Ark. Mat.
DOI: 10.1007/s11512-015-0225-z

Non-conjugacy of maximal abelian diagonalizable subalgebras in extended affine Lie algebras

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Abstract. We construct a counterexample to the conjugacy of maximal abelian diagonalizable subalgebras in extended affine Lie algebras.

Introduction

Consider a finite-dimensional simple Lie algebra $\mathfrak{g}$ over an algebraically closed field k of characteristic zero. In increasing generality, it was proven by Weyl, and later Weil and Chevalley, that all Cartan subalgebras of $\mathfrak{g}$ are conjugate. This result is of great importance for both the structure theory and the representation theory of finite-dimensional simple Lie algebras. While other generalizations of Cartan subalgebras to the infinite-dimensional setting have previously been proposed by Billig and Pianzola in [BP], in this paper we take the viewpoint that a Cartan subalgebra is abelian, acts k-diagonalizable on $\mathfrak{g}$ and is maximal with respect to these properties. To emphasize these defining properties, we will refer to such subalgebras as maximal abelian k-ad-diagonalizable subalgebras (or MADs). A celebrated theorem of Peterson and Kac [PK] states that all MADs of symmetrizable Kac–Moody Lie algebras are conjugate.

In this paper we deal with extended affine Lie algebras (EALAs for short). In the language of EALAs, a finite-dimensional simple Lie algebra is an EALA of nullity 0, while an affine Kac–Moody Lie algebra is an EALA of nullity 1. In this sense EALAs can be thought of as a generalization of these classical Lie algebras to higher nullities. The sequence of papers [P], [CGP1] and [CNPY] led to the

The work of the author was supported by the NSERC grant 385795-2010 of Kirill Zainoulline, the NSERC grant of Erhard Neher and the NSERC grant of Yuly Billig.

Published online: 14 August 2015

conjugacy of all *structure MADs* of EALAs finitely generated over the centroid. However, a natural question was left open: Are all MADs of an EALA conjugate?

The main result of this paper is Theorem 2.6, which answers this question negatively by providing a counterexample to the conjugacy of arbitrary MADs in an EALA. This counterexample is “minimal” in a sense that it is constructed for an EALA of nullity 2, while the conjugacy always holds in nullities 0 and 1.

It should be noted that the non-conjugacy of MADs by a special class of automorphisms for Lie tori was proven in [CGP1, Theorem 7.1 and Section 8.1] (a Lie torus naturally arises as a subquotient of an EALA). However in this paper we prove the non-conjugacy under all automorphisms for the example in [CGP1] and use this in our construction of the counterexample for EALAs (see Lemma 2.9).

The paper is organized as follows. In Section 1 we construct a class of extended affine Lie algebras. Then in Section 2 we construct two MADs of these algebras which are not conjugate. There we give two different proves of the non-conjugacy.

Throughout the paper, we consider k to be an algebraically closed field of characteristic 0 and define $R{=}k[t_1^{\pm 1}, t_2^{\pm 1}]$ to be the ring of Laurent polynomials in 2 variables with coefficients in k.

Acknowledgements. The author would like to thank Vladimir Chernousov, Erhard Neher and Arturo Pianzola for useful discussions. Also the author thanks the referee for very helpful and constructive comments which improved the paper.

1. Example of an extended affine Lie algebra

In this section we give an example of an EALA. We will use it in the next section to construct a counterexample to conjugacy.

Let $Q{=}(t_1, t_2)$ be a quaternion Azumaya algebra over R. Thus, Q is an associative R-algebra with unit 1, the R-basis $1, \imath, \jmath, k$ and the multiplication table given by $\imath\jmath{=}{-}\jmath\imath{=}k$, $\imath k{=}{-}k\imath{=}t_1\jmath$, $\jmath k{=}{-}k\jmath{=}{-}t_2\imath$, $\imath^2{=}t_1$, $\jmath_2{=}t_2$, $k^2{=}{-}t_1t_2$.

One may check that Q has a structure of an associative torus [Ne1, Definition 4.20] of type $\mathbb{Z}^2$. Indeed,

$$Q = \bigoplus_{(a,b)\in\mathbb{Z}^2} Q^{(a,b)}$$

is a $\mathbb{Z}^2$-graded associative algebra over k with one-dimensional graded components $Q^{(a,b)}{=}k\imath^a\jmath^b$, $(a,b){\in}\mathbb{Z}^2$, containing invertible elements $\imath^a\jmath^b$.

Therefore $\mathcal{L}{=}\mathfrak{sl}_2(Q){=}[\mathfrak{gl}_2(Q), \mathfrak{gl}_2(Q)]$ is a Lie torus (see [Ne1, Definition 4.2] for the definition of a Lie torus and [Ne1, Exercise 4.21]). In particular, it has a double grading, i.e.,

$$\mathcal{L}=\bigoplus_{\alpha\in\{0,\pm 2\},(a,b)\in\mathbb{Z}^2}\mathcal{L}_\alpha^{(a,b)},$$

where

$$\mathcal{L}_0^{(a,b)}=k\begin{bmatrix}\imath^a\jmath^b & 0\\ 0 & -\imath^a\jmath^b\end{bmatrix}$$

if a,b are even and

$$\mathcal{L}_0^{(a,b)}=k\begin{bmatrix}\imath^a\jmath^b & 0\\ 0 & -\imath^a\jmath^b\end{bmatrix}\oplus k\begin{bmatrix}\imath^a\jmath^b & 0\\ 0 & \imath^a\jmath^b\end{bmatrix}$$

otherwise; also we have

$$\mathcal{L}_{-2}^{(a,b)}=k\begin{bmatrix}0 & 0\\ \imath^a\jmath^b & 0\end{bmatrix}\quad\text{and}\quad\mathcal{L}_2^{(a,b)}=k\begin{bmatrix}0 & \imath^a\jmath^b\\ 0 & 0\end{bmatrix}.$$

Define a bilinear form $(-,-)$ on $\mathcal{L}$ by

$$\left(\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix},\begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}\right)=(x_{11}y_{11}+x_{12}y_{21}+x_{21}y_{12}+x_{22}y_{22})_0,$$

where a_0 for $a\in Q$ denotes the $Q^{(0,0)}$-component of a. This form is symmetric, nondegenerate, invariant and graded. (Recall, that a form $(-,-)$ is *invariant* if $([a,b],c)=(a,[b,c])$ for all a,b,c, and is *graded* if $(\mathcal{L}_\xi^\lambda,\mathcal{L}_\tau^\mu)=0$ if $\lambda+\mu\neq 0$ or $\xi+\tau\neq 0$.)

Any $\theta\in\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2,k)$ induces a so-called *degree derivation* ∂_θ of $\mathcal{L}$ defined by $\partial_\theta(l^\lambda)=\theta(\lambda)l^\lambda$ for $l^\lambda\in\mathcal{L}^\lambda=\bigoplus_{\alpha\in\{0,\pm 2\}}\mathcal{L}_\alpha^\lambda$. Take $\theta\in\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2,k)$ such that $\theta((1,0))=1$ and $\theta((0,1))$ is not rational. We put $\mathcal{D}=k\partial_\theta$ and $\mathcal{C}=\mathcal{D}^*$.

We set $\mathcal{E}=\mathcal{L}\oplus\mathcal{C}\oplus\mathcal{D}$ and equip it with the multiplication given by

$$[l_1\oplus c_1\oplus\partial_1,l_2\oplus c_2\oplus\partial_2]_{\mathcal{E}}=\big([l_1,l_2]_{\mathcal{L}}+\partial_1(l_2)-\partial_2(l_1)\big)\oplus\sigma(l_1,l_2),$$

where $\sigma:\mathcal{L}\times\mathcal{L}\to\mathcal{C}$ is the *central 2-cocycle* defined by $\sigma(l_1,l_2)(\partial)=(\partial(l_1),l_2)$.

Let $\mathcal{H}=k\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]\oplus\mathcal{C}\oplus\mathcal{D}$. Then $\mathcal{H}$ is a MAD of $\mathcal{E}$, which we will call *standard*, and

$$\mathcal{E}=\mathcal{H}\bigoplus\bigoplus_{(\alpha,\lambda)\in\{0,\pm 2\}\times\mathbb{Z}^2-\{(0,(0,0))\}}\mathcal{L}_\alpha^\lambda$$

is the corresponding weight space decomposition of $\mathcal{E}$.

The symmetric bilinear form $(-,-)$ on $\mathcal{E}$ given by

$$(l_1\oplus c_1\oplus\partial_1,l_2\oplus c_2\oplus\partial_2)=(l_1,l_2)+c_1(\partial_2)+c_2(\partial_1)$$

is nondegenerate and invariant.

It follows from [Ne2, Theorem 6], that $(\mathcal{E},\mathcal{H})$ is an extended affine Lie algebra.

2. Construction of the counterexample

In this section we construct a counterexample to conjugacy of MADs for the extended affine Lie algebra $(\mathcal{E},\mathcal{H})$ of Section 1. Namely, we construct a MAD $\mathcal{H}'$ of $\mathcal{E}$ which is not conjugate to the standard MAD $\mathcal{H}$.

We will need some auxiliary lemmas.

Let $Q=(t_1,t_2)$ be as above. For $H\in\mathfrak{gl}_2(Q)$ let L_H be an R-linear operator on Q^2 given by $L_H(v)=Hv$ for any column vector $v\in Q^2$. In particular, the trace $\mathrm{tr}(L_H)$ is well-defined.

Lemma 2.1. *Let $H\in\mathfrak{gl}_2(Q)$. Then $H\in\mathfrak{sl}_2(Q)$ if and only if $\mathrm{tr}(L_H)=0$.*

Proof. Let $H=\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$, $a,b,c,d\in Q$. Then $H\in\mathfrak{sl}_2(Q)$ iff $a+d\in[Q,Q]$. Since $Q=R\cdot 1\oplus[Q,Q]$, we can write $a=a'+q_1$ and $d=d'+q_2$ where $a',d'\in R\cdot 1$ and $q_1,q_2\in[Q,Q]$. Hence $H\in\mathfrak{sl}_2(Q)$ iff $a'+d'=0$.

On the other hand, $\mathrm{tr}(L_H)=4(a'+d')$ which can be seen if we compute the trace of L_H in the standard R-basis of Q^2. The lemma follows. $\square$

Lemma 2.2. *Let $T\in\mathcal{L}$ be a k-ad-diagonalizable element in $\mathcal{L}$, i.e.,*

$$\mathcal{L}=\bigoplus_{\alpha\in k}\mathcal{L}_\alpha,$$

where as usual

$$\mathcal{L}_\alpha=\big\{x\in\mathcal{L}\mid [T,x]_{\mathcal{L}}=\alpha x\big\}.$$

Then

(a) *T is also k-ad-diagonalizable viewed as an element of $\mathcal{E}_c:=\mathcal{L}\oplus\mathcal{C}$.*

(b) *$[T,l_0]_{\mathcal{E}_c}=0$ for any $l_0\in\mathcal{L}_0$.*

Proof. (a) Since σ is a central 2-cocycle, for $l_\alpha\in\mathcal{L}_\alpha$ and $l_\beta\in\mathcal{L}_\beta$ we have

$$\begin{aligned}0&=\sigma\big(T,[l_\alpha,l_\beta]_{\mathcal{L}}\big)+\sigma\big(l_\alpha,[l_\beta,T]_{\mathcal{L}}\big)+\sigma\big(l_\beta,[T,l_\alpha]_{\mathcal{L}}\big)\\&=\sigma\big(T,[l_\alpha,l_\beta]_{\mathcal{L}}\big)-\beta\sigma(l_\alpha,l_\beta)+\alpha\sigma(l_\beta,l_\alpha).\end{aligned}$$

Thus, $\sigma(T,[l_\alpha,l_\beta]_{\mathcal{L}})=(\alpha+\beta)\sigma(l_\alpha,l_\beta)$. Hence

$$\begin{aligned}\big[T,[l_\alpha,l_\beta]_{\mathcal{E}_c}\big]_{\mathcal{E}_c}&=\big[T,[l_\alpha,l_\beta]_{\mathcal{L}}+\sigma(l_\alpha,l_\beta)\big]_{\mathcal{E}_c}=\big[T,[l_\alpha,l_\beta]_{\mathcal{L}}\big]_{\mathcal{E}_c}\\&=\big[T,[l_\alpha,l_\beta]_{\mathcal{L}}\big]_{\mathcal{L}}+\sigma\big(T,[l_\alpha,l_\beta]_{\mathcal{L}}\big)=(\alpha+\beta)[l_\alpha,l_\beta]_{\mathcal{E}_c}\end{aligned}$$

proving $[\mathcal{L}_\alpha,\mathcal{L}_\beta]_{\mathcal{E}_c}\subset(\mathcal{E}_c)_{\alpha+\beta}$. It then follows from

$$\mathcal{E}_c=[\mathcal{E}_c,\mathcal{E}_c]_{\mathcal{E}_c}=[\mathcal{L},\mathcal{L}]_{\mathcal{E}_c}=\sum_{\alpha,\beta\in k}[\mathcal{L}_\alpha,\mathcal{L}_\beta]_{\mathcal{E}_c}\subset\sum_{\gamma\in k}(\mathcal{E}_c)_\gamma$$

that $\mathcal{E}_c$ is spanned by eigenvectors of $\operatorname{ad}T$, whence the result.

(b) Since $\mathcal{L}=[\mathcal{L},\mathcal{L}]_{\mathcal{L}}$, we can write $l_0=\sum_{i=1}^n[v_i,w_i]_{\mathcal{L}}$, where $v_i\in\mathcal{L}_{\alpha_i}$, $w_i\in\mathcal{L}_{-\alpha_i}$ for some $\alpha_i\in k$. Then

$$\begin{aligned}[T,l_0]_{\mathcal{E}_c}=\sigma(T,l_0)=\sigma\left(T,\sum_{i=1}^n[v_i,w_i]_{\mathcal{L}}\right)&=\sum_{i=1}^n\sigma\big(T,[v_i,w_i]_{\mathcal{L}}\big)\\ &=-\sum_{i=1}^n\big(\sigma\big(v_i,[w_i,T]_{\mathcal{L}}\big)+\sigma\big(w_i,[T,v_i]_{\mathcal{L}}\big)\big)\\ &=-\sum_{i=1}^n\big(\alpha_i\sigma(v_i,w_i)+\alpha_i\sigma(w_i,v_i)\big)=0.\qquad\square\end{aligned}$$

Let $\mathcal{A}$ be the associative algebra of 2×2 matrices with entries in Q. Let $V=Q\oplus Q$ be the free right Q-module of rank 2. We may and will view $\mathcal{A}$ as the algebra $\operatorname{End}_Q(V)$ of Q-endomorphisms of V. Let $m:V=Q\oplus Q\to Q$ be the Q-linear map given by

$$(u,v)\longmapsto(1+\imath)u-(1+\jmath)v.$$

Denote its kernel by W. It was shown in [GP, Proposition 3.20] that m is split and that W is a projective Q-module of rank 1 which is not free. Since m is split there is a decomposition $V=W\oplus U$ where U is a free Q-module of rank 1.

Let $s\in\operatorname{End}_Q(V)$ be the Q-linear endomorphism of V which maps w to $-w$ and u to u for all $w\in W$, $u\in U$.

Lemma 2.3. *The element s lies in $\mathfrak{sl}_2(Q)$.*

Proof. It follows from the definition of s that in the R-basis of Q^2 which is the union of R-bases for U and W the trace of s is 0 (recall that W is a free R-module of rank 4). Hence the claim follows by Lemma 2.1. $\square$

Lemma 2.4. *Considered as an element of the algebra $\mathcal{E}$, s is k-ad-diagonalizable.*

Proof. We will proceed in several steps. First we will show that s is a k-ad-diagonalizable when considered as an element of the Lie algebra $\operatorname{End}_Q(V)^-$.

Let $i_W\colon W\hookrightarrow W\oplus U$, $i_U\colon U\hookrightarrow W\oplus U$ be the canonical inclusions and let $p_W\colon W\oplus U\twoheadrightarrow W$, $p_U\colon W\oplus U\twoheadrightarrow U$ be the canonical projections.

Notice that there is a canonical isomorphism of Q-modules

$$\tau : \mathrm{End}_Q(V) \simeq \mathrm{End}_Q(W)\oplus\mathrm{Hom}_Q(W,U)\oplus\mathrm{Hom}_Q(U,W)\oplus\mathrm{End}_Q(U), \tag{2.1}$$

$$\tau(\phi) = p_W\circ\phi\circ i_W + p_U\circ\phi\circ i_W + p_W\circ\phi\circ i_U + p_U\circ\phi\circ i_U \quad \text{for } \phi\in\mathrm{End}_Q(V).$$

Let $\phi\in\tau^{-1}(\mathrm{End}_Q(W))$. Then

$$[s,\phi] = s\circ\phi - \phi\circ s = -\phi-(-\phi) = 0.$$

Similarly, if $\phi\in\tau^{-1}(\mathrm{End}_Q(U))$ then $[\phi,s]=0$. It follows

$$\tau^{-1}\big(\mathrm{End}_Q(W)\oplus\mathrm{End}_Q(U)\big) \subset \mathrm{End}_Q(V)_0.$$

Let now $\phi\in\tau^{-1}(\mathrm{Hom}_Q(W,U))$. Then

$$[s,\phi] = s\circ\phi - \phi\circ s = \phi-(-\phi) = 2\phi$$

implying

$$\tau^{-1}\big(\mathrm{Hom}_Q(W,U)\big) \subset \mathrm{End}_Q(V)_2.$$

Similarly,

$$\tau^{-1}\big(\mathrm{Hom}_Q(U,W)\big) \subset \mathrm{End}_Q(V)_{-2}.$$

Thus, s is a k-ad-diagonalizable element of $\mathrm{End}_Q(V)^-$ and hence s is a k-ad-diagonalizable element of $\mathfrak{gl}_2(Q)=\mathcal{A}^-$. Moreover, it follows from the proof above that $\mathcal{A}=\mathcal{A}_{-2}\oplus\mathcal{A}_0\oplus\mathcal{A}_2$, where as usual $\mathcal{A}_\alpha=\{x\in\mathfrak{gl}_2(Q)\mid [s,x]_{\mathfrak{gl}_2(Q)}=\alpha x\}$ for $\alpha\in\{0,\pm 2\}$.

Using Lemma 2.3 we get that s is a k-ad-diagonalizable element of $\mathcal{L}$, and $\mathcal{L}=\mathcal{L}_{-2}\oplus\mathcal{L}_0\oplus\mathcal{L}_2$. Therefore, by Lemma 2.2(a), s is a k-ad-diagonalizable element of $\mathcal{E}_c$.

Finally, we will show that s is k-ad-diagonalizable considered as an element of $\mathcal{E}$.

Let $[s,\partial_\theta]_{\mathcal{E}}=y=y_0+y_2+y_{-2}$, where $y_\lambda\in\mathcal{L}_\lambda$. We claim that $y_0=0$.

Assume $y_0\neq 0$. Since $(-,-)|_{\mathcal{L}_0}$ is nondegenerate there is $u\in\mathcal{L}_0$ such that $(u,y_0)\neq 0$. Then taking into consideration that $(\mathcal{L}_0,\mathcal{L}_2)=(\mathcal{L}_0,\mathcal{L}_{-2})=0$ we get

$$0\neq (y_0,u) = (y_0+y_2+y_{-2},u) = \big([s,\partial_\theta]_{\mathcal{E}},u\big) = -\big(\partial_\theta,[s,u]_{\mathcal{E}}\big).$$

But it follows from Lemma 2.2(b), that $[s,u]_{\mathcal{E}}=0$—a contradiction.

Let $\partial' = \partial_\theta - \frac{1}{2}y_2 + \frac{1}{2}y_{-2}$. Observe that

$$\begin{aligned}[s,\partial']_{\mathcal{E}} &= \left[s,\partial_\theta - \frac{1}{2}y_2 + \frac{1}{2}y_{-2}\right]_{\mathcal{E}} \\ &= y - \frac{1}{2}[s,y_2]_{\mathcal{E}} + \frac{1}{2}[s,y_{-2}]_{\mathcal{E}} \\ &= y - y_2 - \frac{1}{2}\sigma(s,y_2) - y_{-2} + \frac{1}{2}\sigma(s,y_{-2}) \\ &= \frac{1}{2}\sigma(s,y_{-2}-y_2) \in \mathcal{C}.\end{aligned}$$

Hence $[s,\partial']_{\mathcal{E}}$ is orthogonal to $\mathcal{L}\oplus\mathcal{C}$.
Also, using the invariance of the form $(-,-)$ we get

$$\left([s,\partial']_{\mathcal{E}},\partial'\right) = \left(s,[\partial',\partial']_{\mathcal{E}}\right) = (s,0) = 0,$$

i.e. $[s,\partial']_{\mathcal{E}}$ is orthogonal to ∂'. We conclude that $([s,\partial']_{\mathcal{E}},x)=0$ for any $x\in\mathcal{E}$. Now it follows from the nondegeneracy of the form $(-,-)$ that $[s,\partial']_{\mathcal{E}}=0$. Since $\mathcal{E}=\mathcal{E}_c\oplus k\partial'$, the assertion of the lemma follows. □

Lemma 2.5. *The 1-dimensional k-vector space $k\cdot s$ is a MAD of $\mathcal{L}$.*

Proof. Let K be the field of fractions of R. Notice that $\mathcal{L}_K\simeq\mathfrak{sl}_2(Q_K)$ is a simple Lie algebra over K containing a K-MAD $\mathfrak{t}$ consisting of matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix}$$

where $x\in K$. By [S, Theorem 2] all the MADs of $\mathcal{L}_K$ are conjugate to $\mathfrak{t}$ and therefore are 1-dimensional over K. Thus, if $\mathfrak{m}$ is a MAD of $\mathcal{L}$ then $\mathfrak{m}\otimes_k K$ is an abelian K-diagonalizable subalgebra of $\mathcal{L}_K$ and hence $\dim_K(\mathfrak{m}\otimes_k K)\leq 1$. By [CGP1, Lemma 7.3(3)] for any MAD $\mathfrak{m}$ of $\mathcal{L}$ there is an equality $\dim_k(\mathfrak{m})=\dim_K(\mathfrak{m}\otimes_k K)$. Hence $\dim_k(\mathfrak{m})=1$. The lemma follows. □

Recall that a MAD $\mathcal{M}$ of $\mathcal{E}$ is called a structure MAD if the pair $(\mathcal{E},\mathcal{M})$ has a structure of an EALA. Let $\mathcal{H}'$ be a MAD of $\mathcal{E}$ which contains s. We now prove the main result of the present paper.

Theorem 2.6. *The MAD $\mathcal{H}'$ is not a structure MAD and therefore is not conjugate to $\mathcal{H}$.*

Proof. Assume that $\mathcal{H}'$ is a structure MAD of $\mathcal{E}$. By [CNPY, Corollary 3.2] the core $\mathcal{E}_c=\mathcal{L}\oplus\mathcal{C}$ of $\mathcal{E}$ does not depend on the choice of the EALA structure on $\mathcal{E}$. Therefore, by [Ne2, Theorem 6] the image $\pi(\mathcal{H}'\cap\mathcal{E}_c)$ under the projection map $\pi:\mathcal{E}_c\to\mathcal{L}$ is a k-ad-diagonalizable subalgebra of $\mathcal{L}$ which is the 0-component of some Lie torus structure on $\mathcal{L}$. It follows from Lemma 2.5 that $\pi(\mathcal{H}'\cap\mathcal{E}_c)=k\cdot s$. By one of the axioms of the Lie torus there exists an $\mathfrak{sl}_2$-triple in $\mathcal{L}$ of the form $(e,f,\gamma s)$, for some $\gamma\in k$. But then, using the identification (2.1), we have $e\in\mathrm{Hom}_Q(U,W)$ and $f\in\mathrm{Hom}_Q(W,U)$. We obtain that $s=[e,\gamma^{-1}f]$. Hence $e\circ\gamma^{-1}f=id_W$ and $\gamma^{-1}f\circ e=id_U$. Hence $U\simeq W$ as Q-modules. But this is impossible, since U is free while W is not. Hence $\mathcal{H}'$ is not a structure MAD and therefore is not conjugate to $\mathcal{H}$. $\square$

Here is an alternative way to prove that the standard MAD $\mathcal{H}$ is not conjugate to the MAD $\mathcal{H}'$ constructed above. We will need two auxiliary lemmas.

Let $^{-}:Q\to Q$ be the involution given by $x+y\imath+z\jmath+t\imath\jmath\mapsto x-y\imath-z\jmath-t\imath\jmath$, $x,y,z,t\in R$.

Lemma 2.7. *Any R-linear automorphism of $\mathfrak{sl}_2(Q)$ is a conjugation with some matrix in $\mathrm{GL}_2(Q)$ or a map*

$$\pi:\begin{bmatrix} a & b\\ c & d\end{bmatrix}\longrightarrow -\begin{bmatrix} \overline{a} & \overline{c}\\ \overline{b} & \overline{d}\end{bmatrix}$$

(*which is a nontrivial outer R-automorphism of $\mathfrak{sl}_2(Q)$*) *followed by a conjugation.*

Proof. Let $S=k[t_1^{\pm\frac{1}{2}},t_2^{\pm\frac{1}{2}}]$. Notice that S is a Galois ring extension of R with the Galois group $\Gamma=\mathrm{Gal}(S/R)\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. It acts naturally on S and this induces a standard componentwise action of Γ on $\mathfrak{sl}_4(S)$. Of course, $\mathfrak{sl}_4(R)=(\mathfrak{sl}_4(S))^\Gamma$.

Let $u\in Z^1(\Gamma,\mathbf{Aut}(\mathfrak{sl}_4)(S))$ be a Galois 1-cocycle such that $u((\overline{1},\overline{0}))$ is a conjugation with $\mathrm{diag}(1,-1,1,-1)$ and $u((\overline{0},\overline{1}))$ is a conjugation with $\mathrm{diag}\left(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\right)$. This cocycle gives rise to a new twisted action of Γ on $\mathfrak{sl}_4(S)$, i.e., $\gamma\in\Gamma$ acts by $x\mapsto u(\gamma)({}^\gamma x)$ for any $x\in\mathfrak{sl}_4(S)$. The direct computation shows that $\mathcal{L}=\mathfrak{sl}_2(Q)$ is isomorphic to the fixed point subalgebra $(\mathfrak{sl}_4(S))^\Gamma=\{x\in\mathfrak{sl}_4(S)\mid x=u(\gamma)({}^\gamma x)\text{ for any }\gamma\in\Gamma\}$ with respect to this new action.

Now let $\tilde{u}$ be the 1-cocycle, defined in [GP, Section 4.4]. By [GP, Proposition 4.10]

$$\mathrm{Aut}_{R-\mathrm{Lie}}(\mathcal{L})=\big({}_{\tilde{u}}\mathbf{Aut}\big(\mathfrak{sl}_4(R)\big)\big)(R),$$

i.e., the automorphisms of the twisted R-algebra $\mathcal{L}$ are precisely the R-points of the corresponding twisted group ${}_{\tilde{u}}\mathbf{Aut}(\mathfrak{sl}_4(R))$. There is a short exact sequence of algebraic groups over R

$$1 \longrightarrow \mathbf{PGL}_{4,R} \longrightarrow \mathbf{Aut}\big(\mathfrak{sl}_4(R)\big) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

[SGA3, Exp. XXIV, Theorem 1.3].

Twisting it with the cocycle $\tilde{u}$ we get the following corresponding exact sequence in the Galois cohomology

$$1 \longrightarrow ({}_{\tilde{u}}\mathbf{PGL}_{4,R})(R) \longrightarrow \big({}_{\tilde{u}}\mathbf{Aut}\big(\mathfrak{sl}_4(R)\big)\big)(R) \longrightarrow ({}_{\tilde{u}}\mathbb{Z}/2\mathbb{Z})(R).$$

It follows that the group of inner automorphisms $({}_{\tilde{u}}\mathbf{PGL}_{4,R})(R)$ of $\mathcal{L}$ has index at most 2 in $\mathrm{Aut}_{R-\mathrm{Lie}}(\mathcal{L})$.

Since $\mathrm{Pic}(S)=0$, it follows along the same lines as in [GP, Section 5.2] that

$${}_{\tilde{u}}\mathbf{GL}_{4,R}(R) \longrightarrow {}_{\tilde{u}}\mathbf{PGL}_{4,R}(R)$$

is surjective.

The direct computations show that ${}_{\tilde{u}}\mathbf{GL}_{4,R}(R) \simeq \mathrm{GL}_2(Q)$. Since π is an outer automorphism the lemma follows. □

Remark 2.8. Lemma 2.7 can be proved using the corresponding result for the Lie algebra $\mathcal{L}_K \simeq \mathfrak{sl}_2(Q_K)$, where K is the field of fractions of R (see the remark after Theorem 10 in [J, Chapter 10.4]).

The following lemma is inspired by [CGP2, Proposition 7.1].

Lemma 2.9. *Let $p = a\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ where $a \in k^\times$ is an arbitrary scalar. Then the elements s and p of $\mathfrak{sl}_2(Q)$ are not conjugate by a k-linear automorphism of $\mathfrak{sl}_2(Q)$.*

Proof. We will prove this in two steps.

Step 1: s and p are not conjugate by an R-linear automorphism of $\mathfrak{sl}_2(Q)$.

Assume that $\phi(p) = s$ for some $\phi \in \mathrm{Aut}_{R-\mathrm{Lie}}(\mathfrak{sl}_2(Q))$. By Lemma 2.7 we should consider two cases.

Case 1: ϕ is a conjugation. Then the eigenspaces in $V = Q \oplus Q$ of the Q-linear transformation $\phi(p)$ are free Q-modules of rank 1 (because they are images of those of p). Since W is an eigenspace of s which is not a free Q-module, we get a contradiction.

Case 2: ϕ is π followed by a conjugation. But $\pi(p) = p$, hence we are reduced to the previous case.

Step 2: s and p are not conjugate by a k-automorphism of $\mathfrak{sl}_2(Q)$.

Recall that for an arbitrary k-algebra A,

$$\mathrm{Ctd}_k(A) = \big\{\chi \in \mathrm{End}_k(A) : \chi(ab) = \chi(a)b = a\chi(b)\ \forall a, b \in A\big\}.$$

By [BN] and [GP] we may identify $R \simeq \mathrm{Ctd}_k(\mathfrak{sl}_2(Q))$, $r \mapsto (\chi : x \mapsto rx)$.

Assume the contrary: Let $\phi \in \mathrm{Aut}_{k-\mathrm{Lie}}(\mathfrak{sl}_2(Q))$ be such that $\phi(p)=s$. It induces an automorphism $C(\phi)$ of $\mathrm{Ctd}_k(\mathfrak{sl}_2(Q))$ defined by $\chi \mapsto \phi^{-1} \circ \chi \circ \phi$ for all $\chi \in \mathrm{Ctd}_k(\mathfrak{sl}_2(Q))$. Consider a new Lie algebra $\mathcal{L}' = \mathfrak{sl}_2(Q) \otimes_{C(\phi)} R$ over R. As a set it coincides with $\mathfrak{sl}_2(Q)$. Also, the Lie bracket in $\mathcal{L}'$ is the same as in $\mathfrak{sl}_2(Q)$, but the action of R on $\mathcal{L}'$ is given by the composition of $C(\phi)$ and the standard action of R on $\mathfrak{sl}_2(Q)$. Thus we have a natural k-linear Lie algebra isomorphism

$$\psi : \mathcal{L}' = \mathfrak{sl}_2(Q) \otimes_{C(\phi)} R \longrightarrow \mathfrak{sl}_2(Q)$$

which sends p to p. It follows from the construction that $\phi \circ \psi : \mathcal{L}' \to \mathfrak{sl}_2(Q)$ is an R-linear isomorphism.

Note that since the action of R on $\mathfrak{sl}_2(Q)$ is componentwise, we have a natural identification

$$\mathcal{L}' = \mathfrak{sl}_2(Q) \otimes_{C(\phi)} R \simeq \mathfrak{sl}_2(Q \otimes_{C(\phi)} R)$$

and it follows from the construction that $Q \otimes_{C(\phi)} R$ is the quaternion algebra $(C(\phi)(t_1), C(\phi)(t_2))$ over R. Thus $\mathfrak{sl}_2(Q)$ and $\mathfrak{sl}_2(Q \otimes_{C(\phi)} R)$ are R-isomorphic and are R-forms of $\mathfrak{sl}_4(R)$, i.e., there exists a finite étale extension S/R such that $\mathfrak{sl}_2(Q) \otimes S \simeq \mathfrak{sl}_4(R) \otimes S$. Moreover they are inner forms, hence correspond to an element $[\xi] \in H^1(R, \mathbf{PGL}_4)$.

The boundary map $H^1(R, \mathbf{PGL}_4) \to H^2(R, \mathbf{G}_m)$ maps $[\xi]$ to the Brauer equivalence class of both Q and $Q \otimes_{C(\phi)} R$. Since $[Q] = [Q \otimes_{C(\phi)} R]$ it follows that there is an R-algebra isomorphism $\overline{\theta} : Q \otimes_{C(\phi)} R \to Q$ which in turn induces a canonical R-Lie algebra isomorphism

$$\theta : \mathfrak{sl}_2(Q \otimes_{C(\phi)} R) \longrightarrow \mathfrak{sl}_2(Q)$$

by componentwise application of $\overline{\theta}$. Clearly, $\theta(p) = p$.

Finally, consider an R-linear automorphism

$$\phi' = \phi \circ \psi \circ \theta^{-1} \in \mathrm{Aut}_{R-\mathrm{Lie}}\big(\mathfrak{sl}_2(Q)\big).$$

We have

$$\phi'(p) = \phi\Big(\psi\big(\theta^{-1}(p)\big)\Big) = \phi\big(\psi(p)\big) = \phi(p) = s$$

which contradicts the claim of the step 1. $\square$

Now we are ready to prove that there is no $\phi \in \mathrm{Aut}_{k-\mathrm{Lie}}(\mathcal{E})$ such that $\phi(\mathcal{H}') = \mathcal{H}$. Assume the contrary. Let $\phi \in \mathrm{Aut}_k(\mathcal{E})$ be such that $\phi(\mathcal{H}') = \mathcal{H}$. Since $\mathcal{E}_c = [\mathcal{E}, \mathcal{E}]_{\mathcal{E}}$ we have that $\mathcal{E}_c$ is ϕ-stable. Hence $\phi(\mathcal{H}' \cap \mathcal{E}_c) = \mathcal{H} \cap \mathcal{E}_c = k \cdot p \oplus \mathcal{C}$. Of course, $k \cdot s \oplus \mathcal{C} \subset$

$\mathcal{H}'\cap\mathcal{E}_c$, because $\mathcal{C}$ is the center of $\mathcal{E}$. Therefore by dimension reasons we have $\phi(k\cdot s\oplus\mathcal{C})=k\cdot p\oplus\mathcal{C}$. The automorphism ϕ induces a k-automorphism $\phi_{\mathcal{L}}:\mathcal{L}\to\mathcal{L}$. Then $\phi_{\mathcal{L}}(k\cdot s)=k\cdot p$. Hence there exists a scalar $\alpha\in k^{\times}$ such that $\phi_{\mathcal{L}}(s)=\alpha\cdot p$. But this contradicts Lemma 2.9.

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Received February 8, 2015
in revised form April 20, 2015