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On improved fractional Sobolev–Poincaré inequalities

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Abstract. We study a certain improved fractional Sobolev–Poincaré inequality on domains, which can be considered as a fractional counterpart of the classical Sobolev–Poincaré inequality. We prove the equivalence of the corresponding weak and strong type inequalities; this leads to a simple proof of a strong type inequality on John domains. We also give necessary conditions for the validity of an improved fractional Sobolev–Poincaré inequality, in particular, we show that a domain of finite measure, satisfying this inequality and a 'separation property', is a John domain.

1. Introduction

It is known that the classical Sobolev–Poincaré inequality holds on a c-John domain G (for the John condition, see Definition 2.1). Namely, if $1\le p<n$, then there exists a constant $C=C(n,p,c)>0$ such that inequality

$$(1)\qquad \left(\int_G |u(x)-u_G|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{np}}\le C\left(\int_G |\nabla u(x)|^p\,dx\right)^{\frac{1}{p}}$$

holds for every $u\in W^{1,p}(G)$. When $1<p<n$ this result was proved independently by Martio [14] and Reshetnyak [16]. The method of Reshetnyak is based on the following potential estimate in a c-John domain: inequality

$$(2)\qquad |u(x)-u_G|\le C(n,c)\int_G \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy,\quad x\in G,$$

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holds whenever u is a Lipschitz function on G. Bojarski extended inequality (1) to the case $p{=}1$ by using the so called Boman chaining technique [4]. Later Hajłasz [8] showed that inequality (1) on John domains for $p{=}1$ follows from the potential estimate (2) together with the Maz'ya's truncation argument [15]. It is also known, that the John condition is necessary and sufficient for the classical Sobolev–Poincaré inequality (1) to hold, if G is of finite measure and satisfies the separation property; this result is due to Buckley and Koskela [5]. For instance, every simply connected planar domain satisfies the separation property.

In this paper, we consider certain fractional counterparts of inequality (1). Let $0<\delta<1$, $1\le p<n/\delta$ and let G be a bounded domain in $\mathbf{R}^n$, $n\ge 2$. The extension results proved by Jonsson and Wallin [9] (and also by Shvartsman [17]) combined with the classical embedding theorems for fractional Sobolev spaces, see e.g. [1, Theorem 7.57], imply that the fractional Sobolev–Poincaré inequality

$$(3)\qquad \left(\int_G |u(x)-u_G|^{\frac{np}{n-\delta p}}\,dx\right)^{\frac{n-\delta p}{np}} \le C\left(\int_G\int_G \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}$$

holds for some $C>0$ and every $u\in L^p(G)$ if G satisfies the measure density condition as in Definition 2.2. Moreover, it follows from the results of Zhou [20, Theorem 1.2] that the measure density condition characterizes the class of domains G on which inequality (3) holds. Recall that John domains satisfy the measure density condition but the converse fails in general. On the other hand, if we assume that G is a c-John domain and $0<\tau<1$ is given, then there exists a constant $C{=}C(n,\delta,c,\tau,p)>0$ such that a stronger inequality

$$(4)\quad \left(\int_G |u(x)-u_G|^{\frac{np}{n-\delta p}}\,dx\right)^{\frac{n-\delta p}{np}} \le C\left(\int_G\int_{B(x,\tau\,\mathrm{dist}(x,\partial G))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}$$

holds for every $u\in L^1(G)$. We call inequality (4) an improved fractional Sobolev–Poincaré inequality, and it is the main object in this paper. These inequalities have applications, e.g., in peridynamics, we refer to [3]. Inequality (4) with $1<p<n/\delta$ is proved in [13] by establishing a fractional analogue of the potential estimate (2) in John domains; see also [19] for the proof of a similar inequality where on the right hand side the Gagliardo–Sobolev type seminorm of a function is replaced by the seminorm in a fractional Hajłasz–Sobolev type space. We note that these two seminorms are, in general, not comparable.

In this paper, we show that inequality (4) is equivalent to a corresponding weak type inequality, see Theorem 4.1. The proof of this result uses the fractional Maz'ya truncation method from [7]. As an application we give a proof of inequality (4) on John domains for the case $p{=}1$, see Section 5.

We also address the necessity of John condition for improved fractional Sobolev–Poincaré inequalities; a simple counterexample shows that the improved inequality (4) does not hold on all bounded domains satisfying the measure density condition, we refer to Section 3. Furthermore, by adapting the method of Buckley and Koskela in Section 6, we show that the John condition is necessary and sufficient for the improved fractional Sobolev–Poincaré inequality (4) to hold, if the domain G has a finite measure and satisfies the separation property; we refer to Theorem 6.1.

When G is a bounded Lipschitz domain and $\tau\in(0,1]$, there exists a constant $C>0$ such that, for every $u\in L^1(G)$, the following inequality holds:

$$\left(\int_G\int_G \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}\le C\left(\int_G\int_{B(x,\tau\,\mathrm{dist}(x,\partial G))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}, \tag{5}$$

see [6, formula (13)]. In particular, the fractional Sobolev–Poincaré inequalities (3) and (4) are equivalent in this case. However, inequality (5) does not hold for John domains in general; we give a counterexample in Proposition 3.4.

2. Notation and preliminaries

Throughout the paper we assume that G is a domain in $\mathbf{R}^n$, $n\ge 2$. The distance from $x\in G$ to the boundary of G is $\mathrm{dist}(x,\partial G)$. The diameter of a set $A\subset\mathbf{R}^n$ is $\mathrm{diam}(A)$. The Lebesgue n-measure of a measurable set $A\subset\mathbf{R}^n$ is denoted by $|A|$. For a measurable set A with a finite and nonzero measure we write $u_A=|A|^{-1}\int_A u(x)\,dx$ whenever the integral is defined. The characteristic function of a set A is written as χ_A. If a function u is defined on $G\subset\mathbf{R}^n$ and occurs in a place where a function defined on $\mathbf{R}^n$ is needed, we understand that u is extended by zero to the whole $\mathbf{R}^n$. We let $C(*,\cdots,*)$ denote a constant which depends on the quantities appearing in the parentheses only.

We use the following definition for John domains; alternative equivalent definitions may be found in [18].

Definition 2.1. A bounded domain G in $\mathbf{R}^n$, $n\ge 2$, is a c-John domain (John domain) with a constant $c\ge 1$, if there exists $x_0\in G$ such that every point x in G can be joined to x_0 by a rectifiable curve $\gamma\colon[0,\ell]\to G$, parametrized by its arc length, for which $\gamma(0)=x$, $\gamma(\ell)=x_0$, and

$$\mathrm{dist}\big(\gamma(t),\partial G\big)\ge t/c,$$

for every $t\in[0,\ell]$. The point x_0 is called a John center of G.

John domains satisfy the measure density condition.

Definition 2.2. A domain G in $\mathbf{R}^n$ is said to satisfy the measure density condition, if there exists a constant $C>0$ such that inequality

$$|G\cap B(x,r)| \ge Cr^n \tag{6}$$

holds for every $x\in G$ and every $r\in(0,1]$.

The domains satisfying the measure density condition are also sometimes called regular; see [17]. Let us remark that this notion of regularity of a domain is a slightly weaker condition than the Ahlfors n-regularity in which case inequality (6) is required to hold for all $0<r<\operatorname{diam}(G)$. Let us also recall the definition of the separation property from [5, Definition 3.2].

Definition 2.3. A proper domain $G\subsetneq\mathbf{R}^n$ with a fixed point $x_0\in G$ satisfies a separation property if there exists a constant $C_0>0$ such that the following holds: for every $x\in G$, there exists a curve $\gamma\colon[0,1]\to G$ with $\gamma(0)=x$, $\gamma(1)=x_0$ so that for each $t\in[0,1]$ either

$$\gamma\big([0,t]\big)\subset B:=B\big(\gamma(t),C_0\operatorname{dist}\big(\gamma(t),\partial G\big)\big)$$

or each $y\in\gamma([0,t])\setminus\overline{B}$ belongs to a different component of $G\setminus\partial B$ than x_0.

Simply connected proper planar domains satisfy the separation property. More generally, if G is quasiconformally equivalent to a uniform domain, then G satisfies the separation property. For the proofs of these statements we refer to [5].

The Riesz δ-potential $\mathcal{I}_\delta$ with $0<\delta<n$ is defined for an appropriate measurable function f on $\mathbf{R}^n$ and $x\in\mathbf{R}^n$ by

$$\mathcal{I}_\delta(f)(x)=\int_{\mathbf{R}^n}\frac{f(y)}{|x-y|^{n-\delta}}\,dy.$$

The Riesz δ-potential satisfies the following weak type estimate, see [2, p. 56] for the proof.

Theorem 2.4. *Let $0<\delta<n$. Then there exists a constant $C=C(n,\delta)>0$ such that inequality*

$$\sup_{t>0}\big|\big\{x\in\mathbf{R}^n:|\mathcal{I}_\delta(f)(x)\big|>t\big\}\big|\,t^{\frac{n}{n-\delta}}\le C\|f\|_1^{\frac{n}{n-\delta}}$$

holds for every $f\in L^1(\mathbf{R}^n)$.

The following theorem gives a fractional potential estimate in a John domain. This result is essentially contained in the proof of [13, Theorem 4.10]. Therein the constants need to be tracked more carefully, but this can be done in a straightforward way.

Theorem 2.5. *Let $0<\tau,\delta<1$ and $M>8/\tau$. Suppose that $G\subset\mathbf{R}^n$ is a c-John domain and $u\in L^1_{\mathrm{loc}}(G)$. Let $x_0\in G$ be the John center of G and write $B_0=B(x_0,\operatorname{dist}(x_0,\partial G)/(Mc))$. Then there exists a constant $C=C(M,n,c,\delta)>0$ such that inequality*

$$\left|u(x)-u_{B_0}\right|\le C\int_G \frac{g(y)}{|x-y|^{n-\delta}}\,dy = C\,\mathcal{I}_\delta(\chi_G g)(x)$$

holds if $x\in G$ is a Lebesgue point of u and the function g is defined by

$$g(y)=\int_{B(y,\tau\operatorname{dist}(y,\partial G))}\frac{|u(y)-u(z)|}{|y-z|^{n+\delta}}\,dz,\qquad y\in G.$$

The following auxiliary result is from [8, Lemma 5].

Lemma 2.6. *Let γ be a positive measure on a set X with $\gamma(X)<\infty$. If $\omega\ge 0$ is a measurable function on X such that $\gamma(\{x\in X:\omega(x)=0\})\ge\gamma(X)/2$, then inequality*

$$\gamma\big(\{x\in X:\omega(x)>t\}\big)\le 2\inf_{a\in\mathbf{R}}\gamma\big(\{x\in X:\left|\omega(x)-a\right|>t/2\}\big)$$

holds for every $t>0$.

3. Counterexamples

We give an illustrative counterexample which shows that the improved Sobolev–Poincaré inequalities are not valid on bounded domains satisfying the measure density condition, in general. Furthermore, we provide a counterexample showing that, for general John domains, the seminorms appearing on right hand sides of (3) and (4) are not comparable.

Theorem 3.1. *Let $0<\delta,\tau<1$, $1<p<n/\delta$ and $q=np/(n-\delta p)$. Then there exists a bounded domain D in $\mathbf{R}^n$ with the following properties.*

(A) *The domain D satisfies the measure density condition; in particular, there exists a constant $C_1>0$ such that inequality*

$$\left(\int_D\left|u(x)-u_D\right|^q dx\right)^{\frac1q}\le C_1\left(\int_D\int_D\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac1p}\tag{7}$$

holds for every $u\in L^p(D)$.

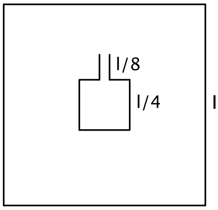

Figure 1. An s-apartment: a room and an s-passage in a unit cube.

(B) *There is no* $C_2>0$ *such that the improved fractional* $(1,p)$*-Poincaré inequality*

$$\int_D \big|u(x)-u_D\big|\,dx \le C_2\left(\int_D\int_{B(x,\tau\,\mathrm{dist}(x,\partial D))}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}} \tag{8}$$

holds for every $u\in L^\infty(D)$. *In particular, the improved fractional* (q,p)*-Poincaré inequality does not hold on* D.

The proof of Theorem 3.1 relies on [13, Theorem 6.9] which we formulate below.

Theorem 3.2. *Let* $s>1$, $p\in(1,\infty)$, $\lambda\in[n-1,n)$, *and* $\delta,\tau\in(0,1)$ *be such that*

$$s<\frac{n+1-\lambda}{1-\delta}\quad \textit{and}\quad p\le\frac{s(n-1)-\lambda+1}{n-s(1-\delta)-\lambda+1}.$$

Then there exists a bounded domain $G_s\subset\mathbf{R}^n$ *satisfying the following properties: the upper Minkowski dimension of* ∂G_s *equals* λ *and the fractional* $(1,p)$*-Poincaré inequality* (8) *does not hold in* $D=G_s$ *for all functions in* $L^\infty(G_s)$. *Moreover, there exists a constant* $c\ge 1$ *and a point* $x_0\in G_s$ *such that every* $x\in G_s$ *can be joined to* x_0 *by a rectifiable curve* $\gamma:[0,\ell]\to G_s$ *such that* $\mathrm{dist}(\gamma(t),\partial G_s)\ge t^s/c$ *for every* $t\in[0,\ell]$.

In the proof of Theorem 3.2 one modifies the usual rooms and s-passages construction by placing a room and a passage of width $2\ell(Q)^s/8^s$ to each Whitney cube Q of an appropriate John domain G, we refer to Figure 1 from [11].

Remark 3.3. The domain G_s given by Theorem 3.2 is a bounded domain satisfying the measure density condition. Indeed, the construction begins with a fixed John domain G; by the John condition, G is a bounded domain and it satisfies inequality (6). The domain G_s is then obtained by removing a set of measure zero from G. We also remark that the usual rooms and s-passages construction, as described in [10, Section 3], does not yield a domain satisfying the measure density condition.

Proof of Theorem 3.1. Let us fix $\lambda=n-1$ and choose $1<s<2/(1-\delta)$ such that

$$p\le \frac{1}{n-s(1-\delta)-\lambda+1}\le \frac{s(n-1)-\lambda+1}{n-s(1-\delta)-\lambda+1}.$$

Theorem 3.2 implies that there exists a bounded domain $D:=G_s$ such that the fractional $(1,p)$-Poincaré inequality (8) does not hold for all functions in $L^\infty(D)$. Since $q>1$, the claim (B) follows by Hölder's inequality.

Let us now prove claim (A). By Remark 3.3, the bounded domain G_s satisfies the measure density condition and inequality (7) is a consequence of this fact. Indeed, since G_s satisfies the measure density condition, the embedding $W^{\delta,p}(G_s)\hookrightarrow L^q(G_s)$ is bounded, see e.g. [20, Theorem 1.2]. In particular, there exists a constant $C>0$ such that inequality

(9)
$$\left(\int_{G_s}\left|u(x)-u_{G_s}\right|^q dx\right)^{\frac{1}{q}}\le C\left(\int_{G_s}\int_{G_s}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx+\|u-u_{G_s}\|^p_{L^p(G_s)}\right)^{\frac{1}{p}}$$

holds for each $u\in L^p(G_s)$. Inequality (7) follows from (9) and the estimate

$$\begin{aligned}\|u-u_{G_s}\|^p_{L^p(G_s)}&=\int_{G_s}\left|u(x)-u_{G_s}\right|^p dx\le\int_{G_s}\fint_{G_s}\left|u(x)-u(y)\right|^p dy\,dx\\&\le\frac{\operatorname{diam}(G_s)^{n+\delta p}}{|G_s|}\int_{G_s}\int_{G_s}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx.\qquad\square\end{aligned}$$

Next we show that inequality (5) fails for some John domains.

Proposition 3.4. *Let $1\le p<\infty$ and $0<\delta<1$ with $p\delta\ge 1$, and let $\tau=1$. Then there exists a John domain G for which inequality (5) fails.*

Proof. Let $G=(-1,1)^2\setminus((0,1)\times\{0\})$. Let $u:G\to[0,1]$ be defined by $u(x)=x_1$ for $x\in(0,1)^2$, and $u=0$ otherwise.

We observe that if $x\in G$ and $y\in B(x,\operatorname{dist}(x,\partial G))$, then $|u(x)-u(y)|\le|x-y|$, hence the right hand side of (5) is finite.

To deal with the left hand side of (5), we denote $L=(1/2,1)\times(-1/4,0)$, and for $x\in L$ we denote $E(x)=(x_1-|x_2|,x_1)\times(0,|x_2|)$. Then

$$\int_G\int_G\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\ge 4^{-p}\int_L\int_{E(x)}|x-y|^{-n-\delta p}\,dy\,dx\ge c\int_L|x_2|^{-\delta p}\,dx=\infty.$$

Thus, inequality (5) fails. $\square$

4. From weak to strong

The following theorem shows that an improved fractional Poincaré inequality of weak type is equivalent to the corresponding inequality of strong type if $q \ge p$.

Theorem 4.1. *Let μ be a positive Borel measure on an open set $G \subset \mathbf{R}^n$ so that $\mu(G)<\infty$. Let $0<\delta<1$, $0<\tau\le\infty$, and $0<p\le q<\infty$. Then the following conditions are equivalent (with the understanding that $B(y,\tau\,\mathrm{dist}(y,\partial G)):=\mathbf{R}^n$ whenever $y\in G$ and $\tau=\infty$):*

(A) *There is a constant $C_1>0$ such that inequality*

$$\inf_{a\in\mathbf{R}} \sup_{t>0} \mu\big(\{x\in G : |u(x)-a| > t\}\big) t^q \le C_1 \bigg(\int_G \int_{G\cap B(y,\tau\,\mathrm{dist}(y,\partial G))} \frac{|u(y)-u(z)|^p}{|y-z|^{n+\delta p}}\, d\mu(z)\, d\mu(y) \bigg)^{\frac{q}{p}}$$

holds, for every $u\in L^\infty(G;\mu)$.

(B) *There is a constant $C_2>0$ such that inequality*

$$\inf_{a\in\mathbf{R}} \int_G |u(x)-a|^q\, d\mu(x) \le C_2 \bigg(\int_G \int_{G\cap B(y,\tau\,\mathrm{dist}(y,\partial G))} \frac{|u(y)-u(z)|^p}{|y-z|^{n+\delta p}}\, d\mu(z)\, d\mu(y) \bigg)^{\frac{q}{p}}$$

holds, for every $u\in L^1(G;\mu)$.

In the implication from (A) *to* (B) *the constant C_2 is of the form $C(p,q)C_1$. In the converse implication $C_1=C_2$.*

Remark 4.2. Theorem 4.1 extends [8, Theorem 4] to the fractional setting. The proof is a combination of an argument in [8, Theorem 4] and a fractional Maz'ya truncation method from the proof of [7, Proposition 5].

Proof of Theorem 4.1. The implication from (B) to (A) is immediate. Let us assume that condition (A) holds for all bounded μ-measurable functions. Fix $u\in L^1(G;\mu)$ and let $b\in\mathbf{R}$ be such that

$$\mu\big(\{x\in G : u(x)\ge b\}\big) \ge \frac{\mu(G)}{2} \quad\text{and}\quad \mu\big(\{x\in G : u(x)\le b\}\big) \ge \frac{\mu(G)}{2}.$$

We write $v_+=\max\{u-b,0\}$ and $v_-=-\min\{u-b,0\}$. In the sequel v denotes either v_+ or v_-; all the statements are valid in both cases. Moreover, without loss of generality, we may assume that $v\ge 0$ is defined and finite everywhere in G.

For $0<t_1<t_2<\infty$ and every $x\in G$, we define

$$v_{t_1}^{t_2}(x)=\begin{cases} t_2-t_1, & \text{if } v(x)\geq t_2,\\ v(x)-t_1, & \text{if } t_1<v(x)<t_2,\\ 0, & \text{if } v(x)\leq t_1. \end{cases}$$

Observe that, if $0<t_1<t_2<\infty$, then

$$\mu\big(\{x\in G\colon v_{t_1}^{t_2}(x)=0\}\big)\geq \mu(G)/2.$$

For $y\in G$ we write $B_{y,\tau}=B(y,\tau\,\mathrm{dist}(y,\partial G))$. By Lemma 2.6 and condition (A), applied to the function $v_{t_1}^{t_2}\in L^\infty(G;\mu)$,

$$\begin{aligned}\sup_{t>0}\mu\big(\{x\in G\colon v_{t_1}^{t_2}(x)>t\}\big)t^q &\leq 2^{1+q}\inf_{a\in\mathbf{R}}\sup_{t>0}\mu\big(\{x\in G\colon |v_{t_1}^{t_2}(x)-a|>t\}\big)t^q\\ &\leq 2^{1+q}C_1\bigg(\int_G\int_{G\cap B_{y,\tau}}\frac{|v_{t_1}^{t_2}(y)-v_{t_1}^{t_2}(z)|^p}{|y-z|^{n+\delta p}}\,d\mu(z)\,d\mu(y)\bigg)^{\frac{q}{p}}. \end{aligned}\tag{10}$$

We write $E_k=\{x\in G\colon v(x)>2^k\}$ and $A_k=E_{k-1}\setminus E_k$, where $k\in\mathbf{Z}$. Since $v\geq 0$ is finite everywhere, we can write

$$G=\{x\in G\colon 0\leq v(x)<\infty\}=\bigcup_{i\in\mathbf{Z}}A_i\cup\underbrace{\{x\in G\colon v(x)=0\}}_{=:A_{-\infty}}.\tag{11}$$

Hence, by inequality (10) and the fact that $\sum_{k\in\mathbf{Z}}|a_k|^{q/p}\leq(\sum_{k\in\mathbf{Z}}|a_k|)^{q/p}$, we obtain that

$$\begin{aligned}\int_G|v(x)|^q\,d\mu(x)&\leq\sum_{k\in\mathbf{Z}}2^{(k+1)q}\mu(A_{k+1})\\ &\leq\sum_{k\in\mathbf{Z}}2^{(k+1)q}\mu\big(\{x\in G\colon v_{2^{k-1}}^{2^k}(x)\geq 2^{k-1}\}\big)\\ &\leq 2^{1+4q}C_1\bigg(\sum_{k\in\mathbf{Z}}\int_G\int_{G\cap B_{y,\tau}}\frac{|v_{2^{k-1}}^{2^k}(y)-v_{2^{k-1}}^{2^k}(z)|^p}{|y-z|^{n+\delta p}}\,d\mu(z)\,d\mu(y)\bigg)^{\frac{q}{p}}.\end{aligned}$$

By (11) we can estimate

$$\begin{aligned}&\sum_{k\in\mathbf{Z}}\int_G\int_{G\cap B_{y,\tau}}\frac{|v_{2^{k-1}}^{2^k}(y)-v_{2^{k-1}}^{2^k}(z)|^p}{|y-z|^{n+\delta p}}\,d\mu(z)\,d\mu(y)\\ &\quad\leq\bigg\{\sum_{k\in\mathbf{Z}}\sum_{-\infty\leq i\leq k}\sum_{j\geq k}\int_{A_i}\int_{A_j\cap B_{y,\tau}}\\ &\quad\quad+\sum_{k\in\mathbf{Z}}\sum_{i\geq k}\sum_{-\infty\leq j\leq k}\int_{A_i}\int_{A_j\cap B_{y,\tau}}\bigg\}\frac{|v_{2^{k-1}}^{2^k}(y)-v_{2^{k-1}}^{2^k}(z)|^p}{|y-z|^{n+\delta p}}\,d\mu(z)\,d\mu(y).\end{aligned}\tag{12}$$

Let $y \in A_i$ and $z \in A_j$, where $j-1 \ge i \ge -\infty$. Then $|v(y)-v(z)| \ge |v(z)|-|v(y)| \ge 2^{j-2}$. Hence,

$$\big|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)\big| \le 2^k \le 4 \cdot 2^{k-j}\big|v(y)-v(z)\big|. \tag{13}$$

Since the estimate

$$\big|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)\big| \le \big|v(y)-v(z)\big|$$

holds for every $k \in \mathbf{Z}$, inequality (13) is valid whenever $-\infty \le i \le k \le j$ and $(y,z) \in A_i \times A_j$. By inequality (13):

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} \sum_{-\infty\le i\le k} \sum_{j\ge k} \int_{A_i} \int_{A_j\cap B_{y,\tau}} \frac{|v_{2^{k-1}}^{2^k}(y) - v_{2^{k-1}}^{2^k}(z)|^p}{|y-z|^{n+\delta p}}\, d\mu(z)\, d\mu(y) \\
&\quad \le 4^p \sum_{k\in\mathbf{Z}} \sum_{-\infty\le i\le k} \sum_{j\ge k} 2^{p(k-j)} \int_{A_i} \int_{A_j\cap B_{y,\tau}} \frac{|v(y)-v(z)|^p}{|y-z|^{n+\delta p}}\, d\mu(z)\, d\mu(y).
\end{aligned} \tag{14}$$

Since $\sum_{k=i}^{j} 2^{p(k-j)} \le (1-2^{-p})^{-1}$, changing the order of the summation yields that the right hand side of inequality (14) is bounded by

$$\frac{4^p}{1-2^{-p}} \int_G \int_{G\cap B_{y,\tau}} \frac{|v(y)-v(z)|^p}{|y-z|^{n+\delta p}}\, d\mu(z)\, d\mu(y).$$

The estimation of the second term in (12) is also performed as above. To conclude that (B) holds with $C_2 = C(q,p)C_1$ it remains to recall that $|u-b| = v_+ + v_-$ and $q>0$. Observe also that $|v_\pm(y) - v_\pm(z)| \le |u(y)-u(z)|$ for all $y,z \in G$. □

Remark 4.3. If $q \ge 1$ in Theorem 4.1, then we may replace the infimum on the left hand side of the inequality appearing in condition (B) by $\int_G |u(x) - u_{G;\mu}|^q\, d\mu(x)$. Indeed, by Hölder's inequality,

$$\int_G \big|u(x) - u_{G;\mu}\big|^q\, d\mu(x) \le 2^q \inf_{a\in\mathbf{R}} \int_G \big|u(x)-a\big|^q\, d\mu(x).$$

Here we have written $u_{G;\mu} = \frac{1}{\mu(G)} \int_G u(y)\, d\mu(y)$.

5. Improved fractional Sobolev–Poincaré inequality

Hurri-Syrjänen and Vähäkangas prove in [13, Theorem 4.10] an improved fractional Sobolev–Poincaré inequality on a given c-John domain G. Namely, let us fix

$0<\delta,\tau<1$ and $1<p<n/\delta$. Then there exists a constant $C=C(n,\delta,c,\tau,p)$ such that inequality

(15)
$$\left(\int_G |u(x)-u_G|^{\frac{np}{n-\delta p}}\,dx\right)^{\frac{n-\delta p}{np}} \le C\left(\int_G\int_{B(x,\tau\,\mathrm{dist}(x,\partial G))}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}$$

holds for every $u\in L^1(G)$.

The proof in [13] is based on the fractional potential estimate in a John domain. The equivalence of inequality (15) to the corresponding weak type inequality, Theorem 4.1, allows to employ the potential estimate while proving inequality (15) with $p=1$ also.

Theorem 5.1. *Suppose that G is a c-John domain in $\mathbf{R}^n$ and let $\tau,\delta\in(0,1)$ be given. Then there exists a constant $C=C(n,\delta,c,\tau)>0$ such that inequality*

$$\left(\int_G |u(x)-u_G|^{\frac{n}{n-\delta}}\,dx\right)^{\frac{n-\delta}{n}} \le C\int_G\int_{B(x,\tau\,\mathrm{dist}(x,\partial G))}\frac{|u(x)-u(y)|}{|x-y|^{n+\delta}}\,dy\,dx$$

holds for every $u\in L^1(G)$.

Proof. By Theorem 4.1 and Remark 4.3, it suffices to prove that there exists a constant $C=C(n,\delta,c,\tau)>0$ such that inequality

$$\inf_{a\in\mathbf{R}}\sup_{t>0}\Big|\big\{x\in G:\,\big|u(x)-a\big|>t\big\}\Big|\,t^{\frac{n}{n-\delta}}$$
(16)
$$\le C\left(\int_G\int_{B(y,\tau\,\mathrm{dist}(y,\partial G))}\frac{|u(y)-u(z)|}{|y-z|^{n+\delta}}\,dz\,dy\right)^{\frac{n}{n-\delta}}$$

holds for every $u\in L^\infty(G)$. Let us denote by $x_0\in G$ the John center of G, and let

$$B_0:=B\big(x_0,\mathrm{dist}(x_0,\partial G)/(Mc)\big),$$

where $M=9/\tau$. We also write

$$g(y)=\int_{B(y,\tau\,\mathrm{dist}(y,\partial G))}\frac{|u(y)-u(z)|}{|y-z|^{n+\delta}}\,dz$$

for every $y\in G$. By Theorem 2.5, for each Lebesgue point $x\in G$ of u,

(17)
$$|u(x)-u_{B_0}|\le C(n,c,\delta,\tau)\int_G\frac{g(y)}{|x-y|^{n-\delta}}\,dy=C(n,c,\delta,\tau)\,\mathcal{I}_\delta(\chi_G g)(x).$$

By inequality (17) and Theorem 2.4, there exists a constant $C = C(n, c, \delta, \tau)$ such that

$$\big|\{x \in G : |u(x) - u_{B_0}| > t\}\big| t^{\frac{n}{n-\delta}} \le C\bigg(\int_G \int_{B(y,\tau\,\mathrm{dist}(y,\partial G))} \frac{|u(y) - u(z)|}{|y - z|^{n+\delta}}\, dz\, dy\bigg)^{\frac{n}{n-\delta}}$$

for every $t > 0$. Inequality (16) follows. □

Remark 5.2. Inequality (15) makes sense only if the domain G has a finite measure. If we replace the left hand side of inequality (15) by

$$\bigg(\int_G |u(x)|^{\frac{np}{n-\delta p}}\, dx\bigg)^{\frac{n-\delta p}{np}},$$

then the resulting inequality is valid on so-called unbounded John domains G that are of infinite measure, we refer to [12, Section 5].

6. Necessary conditions for the improved inequality

In this section, we obtain necessary conditions for the improved Poincaré inequalities. Theorem 6.1 is parallel to the result of Buckley and Koskela on the classical Sobolev–Poincaré inequality (1), see [5, Theorem 1.1]. See also [19], where the geometric conditions of the same spirit are used to obtain a criteria for a domain G to support the embedding of Hajłasz–Sobolev type spaces $\dot{M}^{s,p}_{\mathrm{ball}}(G)$ into $L^q(G)$, for $s \in (0, 1]$, $p \in (n(n+s), n/s)$ and $q = np/(n - ps)$.

Theorem 6.1. *Assume that G is a domain of finite measure in $\mathbf{R}^n$ which satisfies the separation property. Let $\delta \in (0, 1)$ and $1 \le p < n/\delta$ be given. If there exists a constant $C_1 > 0$ such that the improved fractional Sobolev–Poincaré inequality*

$$\bigg(\int_G |u(x) - u_G|^{\frac{np}{n-\delta p}}\, dx\bigg)^{\frac{n-\delta p}{np}} \le C_1\bigg(\int_G \int_{B(x,\mathrm{dist}(x,\partial G))} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}}\, dy\, dx\bigg)^{\frac{1}{p}} \tag{18}$$

holds for every $u \in L^\infty(G)$, then G is a John domain.

To prove Theorem 6.1 it suffices to prove Proposition 6.2, and then follow the geometric arguments given in [5, pp. 6–7]. Observe that $(1/p - 1/q)/\delta = 1/n$ and $(n - \delta p)q/(np) = 1$ if $q = np/(n - \delta p)$.

Proposition 6.2. *Suppose that $G \subset \mathbf{R}^n$ is a domain of finite measure. Let $\delta \in (0,1)$ and $1 \le p < q < \infty$ be given. Assume that there exists a constant $C_1 > 0$ such that inequality*

$$\left(\int_G |u(x) - u_G|^q \, dx\right)^{\frac{1}{q}} \le C_1 \left(\int_G \int_{B(x, \operatorname{dist}(x, \partial G))} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} \, dy \, dx\right)^{\frac{1}{p}} \tag{19}$$

holds for every $u \in L^\infty(G)$. Fix a ball $B_0 \subset G$, and let $d > 0$ and $w \in G$. Then there exists a constant $C > 0$ such that

$$\operatorname{diam}(T) \le C\left(d + |T|^{(\frac{1}{p} - \frac{1}{q})\frac{1}{\delta}}\right) \quad \textit{and} \quad |T|^{\frac{1}{n}} \le C\left(d + d^{\frac{(n-\delta p)q}{np}}\right) \tag{20}$$

if T is the union of all components of $G \setminus B(\omega, d)$ that do not intersect the ball B_0. The constant C depends on C_1, $|B_0|$, $|G|$, n, δ, q, and p only.

Notice that inequalities in (20) extend [5, Theorem 2.1] to the fractional case.

Proof. We start by proving the first inequality in (20). Without loss of generality, we may assume that $T \neq \varnothing$. Let $T(r) = T \setminus B(\omega, r)$, we will later prove inequality

$$\left|T(r)\right|^{\frac{p}{q}} \le \frac{c|T(\rho)|}{(r-\rho)^{\delta p}}, \tag{21}$$

provided $d \le \rho < r$. Assuming that this inequality holds, one proceeds as follows. Define $r_0 = d$ and for $j \ge 1$ pick $r_j > r_{j-1}$ such that

$$\left|A(r_{j-1}, r_j)\right| = \left|T \cap B(w, r_j) \setminus B(w, r_{j-1})\right| = 2^{-j}|T|.$$

Then $|T(r_j)| = |T \setminus B(w, r_j)| = 2^{-j}|T|$. Hence, by inequality (21)

$$\begin{aligned}
\operatorname{diam}(T) &\le 2d + \sum_{j=1}^{\infty} 2|r_j - r_{j-1}| \\
&\le 2d + c\sum_{j=1}^{\infty}\left(\left|T(r_{j-1})\right|\left|T(r_j)\right|^{-\frac{p}{q}}\right)^{\frac{1}{\delta p}} \\
&= 2d + c\sum_{j=1}^{\infty}\left(2^{-j+1}|T|2^{j\frac{p}{q}}|T|^{-\frac{p}{q}}\right)^{\frac{1}{\delta p}} \\
&= 2d + c|T|^{(\frac{1}{p} - \frac{1}{q})\frac{1}{\delta}} \sum_{j=1}^{\infty} 2^{-j(\frac{1}{p} - \frac{1}{q})\frac{1}{\delta}} \le 2d + c|T|^{(\frac{1}{p} - \frac{1}{q})\frac{1}{\delta}}
\end{aligned}$$

and this concludes the main line of the argument.

It remains to prove inequality (21). We assume that $T(r)\neq\varnothing$ and define a bounded function u on G as follows

$$u(x)=\begin{cases}1, & x\in T(r),\\ \frac{\operatorname{dist}(x,B(\omega,\rho))}{r-\rho}, & x\in A(\rho,r)=T(\rho)\setminus T(r),\\ 0, & x\in G\setminus T(\rho).\end{cases}$$

For $x\in G$, let us denote $B_{x,1}=B(x,\operatorname{dist}(x,\partial G))$. By the fact that $u=0$ on B_0 and inequality (19) we obtain

$$\begin{aligned}\bigl|T(r)\bigr|^{\frac{p}{q}} &\le \Bigl(\int_G |u(x)|^q\,dx\Bigr)^{\frac{p}{q}} \le c\Bigl(\int_G \bigl|u(x)-u_G\bigr|^q\,dx\Bigr)^{\frac{p}{q}}\\ &\le c\int_G\int_{B_{x,1}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx. \end{aligned}\tag{22}$$

For all measurable $E,F\subset G$, denote

$$I(E,F)=\int_E\int_{B_{x,1}\cap F}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx.$$

Since $u=0$ on $G\setminus T(\rho)$ and $u=1$ on $T(r)$, we can write the right hand side of (22) as

$$\begin{aligned}I(G,G)&=I\bigl(T(r),A(\rho,r)\bigr)+I\bigl(T(r),G\setminus T(\rho)\bigr)\\ &\quad+I\bigl(A(\rho,r),T(r)\bigr)+I\bigl(A(\rho,r),A(\rho,r)\bigr)+I\bigl(A(\rho,r),G\setminus T(\rho)\bigr)\\ &\quad+I\bigl(G\setminus T(\rho),T(r)\bigr)+I\bigl(G\setminus T(\rho),A(\rho,r)\bigr).\end{aligned}\tag{23}$$

For the first and the third term of (23) we use the following estimate

$$I\bigl(T(r),A(\rho,r)\bigr)+I\bigl(A(\rho,r),T(r)\bigr)\le 2\int_{A(\rho,r)}\int_{T(r)}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx.$$

We observe that, for every $x\in A(\rho,r)$,

$$\bigl|\operatorname{dist}\bigl(x,B(\omega,\rho)\bigr)-(r-\rho)\bigr|\le\min\bigl\{\operatorname{dist}\bigl(x,T(r)\bigr),r-\rho\bigr\}=m(x).$$

By the definition of function u,

$$\begin{aligned}&\int_{A(\rho,r)}\int_{T(r)}\frac{|\operatorname{dist}(x,B(\omega,\rho))-(r-\rho)|^p}{(r-\rho)^p|x-y|^{n+\delta p}}\,dy\,dx\\ &\qquad\le\int_{A(\rho,r)}\int_{T(r)}\frac{m^p(x)}{(r-\rho)^p|x-y|^{n+\delta p}}\,dy\,dx\end{aligned}$$

$$\le \int_{A(\rho,r)} \int_{\mathbf{R}^n \setminus B(x,m(x))} \frac{m^p(x)}{(r-\rho)^p |x-y|^{n+\delta p}}\, dy\, dx$$
$$= c \int_{A(\rho,r)} \frac{(m(x))^{p-\delta p}}{(r-\rho)^p}\, dx \le \frac{c|A(\rho,r)|}{(r-\rho)^{\delta p}}.$$

We estimate the second term $I(T(r), G\setminus T(\rho))$. Let us show that, for every $x\in T(r)$,

$$B_{x,1}\cap\big(G\setminus T(\rho)\big) \subset \mathbf{R}^n\setminus B(x, r-\rho). \tag{24}$$

If $y\in G\setminus T(\rho)$, then the point y belongs to the ball $B(\omega,\rho)$ or to a component of $G\setminus B(\omega,d)$ that intersects the ball B_0. At the same time, if $y\in B_{x,1}$, then $B(x, |x-y|) \subset G$ which means that the situation when x and y are in different components of $G\setminus B(\omega,d)$ is not possible. Hence, $y\in B(\omega,\rho)$, and indeed $|x-y| \ge |x-w| - |w-y| \ge r-\rho$.

By (24), for each $x\in T(r)$, we have

$$\int_{B_{x,1}\cap(G\setminus T(\rho))} \frac{1}{|x-y|^{n+\delta p}}\, dy \le \int_{\mathbf{R}^n\setminus B(x,r-\rho)} \frac{1}{|x-y|^{n+\delta p}}\, dy = c(r-\rho)^{-\delta p},$$

and hence

$$I\big(T(r), G\setminus T(\rho)\big) \le c\frac{|T(r)|}{(r-\rho)^{\delta p}} \le c\frac{|T(\rho)|}{(r-\rho)^{\delta p}}.$$

Next we consider $I(A(\rho,r), A(\rho,r))$. Notice that, for every $x\in A(\rho,r)$,

$$\int_{B_{x,1}\cap A(\rho,r)} \frac{|\operatorname{dist}(x,B(\omega,\rho)) - \operatorname{dist}(y,B(\omega,\rho))|^p}{(r-\rho)^p|x-y|^{n+\delta p}}\, dy$$
$$\le (r-\rho)^{-p} \int_{A(\rho,r)\cap B(x,r-\rho)} \frac{1}{|x-y|^{n+\delta p-p}}\, dy + \int_{A(\rho,r)\setminus B(x,r-\rho)} \frac{1}{|x-y|^{n+\delta p}}\, dy$$
$$\le \frac{c(r-\rho)^{p-\delta p}}{(r-\rho)^p} + \frac{c}{(r-\rho)^{\delta p}}.$$

Hence, we obtain that

$$I\big(A(\rho,r), A(\rho,r)\big) \le c\frac{|A(\rho,r)|}{(r-\rho)^{\delta p}}.$$

Then we focus on $I(A(\rho,r), G\setminus T(\rho))$. Let us first observe that, for every $x\in A(\rho,r)$,

$$B_{x,1}\cap\big(G\setminus T(\rho)\big) \subset \mathbf{R}^n\setminus B\big(x, \operatorname{dist}\big(x, B(\omega,\rho)\big)\big).$$

To verify this, we fix $y \in B_{x,1} \cap (G \setminus T(\rho))$. By repeating the argument used in the proof of inclusion (24) we obtain that $y \in B(\omega,\rho)$ and $|y - x| \geq \operatorname{dist}(x, B(\omega,\rho))$. Thus, for every $x \in A(\rho,r)$,

$$\begin{aligned}
\int_{B_{x,1}\cap(G\setminus T(\rho))} \frac{1}{|x-y|^{n+\delta p}}\,dy &\leq \int_{\mathbf{R}^n\setminus B(x,\operatorname{dist}(x,B(\omega,\rho)))} \frac{1}{|x-y|^{n+\delta p}}\,dx \\
&\leq c\big(\operatorname{dist}\big(x,B(\omega,\rho)\big)\big)^{-\delta p}.
\end{aligned}$$

Therefore, we have

$$I\big(A(\rho,r), G\setminus T(\rho)\big) \leq c\int_{A(\rho,r)} \frac{(\operatorname{dist}(x,B(\omega,\rho)))^{p-\delta p}}{(r-\rho)^p}\,dx \leq \frac{c|A(\rho,r)|}{(r-\rho)^{\delta p}}.$$

In order to estimate the terms $I(G\setminus T(\rho), T(r))$ and $I(G\setminus T(\rho), A(\rho,r))$ we observe that, if $x \in G\setminus T(\rho)$ and $B_{x,1}\cap T(\rho) \neq \varnothing$, then $x \in B(w,\rho)$. This follows from the fact that, if $y \in B_{x,1}\cap T(\rho)$ then $B(x,|x-y|) \subset G$ and, hence, x and y cannot belong to different components of $G\setminus B(\omega,\rho)$.

Using the observation above and adapting the estimates for the term $I(T(r), G\setminus T(\rho))$, we obtain

$$\begin{aligned}
I\big(G\setminus T(\rho), T(r)\big) &= I\big(B(\omega,\rho)\cap G, T(r)\big) \\
&\leq \int_{T(r)}\int_{B(\omega,\rho)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx \leq c\frac{|T(\rho)|}{(r-\rho)^{\delta p}}.
\end{aligned}$$

Following the same argument and adapting the estimates for $I(A(\rho,r), G\setminus T(\rho))$ we obtain that $I(G\setminus T(\rho), A(\rho,r)) \leq c|A(\rho,r)|(r-\rho)^{-\delta p}$.

We proceed to the proof of the second part of Proposition 6.2. We first observe that $|T| \leq Cd^n + |T(2d)|$. Hence, it remains to show that

$$|T(2d)| \leq Cd^{\frac{(n-\delta p)q}{p}}. \tag{25}$$

In order to do this, we use a slightly modified proof of the first inequality. More precisely, by inequality (22), for $d \leq \rho < r$, we have

$$\big|T(r)\big|^{\frac{p}{q}} \leq I(G,G),$$

where $I(G,G)$ can be written as in (23). From the reasoning above it is seen that all the terms in (23) except $I(T(r), G\setminus T(\rho))$ and $I(G\setminus T(\rho), T(r))$ are bounded from above by $c|A(\rho,r)|(r-\rho)^{-\delta p}$. Furthermore, for the remaining terms, we have

$$I\big(T(r),G\setminus T(\rho)\big)+I\big(G\setminus T(\rho),T(r)\big)=I\big(T(r),B(\omega,\rho)\cap G\big)+I\big(B(\omega,\rho)\cap G,T(r)\big)$$
$$\le 2\int_{B(\omega,\rho)}\int_{T(r)}\frac{dy\,dx}{|x-y|^{n+\delta p}}\le 2\int_{B(\omega,\rho)}\int_{\mathbf{R}^n\setminus B(x,r-\rho)}\frac{dy\,dx}{|x-y|^{n+\delta p}}\le c\frac{|B(\omega,\rho)|}{(r-\rho)^{\delta p}}.$$

Thus,

$$\big|T(r)\big|^{\frac{p}{q}}\le\frac{c}{(r-\rho)^{\delta p}}\big(|A(\rho,r)|+|B(\omega,\rho)|\big).$$

Next we set $\rho=d$ and $r=2d$ in the inequality above, and using the trivial estimates for the measures of a ball and of an annulus, we obtain (25). $\square$

References

1. Adams, R. A., *Sobolev Spaces*, Pure and Applied Mathematics **65**, Academic Press, New York, 1975.
2. Adams, D. R. and Hedberg, L. I., *Function Spaces and Potential Theory*, Springer, Berlin, 1996.
3. Bellido, J. C. and Mora-Corral, C., Existence for nonlocal variational problems in peridynamics, *SIAM J. Math. Anal.* **46** (2014), 890–916.
4. Bojarski, B., Remarks on Sobolev imbedding inequalities, in *Complex Analysis*, *Joensuu* 1987, Lecture Notes in Math. **1351**, pp. 52–68, Springer, Berlin, 1988.
5. Buckley, S. and Koskela, P., Sobolev–Poincaré implies John, *Math. Res. Lett.* **2** (1995), 577–593.
6. Dyda, B., On comparability of integral forms, *J. Math. Anal. Appl.* **318** (2006), 564–577.
7. Dyda, B. and Vähäkangas, A. V., Characterizations for fractional Hardy inequality, *Adv. Calc. Var.* **8** (2015), 173–182.
8. Hajłasz, P., Sobolev inequalities, truncation method, and John domains, in *Papers on Analysis*, Rep. Univ. Jyväskylä Dep. Math. Stat. **83**, pp. 109–126, Univ. Jyväskyä, Jyväskylä, 2001.
9. Jonsson, A. and Wallin, H., A Whitney extension theorem in L^p and Besov spaces, *Ann. Inst. Fourier* **28** (1978), 139–192.
10. Harjulehto, P. and Hurri-Syrjänen, R., On a (q,p)-Poincaré inequality, *J. Math. Anal. Appl.* **337** (2008), 61–68.
11. Harjulehto, P., Hurri-Syrjänen, R. and Vähäkangas, A. V., On the $(1,p)$-Poincaré inequality, *Illinois J. Math.* **56** (2012), 905–930.
12. Hurri-Syrjänen, R. and Vähäkangas, A. V., Fractional Sobolev–Poincaré and fractional Hardy inequalities in unbounded John domains, *Mathematika* **61** (2015), 385–401.
13. Hurri-Syrjänen, R. and Vähäkangas, A. V., On fractional Poincaré inequalities, *J. Anal. Math.* **120** (2013), 85–104.
14. Martio, O., John domains, bi-Lipschitz balls and Poincaré inequality, *Rev. Roumaine Math. Pures Appl.* **33** (1988), 107–112.
15. Maz'ya, V. G., *Sobolev Spaces*, Springer, Berlin, 1985.

16. Reshetnyak, Y. G., Integral representations of differentiable functions, in *Partial Differential Equations*, pp. 173–187, Nauka, Novosibirsk, 1986.
17. Shvartsman, P., Local approximations and intrinsic characterization of spaces of smooth functions on regular subsets of $\mathbf{R}^n$, *Math. Nachr.* **279** (2006), 1212–1241.
18. Väisälä, J., Exhaustions of John domains, *Ann. Acad. Sci. Fenn., Ser. A* 1 *Math.* **19** (1994), 47–57.
19. Zhou, Y., Criteria for optimal global integrability of Hajłasz–Sobolev functions, *Illinois J. Math.* **55** (2011), 1083–1103.
20. Zhou, Y., Fractional Sobolev extension and imbedding, *Trans. Amer. Math. Soc.* **367** (2015), 959–979.

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