

# Interface asymptotics of Partial Bergman kernels around a critical level

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**Abstract.** In a recent series of articles, the authors have studied the transition behavior of partial Bergman kernels  $\Pi_{k,[E_1,E_2]}(z,w)$  and the associated DOS (density of states)  $\Pi_{k,[E_1,E_2]}(z)$  across the interface  $\mathcal{C}$  between the allowed and forbidden regions. Partial Bergman kernels are Toeplitz Hamiltonians quantizing Morse functions  $H:M\rightarrow\mathbb{R}$  on a Kähler manifold. The allowed region is  $H^{-1}([E_1,E_2])$  and the interface  $\mathcal{C}$  is its boundary. In prior articles it was assumed that the endpoints  $E_j$  were regular values of  $H$ . This article completes the series by giving parallel results when an endpoint is a critical value of  $H$ . In place of the Erf scaling asymptotics in a  $k^{-\frac{1}{2}}$  tube around  $\mathcal{C}$  for regular interfaces, one obtains  $\delta$ -asymptotics in  $k^{-\frac{1}{4}}$ -tubes around singular points of a critical interface. In  $k^{-\frac{1}{2}}$  tubes, the transition law is given by the osculating metaplectic propagator.

## 1. Introduction

This note is a continuation of our analysis in [ZZ19b] of the pointwise asymptotics of partial Bergman kernel densities  $\Pi_{k,I}(z)$  around interfaces between allowed and forbidden regions. Let  $(L,h)\rightarrow(M,\omega,J)$  be a polarized Kähler manifold,  $\omega=-i\partial\bar{\partial}\log h$ , and let  $H^0(M,L^k)$  denote the space of holomorphic sections of the  $k$ -th power of the positive Hermitian line bundle  $L$ . Let  $H:M\rightarrow\mathbb{R}$  be a smooth function with Morse critical point, called the Hamiltonian function. The Berezin-Toeplitz quantization of  $H$  is an operator acting on  $H^0(M,L^k)$ :

$$(1) \quad \widehat{H}_k := \Pi_k \circ (H + \frac{i}{k} \nabla_{\xi_H}) \circ \Pi_k : H^0(M, L^k) \longrightarrow H^0(M, L^k).$$

where  $\Pi_k = \Pi_{h^k} : L^2(M, L^k) \rightarrow H^0(M, L^k)$  is the orthogonal projection,  $H$  acts by multiplication and  $\nabla_{\xi_H}$  is the Chern covariant derivative along the Hamiltonian

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flow  $\xi_H$ .<sup>(1)</sup> We denote the eigenvalues (repeated with multiplicity) of  $\widehat{H}_k$  by

$$(2) \quad \mu_{k,1} \leq \mu_{k,2} \leq \dots \leq \mu_{k,N_k},$$

where  $N_k = \dim H^0(M, L^k)$ , and the corresponding orthonormal eigensections in  $H^0(M, L^k)$  by  $s_{k,j}$ .

Given the spectral interval  $I \subset \mathbb{R}$  we define the partial Bergman kernels to be the orthogonal projections,

$$(3) \quad \Pi_{k,I} : H^0(M, L^k) \longrightarrow \mathcal{H}_{k,I},$$

onto the spectral subspace,

$$(4) \quad \mathcal{H}_{k,I} := \text{span}\{s_{k,j} : \mu_{k,j} \in I\}$$

Its (Schwartz) kernel is defined by

$$(5) \quad \Pi_{k,I}(z, w) = \sum_{\mu_{k,j} \in I} s_{k,j}(z) \overline{s_{k,j}(w)}.$$

and the metric contraction of (5) on the diagonal with respect to  $h^k$  is the partial density of states,

$$\Pi_{k,I}(z) = \sum_{\mu_{k,j} \in I} \|s_{k,j,\alpha}(z)\|^2.$$

We denote by  $\Pi_k(z, w)$  and  $\Pi_k(z)$  the (full) Bergman kernel and density function. Here and throughout the article, we use the notation  $K(z)$  for the metric contraction of the diagonal values  $K(z, z)$  of a kernel.

We define the classical allowed region  $\mathcal{A}$  and forbidden region  $\mathcal{F}$  as open subsets

$$\mathcal{A} := \text{Int}(H^{-1}(I)), \quad \mathcal{F} = \text{Int}(M \setminus \mathcal{A}),$$

and the interface as

$$\mathcal{C} = \partial\mathcal{A} = \partial\mathcal{F}.$$

In [ZZ19b] it is proved that

$$\frac{\Pi_{k,I}(z)}{\Pi_k(z)} = \begin{cases} 1 & \text{if } z \in \mathcal{A} \\ 0 & \text{if } z \in \mathcal{F} \end{cases} \quad \text{mod } O(k^{-\infty}),$$

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<sup>(1)</sup> We note that as far as leading term in the asymptotic expansion is concerned, we may replace  $\widehat{H}_k$  by  $T = \Pi_k \circ H \circ \Pi_k$ , which has the same principal symbol as  $\widehat{H}_k$ . See [ZZ19b] Remark 4.4.

and moreover if the interface  $\mathcal{C}$  is a smooth hypersurface (with possibly several components), then the scaled density decay profile of  $\frac{\Pi_{k,I}(z)}{\Pi_k(z)}$  in a tube of radius  $\frac{1}{\sqrt{k}}$  around  $\mathcal{C}$  has the shape of the Gaussian error function  $\text{Erf}(x) = \mathbb{P}_{X \sim N(0,1)}(X < x)$ :

$$(6) \quad \frac{\Pi_{k,I}(z)}{\Pi_k(z)} \Big|_{z=\exp_{z_0}(t\nu/\sqrt{k})} = \text{Erf}(2\sqrt{\pi}t) + O(k^{-1/2})$$

where  $z_0 \in \mathcal{C}$ ,  $\nu$  is the unit normal vector to  $\mathcal{C}$  at  $z_0$  pointing towards allowed region, and  $\exp$  is the exponential map with respect to the Kähler metric.

To be precise, let  $\{H=E\}$  be a regular level of  $H$  and let  $z \in \{H=E\}$ . Let  $F^t$  denote the gradient flow  $\nabla H$  for time  $t$ .<sup>(2)</sup> Then, for any Schwartz class function  $f \in \mathcal{S}(\mathbb{R})$ ,

$$(7) \quad \begin{aligned} & \sum_j f(\sqrt{k}(\mu_{k,j} - E)) \|s_{k,j}(F^{\beta/\sqrt{k}}(z))\|_h^2 \\ & \simeq \left(\frac{k}{2\pi}\right)^m \int_{-\infty}^{\infty} f(x) e^{-\left(\frac{x}{|\nabla H|(z)} - \beta|\nabla H(z)|\right)^2} \frac{dx}{\sqrt{\pi}|\nabla H(z)|}. \end{aligned}$$

Thus, in the scaling limit, Erf smoothly interpolates between the value 1 on the allowed region  $\mathcal{A}_{[E_1, E_2]}$  and the value 0 on the forbidden region  $M \setminus \mathcal{A}_{[E_1, E_2]}$ .

Henceforth, to simplify notation, we use Kähler local coordinates  $u$  centered at  $z_0$  to write points in the  $k^{-\varepsilon}$  tube around  $\mathcal{C}$  by

$$z = z_0 + k^{-\varepsilon}u := \exp_{z_0}(k^{-\varepsilon}u), \quad u \in T_{z_0}\mathcal{C}$$

The abuse of notation in dropping the higher order terms of the normal exponential map is harmless since we are working so close to  $\mathcal{C}$ . At regular points  $z_0$  we may use the exponential map along  $N_{z_0}\mathcal{C}$  but we also want to consider critical points. More generally we write  $z_0 + u$  for the point with Kähler normal coordinate  $u$ . In these coordinates,

$$\omega(z_0 + u) = i \sum_{j=1}^m du_j \wedge d\bar{u}_j + O(|u|).$$

We also choose a local frame  $e_L$  of  $L$  near  $z$ , such that the corresponding  $\varphi = -\log h(e_L, e_L)$  is given by

$$\varphi(z_0 + u) = |u|^2 + O(|u|^3).$$

See [ZZ19b] for more on such adapted frames and Heisenberg coordinates.

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<sup>(2)</sup> We use gradient flow of  $H$  in (7) and the exponential map in (6). They give the same leading term since the difference between  $F^{\beta/\sqrt{k}}(z)$  and  $\exp(\beta|\nabla H|(z)/\sqrt{k})(z)$  is of higher order in the  $O(k^{-1/2})$  expansion.

Clearly, the formula (7) breaks down at critical points and near such points on critical levels. Our main goal in this paper is to generalize the interface asymptotics to the case when the Hamiltonian is a Morse function and the interface  $\mathcal{C}=\{H=E\}$  is a critical level, so that  $\mathcal{C}$  contains a non-degenerate critical point  $z_c$  of  $H$ . To allow for non-standard scaling asymptotics, we study the smoothed partial Bergman density near the critical value  $E=H(z_c)$ ,

$$\Pi_{k,E,f,\delta}(z) := \sum_j \|s_{k,j}(z)\|^2 \cdot f(k^\delta(\mu_{k,j}-E))$$

where  $f \in \mathcal{S}(\mathbb{R})$  with Fourier transform  $\hat{f} \in C_c^\infty(\mathbb{R})$ , and  $0 \leq \delta \leq 1$ . This is the smooth analog of summing over eigenvalues within  $[E-k^{-\delta}, E+k^{-\delta}]$ .

The behavior of the scaled density of states is encoded in the following measures,

$$(8) \quad \begin{cases} d\mu_k^z(x) = \sum_j \|s_{k,j}(z)\|^2 \delta_{\mu_{k,j}}(x), \\ d\mu_k^{z,\delta}(x) = \sum_j \|s_{k,j}(z)\|^2 \delta_{k^\delta(\mu_{k,j}-H(z))}(x), \\ d\mu_k^{(z,u,\varepsilon),\delta}(x) = \sum_j \|s_{k,j}(z+k^{-\varepsilon}u)\|^2 \delta_{k^\delta(\mu_{k,j}-H(z))}(x). \end{cases}$$

For each measure  $\mu$  we denote by  $d\hat{\mu}$  the normalized probability measure

$$d\hat{\mu}(x) = \mu(\mathbb{R})^{-1} d\mu(x).$$

For all  $z \in M$ , we have the following weak limit, reminiscent of the law of large numbers;

$$\hat{\mu}_k^z(x) \rightharpoonup \delta_{H(z)}(x).$$

For  $z \in M$  with  $dH(z) \neq 0$ , (7) shows that

$$\hat{\mu}_k^{z,1/2}(x) \rightharpoonup e^{-\frac{x^2}{|dH(z)|^2}} \frac{dx}{\sqrt{\pi}|dH(z)|}.$$

### 1.1. Main results

The first main result is the generalization of (6) to the critical point case. We use the following setup: Let  $z_c$  be a non-degenerate Morse critical point of  $H$ , then for small enough  $u \in \mathbb{C}^m$ , we denote the Taylor expansion components by

$$H(z_c+u) = E + H_2(u) + O(|u|^3).$$

where

$$E = H(z_c), \quad H_2(u) = \frac{1}{2} \text{Hess}_{z_c} H(u, u).$$

**Theorem 1.1.** *For any  $f \in \mathcal{S}(\mathbb{R})$  with  $\hat{f} \in C_c^\infty(\mathbb{R})$ , we have*

$$\begin{aligned} \Pi_{k,E,f,1/2}(z_c+k^{-1/4}u) &:= \sum_j \|s_{k,j}(z_c+k^{-1/4}u)\|^2 \cdot f(k^{1/2}(\mu_{k,j}-E)) \\ &= \left(\frac{k}{2\pi}\right)^m f(H_2(u)) + O_f(k^{m-1/4}). \end{aligned}$$

More over, the normalized rescaled pointwise spectral measure

$$d\hat{\mu}_k^{(z_c,u,1/4),1/2}(x) := \frac{\sum_j \|s_{k,j}(z_c+k^{-1/4}u)\|^2 \delta_{k^{1/2}(\mu_{k,j}-E)}(x)}{\sum_j \|s_{k,j}(z_c+k^{-1/4}u)\|^2}$$

converges weakly

$$\hat{\mu}_k^{(z_c,u,1/4),1/2}(x) \rightharpoonup \delta_{H_2(u)}(x).$$

We notice that the scaling width has changed from  $k^{-\frac{1}{2}}$  to  $k^{-1/4}$  due to the critical point. The fact that we obtained a ‘delta function’ in the limit is less surprising since it is simply a degenerate Gaussian. The techniques of this article allow for the generalization to Bott-Morse Hamiltonians with non-degenerate critical manifolds; since it is rather routine, we restrict to Morse functions to simplify the exposition.

The difference in scalings raises the question of what happens if we scale by  $k^{-\frac{1}{2}}$  around a critical point. The result is stated in terms of the metaplectic representation on the osculating Bargmann-Fock space at  $z_c$ . These notions are reviewed in Section 3. The key points are summarized in the statement of:

**Theorem 1.2.** *Let  $1 \gg T > 0$  be small enough, such that there is no non-constant periodic orbit with periods less than  $T$ . Then for any  $f \in \mathcal{S}(\mathbb{R})$  with  $\hat{f} \in C_c^\infty((-T, T))$ , we have*

$$\Pi_{k,E,f,1}(z_c+k^{-1/2}u) = \left(\frac{k}{2\pi}\right)^m \int_{\mathbb{R}} \hat{f}(t)\mathcal{U}(t,u) \frac{dt}{2\pi} + O(k^{m-1/2})$$

where  $\mathcal{U}(t,u)$  is the metaplectic quantization of the Hamiltonian flow of  $H_2(u)$  defined as

$$\mathcal{U}(t,u) = (\det P)^{-1/2} \exp(\bar{u}(P^{-1}-1)u + u\bar{Q}P^{-1}u/2 - \bar{u}P^{-1}Q\bar{u}/2).$$

Here  $P=P(t), Q=Q(t)$  are complex  $m \times m$  matrices such that if  $u(t) = \exp(t\xi_{H_2})u$ , then

$$\begin{pmatrix} u(t) \\ \bar{u}(t) \end{pmatrix} = \begin{pmatrix} P(t) & Q(t) \\ \bar{Q}(t) & \bar{P}(t) \end{pmatrix} \begin{pmatrix} u \\ \bar{u} \end{pmatrix}.$$

*Remark 1.3.* Unlike the universal Erf decay profile in the  $1/\sqrt{k}$ -tube around the smooth part of  $\mathcal{C}$ , we cannot give the decay profile of  $\Pi_{k,I}(z)$  near the critical point  $z_c$ . The reason is that there are eigensections that highly peak near  $z_c$  and with eigenvalues clustering around  $H(z_c)$ . Hence it even matters whether we use  $[E_1, E_2]$  or  $(E_1, E_2)$ . See the following case where the Hamiltonian action is holomorphic, where the peak section at  $z_c$  is an eigensection, and all other eigensections vanishes at  $z_c$ .

The next result pertains to Hamiltonians generating holomorphic  $\mathbb{R}$  actions, as studied in [RS], [ZZ19a]. The Hamiltonian flow always extends to a holomorphic  $\mathbb{C}$  action.

**Proposition 1.4.** *Assume  $H$  generate a holomorphic Hamiltonian  $\mathbb{R}$  action. The pointwise spectral measure  $d\mu_k^{z_c}(x)$  is always a delta-function*

$$\mu_k^{z_c} = \delta_{H(z_c)}(x), \quad \forall k = 1, 2, \dots$$

*Equivalently, for any spectral interval  $I$ ,*

$$\lim_{k \rightarrow \infty} \Pi_{k,I}(z_c) = \begin{cases} 1 & E \in I \\ 0 & E \notin I \end{cases}.$$

The above result follows immediately from:

**Proposition 1.5.** *Let  $z_c$  be a Morse critical point of  $H$ ,  $E = H(z_c)$ . Then*

(1) *The  $L^2$ -normalized peak section  $s_{k,z_c}(z) = C(z_c)\Pi_k(z, z_c)$  is an eigensection of  $\widehat{H}_k$  with eigenvalue  $H(z_c)$ . And all other eigensections orthogonal to  $s_{k,z_c}$  vanishes at  $z_c$ .*

(2) *If  $s_{k,j} \in H^0(M, L^k)$  is an eigensection of  $\widehat{H}_k$  with eigenvalue  $\mu_{k,j} < E$ , then  $s_{k,j}$  vanishes on  $W^+(z_c)$ .*

(3) *If  $s_{k,j} \in H^0(M, L^k)$  is an eigensection of  $\widehat{H}_k$  with eigenvalue  $\mu_{k,j} > E$ , then  $s_{k,j}$  vanishes on  $W^-(z_c)$ .*

In the above statement,  $W^\pm(z_c)$  refers to the stable (resp. unstable) manifold for critical point  $z_c$  and flow  $\nabla H$ . See Section 6 for more details and a proof.

In particular, this shows the concentration of eigensection near  $z_c$ . Depending on whether the spectral interval  $I$  includes boundary point  $H(z_c)$  or not, the partial Bergman density will differ by a large Gaussian bump of height  $\sim k^m$ .

### 1.2. Sketch of proof

As in [ZZ19b], [ZZ18] the proofs involve rescaling parametrices for the propagator

$$(9) \quad U_k(t) = \exp itk\widehat{H}_k$$

of the Hamiltonian (1). The parametrix construction is reviewed in Section 2. We begin by observing that for all  $z \in M$ , the time-scaled propagator has pointwise scaling asymptotics with the  $k^{-\frac{1}{2}}$  scaling:

**Proposition 1.6.** ([ZZ19b] Proposition 5.3) *If  $z \in M$ , then for any  $\tau \in \mathbb{R}$ ,*

$$\widehat{U}_k(t/\sqrt{k}, \widehat{z}, \widehat{z}) = \left(\frac{k}{2\pi}\right)^m e^{it\sqrt{k}H(z)} e^{-t^2 \frac{\|dH(z)\|^2}{4}} (1 + O(|t|^3 k^{-1/2})),$$

where the constant in the error term is uniform as  $t$  varies over compact subset of  $\mathbb{R}$ .

The condition  $dH(z) \neq 0$  in the original statement in [ZZ19b] is never used in the proof, hence both statement and proof carry over to the critical point case. We therefore omit the proof of this Proposition.

We also give asymptotics for the trace of the scaled propagator  $U_k(t/\sqrt{k})$ . It is based on stationary phase asymptotics and therefore also reflects the structure of the critical points.

**Theorem 1.7.** *If  $t \neq 0$ , the trace of the scaled propagator  $U_k(t/\sqrt{k}) = e^{i\sqrt{kt}\widehat{H}_k}$  admits the following asymptotic expansion*

$$\begin{aligned} \int_{z \in M} U_k(t/\sqrt{k}, z) d\text{Vol}_M(z) &= \left(\frac{k}{2\pi}\right)^m \left(\frac{t\sqrt{k}}{4\pi}\right)^{-m} \\ &\cdot \sum_{z_c \in \text{crit}(H)} \frac{e^{it\sqrt{k}H(z_c)} e^{(i\pi/4) \text{sgn}(\text{Hess}_{z_c}(H))}}{\sqrt{|\det(\text{Hess}_{z_c}(H))|}} \\ &\cdot (1 + O(|t|^3 k^{-1/2})) \end{aligned}$$

where  $\text{sgn}(\text{Hess}_{z_c}(H))$  is the signature of the Hessian, i.e. the number of its positive eigenvalues minus the number of its negative eigenvalues.

To avoid duplication of the background sections in [ZZ19b], [ZZ18], we refer to those papers for discussions of osculating Bargmann-Fock spaces, for the Boutet-de-Monvel-Sjostrand parametrix for the Bergman kernel, and the corresponding parametrix for the propagator. This requires background on lifting Hamiltonian flows to contact flows on the unit frame bundle of  $L^*$  and its quantization as the Toeplitz operator (1). All the necessary background for this article is contained in the early sections of [ZZ19b], [ZZ18].

### 1.3. Related results

Some results on the distribution of eigenvalues around critical levels of Schrödinger operators on Riemannian manifolds may be found in [BPU95], [Cam04], [Cam04b], [Cam08]. As far as we saw, the articles do not study the spectral projections kernels pointwise in that setting. Most relevant to the present articles are results of Deleporte on ground states of Toeplitz Hamiltonians around minimum points, both non-degenerate [D16a] and Morse-Bott degenerate manifolds of minima [D16b]. The authors plan to extend the results of the present article to the case of Morse-Bott minima in a subsequent article.

## 2. Toeplitz quantization of Hamiltonian flows

In this section we briefly review the construction of a Toeplitz parametrix for the propagator  $U_k(t)$  of the quantum Hamiltonian (1). For a detailed presentation we refer to [ZZ19b], [ZZ18], and for a more general background on Toeplitz operator we refer to [BG], [MeSj].

Let  $(M, \omega, L, h)$  be a polarized Kähler manifold, and  $\pi: X \rightarrow M$  the unit circle bundle in the dual bundle  $(L^*, h^*)$ .  $X$  is a contact manifold, equipped with the Chern connection contact one-form  $\alpha$ , whose associated Reeb flow  $R$  is the rotation  $\partial_\theta$  in the fiber direction of  $X$ . Any Hamiltonian vector field  $\xi_H$  on  $M$  generated by a smooth function  $H: M \rightarrow \mathbb{R}$  can be lifted to a contact Hamiltonian vector field  $\hat{\xi}_H$  on  $X$ , which generates a contact flow  $\hat{g}^t$ . The following Proposition expresses the lift of (9) to  $\mathcal{H}(X) = \bigoplus_{k \geq 0} \mathcal{H}_k(X)$ .

**Proposition 2.1.** *There exists a semi-classical symbol  $\sigma_k(t)$  so that the unitary group (9) has the form*

$$(10) \quad \hat{U}_k(t) = \hat{\Pi}_k(\hat{g}^{-t})^* \sigma_k(t) \hat{\Pi}_k$$

*modulo smooth kernels of order  $k^{-\infty}$ .*

It follows from Proposition 2.1 and from the Boutet de Monvel–Sjöstrand parametrix construction for the Szegő kernel that  $\hat{U}_k(t, x, x)$  admits an oscillatory integral representation of the form,

$$(11) \quad \begin{aligned} \hat{U}_k(t, x, x) &\simeq \int_X \int_0^\infty \int_0^\infty \int_{S^1} \int_{S^1} e^{\sigma_1 \hat{\psi}(r_{\theta_1} x, \hat{g}^t y) + \sigma_2 \hat{\psi}(r_{\theta_2} y, x) - ik\theta_1 - ik\theta_2} \\ &\times S_k d\theta_1 d\theta_2 d\sigma_1 d\sigma_2 dy \end{aligned}$$

where  $S_k$  is a semi-classical symbol, and the asymptotic symbol  $\simeq$  means that the difference of the two sides is rapidly decaying in  $k$ . The phase function  $\psi$  is that

of the Szegő kernel, i.e. is the (almost)-analytic extension of the defining function of the strictly pseudo-convex domain  $D_h^* \subset L^*$  and is closely related to the analytic extension of the Kähler potential  $\phi(z, \bar{z})$  to the off-diagonal.

We use the notation  $\hat{w} = (w, \theta_w)$  for points such that  $\pi(\hat{w}) = w$ , and  $\hat{g}^t \hat{w} = (w(t), \theta_w(t))$  for  $|w| < \varepsilon, |g^t w| < \varepsilon$ . The Taylor expansion of the phase function  $\hat{\psi}$  around the diagonal has the form

$$\hat{\psi}(0, \hat{g}^t \hat{w}) + \hat{\psi}(\hat{w}, 0) = i(\theta_w(0) - \theta_w(t)) - |w(0)|^2/2 - |w(t)|^2/2 + O(|w|^3 + |w(t)|^3), \tag{12}$$

If we scale the variables by

$$w = u/\sqrt{k}, \quad t = \tau/\sqrt{k},$$

(12) becomes

$$k^{-1/2}[iH(0)\tau] + k^{-1}[i\frac{1}{2}H_1(u)\tau - |u|^2/2 - |u + \xi_{H_1}\tau|^2/2] + O(k^{-3/2}(|u|^3 + |\tau|^3)).$$

We will be scaling with other powers of  $k$  but the general expansion is similar.

We refer to [ZZ19b], [ZZ18] for detailed discussions of this parametrix and references to the literature.

### 3. Model case: Bargman-Fock space

We now discuss the linear (Bargmann-Fock) model in detail, since it is used to reduce nonlinear settings to the linear one.

Let  $M = \mathbb{C}^m$  with coordinate  $z_i = x_i + \sqrt{-1}y_i$ ,  $L \rightarrow M$  be the trivial line bundle. We fix a trivialization and identify  $L \cong \mathbb{C}^m \times \mathbb{C}$ . We use Kähler form<sup>(3)</sup>

$$\omega = i \sum_i dz_i \wedge d\bar{z}_i = 2 \sum_i dx_i \wedge dy_i$$

and Kähler potential

$$\varphi(z) = |z|^2 := \sum_i |z_i|^2.$$

The Bargmann-Fock space of degree  $k$  on  $\mathbb{C}^m$  is defined by

$$\mathcal{H}_k = \{f(z)e^{-k|z|^2/2} \mid f(z) \text{ holomorphic function on } \mathbb{C}^m, \int_{\mathbb{C}^m} |f(z)|^2 e^{-k|z|^2} d\text{Vol}_{\mathbb{C}^m}(z) < \infty\}.$$

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<sup>(3)</sup> We warn the reader that the normalization of  $\omega$  may differ by factor of 2 or  $\pi$  from other references. In particular, our metric  $g$  on  $\mathbb{R}^{2m}$  is twice the Euclidean metric.

The volume form on  $\mathbb{C}^m$  is  $d\text{Vol}_{\mathbb{C}^m} = \omega^m / m!$ .

The circle bundle  $\pi: X \rightarrow M$  can be trivialized as  $X \cong \mathbb{C}^m \times S^1$ . The contact form on  $X$  is

$$\alpha = d\theta + (i/2) \sum_j (z_j d\bar{z}_j - \bar{z}_j dz_j).$$

and the Reeb flow is  $R = \partial_\theta$ . If  $s(z)$  is a holomorphic function (section of  $L^k$ ) on  $\mathbb{C}^m$ , then its CR-holomorphic lift to  $X$  is

$$\hat{s}(z, \theta) = e^{k(i\theta - \frac{1}{2}|z|^2)} s(z).$$

Indeed, the horizontal lift of  $\partial_{\bar{z}_j}$  is  $\partial_{\bar{z}_j}^h = \partial_{\bar{z}_j} - \frac{i}{2} z_j \partial_\theta$ , and  $\partial_{\bar{z}_j}^h \hat{s}(z, \theta) = 0$ . The volume form on  $X$  is  $d\text{Vol}_X = (d\theta/2\pi) \wedge \omega^m / m!$ .

More invariantly, let  $(V, \omega)$  be a real  $2m$  dimensional symplectic vector space. Let  $J: V \rightarrow V$  be a  $\omega$  compatible linear complex structure, that is  $g(v, w) := \omega(v, Jw)$  is a positive-definite bilinear form and  $\omega(v, w) = \omega(Jv, Jw)$ . There exists a canonical identification of  $V \cong \mathbb{C}^m$  up to  $U(m)$  action, identifying  $\omega$  and  $J$ . We denote the BF space for  $(V, \omega, J)$  by  $\mathcal{H}_{k,J}$ .

### 3.1. Linear Hamiltonian function and Heisenberg representation

A linear Hamiltonian function  $H$  on  $\mathbb{C}^m$  has the form,

$$(13) \quad H(x, y) = \text{Re}(\alpha \cdot \bar{z}) = \frac{1}{2}(\alpha \bar{z} + \bar{\alpha} z),$$

for some  $0 \neq \alpha \in \mathbb{C}^m$ . Then the contact vector field generated by  $H$  is

$$\hat{\xi}_H = \sum_j (-i/2)(\alpha_j \partial_{z_j} - \bar{\alpha}_j \partial_{\bar{z}_j}) - \frac{1}{2} H \partial_\theta.$$

The contact lifted Hamiltonian flow  $\hat{g}^t(\hat{z}) = \exp(t\hat{\xi}_H)$  is then

$$(14) \quad \hat{g}^t(\hat{z}) = (z + \frac{\alpha t}{2i}, \theta - \frac{t}{4}(\alpha \bar{z} + \bar{\alpha} z)), \quad \hat{z} = (z, \theta).$$

**Proposition 3.1.** ([ZZ19b], Proposition 5.1) *The kernel for the propagator  $\hat{U}_k(t) = \hat{\Pi}_k e^{ikt\hat{H}_k} \hat{\Pi}_k$ , is then given by*

$$(15) \quad \hat{U}_k(t, \hat{z}, \hat{w}) = \hat{\Pi}_k(\hat{g}^{-t}\hat{z}, \hat{w}) = \left(\frac{k}{2\pi}\right)^m e^{k\hat{\psi}(\hat{g}^{-t}\hat{z}, \hat{w})}.$$

where the function  $\hat{\psi}(\hat{z}, \hat{w})$  is given by

$$\hat{\psi}(\hat{z}, \hat{w}) = i(\theta_z - \theta_w) + z\bar{w} - |z|^2/2 - |w|^2/2, \quad \hat{z} = (z, \theta_z), \hat{w} = (w, \theta_w).$$

In particular, if  $\hat{z}=\hat{w}$ , we have

$$(16) \quad \hat{U}_k(t, \hat{z}, \hat{z}) = \left(\frac{k}{2\pi}\right)^m e^{ikH(z)t} e^{-kt^2 \frac{\|dH(z)\|^2}{4}},$$

where  $\|dH(z)\|^2=|\alpha|^2/2$ .

### 3.2. Quadratic Hamiltonian function and metaplectic representation

Identify  $\mathbb{C}^m$  with  $\mathbb{R}^{2m}$ . The space  $Sp(m, \mathbb{R})$  consists of linear transformation  $S:\mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$ , such that  $S^*\omega=\omega$ . In coordinates, we write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In complex coordinates  $z_i=x_i+iy_i$ , we have then

$$\begin{pmatrix} z' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} =: \mathcal{A} \begin{pmatrix} z \\ \bar{z} \end{pmatrix},$$

where

$$(17) \quad \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} = \mathcal{W}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathcal{W}, \quad \mathcal{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}.$$

The choice of normalization of  $\mathcal{W}$  is such that  $\mathcal{W}^{-1}=\mathcal{W}^*$ . Thus,

$$P = \frac{1}{2}(A+D+i(C-B)).$$

We say such  $\mathcal{A} \in Sp_c(m, \mathbb{R}) \subset M(2n, \mathbb{C})$ . The following identities are often useful.

**Proposition 3.2.** ([F89] Proposition 4.17) *Let  $\mathcal{A} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in Sp_c$ , then*

- (1)  $\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}^{-1} = \begin{pmatrix} P^* & -Q^t \\ -Q^* & P^t \end{pmatrix} = K\mathcal{A}^*K$ , where  $K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ .
- (2)  $PP^* - QQ^* = I$  and  $PQ^t = QP^t$ .
- (3)  $P^*P - Q^t\bar{Q} = I$  and  $P^t\bar{Q} = Q^*P$ .

The (double cover) of  $Sp(m, \mathbb{R})$  acts on the (downstairs) BF space  $\mathcal{H}_k$  via kernel: given  $M = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in Sp_c$ , we have

$$\mathcal{K}_{k,M}(z, w) = \left(\frac{k}{2\pi}\right)^m (\det P)^{-1/2} \exp \left\{ k \left( z\bar{Q}P^{-1}z/2 + \bar{w}P^{-1}z - \bar{w}P^{-1}Q\bar{w}/2 \right) \right\}$$

where the ambiguity of the sign the square root  $(\det P)^{-1/2}$  is determined by the lift to the double cover. When  $\mathcal{A}=Id$ , then  $\mathcal{K}_{k,\mathcal{A}}(z, \bar{w})=\Pi_k(z, \bar{w})$ .

The associated density of states is thus given by the metric contraction,

$$\mathcal{K}_{k,M}(z) = \left(\frac{k}{2\pi}\right)^m (\det P)^{-1/2} \exp \{k(z\bar{Q}P^{-1}z/2 + \bar{z}P^{-1}z - \bar{z}P^{-1}Q\bar{z}/2) - k|z|^2\}.$$

Another useful expression for  $\mathcal{K}_{k,M}$  in the spirit of Proposition 2.1 is the following:

**Proposition 3.3.** ([ZZ18] Proposition 2.4) *Let  $\mathcal{A}:\mathbb{C}^m \rightarrow \mathbb{C}^m$  be a linear symplectic map,  $\mathcal{A} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$ , and let  $\hat{\mathcal{A}}:X \rightarrow X$  be the contact lift that fixes the fiber over 0, then*

$$\hat{\mathcal{K}}_{k,\mathcal{A}}(\hat{z}, \hat{w}) = (\det P^*)^{1/2} \int_X \hat{\Pi}_k(\hat{z}, \hat{\mathcal{A}}\hat{u}) \hat{\Pi}_k(\hat{u}, \hat{w}) d\text{Vol}_X(\hat{u})$$

*Remark 3.4.* The point of the above proposition is that, the symbol  $\sigma_k(t)$  in (10) is given by  $(\det P^*)^{1/2}$ .

Consider quadratic Hamiltonian  $H:\mathbb{R}^{2m} = \mathbb{C}^m \rightarrow \mathbb{R}$ ,

$$H = \sum_{i,j} (1/2)a_{ij}z_i z_j + (1/2)\bar{a}_{ij}\bar{z}_i \bar{z}_j + b_{ij}z_i \bar{z}_j$$

where  $a_{ij}=a_{ji} \in \mathbb{C}$  and  $b_{ij}=\bar{b}_{ji} \in \mathbb{C}$ . Then the Hamiltonian vector field with respect to  $\omega=i \sum_j dz_j d\bar{z}_j$  is

$$\xi_H = \sum_{i,j} (a_{ij}z_j + b_{ij}\bar{z}_j)(i\partial_{\bar{z}_i}) + (a_{ij}\bar{z}_i + b_{ij}z_i)(-i\partial_{z_j}).$$

Hence, if  $\xi_H$  generates the flow  $P(t), Q(t)$ , then

$$\frac{d}{dt} \begin{pmatrix} P(t) & Q(t) \\ \bar{Q}(t) & \bar{P}(t) \end{pmatrix} = \begin{pmatrix} -i\bar{b} & -i\bar{a} \\ ia & ib \end{pmatrix} \begin{pmatrix} P(t) & Q(t) \\ \bar{Q}(t) & \bar{P}(t) \end{pmatrix}.$$

In particular, since  $P(0)=Id, Q(0)=0$ , we have

$$\dot{P}(0) = -i\bar{b}, \quad \dot{Q}(0) = -i\bar{a}.$$

*Remark 3.5.*  $\xi_H$  preserves the holomorphic structure, if and only if  $a_{ij}=0$ . Thus  $Q(t)=0, P(t)=e^{-it\bar{b}} \in U(m)$ , and  $P(t)^{-1}=P(-t)=P(t)^* = e^{it\bar{b}}$ .

**4. Smoothed partial Bergman density with spectrum width  $k^{-1/2}$ :  
proof of Theorem 1.1**

To prove Theorem 1.1 we first consider smoothed sums over eigenvalues in a  $k^{-1/2}$  neighborhood of an energy. We first state a lemma about localization of sum.

**Lemma 4.1.** *For any  $1 \gg \varepsilon > 0$  and any  $z \in M$ , we can find  $R > 1$  large enough, such that*

$$\liminf_{k \rightarrow \infty} \frac{\sum_j 1_{[-1,1]}(\sqrt{k}(\mu_{k,j} - H(z))/R) \|s_{k,j}(z)\|^2}{\sum_j \|s_{k,j}(z)\|^2} > 1 - \varepsilon.$$

*Proof.* Let  $\chi: \mathbb{R} \rightarrow [0, 1]$  be a smooth function, such that  $\chi(x) = 1$  and  $\chi(x) = 0$  for  $|x| > 1$ . Furthermore, we may require its Fourier transform  $\widehat{\chi}(t) \geq 0$ , e.g. choose  $\chi(x) = (\eta \star \eta)(x)$  for some  $\eta \in C_c^\infty(\mathbb{R})$ . Since  $1_{[-1,1]}(\sqrt{k}(\mu_{k,j} - H(z))/R) \geq \chi(\sqrt{k}(\mu_{k,j} - H(z))/R)$ , hence it suffices to prove for any  $\varepsilon > 0$ , one can find  $R > 0$  large enough that

$$\liminf_{k \rightarrow \infty} \frac{\sum_j \chi(\sqrt{k}(\mu_{k,j} - H(z))/R) \|s_{k,j}(z)\|^2}{\sum_j \|s_{k,j}(z)\|^2} > 1 - \varepsilon.$$

or

$$\lim_{R \rightarrow \infty} \liminf_{k \rightarrow \infty} \int_{\mathbb{R}} \widehat{\chi}_R(t) V_k(t/\sqrt{k}, z) \frac{dt}{2\pi} = 1, \text{ where } V_k(t, z) := \frac{e^{-iktH(z)} U_k(t, z)}{U_k(0, z)},$$

where  $\chi_R(x) := \chi(x/R)$ , and its Fourier transformation is  $\widehat{\chi}_R(t) = R\widehat{\chi}(Rt)$ .

First, we note that  $\widehat{\chi}_R(t) \geq 0$  and  $\int \widehat{\chi}_R(t) \frac{dt}{2\pi} = \chi_R(0) = 1$ . Since  $\widehat{\chi}(t)$  is a Schwartz function, for any positive integer  $N$ , we have constant  $C_N$ , such that for  $|t| > 1$ ,  $|\widehat{\chi}(t)| < C_N |t|^{-N}$ . Hence, for any smooth bounded function  $f(t)$ , we have

$$\lim_{R \rightarrow \infty} \widehat{\chi}_R(t) f(t) \frac{dt}{2\pi} = f(0).$$

Next, we claim that  $|V_k(t, z)| \leq 1 = V_k(0, z)$  for all  $t \in \mathbb{R}$ . Indeed, let  $E_k^z \in H^0(M, L^k)$  be an  $L^2$  normalized peak section (or coherent state) at  $z$ , i.e. the  $L^2$  normalization of the section  $z \rightarrow \Pi_{h^k}(\cdot, z)$ . Then the  $L^2$ -normalized section  $e^{itk\widehat{H}_k} E_k^z$  satisfies

$$\left| \frac{\langle e^{itk\widehat{H}_k} E_k^z, E_k^z \rangle}{\|E_k^z\|^2} \right| = \frac{\| \langle e^{itk\widehat{H}_k} E_k^z, E_k^z \rangle \|}{\|E_k^z\|^2} \leq 1.$$

Hence we have

$$|V_k(t, z)| = \left| \frac{\langle e^{itk\widehat{H}_k} E_k^z, E_k^z \rangle}{\langle E_k^z, E_k^z \rangle} \right| \leq \frac{\| \langle e^{itk\widehat{H}_k} E_k^z, E_k^z \rangle \|}{\|E_k^z\|^2} \leq 1.$$

If we choose cut-off function  $\eta(t)$ , that  $\eta(t)=1$  for  $|t|<1$  and  $\eta(t)=0$  for  $|t|>2$ . Then for any  $T>1$ , we have

$$\int \chi_R(t)\eta(t/T)V_k(t/\sqrt{t}, z) \frac{dt}{2\pi} = \int \chi_R(t)\eta(t/T)e^{-t^2\|dH(z)\|^2/4} \frac{dt}{2\pi} + O(k^{-1/2})$$

and for each integer  $N \geq 1$ , we have constant  $c_N$  independent of  $R, T$ , that

$$\begin{aligned} \int \chi_R(t)(1-\eta(t/T))|V_k(t/\sqrt{t}, z)| \frac{dt}{2\pi} &\leq \int_{|t|>T} \hat{\chi}_R(t) \frac{dt}{2\pi} = \int_{|t|>RT} \hat{\chi}(t) \frac{dt}{2\pi} \\ &= c_N |RT|^{1-N}. \end{aligned}$$

Hence

$$1 \geq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}} \hat{\chi}_R(t)V_k(t/\sqrt{k}, z) \frac{dt}{2\pi} \geq \int \chi_R(t)\eta(t/T)e^{-t^2\|dH(z)\|^2/4} \frac{dt}{2\pi} - c_N(RT)^{1-N}$$

Taking limit  $R \rightarrow \infty$ , we get

$$1 \geq \lim_{R \rightarrow \infty} \liminf_{k \rightarrow \infty} \int_{\mathbb{R}} \hat{\chi}_R(t)V_k(t/\sqrt{k}, z) \frac{dt}{2\pi} \geq \eta(0) = 1.$$

This finishes the proof of the Lemma.  $\square$

### 4.1. Proof of Theorem 1.1

*Proof.* We consider Fourier transform of  $f$  in the definition of  $\Pi_{k,E,f,1/2}(z_c + k^{-1/4}u)$ . Write  $z_c + k^{-1/4}u = z$ . Using the parametrix (11) for the propagator (9), and Taylor expanding the phase as in (12),

$$\begin{aligned} \Pi_{k,E,f,1/2}(z) &= \sum_j \|s_{k,j}(z)\|^2 \cdot \int_{\mathbb{R}} e^{itk^{1/2}(\mu_{k,j}-E)} \hat{f}(t) dt \\ &= \int_{\mathbb{R}} e^{itk^{1/2}(-E)} \hat{f}(t)U_k(t/\sqrt{k}, z) dt \\ &= \left(\frac{k}{2\pi}\right)^m \int_{\mathbb{R}} \hat{f}(t)e^{itk^{1/2}(H(z)-H(z_c))} e^{-t^2\|dH(z)\|^2/4} (1+O(|t|^3k^{-1/2})) dt \\ &= \left(\frac{k}{2\pi}\right)^m \int_{\mathbb{R}} \hat{f}(t)e^{it(\text{Hess}_{z_c}H)(u,u)/2} [1+O(|t|^3k^{-1/2})+O(|t|^2k^{-1/4})] dt \\ &= \left(\frac{k}{2\pi}\right)^m f((\text{Hess}_{z_c}H)(u,u)/2) + O_f(k^{m-1/4}). \end{aligned}$$

where in the last step, we use the fast decay of  $\hat{f}(t)$  to bound the error term that grows as power law in  $|t|$ .

To show the weak convergence, suffice to test again all continuous bounded function  $f \in C_b(\mathbb{R})$ . It is not hard to see that this sequence of measures  $\{\hat{\mu}_k^{(z_c, u, 1/4), 1/2}\}_k$  is tight, hence it suffices to test against only compactly supported continuous functions  $f \in C_c(\mathbb{R})$ . Finally, since  $d\hat{\mu}_k$  all has unit mass, suffice to test against  $f \in C_c^\infty(\mathbb{R})$ .

Now we show this sequence of measures  $\{\hat{\mu}_k^{(z_c, u, 1/4), 1/2}\}_k$  is tight. Suffice to show for any  $\varepsilon > 0$ , exists  $R > 0$ , such that

$$\left(\frac{k}{2\pi}\right)^{-m} \sum_{|\mu_{k,j} - H(z)| > k^{-1/2}R} \|s_{k,j}(z)\|^2 < \varepsilon.$$

This follows from Lemma 4.1.  $\square$

*Remark 4.2.* The proof of Theorem 1.1 is similar to the proof of (7) in [ZZ19b]. The only change is that the linear term vanishes and one has a quadratic term instead. This accounts for the different scaling.

### 5. Smoothed partial Bergman density with spectrum width $k^{-1}$ : proof of Theorem 1.2

Recall that  $z_c \in M$  is a non-generate critical point of  $H$  and  $E = H^{-1}(z_c)$ . For simplicity of notation, we may assume  $z_c$  is the only critical point on  $H^{-1}(E)$ .

Assume that for each  $T > 0$ , there are finitely many closed Hamiltonian orbit with primitive period less than  $T$ . In particular, there exists  $1 \gg T > 0$ , such that there is no closed Hamiltonian orbit with primitive period less than  $T$  except for constant orbit at critical points.

For  $z_0 \in H^{-1}(E)$ , we consider the following partial Bergman density

$$\Pi_{k,E,f,1}(z_0 + k^{-1/2}u) := \sum_j \|s_{k,j}(z)\|^2 \cdot f(k(\mu_{k,j} - E))$$

where we used Kähler normal coordinate to identify a neighborhood of  $z_0$  with  $T_{z_0}M$

$$z_0 + k^{-1/2}u := \exp_{z_0}(k^{-1/2}u), \quad u \in T_{z_0}M,$$

and we choose test function  $f$  that

$$f \in \mathcal{S}(\mathbb{R}) \text{ with } \hat{f}(t) \in C_c^\infty(-T, T).$$

**Proposition 5.1.** *If  $z_0$  is not a critical point of  $H$ , then*

$$\Pi_{k,E,f,1}(z_0 + k^{-1/2}u) = \left(\frac{k}{2\pi}\right)^{m-1/2} \frac{\sqrt{2}\hat{f}(0)}{2\pi\|dH(z_0)\|} e^{-|\langle dH(z_0), u \rangle|^2 / \|dH(z_0)\|^2} + O(k^{m-1})$$

*Proof.* A similar case is considered in [ZZ19b] Theorem 3; we repeat the proof here for completeness.

$$\begin{aligned}
 & \Pi_{k,E,f,1}(z_0+k^{-1/2}u) \\
 &= \int_{-T}^T \hat{f}(t)e^{-itkE}U_k(t, z_0+k^{-1/2}u) \frac{dt}{2\pi} \\
 &= \left(\frac{k}{2\pi}\right)^m \int_{-T}^T \hat{f}(t)e^{itk(H(z_0+k^{-1/2}u)-H(z_0))} e^{-kt^2\|dH(z_0+k^{-1/2}u)\|^2/4} \frac{dt}{2\pi} \\
 &\quad \times (1+O(k^{-1/2})) \\
 &= \left(\frac{k}{2\pi}\right)^m \int_{-T\sqrt{k}}^{T\sqrt{k}} \hat{f}(\tau/\sqrt{k})e^{i\tau\langle dH(z_0),u\rangle} e^{-\tau^2\|dH(z_0)\|^2/4} \frac{d\tau}{2\pi\sqrt{k}}(1+O(k^{-1/2})) \\
 &= \left(\frac{k}{2\pi}\right)^{m-1/2} \hat{f}(0)\frac{\sqrt{2}}{2\pi\|dH(z_0)\|} e^{-|\langle dH(z_0),u\rangle|^2/\|dH(z_0)\|^2} +O(k^{m-1}) \quad \square
 \end{aligned}$$

**5.1. Proof of Theorem 1.2**

We now complete the proof of Theorem 1.2.

*Proof.* We first use the Fourier transform to write,

$$(18) \quad \Pi_{k,E,f,1}(z_c+k^{-1/2}u) = \int_{-T}^T \hat{f}(t)e^{-itkE}U_k(t, z_c+k^{-1/2}u) \frac{dt}{2\pi}.$$

Next, we make a linear (Bargmann-Fock) approximation of  $U_k(t, z_c+k^{-1/2}u)$  for  $t \in (-T, T)$ .<sup>(4)</sup>

We lift the propagator to the unit frame bundle, where as in Section 2,

$$(19) \quad U_k(t, z) = \widehat{U}_k(t, \hat{z}, \hat{z}) = \int_X \widehat{\Pi}_k(\hat{z}, \hat{g}^t \hat{w}) \widehat{\Pi}_k(\hat{w}, \hat{z}) \eta_k(t, \hat{z}, \hat{w}) d\hat{w} + R_k(t, z).$$

First, we may cut-off the integral of  $w$ , such that  $w$  and  $g^t w$  are within  $k^{-1/2+\varepsilon}$  neighborhood of  $z$ . This will introduce  $O(k^{-\infty})$  error term. Next, we set  $z = z_c+k^{-1/2}u$  where  $u$  is in a compact set  $K \subset \mathbb{C}^m$ , and use Kähler normal coordinate at  $z_c$ . We write

$$w = z_c+k^{-1/2}v, \quad v \in B(k^{-\varepsilon}).$$

Then, the Bergman kernel can be approximated as

$$\widehat{\Pi}_k(\hat{w}, \hat{z}) \widehat{\Pi}_k(\hat{z}, \hat{g}^t \hat{w}) = \left(\frac{k}{2\pi}\right)^{2m} e^{k\psi(\hat{w}, \hat{z})+k\psi(\hat{z}, \hat{w}(t))} (1+O(k^{-1/2}))$$

---

<sup>(4)</sup> We warn the reader that, even though there is no periodic orbit for  $\xi_H$  within time  $t \in (-T, T)$ , there might be periodic orbit for the linearized flow on  $T_{z_c}M$ .

where we write  $\hat{g}^t \hat{w} =: \hat{w}(t)$ , and as in (12),

$$\psi(\hat{w}, \hat{z}) = i(\theta_w - \theta_z) + k^{-1}(v\bar{u} - \frac{1}{2}|u|^2 - \frac{1}{2}|v|^2) + O((|v|^3 + |u|^3)k^{-3/2})$$

and

$$\psi(\hat{z}, \hat{w}(t)) = i(\theta_z - \theta_w(t)) + k^{-1}(u\bar{v}(t) - \frac{1}{2}|u|^2 - \frac{1}{2}|v(t)|^2) + O((|v(t)|^3 + |u|^3)k^{-3/2}).$$

We Taylor expand the remainder in the exponent, and get

$$\begin{aligned} & \widehat{\Pi}_k(\hat{w}, \hat{z})\Pi_k(\hat{z}, \hat{g}^t \hat{w}) \\ &= \left(\frac{k}{2\pi}\right)^{2m} e^{ikt(\theta_w - \theta_w(t))} e^{\psi_{BF}(u, v(t)) + \psi_{BF}(v, u)} (1 + O((|v(t)|^3 + |v|^3 + |u|^3)k^{-1/2})) \end{aligned}$$

We claim that the evolution of  $\hat{w}(t) = (w(t), \theta_w(t))$  can be computed using the oscillating Bargmann-Fock approximation  $z_c$  with

$$H_{BF}(z_c + u) := H(z_c) + H_2(u)$$

and

$$\omega_{BF}(z_c + u) := \omega(z_c), \varphi_{BF}(z_c + u) = |u|^2.$$

The non-obvious part is about the term  $e^{ikt(\theta_w - \theta_w(t))}$  where we refer to ([ZZ18], Proposition 3.5) for more detail.

Hence, we reduce the evolution of  $w = z_c + k^{-1/2}v$  to evolution of  $v$  in the Bargmann-Fock approximation, where the orbit is denoted as  $\hat{v}_{BF}(t) = (v_{BF}(t), \theta_v^{BF}(t))$ . Note that the factor of  $k$  in the phase is cancelled by  $(k^{-1/2}v)^2$  from the quadratic expansion.

$$\begin{aligned} & \widehat{\Pi}_k(\hat{w}, \hat{z})\Pi_k(\hat{z}, \hat{g}^t \hat{w}) \\ &= \left(\frac{k}{2\pi}\right)^{2m} e^{iktE + it(\theta_v - \theta_v^{BF}(t))} e^{\psi_{BF}(u, v(t)) + \psi_{BF}(v, u)} (1 + T|v|^3 O(k^{-1/2})) \end{aligned}$$

Now, we may plug back in (19), and do the  $dv$  integral. The integral can be computed in purely the Bargmann-Fock model, using Proposition 3.3.

$$U_k(t, z) = \left(\frac{k}{2\pi}\right)^m e^{itkE} \mathcal{U}(t, u) (1 + O(k^{-1/2})).$$

Finally, we plug back in to (18) and finish the proof of Theorem 1.2.  $\square$

**6. Holomorphic Hamiltonian action: proof of Proposition 1.5**

We recall the setup in from [ZZ19a]. Let  $(L, h) \rightarrow (M, \omega, J)$  be a holomorphic Hermitian line bundle, such that  $\omega = -i\partial\bar{\partial} \log h$ . Let  $H: M \rightarrow \mathbb{R}$  be a smooth Hamiltonian, such that  $\xi_H$  preserves the complex structure  $J$ . The Berezin-Toeplitz quantization reduces to the Kostant quantization

$$\widehat{H}_k := ik^{-1}\nabla_{\xi_H} + H : H^0(M, L^k) \longrightarrow H^0(M, L^k).$$

And the unitary operator  $U_k(t) = \Pi_k e^{i\widehat{H}_k t} \Pi_k$  simplifies as

$$U_k(t) := e^{ikt\widehat{H}_k} : H^0(M, L^k) \longrightarrow H^0(M, L^k).$$

*Example 6.1.* Let  $(L, h) \rightarrow (M, \omega)$  be a smooth projective toric variety with positive equivariant line bundle  $L$ , and let  $T = (S^1)^m$  be the compact torus acting on  $M$  and  $L$ . Let  $\mu: M \rightarrow \mathbb{R}^n$  be the moment map, with  $P = \mu(M)$ , such that lattice points  $kP \cap \mathbb{Z}^n$  is the weights in the weight decomposition of  $T$  on  $H^0(M, L^k)$ . Let  $x_1, \dots, x_n$  be coordinates on  $\mathbb{R}^n$ , then a non-zero linear function  $l = \sum_i a_i x_i$  defines Hamiltonian function on  $M$

$$H = l \circ \mu : M \longrightarrow \mathbb{R}.$$

And critical points (submanifolds) of  $H$  on  $M$  are intersection of toric boundary divisors on which  $H$  is constant, or faces of  $P$  where  $l$  is constant. If the coefficients  $a_i$  are generic, then only vertices of  $P$  are fixed point. If there exists  $c \neq 0$ , such that  $a_i = cn_i, n_i \in \mathbb{Z}$  for all  $i$ , then  $\xi_H$  integrate to a holomorphic  $S^1$ -action.

If  $\xi_H$  acts holomorphically, then we have a holomorphic  $\mathbb{R}$ -action, which extends to a holomorphic  $\mathbb{C}$ -action with the other generator  $\nabla H = J\xi_H$ .<sup>(5)</sup> If  $z_c$  is a critical point, we denote the stable / unstable manifolds by

$$W^\pm(z_c) = \{z \in M : \lim_{t \rightarrow \infty} \exp(\mp t \nabla H)z = z_c\}.$$

Thus  $H|_{W^-(z_c)} \leq H(z_c) \leq H|_{W^+(z_c)}$ .

Let  $e_L$  be a local non-vanishing section of  $L$ , invariant under the  $\mathbb{R}$  action, i.e.,  $\widehat{H}_1(e_L) = 0$ . Define  $\varphi$  by  $\|e_L(z)\|^2 = e^{-\varphi}$ . We recall the following easy lemma.

**Lemma 6.2.** ([ZZ19a], Lemma 2.2)

$$\nabla H(\varphi(z)) = 2H(z)$$

---

<sup>(5)</sup> Our convention for sign is that  $g(X, Y) = \omega(X, JY), dH(Y) = \omega(\xi_H, Y) = g(\nabla H, Y)$ .

*Proof.*

$$0 = \widehat{H}e_L = i\nabla_{\xi_H}e_L + He_L = i\langle A, \xi_H \rangle e_L + He_L = i\langle -\partial\varphi, \xi_H \rangle e_L + He_L$$

where  $A$  is the Chern connection one-form with respect to the trivialization  $e_L$ . Since  $e_L$  is non-vanishing, we have

$$H = i\langle \xi_H, \partial\varphi \rangle = \langle \xi_H, \frac{i}{2}(\partial - \bar{\partial})\varphi \rangle = \frac{1}{2}\langle \xi_H, d^c\varphi \rangle = \frac{1}{2}\langle \nabla H, d\varphi \rangle. \quad \square$$

**Lemma 6.3.** ([GS], Eq. (5.5))

$$\nabla H(\|s_{k,j}(z)\|^2) = -2k(H(z) - \mu_{k,j})\|s_{k,j}(z)\|^2$$

*Proof.* Let  $s_{k,j} = f_{k,j}e_L$ , then

$$\begin{aligned} \widehat{H}_k(s_{k,j}) &= (i/k)\nabla_{\xi_H}(f_{k,j}e_L) + H(f_{k,j}e_L) = (i/k)\xi_H(f_{k,j})e_L + f_{k,j}\widehat{H}_k(e_L) \\ &= (i/k)\xi_H(f_{k,j})e_L \end{aligned}$$

Hence,  $\mu_{k,j}f_{k,j}e_L = (i/k)\xi_H(f_{k,j})e_L$ , we have  $\xi_H(f_{k,j}) = -ik\mu_{k,j}f_{k,j}$ , hence

$$\langle \nabla H, df_{k,j} \rangle = \langle J\xi_H, \partial f_{k,j} \rangle = i\langle \xi_H, \partial f_{k,j} \rangle = k\mu_{k,j}f_{k,j}.$$

Since  $\nabla H$  is a real vector field, we can take complex conjugation to get

$$\langle \nabla H, d\overline{f_{k,j}} \rangle = k\mu_{k,j}\overline{f_{k,j}}.$$

Now, we can finish the proof by apply previous lemma and above results to  $\|s_{k,j}(z)\|^2 = e^{-k\varphi}|f_{k,j}(z)|^2$ .  $\square$

### 6.1. Proof of Proposition 1.5

*Proof.* (1) Let  $s_{k,z_c}(z) := \Pi_k(z, z_c)$  be the peak section at  $z_c$ , and  $\widehat{s}_{k,z_c}$  be the CR holomorphic function on the circle bundle  $X$  of  $L^*$ . Since the lifted contact flow  $\widehat{\xi}_H = \xi_H^h - H\partial_\theta$  on  $X$  preserves the fiber over  $z_c$  and acts by rotation, and

$$e^{ikt\widehat{H}_k}s_k = \exp(-t\widehat{\xi}_H)^*(\widehat{s}_k)$$

hence

$$e^{ikt\widehat{H}_k}s_{k,z_c}(z_c) = e^{iktH(z_c)}s_{k,z_c}(z_c).$$

Since the peak section is unique up to scaling, we have

$$e^{ikt\widehat{H}_k - iktH(z_c)}s_{k,z_c}(z) = s_{k,z_c}(z), \forall t \in \mathbb{R}, z \in M.$$

This shows  $\widehat{H}_k s_{k,z_c} = H(z_c) s_{k,z_c}$ . If any other  $L^2$  normalized section  $s_k$  orthogonal to  $s_{k,z_c}$  does not vanish on  $z_c$ , then we can find another  $L^2$  normalized section

$$\tilde{s}_{k,z_c} = s_{k,z_c} \cos(\theta) + e^{-i \arg(s_k(z_c)/s_{k,z_c}(z_c))} s_k \sin(\theta), \quad \tan(\theta) = |s_k(z_c)|/|s_{k,z_c}(z_c)|$$

with higher peak  $\sqrt{|s_k(z_c)|^2 + |s_{k,z_c}(z_c)|^2} > |s_{k,z_c}(z_c)|$  at  $z_c$ .

We prove (2), and (3) is similar. Suppose  $v \in W^+(z_c)$ , and  $v(t) = \exp(-t \nabla H)(v)$ , then from Lemma 6.3, we have

$$\frac{d}{dt} \|s_{k,j}(v(t))\|^2 = 2(H(v(t)) - \mu_{k,j}) \|s_{k,j}(v(t))\|^2.$$

Since  $H(v(t)) - \mu_{k,j} > H(z_c) - \mu_{k,j} =: C > 0$ , we get

$$\|s_{k,j}(v) \leq e^{-2Ct} \|s_{k,j}(v(t))\|^2,$$

for all  $t > 0$ . Taking limit  $t \rightarrow +\infty$  gives the result. This completes the proof of Proposition 1.5.  $\square$

### 7. Trace asymptotics: proof of Theorem 1.7

In this section, we prove Theorem 1.7. We start from Proposition 1.6 and integrate over  $M$  to get

$$\begin{aligned} \text{Tr } \widehat{U}_k\left(\frac{t}{\sqrt{k}}\right) &= \int_X \widehat{U}_k\left(\frac{t}{\sqrt{k}}, x, x\right) dV(x) \\ (20) \quad &= \left(\frac{k}{2\pi}\right)^m \int_M e^{it\sqrt{k}H(z)} e^{-t^2 \frac{\|dH(z)\|^2}{4}} d\text{Vol}_M(z) (1 + O(|t|^3 k^{-1/2})) \end{aligned}$$

Applying stationary phase in the large parameter  $\sqrt{k}$  gives,

$$\begin{aligned} &\int_{z \in M} U_k(t/\sqrt{k}, z) d\text{Vol}_M(z) \\ &= \left(\frac{k}{2\pi}\right)^m \left(\frac{t\sqrt{k}}{4\pi}\right)^{-m} \sum_{z_c \in \text{crit}(H)} \frac{e^{it\sqrt{k}H(z_c)} e^{(i\pi/4) \text{sgn}(\text{Hess}_{z_c}(H))}}{\sqrt{|\det(\text{Hess}_{z_c}(H))|}} (1 + O(|t|^3 k^{-1/2})) \end{aligned}$$

Here we use that  $\dim_{\mathbb{R}} M = 2m$  and that  $e^{-t^2 \frac{\|dH(z)\|^2}{4}} = 1$  on the critical set.

*Remark 7.1.* The asymptotics are non-uniform around  $t=0$  since the phase vanishes at  $t=0$  and thus has a larger critical point set.

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