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Flexible and inflexible CR submanifolds

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Abstract. In this paper we prove new embedding results for compactly supported deformations of CR submanifolds of $\mathbb{C}^{n+d}$: We show that if M is a 2-pseudoconcave CR submanifold of type (n,d) in $\mathbb{C}^{n+d}$, then any compactly supported CR deformation stays in the space of globally CR embeddable in $\mathbb{C}^{n+d}$ manifolds. This improves an earlier result, where M was assumed to be a quadratic 2-pseudoconcave CR submanifold of $\mathbb{C}^{n+d}$. We also give examples of weakly 2-pseudoconcave CR manifolds admitting compactly supported CR deformations that are not even locally CR embeddable.

1. Introduction

In a previous paper [BH2] we introduced the concept of *flexible* versus *inflexible* CR submanifolds. This is related to the CR embeddability of deformations of CR structures. Roughly speaking a *flexible* submanifold admits a compactly supported CR deformation that "pops out" of the space of globally CR embeddable manifolds. On the other hand, for an *inflexible* CR submanifold, any compactly supported CR deformation stays in the space of globally CR embeddable manifolds.

Much work has been concentrated on CR manifolds M of hypersurface type which form the boundaries of strictly pseudoconvex domains. In that situation, M is *inflexible* when $\dim_{CR} M \geq 2$, and M is *flexible* when $\dim_{CR} M = 1$ (even without the assumption of strict pseudoconvexity). See Example 1 in Section 4.

Even in the situation of $\operatorname{codim}_{CR} M = 1$ (hypersurface type) it is of interest to study what happens for split signature of the Levi form. In that hypersurface case, 1-pseudoconcavity means that the Levi form has at least 1 negative eigenvalue, and at least 1 positive eigenvalue; 2-pseudoconcavity means that the Levi form has at least 2 negative and at least 2 positive eigenvalues, etc.

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And CR manifolds can have higher CR-codimension, in which case *q-pseudoconcavity* also seems to be a fruitful concept. It means that for every $x \in M$ and every characteristic conormal direction ξ at x, the scalar Levi form $\mathcal{L}_x(\xi, \cdot)$ in this conormal direction has at least q positive and q negative eigenvalues. (See Section 2 for the precise definitions.)

The theory of pseudoconcave CR manifolds was initiated approximately 25 years ago (see [HN]). Since that time it has slowly come to light that CR manifolds of higher codimension arise naturally in mathematics; i.e., such manifolds abound, but for a long time it was ignored that they have a natural CR structure. Besides typical examples of quadratic CR submanifolds of $\mathbb{C}^{n+d}$, they also arise naturally as minimal orbits for the holomorphic action of real Lie groups on flag manifolds. These are even homogeneous and almost always are q-pseudoconcave, for some q. In fact in [MN] the authors follow the general method initiated by N. Tanaka of investigating manifolds endowed with partial complex structures that come from Levi-Tanaka algebras which are the canonical prolongations of pseudocomplex fundamental graded Lie algebras. A lot of explicit such examples can be found in [MN], [HN] or [HN1].

When M is of hypersurface type, there are some hints that the 1-pseudoconcave case (Lorentzian case) and the q-pseudoconcave ($q \geq 2$) differ. For example, it is in the Lorentzian signature case where it is possible to generalize Nirenberg's example [Ni] to $\dim_{CR} > 1$, as was done in [JT]. But when $q \geq 2$, that construction does not work. Indeed in example 6 of section 4 we present a CR manifold N, of any CR-codimension, which is only 1-pseudoconcave (but weakly 2-pseudoconcave) and it is *flexible*. This shows that our Theorem 1.1 below is almost optimal. However, our N is not globally CR embedded into Euclidean space. Therefore it remains an open problem to find a 1-pseudoconcave CR submanifold of some Euclidean space that is flexible.

The main result obtained in [BH2] was that any 2-pseudoconcave *quadratic* CR submanifold of type (n, d) in $\mathbb{C}^{n+d}$ is inflexible. In the present paper we obtain the same result for CR submanifolds that are not necessarily assumed to be quadratic. More precisely,

Theorem 1.1. *Let M be a CR submanifold of type (n, d) in $\mathbb{C}^{n+d}$ that is 2-pseudoconcave. Let $(M_a, HM_a, J_a)_{|a|<a_o}$ be a compactly supported CR deformation of (M, HM, J). Then, provided a is sufficiently small, given any smooth CR function $f: (M, HM, J) \longrightarrow \mathbb{C}$, there is a CR function $f_a: (M_a, HM_a, J_a) \longrightarrow \mathbb{C}$ such that for any given $\ell \in \mathbb{N}$, any given compact K of M and arbitrary small $\varepsilon > 0$, one can find a CR function $f_a: (M_a, HM_a, J_a) \longrightarrow \mathbb{C}$ such that the $\mathcal{C}^\ell$ norm of $f - f_a$ on K is less than ε.*

Moreover, f_a can be chosen to coincide with the given f outside a compact of M. In particular, (M_a, HM_a, J_a) is CR embeddable into $\mathbb{C}^{n+d}$ for a sufficiently close to 0.

Corollary 1.2. *Let M be a 2-pseudoconcave CR submanifold of type (n,d) in $\mathbb{C}^{n+d}$. Then M is inflexible.*

Remark. The same result holds, with the same proof, if $\mathbb{C}^{n+d}$ is replaced by a strictly pseudoconvex domain in $\mathbb{C}^{n+d}$. We conjecture that it also holds with $\mathbb{C}^{n+d}$ is replaced by an $(n+d)$-dimensional Stein manifold X. However, our proof relies on the results from [LS]; and it is not clear if these results also hold in the more general setting of Stein manifolds.

In the proof of Theorem 1.1 we use an L^2 vanishing result obtained in [LS], which involved heavy use of integral formulas. In [BH2] we were able to obtain the analogous result by employing partial Fourier transform techniques, because of the quadratic nature of M. However in both [BH2] and in the present paper we also need certain subelliptic estimates, from [FK] in codimension one, and from [HN] in higher codimension. Thus, although the results of [BH2] were restricted to the quadratic case, the proofs there are more self-contained, since they do not rely on the rather complicated integral formulas upon which [LS] is based.

2. Definitions

An abstract CR manifold of type (n,d) is a triple (M, HM, J), where M is a smooth real manifold of dimension $2n+d$, HM is a subbundle of rank $2n$ of the tangent bundle TM, and $J: HM \to HM$ is a smooth fiber preserving bundle isomorphism with $J^2 = -\mathrm{Id}$. We also require that J be formally integrable; i.e. that we have

$$[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$$

where

$$T^{0,1}M = \{X + iJX \mid X \in \Gamma(M, HM)\} \subset \Gamma(M, \mathbb{C}TM),$$

with Γ denoting smooth sections.

The CR dimension of M is $n \geq 1$ and the CR codimension is $d \geq 1$.

M admits a CR embedding into some complex manifold X if one can find a smooth embedding φ of M into X such that the induced CR structure $\varphi_*(T^{0,1}M)$ on $\varphi(M)$ coincides with the CR structure $T^{0,1}(X) \cap \mathbb{C}T(\varphi(M))$ from the ambient complex manifold X.

Let (M, HM, J) be a CR manifold of type (n,d) globally CR embedded into some complex manifold X. We say that (M, HM, J) admits a *compactly sup-*

ported CR deformation if there exists a family $(M_a, HM_a, J_a)_{|a|<a_o}$ of abstract CR manifolds depending smoothly on a real parameter a, $|a|<a_o$ and converging to (M, HM, J) as a tends to 0 in the usual $\mathcal{C}^\infty$ topology; we also require that $(M_a, HM_a, J_a)=(M, HM, J)$ for every $a\neq 0$ outside some compact K of M not depending on a.

Note that when (M, HM, J) is CR embedded into some complex manifold, then one can always "punch" M as to obtain compactly supported CR deformations (at least locally). With the exception of $n=1$ (when the formal integrability condition is always satisfied), it can be difficult, however, to find compactly supported CR deformations in the absence of local CR embeddability.

We say that (M, HM, J) is a *flexible* CR submanifold of X if it admits a compactly supported CR deformation $(M_a, HM_a, J_a)_{|a|<a_o}$ such that for every sufficiently small $a\neq 0$, the CR structure (M_a, HM_a, J_a) is not globally CR embeddable into X. So, for example, the Heisenberg CR structure $\mathbb{H}^2$ in $\mathbb{C}^2$ is flexible. This follows from Nirenberg's famous local nonembeddability examples [Ni], which can be interpreted as small (local) deformations of the Heisenberg structure on $\mathbb{H}^2$. More examples will be discussed in the last section.

We say that (M, HM, J) is an *inflexible* CR submanifold of X if it is not flexible. That means that (M, HM, J) is inflexible if and only if for every compactly supported CR deformation $(M_a, HM_a, J_a)_{|a|<a_o}$ of (M, HM, J), the CR manifold (M_a, HM_a, J_a) is globally CR embeddable into X.

We denote by $H^oM=\{\xi\in T^*M \,|\, <X,\xi>=0, \forall X\in H_{\pi(\xi)}M\}$ the *characteristic conormal bundle* of M. Here $\pi: TM\longrightarrow M$ is the natural projection. To each $\xi\in H^o_pM\setminus\{0\}$, we associate the Levi form at p in the codirection ξ:

$$\mathcal{L}_p(\xi, X) = \xi([J\widetilde{X}, \widetilde{X}]) = d\tilde{\xi}(X, JX) \text{ for } X \in H_pM$$

which is Hermitian for the complex structure of H_pM defined by J. Here $\tilde{\xi}$ is a section of H^oM extending ξ and $\widetilde{X}$ a section of HM extending X.

Following [HN] M is called q-pseudoconcave, with $0\leq q\leq \frac{n}{2}$, if for every $p\in M$ and every characteristic conormal direction $\xi\in H^o_pM\setminus\{0\}$, the Levi form $\mathcal{L}_p(\xi,\cdot)$ has at least q negative and q positive eigenvalues.

For other standard definitions related to CR structures we also refer the reader to [HN] or [HN1].

3. Proofs

The idea of the proof of Theorem 1.1 is as follows: For a given CR function f on M we want to find a CR function f_a on M_a which is very close to the given

f on M. Therefore we want to solve the Cauchy-Riemann equations $\overline{\partial}_{M_a}u=\overline{\partial}_{M_a}f$ with u having compact support and the $\mathcal{C}^k$-norms of u being controlled by some $\mathcal{C}^l$-norms of $\overline{\partial}_{M_a}$ (uniformly with respect to a). Setting $f_a=f-u_a$ then gives the desired CR function on M_a.

Let M be as in Theorem 1.1, and let B be a sufficiently large Euclidean ball containing the compact K that is the support of the CR deformation of $M\subset\mathbb{C}^{n+d}$. Recalling that M is 2-pseudoconcave, we have the following result from [LS, Theorem 1.0.2]:

Proposition 3.1. *Let $q=n-1$ or $q=n$, and assume $f\in L^2_{n+d,q}(M\cap B)$ satisfies $\overline{\partial}_M f=0$. Then there exists $u\in L^2_{n+d,q-1}(M\cap B)$ satisfying $\overline{\partial}_M u=f$.*

Here we are considering (unweighted) L^2 spaces with respect to the induced metrics from the Euclidean metric on $\mathbb{C}^{n+d}$. By classical Hilbert space theory (see e.g. [H, Theorem 1.1.2]), one deduces from Proposition 3.1 the following

Proposition 3.2. *Let $q=n-1$ or $q=n$. Then there exists a constant $C>0$ such that*

$$\|u\|^2\leq C(\|\overline{\partial}_M u\|^2+\|\overline{\partial}^*_M u\|^2)$$

for all $u\in L^2_{n+d,q}(M\cap B)\cap\mathrm{Dom}(\overline{\partial}_M)\cap\mathrm{Dom}(\overline{\partial}^*_M)$.

Next, we use again that M is 2-pseudoconcave. 2-pseudoconcavity is clearly stable under smooth, small perturbations. Therefore M_a is also 2-pseudoconcave for a sufficiently small, and the 2 positive resp. 2 negative eigenvalues of the Levi form in sufficiently close characteristic conormal directions can be bounded from below resp. above independent of a. Therefore one obtains a uniform subelliptic estimate in degrees $q\in\{0,1,n-1,n\}$ (by closely looking at the proofs in [FK] for $d=1$ and [HN] for higher codimensions): There exists $\varepsilon>0$ such that for every compact K of M, there exists a constant $C_K>0$ independent of a such that

$$\|u\|^2_\varepsilon\leq C_K(\|\overline{\partial}_{M_a}u\|^2+\|\overline{\partial}^*_{M_a}u\|^2+\|u\|^2) \tag{3.1}$$

for all smooth forms $u\in\mathcal{D}^{p,q}_K(M_a)$ with support contained in K, $0\leq p\leq n+d$, $q\in\{0,1,n-1,n\}$.

Combining Proposition 3.2 and (3.1), we can establish an L^2 a priori estimate in degree $(n+d,n-1)$ and $(n+d,n)$, which is uniform with respect to a (in the sense that the constant involved does not depend on a).

Proposition 3.3. *There is $a_0>0$ and a constant $C>0$ such that for $q\in\{n-1,n\}$ we have*

$$\|u\|^2\leq C(\|\overline{\partial}_{M_a}u\|^2+\|\overline{\partial}^*_{M_a}u\|^2)$$

for all $u\in L^2_{n+d,q}(M\cap B)\cap\mathrm{Dom}(\overline{\partial}_{M_a})\cap\mathrm{Dom}(\overline{\partial}^*_{M_a})$, $|a|<a_0$.

Proof. Assume by contradiction that there is a sequence $\{u_{a_\nu}\}\in L^2_{n+d,q}(M_{a_\nu}\cap B)\cap\mathrm{Dom}(\overline{\partial}_{M_{a_\nu}})\cap\mathrm{Dom}(\overline{\partial}^*_{M_{a_\nu}})$, $a_\nu\to 0$, such that

$$\|u_{a_\nu}\|=1, \tag{3.2}$$

whereas

$$\|\overline{\partial}_{M_{a_\nu}}u_{a_\nu}\|^2+\|\overline{\partial}^*_{M_{a_\nu}}u_{a_\nu}\|^2<a_\nu. \tag{3.3}$$

We now want to show that $\{u_{a_\nu}\}$ is a Cauchy sequence.

Remember that $M_{a_\nu}=M$ outside K. We now choose a slightly larger compact K_1 containing K in its interior, and a smooth cut-off function χ such that $\chi\equiv 1$ outside K_1 and $\chi\equiv 0$ in a neighborhood of K. Since $\overline{\partial}_{M_{a_\nu}}$, $\overline{\partial}^*_{M_{a_\nu}}$ coincide with $\overline{\partial}_M$, $\overline{\partial}^*_M$ outside K, we obtain from Proposition 3.2

$$\|\chi u\|^2\le C(\|\overline{\partial}_M(\chi u)\|^2+\|\overline{\partial}^*_M(\chi u)\|^2)$$

for all $u\in L^2_{n+d,q}(M_a\cap B)\cap\mathrm{Dom}(\overline{\partial}_{M_{a_\nu}})\cap\mathrm{Dom}(\overline{\partial}^*_{M_{a_\nu}})$, which implies

$$\|\chi u\|^2\le C'(\|\overline{\partial}_M u\|^2+\|\overline{\partial}^*_M u\|^2+\int_{K_1\setminus K}|u|^2dV) \tag{3.4}$$

for some constant $C'>0$.

On the other hand, let η be a smooth cut-off function so that $\eta\equiv 1$ in a neighborhood of K_1. Then $\|\eta u_{a_\nu}\|_\varepsilon$ is bounded by (3.1), so the generalized Rellich lemma implies that the sequence $\{u_{a_\nu}\}$ restricted to K_1 is precompact in $L^2_{n+d,q}(K_1)$. Thus it is no loss of generality to assume that the restriction of $\{u_{a_\nu}\}$ to K_1 is a Cauchy sequence. But this combined with (3.4) implies that $\{u_{a_\nu}\}$ is a Cauchy sequence in $L^2_{n+d,q}(M\cap B)$.

Denote by u_0 the limit of this sequence. From (3.3) it follows that $\overline{\partial}_M u_0$ and $\overline{\partial}^*_M u_0$, defined in the distribution sense, both vanish. But from (3.2) it also follows that $\|u_0\|=1$. This contradicts Proposition 3.2 and therefore completes the proof of the proposition. □

By duality, we obtain from Proposition 3.3 that one can solve the $\overline{\partial}_{M_a}$-equation with support in $M\cap\overline{B}$ in degree $(0,1)$ with a uniform constant. For this, we consider an L^2 variant of $\overline{\partial}_{M_a}$ defined in the following way: Let $u\in L^2_{p,q}(M_a\cap B)$. We say that $u\in\mathrm{Dom}(\overline{\partial}^c_{M_a})$ and $\overline{\partial}^c_{M_a}u=f$ if there exists a sequence of test forms $u_j\in\mathcal{D}^{p,q}(M_a\cap B)$ such that $u_j\to u$ in L^2 and $\overline{\partial}_{M_a}u_j\to f$ in L^2.

Proposition 3.4. *There is $a_0>0$ and a constant $C>0$ independent of a such that for every $f\in L^2_{0,1}(M_a)$ with $\overline{\partial}_{M_a}f=0$ and f compactly supported in $M\cap B$, one can find $u\in L^2_{0,0}(M_a)$ such that $\overline{\partial}^c_{M_a}u=f$ and $\|u\|\le C\|f\|$.*

Proof. Consider the operator

$$T_f\colon L^2_{n+d,n}(M_a\cap B) \longrightarrow \mathbb{C}$$
$$\psi \mapsto \int_{M_a\cap B} f\wedge\varphi,$$

where $\varphi\in L^2_{n+d,n-1}(M_a\cap B)$ satisfies $\overline{\partial}_{M_a}\varphi=\psi$ in the weak sense and $\|\varphi\|\le C\|\psi\|$ (such a φ exists by Proposition 3.3). T_f is well defined. Indeed, if $\overline{\partial}_{M_a}\varphi=0$, then we may apply Proposition 3.3 again and conclude that there exists $h\in L^2_{n+d,n-2}(M_a\cap B)$ satisfying $\overline{\partial}_{M_a}h=\varphi$. By Stokes' theorem this implies

$$\int_{M_a\cap B} f\wedge\varphi=\int_{M_a\cap B} f\wedge\overline{\partial}_{M_a}h=\int_{M_a\cap B}\overline{\partial}_{M_a}(f\wedge h)=0. \tag{3.5}$$

Note also that T_f is continuous of norm $\le C$. Using Riesz's theorem, we conclude that there exists $u\in L^2_{0,0}(M_a)$ satisfying

$$\int_{M_a\cap B} u\wedge\overline{\partial}_{M_a}\varphi=T_f(\overline{\partial}_{M_a}\varphi)=\int_{M_a\cap B} f\wedge\varphi$$

for all $\varphi\in L^2_{n+d,n}(M_a\cap B)$. Let ϑ_a be the formal adjoint of $\overline{\partial}_{M_a}$ on $L^2_{\cdot,\cdot}(M_a\cap B)$. It is easy to see that $\overline{\partial}^c_{M_a}$ and ϑ_a are adjoint operators on $L^2_{\cdot,\cdot}(M_a\cap B)$. (3.5) implies that $(u,\vartheta_a\varphi)=(f,\varphi)$ for any $\varphi\in\mathrm{Dom}(\vartheta_a)$, which is equivalent to $\overline{\partial}^c_{M_a}u=f$. $\square$

Proof of Theorem 1.1. Let f be a CR function on M. Then $\overline{\partial}_{M_a}f$ has compact support and tends to zero when a tends to zero. Proposition 3.4 implies that we can solve the equation $\overline{\partial}_{M_a}u_a=\overline{\partial}_{M_a}f$ with $\|u_a\|\le C\|\overline{\partial}_{M_a}f\|$ and u_a supported in $M\cap\overline{B}$. Hence u_a is as small as we wish in $L^2_{0,0}(M_a)$, provided a is small enough. It is well-known that the subelliptic estimate (3.1) in degree $q=0$ implies also the following: Suppose given a compact $K'\subset M_a$ and two smooth real functions ζ, ζ_1 with $\operatorname{supp}\zeta\subset\operatorname{supp}\zeta_1\subset K'$ and $\zeta_1=1$ on $\operatorname{supp}\zeta$, then for any integer $m\in\mathbb{N}$ there exists a constant $C_{K,m}$ such that

$$\|\zeta u\|^2_{m+\varepsilon}\le C_{K,m}(\|\zeta_1\overline{\partial}_{M_a}u\|^2_m+\|\zeta_1 u\|^2)$$

Here $\|\ \|_m$ denotes the Sobolev norm of order m. But then also the $\mathcal{C}^\ell$-norm of u_a over a given compact $K'\subset M_a$ can be controlled by some $\mathcal{C}^m$-norm of $\overline{\partial}_{M_a}u_a=f$, and hence made small when letting a tend to zero. Setting $f_a=f-u_a$ proves the theorem. $\square$

4. Examples of flexible CR submanifolds

The aim of this section is to provide known and new examples of flexible CR submanifolds.

1. Rossi [R] constructed small real analytic deformations of the standard CR structure on the 3-sphere S^3 in $\mathbb{C}^2$, and such that the resulting abstract CR structures fail to CR embed globally into $\mathbb{C}^2$. Hence S^3 is a flexible CR submanifold of $\mathbb{C}^2$.

This is in contrast to higher dimensions: Any strictly pseudoconvex CR manifold M of CR dimension $n\geq 2$ is globally CR embeddable into some $\mathbb{C}^N$ by [BdM]. If, in addition, M is the boundary of a strictly pseudoconvex domain in $\mathbb{C}^{n+1}$, then M is inflexible. This follows from a result by [T], since in this situation we have $H^{0,1}(M)=0$.

2. Nirenberg's famous local nonembeddability examples [Ni] can be interpreted as small (local) deformations of the Heisenberg structure on $\mathbb{H}^2\subset\mathbb{C}^2$. Since the formal integrability condition for CR structures is always satisfied in dimension 3, one can use a cut-off function to make the local deformations a compactly supported deformation of the global Heisenberg group.

3. More generally, *any* 3-dimensional CR submanifold is flexible. Indeed, if M has a point of strict pseudoconvexity, then one can use the local nonembeddability result of [JT] to produce a small, non-locally embeddable CR deformation which is compactly supported near that point.

If M is Levi-flat, then one first makes arbitrary small bumps near a fixed point to get points of strict pseudoconvexity and proceeds as before.

4. $S^3\times S^3\in\mathbb{C}^4$ is an example of a flexible CR submanifold of codimension 2 (because each factor is flexible). Depending on the conormal direction, its Levi-forms have signature $++$, $--$, $+0$ or -0. By adding more products one can obtain flexible CR submanifolds of any CR codimension.

5. Let X be any compact Riemann surface. Then $S^3\times X$ is flexible.

6. Let M be a compact 1-pseudoconcave CR submanifold of type $(2,d)$ of some complex manifold X, d arbitrary. Then $N=M\times\mathbb{CP}^1$ is again 1-pseudoconcave, and even weakly 2-pseudoconcave (the Levi form has signature $(+-0)$ in every nonzero conormal direction). Using ideas from [Hi1] we will now show that N is flexible, which indicates that Theorem 1.1 is close to being optimal. Indeed, by Theorem 3.2 of [BH1] there exists a smooth $(0,1)$-form ω on M satisfying $\overline{\partial}_M\omega=0$ on M such that ω is not $\overline{\partial}_M$-exact on any neighborhood of any point p on M. We will use this form ω to deform the CR structure on $M\times\mathbb{CP}^1$.

On $\mathbb{CP}^1$ we use the two standard holomorphic charts $V_+=\mathbb{CP}^1\setminus\{\infty\}$ and $V_-=\mathbb{CP}^1\setminus\{\infty\}$ given by the stereographic projection, with coordinates $z_+\in V_+\simeq\mathbb{C}$ and

$z_-\in V_-\simeq\mathbb{C}$, where $z_-z_+=1$ on $V_-\cap V_+$. Then the usual complex structure on $\mathbb{CP}^1$ is given by $\frac{\partial}{\partial\overline{z}_\beta}$ on V_β for $\beta=(+,-)$.

Let U be an open set of M such that $T^{0,1}M$ is spanned over U by $\overline{L}_1,\overline{L}_2$. We define $T^{0,1}N_a$ to be spanned over $U\times V_\beta$ by the basis

$$\begin{cases}\overline{X}_0 = & \dfrac{\partial}{\partial\overline{z}_\beta}\\ \overline{X}_j = & \overline{L}_j+\beta a\omega(\overline{L}_j)z_\beta\dfrac{\partial}{\partial z_\beta},\ \ j=1,2\end{cases}\tag{4.1}$$

This gives a well defined CR structure on N. To see that the integrability condition is valid, first note that $[\overline{X}_0,\overline{X}_j]=0$ for $j=1,2$. Moreover, by assumption on ω we have

$$0=\overline{\partial}_M(\overline{L}_1,\overline{L}_2)=\overline{L}_1(\omega(\overline{L}_2))-\overline{L}_2(\omega(\overline{L}_1))-\omega([\overline{L}_1,\overline{L}_2]),$$

thus

$$\begin{aligned}[\overline{X}_1,\overline{X}_2] &= [\overline{L}_1,\overline{L}_2]+[\overline{L}_1,\beta a\omega(\overline{L}_2)z_\beta\frac{\partial}{\partial z_\beta}]+[\beta a\omega(\overline{L}_1)z_\beta\frac{\partial}{\partial z_\beta},\overline{L}_2]\\ &= [\overline{L}_1,\overline{L}_2]+\beta a\big(\overline{L}_1(\omega(\overline{L}_2))z_\beta\frac{\partial}{\partial z_\beta}-\overline{L}_2(\omega(\overline{L}_1))z_\beta\frac{\partial}{\partial z_\beta}\big)\\ &= [\overline{L}_1,\overline{L}_2]+\beta a\omega([\overline{L}_1,\overline{L}_2])z_\beta\frac{\partial}{\partial z_\beta},\end{aligned}$$

thus $T^{0,1}N_a$ is stable under the Lie bracket.

However, for $a\neq 0$, local CR embeddability of N_a implies the local $\overline{\partial}_M$-exactness of ω. The argument follows [Hi2] or [Hi3]. In fact the argument shows that N_a is not even locally CR embeddable at any point (t^o,z^o_β) of $N_a=M\times\mathbb{CP}^1$: Near t^o, M is locally CR embeddable into $\mathbb{C}^{2+d}$ with coordinate functions $\zeta_1,\dots,\zeta_{2+d}$. We may assume that $t=(t_1,\dots,t_{4+d})=(\mathrm{Re}\zeta_1,\dots,\mathrm{Re}\zeta_{2+d},\mathrm{Im}\zeta_1,\mathrm{Im}\zeta_2)$ are real coordinates on M with $t^o=0$ in these coordinates.

Suppose now that we have a local CR embedding of N_a near (t^o,z^o_β) by CR functions $u_1(t,z_\beta),u_2(t,z_\beta),\dots,u_{3+d}(t,z_\beta)$ with $du_1\wedge\dots\wedge du_{3+d}\neq 0$ at (t^o,z^o_β). Then each u_j is holomorphic in z_β in view of (4.1).

It is then not difficult to see that $\frac{\partial u_j}{\partial z_\beta}\neq 0$ for some j at (t^o,z^o_β). By renaming, we may assume $\frac{\partial u_{3+d}}{\partial z_\beta}\neq 0$.

The coordinates on $(z_1,\dots,z_{3+d})$ on $\mathbb{C}^{3+d}$ also define CR functions on N_a and $dz_1\wedge\dots\wedge dz_{2+d}\wedge du_{3+d}\neq 0$ at (t^o,u^o_β). So we arrive at a new local embedding map

$$\varphi:(t,z_\beta)\longmapsto(z_1(t,z_\beta),\dots,z_{2+d}(t,z_\beta),u_{3+d}(t,z_\beta))$$

of some neighborhood W of (t^o, z^o_β) into $\mathbb{C}^{3+d}$. $\varphi(W)$ is a piece of a real hypersurface in $\mathbb{C}^{3+d}$. Let $w{=}(w_1, ..., w_{3+d})$ denote the coordinates in $\mathbb{C}^{3+d}$, and consider, for points on $\varphi(W)$, the function

$$F(w) = \varphi_*\big(-[\frac{\partial u_{3+d}}{\partial z_\beta}]^{-1}\big),$$

where φ_* is the push-forward by the diffeomorphism φ of W onto $\varphi(W)$. It follows that F is a CR function on $\varphi(W)$; in particular, it is holomorphic in w_{3+d} by the inverse mapping theorem for holomorphic functions of one variable. On $\varphi(W)$ we may define the function

$$G(w) = \int_0^{w_{3+d}} F(w_1, ..., w_{2+d}, \eta) d\eta \tag{4.2}$$

by a contour integral in the w_{3+d}-plane. This is well defined by the open mapping theorem from one complex variable. We now pull back to get a function $g(t, z_\beta){=}\varphi^* G$ on V, which is a CR function there. This can be seen by replacing F in (4.2) by a smooth extension $\widetilde{F}$ of F off of $\varphi(W)$ such that $\overline{\partial}\widetilde{F}_{|\varphi(V)}{=}0$ and differentiating. Next we have

$$\frac{\partial g}{\partial z_\beta} = F(w_1, ..., w_{2+d}, u_{3+d}(t, z_\beta)) \frac{\partial u_{3+d}}{\partial z_\beta}(t, z_\beta) = -1, \tag{4.3}$$

so $g(t, z_\beta){=}{-}z_\beta{+}\chi(t)$, where $\chi(t)$ is a smooth "constant of integration". Now the fact that g is a CR function implies that $\overline{X}_j g{=}0$, hence $\overline{L}_j\chi{-}\beta a\omega(\overline{L}_j){=}0$ for $j{=}1, 2$. But for $a{\neq}0$, this means that there exists a neighborhood V' of t^o on M such that ω is $\overline{\partial}_M$-exact on V'. This is a contradiction to the assumption on ω. Therefore for $a{\neq}0$, N_a is not locally CR embeddable on any open neighborhood of (t^o, z^o_β) on N_a. □

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