_2)(E\times\{0\}){=}\nu_1b_1^2\frac{r}{\delta_1}{+}\nu_2b_2^2\frac{r}{\delta_2}$. In particular, we have

$$\nu_1a_1+\nu_2a_2\leq d,\quad d'(\nu_1b_1^2+\nu_2b_2^2)\leq d, \tag{9}$$

since $\frac{r}{\delta_i}$ is an integer and $\frac{r}{\delta_i}{\geq}d'\frac{d}{a_i}{-}1{>}d'{-}1$.

Step 4. We cannot have $b_i > 0$. Indeed, in this case, since $0 \leq \nu_i a_i, \nu_i b_i < d$ by (9) and d divides the difference $\nu_i a_i - \nu_i b_i$ by (8), then we necessarily have $a_i = b_i$ contradicting Step 2.

Step 5. We cannot have $b_i \leq 0$. Indeed the same argument as in the previous step shows that we would necessarily have $\nu_i b_i = \nu_i a_i - d$ for $i = 1, 2$. By (9) we may assume, say, $\nu_1 a_1 \leq \frac{d}{2}$ and thus by the above equality $\nu_1 |b_1| = -\nu_1 b_1 \geq \frac{d}{2}$. Then (9) gives:

$$d \geq d' \nu_1 b_1^2 \geq |b_1| \frac{dd'}{2},$$

a contradiction since $d' > 2$.

Combining the last two steps we arrive at a contradiction and have thus proved that the polarisation is irreducible and hence is a smooth (d, d')-elliptic curve of genus two. $\square$

4. bi-tri-elliptic curves

For the applications to the classification of Gorenstein stable Godeaux surfaces the case of bi-tri-elliptic configurations is of particular interest. In this section we first formulate Theorem 3.14 in this case and then analyse reducible bi-tri-elliptic curves in more detail.

Indeed, by Theorem 3.14 we have the following characterization of bi-tri-elliptic configurations.

Corollary 4.1. *Let (C, f, g) be a bi-tri-elliptic configuration on a stable curve of arithmetic genus two. Then the twisting number m defined in (5) satisfies $1 \leq m \leq 5$ and there are the following possibilities*:

(a) *m is odd and the configuration arises as in Example 3.9 with* $\deg \varphi = m$;

(b) *$m = 2\mu$ is even and the configuration arises as in Example 3.11 with* $\deg \varphi = \mu$.

Remark 4.2. Counting parameters we see that the space of bi-tri-elliptic configurations is one-dimensional, but we did not consider its finer structure, e.g., the number of irreducible or connected components.

Note that the image in the moduli space A_2 of principally polarized abelian varieties of these configurations is given by the intersection of the Humbert surfaces H_4 and H_9 (see e.g. [HKW93]).

Now let us consider an elliptic curve E with a degree 2 endomorphism $\psi \colon E \to E$ and let $C \cong E \cup_0 E$ i.e., C is given by two copies of E meeting transversally at the origin. Then we can build a natural bi-tri-elliptic configuration

$$
(10) \qquad \begin{array}{ccccc} & & E\cup_0 E & & \\ & \overset{f=\mathrm{id}\cup\mathrm{id}}{\underset{2:1}{\swarrow}} & & \overset{g=\mathrm{id}\cup\psi}{\underset{3:1}{\searrow}} & \\ E & & & & E \end{array} .
$$

We will now show that every bi-tri-elliptic configuration (C,f,g) with C reducible is of this form. Indeed, by Lemma 2.4 the bi-elliptic map f on the reducible curve C is isomorphic to the composition of horizontal arrows in the diagram

$$
\begin{array}{ccccc} & & \overline{F} & & \\ & & \downarrow & & \\ C=E\times\{0\}\cup\{0\}\times E & \hookrightarrow & E\times E & \xrightarrow{+} & E \\ & \searrow^{g} & \downarrow & & \\ & & D & & \end{array} ,
$$

and the tri-elliptic map is uniquely determined by the subgroup $\overline{F}$. Note that the covering involution of f exchanges the components of C.

We have $\overline{F}C=3$ and without changing f we can assume that $\overline{F}(\{0\}\times E)=1$ and $\overline{F}(E\times\{0\})=2$. In other words, $\overline{F}$ is the graph of a degree 2 endomorphism $\psi\colon E\to E$ and $E\times\{0\}$ is identified with the second elliptic curve D by the restriction of g. Therefore, the bi-tri-elliptic configurations is as in (10).

An isomorphism from (C,f,g) to another bi-tri-elliptic configuration $(\widetilde{C}=\widetilde{E}\cup_0 \widetilde{E},\tilde{f},\tilde{g})$ such that $\tilde{f}$ is the identity on each component of $\widetilde{C}$ is uniquely determined by an isomorphism $E\cong\widetilde{E}$ and thus we have proved the first part of the following

Proposition 4.3. *The above construction induces a bijection on the set of iso morphism-classes of bi-tri-elliptic configurations (C,f,g) with C a reducible stable curve of genus two and the set $\{(E,\psi)\}$ of elliptic curves together with an endomorphism of degree* 2.

For every $1\le m\le 5$ there are exactly two such pairs (E,ψ), which are listed in Table 1, thus in total there are 10 *isomorphism classes of bi-tri-elliptic configurations with C a reducible stable curve of genus two.*

Table 1. Endomorphisms of degree 2 on elliptic curves.

$E=\mathbb{C}/\Gamma$	$\Gamma=\mathrm{End}(E)$	ξ	m
E_1	$\mathbb{Z}[\mathrm{i}]$	$-1\pm\mathrm{i}$	1
		$1\pm\mathrm{i}$	5
E_2	$\mathbb{Z}\left[\mathrm{i}\sqrt{2}\right]$	$\pm\mathrm{i}\sqrt{2}$	3
E_3	$\mathbb{Z}\left[\frac{1}{2}(1+\mathrm{i}\sqrt{7})\right]$	$-\frac{1}{2}(1\pm\mathrm{i}\sqrt{7})$	2
		$\frac{1}{2}(1\pm\mathrm{i}\sqrt{7})$	4

Proof. We need to recall some elementary facts about endomorphisms of elliptic curves. Details can be found for example in [Sil09, Chapter 11] or [Sil94, Chapter II]. Any endomorphism ψ of an elliptic curve E is given by multiplication by a complex number ξ and this embeds $\operatorname{End} E \hookrightarrow \mathbb{C}$ as an order in an imaginary quadratic number field $K \cong \operatorname{End}(E) \otimes \mathbb{Q}$.

Moreover, the degree of the endomorphism ψ coincides with the norm $N_{K/\mathbb{Q}}(\xi)$. Thus elements inducing an endomorphism of degree 2 are characterised as those $\xi \in \mathbb{C} \setminus \mathbb{R}$ that are integral over $\mathbb{Z}$ with characteristic polynomial

$$p_\xi(t) = t^2 - \operatorname{trace}_{K/\mathbb{Q}}(\xi)t + N_{K/\mathbb{Q}}(\xi) = t^2 - 2\operatorname{Re}(\xi)t + 2 \in \mathbb{Z}[t].$$

This gives exactly the elements listed in Table 1 and each one of them is contained in a unique maximal order by [Sil09, Example 11.3.1] (see also [Sil94, Proposition 2.3.1]).

It remains to compute the invariant m, which is in our case the intersection of $\Gamma_\psi = \overline{F} \subset E \times E$ with the kernel of the addition map, that is, the anti-diagonal. Thus m equals the number of fixed points of the endomorphism $-\psi$, which by the Lefschetz fixed-point formula [GH78, Chapter 3.4] gives

$$m = \sum_{i=0}^{2} (-1)^i \operatorname{trace}\left(-\psi_*|_{H_i(E,\mathbb{Q})}\right) = 1 - \operatorname{trace}_{K/\mathbb{Q}}(-\xi) + N_{K/\mathbb{Q}}(-\xi) = p_\xi(-1),$$

because every fixed point of ψ is simple. □

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