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The Loewner equation for multiple slits, multiply connected domains and branch points

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Abstract. Let $\mathbb{D}\subset\mathbb{C}$ be the unit disk and let $\gamma_1,\gamma_2:[0,T]\to\overline{\mathbb{D}}\setminus\{0\}$ be parametrizations of two slits $\Gamma_1:=\gamma(0,T],\Gamma_2:=\gamma_2(0,T]$ such that Γ_1 and Γ_2 are disjoint.

Let g_t be the unique normalized conformal mapping from $\mathbb{D}\setminus(\gamma_1[0,t]\cup\gamma_2[0,t])$ onto $\mathbb{D}$ with $g_t(0)=0$, $g_t'(0)>0$. Furthermore, for $k=1,2$, denote by $h_{k;t}$ the unique normalized conformal mapping from $\mathbb{D}\setminus\gamma_k[0,t]$ onto $\mathbb{D}$ with $h_{k;t}(0)=0$, $h_{k;t}'(0)>0$.

Loewner's famous theorem (1923) can be stated in the following way: The function $t\mapsto h_{k;t}$ is differentiable at t_0 if and only if $t\mapsto\log(h_{k;t}'(0))$ is differentiable at t_0.

In this paper we compare the differentiability of $t\mapsto h_{k;t}$ with that of $t\mapsto g_t$. We show that the situation is more complicated in the case $t_0=0$ with $\gamma_1(0)=\gamma_2(0)$.

Furthermore, we also look at this problem in the case of a multiply connected domain with its corresponding Komatu–Loewner equation.

1. Introduction and results

1.1. The main results

The simply connected case

By $\mathbb{D}:=\{z\in\mathbb{C}\,|\,|z|<1\}$ we denote the unit disk.

Let γ: $[0,T]\to\overline{\mathbb{D}}$ be a simple curve (i.e. γ is continuous and injective) with $\Gamma:=\gamma(0,T]\subset\mathbb{D}\setminus\{0\}$ and $\gamma(0)\in\partial\mathbb{D}$. In the following such a set Γ will be called *slit*.

For every $t\in[0,T]$, the domain $\Omega_t:=\mathbb{D}\setminus\gamma[0,t]$ is simply connected and it can be mapped onto $\mathbb{D}$ by a conformal map $g_t:\Omega_t\to\mathbb{D}$.

This mapping is unique if we require the normalization $g_t(0)=0$, $g_t'(0)>0$. The function $g_t'(0)$ is increasing and $g_t'(0)\geq 1$ for all t as a consequence of the Schwarz lemma. The *logarithmic mapping radius* is defined as $\operatorname{lmr}(g_t):=\log(g_t'(0))$.

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In his much celebrated paper from 1923 [10], Loewner considered the question whether the function $t \mapsto g_t$ could be differentiable, even though there are no smoothness assumptions on Γ. Loewner's famous theorem can be stated in the following way:

The differentiability of $t \mapsto \text{lmr}(g_t)$ is equivalent to the differentiability of the function $t \mapsto g_t$, more precisely the following statement holds; see, e.g., Theorem 2 in [2].

Theorem A. *The function* $c(t) := \text{lmr}(g_t)$ *is differentiable at* $t = t_0$ *if and only if the family* $\{g_t\}_{t \in [0,T]}$ *is differentiable at* $t = t_0$*, i.e. for every* $z \in \mathbb{D} \setminus \Gamma$*, the function* $t \mapsto g_t(z)$ *is differentiable at* $t = t_0$*. In this case,* $g_t(z)$ *satisfies the following differential equation*:

$$\dot{g}_{t_0}(z) = \dot{c}(t_0) \cdot g_{t_0}(z) \cdot \frac{\xi(t_0) + g_{t_0}(z)}{\xi(t_0) - g_{t_0}(z)}, \tag{1}$$

where $\xi(t_0) = \lim_{z \to \gamma(t_0)} g_{t_0}(z)$.

In the following, we will call γ a $\mathbb{D}$-*Loewner parametrization for* Γ *at* t_0, if the two equivalent conditions in Theorem A hold.

Remark 1.1. Usually, the parametrization of Γ is chosen in such a way that $\text{lmr}(g_t) = t$. In this case, the mappings $\{g_t\}$ are (continuously) differentiable for all $t \in [0, T]$. Thus, an arbitrary slit Γ can be described by a differential equation for the family $\{g_t\}$. This celebrated idea of Loewner turned out to be quite useful for the theory of univalent mappings and its most prominent application nowadays is the stochastic Loewner evolution invented by Schramm in 2000.

Now let $\gamma_1, \gamma_2 \colon [0, T] \to \overline{\mathbb{D}} \setminus \{0\}$ be parametrizations of two slits $\Gamma_1 := \gamma_1(0, T]$, $\Gamma_2 := \gamma_2(0, T]$ such that Γ_1 and Γ_2 are disjoint.

For a fixed time t_0 we will distinguish between two cases:

Either $\gamma_1(t_0) \neq \gamma(t_0)$ ("*disjoint case*") or $\gamma_1(t_0) = \gamma_2(t_0)$ ("*branch point case*"), which is only possible for $t_0 = 0$.

Again, we can define g_t to be the unique normalized conformal mapping from $\Omega_t := \mathbb{D} \setminus (\gamma_1[0, t] \cup \gamma_2[0, t])$ onto $\mathbb{D}$ with $g_t(0) = 0$, $g_t'(0) > 0$.

We are interested in the question, under which conditions the family $\{g_t\}_{t \in [0,T]}$ is differentiable at a point $t_0 \in [0, T]$.

Again, a necessary condition is that $c(t) := \text{lmr}(g_t) := \log(g_t'(0))$ is differentiable at $t = t_0$. However, this condition is not sufficient anymore, see Example 2.5.

On the other hand, the two statements are equivalent in the branch point case; see Theorem 1.7.

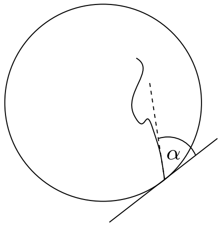


Figure 1. A slit *approaching $\partial\mathbb{D}$ in α-direction.*

In the disjoint case, differentiability of $t \mapsto g_t$ is guaranteed if both slits are $\mathbb{D}$-Loewner parametrized. More precisely, the following equivalence holds.

Theorem 1.2. *Suppose that $t_0 \in [0,T]$ such that $\gamma_1(t_0) \neq \gamma_2(t_0)$. Then the following two conditions are equivalent:*
1. *For $j=1,2$, γ_j is a $\mathbb{D}$-Loewner parametrization for Γ_j at t_0.*
2. *The function $t \mapsto g_t(z)$ is differentiable at t_0 for every $z \in \Omega_{t_0}$.*

For $j=1,2$, let $h_{j;t}$ be the unique conformal mapping from $\mathbb{D} \setminus \gamma_j[0,t]$ onto $\mathbb{D}$ with $h_{j;t}(0)=0$, $h'_{j;t}(0)>0$ and let $c_j(t) := \operatorname{lmr}(h_{j;t}) = \log(h'_{j;t}(0))$. We will also derive a relation between $\dot{c}$ and $\dot{c}_j$. Here we note the simplest case $t_0=0$:

If the two equivalent statements in Theorem 1.2 hold for $t_0=0$, then $c(t)$ is differentiable at $t=0$ with

$$\dot{c}(0) = \dot{c}_1(0) + \dot{c}_2(0). \tag{2}$$

A general relation between $\dot{c}$ and $\dot{c}_j$ if $t_0>0$ is given by Theorem 1.12.

The situation is different for the branch point case.

Theorem 1.3. *There exist two slits Γ_1, Γ_2 in $\mathbb{D}$ with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \{p\} \subset \partial\mathbb{D}$ with $\mathbb{D}$-Loewner parametrizations $\gamma_k \colon [0,T] \to \Gamma_k$ in $[0,T]$, such that the function $t \mapsto g_t(z)$, $z \in \Omega_T$, is not differentiable at $t=0$.*

On the other hand, we also give a condition ensuring differentiability of $t \mapsto g_t(z)$ at $t=0$ in this case.

Definition 1.4. Let $\alpha \in (0,\pi)$. We say that a simple curve $\gamma\colon [0,T] \to \overline{\mathbb{D}}$, $\gamma(0) \in \partial\mathbb{D}$, $\gamma(0,T] \subset \mathbb{D}$, *approaches $\partial\mathbb{D}$ in α-direction* (see Figure 1) if for every $\varepsilon>0$ there exists $s>0$ such that

$$\gamma(0,s] \subset \big\{ z \in \mathbb{D} \mid \alpha - \varepsilon < \arg\big(\gamma(0) - z\big) + \arg\big(\gamma(0)\big) - \pi/2 < \alpha + \varepsilon \big\}.$$

Theorem 1.5. *Let $b_1, b_2 \geq 0$, $\gamma_1(0) = \gamma_2(0)$ and assume that Γ_j approaches $\partial\mathbb{D}$ in α_j-direction with $\alpha_1 \leq \alpha_2$. Let γ_j be a $\mathbb{D}$-Loewner parametrization for Γ_j at $t=0$ for $j=1$ and $j=2$ with $b_1 = \dot{c}_1(0), b_2 = \dot{c}_2(0)$. Then the function $t \mapsto g_t(z)$ is differentiable at $t=0$ for every $z \in \Omega$.*

- *If $b_1 = 0$ or $b_2 = 0$, then $\dot{c}(0) = \max\{b_1, b_2\}$.*

If $b_1, b_2 > 0$, then

- $\max\{b_1, b_2\} \leq \dot{c}(0) < b_1 + b_2$,
- $\dot{c}(0) = \max\{b_1, b_2\}$ *if and only if* $\alpha_1 = \alpha_2$, *and*
- $\dot{c}(0) \to b_1 + b_2$ *as* $(\alpha_1, \alpha_2) \to (0, \pi)$.

Note that the very last statement says that the branch point case behaves like the disjoint case when $(\alpha_1, \alpha_2) \to (0, \pi)$, see (2).

Finally it is worth mentioning that the converse of Theorem 1.5 is wrong; see Example 2.6.

The multiply connected case

A *circular slit disk* D is an n-connected domain of the form $D = \mathbb{D} \setminus (C_1 \cup \ldots \cup C_{n-1})$, where the C_j's are proper disjoint circular arcs in $\mathbb{D}$ centered at 0. For any circular slit disk D and any $u \in \partial\mathbb{D}$, we denote by $w \mapsto \Phi(u, w; D)$ the unique conformal mapping from D onto the right half-plane minus slits parallel to the imaginary axis with $\Phi(u, u; D) = \infty$ and $\Phi(u, 0; D) = 1$. For example, $\Phi(u, w; \mathbb{D}) = \frac{u+w}{u-w}$.

Now let Ω be an n-connected circular slit disk and let $\gamma\colon [0, T] \to \overline{\mathbb{D}}$ be a simple curve with $\Gamma := \gamma(0, T] \subset \Omega \setminus \{0\}$ and $\gamma(0) \in \partial\mathbb{D}$. In this case, $\Omega_t := \Omega \setminus \gamma[0, t]$ is an n-connected domain for every $t \in [0, T]$ and it can be mapped onto a circular slit disk D_t by a conformal map $g_t : \Omega_t \to D_t$. This mapping is unique if we require the normalization $g_t(0) = 0$, $g_t'(0) > 0$, $g_t(\partial\mathbb{D}) \subset \partial\mathbb{D}$; see [4], Chapter 15.6. In the following, we will call mappings *normalized* if they satisfy these three conditions.

Again we define the *logarithmic mapping radius* $\operatorname{lmr}(g_t) := \log(g_t'(0))$. The analog of Theorem A is given by the following theorem; see Theorem 5.1 in [1] or Theorem 2 in [2]. Loewner equations for multiply connected domains were first studied by Komatu; see [6] and [5].

Theorem B. *The function $c(t) := \operatorname{lmr}(g_t)$ is differentiable at $t = t_0$ if and only if the family $\{g_t\}_{t \in [0,T]}$ is differentiable at $t = t_0$, i.e. for every $z \in \Omega \setminus \Gamma$, the function $t \mapsto g_t(z)$ is differentiable at $t = t_0$. In this case, $g_t(z)$ satisfies the following differential equation:*

$$\dot{g}_{t_0}(z) = \dot{c}(t_0) \cdot g_{t_0}(z) \cdot \Phi\big(\xi(t_0), g_{t_0}(z); D_{t_0}\big), \tag{3}$$

where $\xi(t_0) = \lim_{z \to \gamma(t_0)} g_{t_0}(z)$.

In the following, we will call γ an Ω*-Loewner parametrization for* Γ *at* t_0, if the two equivalent conditions in Theorem B hold.

The following relation to $\mathbb{D}$-Loewner parametrizations is not surprising.

Theorem 1.6. *Let* $t_0 \in [0,T]$. *Then* γ *is an* Ω*-Loewner parametrization for* Γ *at* t_0 *if and only if it is a* $\mathbb{D}$*-Loewner parametrization for* Γ *at* t_0.

Now we pass again to the case of two slits: Let $\gamma_1, \gamma_2 \colon [0,T] \to \overline{\mathbb{D}}$ be parametrizations of two slits $\Gamma_1 = \gamma(0,T]$ and $\Gamma_2 = \gamma_2(0,T]$ such that Γ_1 and Γ_2 are disjoint, $\Gamma_1, \Gamma_2 \subset \Omega \setminus \{0\}$ and $\gamma_1(0) \neq \gamma_2(0)$ or $\gamma_1(0) = \gamma_2(0)$.

Again, we define g_t to be the unique normalized mapping from $\Omega_t := \Omega \setminus (\gamma_1[0,t] \cup \gamma_2[0,t])$ onto a circular slit disk $\mathbb{D}_t$ and $\operatorname{lmr}(g_t) := \log(g_t'(0))$.

Furthermore, let h_t be the unique normalized mapping from $\Psi_t := \mathbb{D} \setminus (\gamma_1[0,t] \cup \gamma_2[0,t])$ onto $\mathbb{D}$.

Theorem 1.7. *Let* $t_0 \in [0,T]$. *Then the following two statements are equivalent.*

1. *The function* $t \mapsto g_t(z)$ *is differentiable at* t_0 *for every* $z \in \Omega_{t_0}$.
2. *The function* $t \mapsto h_t(z)$ *is differentiable at* t_0 *for every* $z \in \Psi_{t_0}$.

In the branch point case, i.e. $\gamma_1(0) = \gamma_2(0)$ *and* $t_0 = 0$, *the above statements are equivalent to each of the following two statements.*

3. *The function* $t \mapsto \operatorname{lmr}(h_t)$ *is differentiable at* 0.
4. *The function* $t \mapsto \operatorname{lmr}(g_t)$ *is differentiable at* 0.

As a direct consequence of the last two theorems, we can state Theorem 1.2 and Theorem 1.5 for the multiply connected case.

Corollary 1.8. *Suppose that* $t_0 \in [0,T]$ *such that* $\gamma_1(t_0) \neq \gamma_2(t_0)$. *Then the following conditions are equivalent*:

1. *The function* $t \mapsto g_t(z)$ *is differentiable at* t_0 *for every* $z \in \Omega_{t_0}$.
2. *For* $j = 1, 2$, γ_j *is a* $\mathbb{D}$*-Loewner parametrization for* Γ_j *at* t_0.

Corollary 1.8 shows that the question whether the function $t \mapsto g_t(z)$ is differentiable at t_0 can be reduced to the corresponding question for each single slit with respect to the simply connected domain $\mathbb{D}$.

Corollary 1.9. *Suppose* $\gamma_1(0) = \gamma_2(0)$ *and that* Γ_j *approaches* $\partial\mathbb{D}$ *in* α_j*-direction with* $\alpha_1 \leq \alpha_2$. *If* γ_j *is a* $\mathbb{D}$*-Loewner parametrization for* Γ_j *at* $t = 0$ *for* $j = 1$ *and* $j = 2$, *then the function* $t \mapsto g_t(z)$ *is differentiable at* $t = 0$ *for every* $z \in \Omega$.

Remark 1.10. All statements presented here can be easily generalized to the case of $m>2$ slits and to slits that are branched within the unit disc.

We only consider the case of two slits and of one branch point on $\partial\mathbb{D}$ in order to simplify the notation in the proofs.

1.2. Organization of the paper

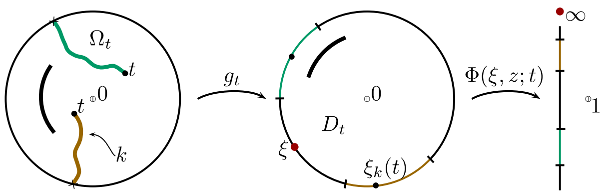
Before we pass on to the proofs of Theorems 1.2, 1.3, 1.5, 1.6 and 1.7, we will explain how Theorems 1.2, 1.6 and 1.7 follow from a more technical statement. To this end, we first introduce some further notations.

We denote by Ω an arbitrary circular slit disk.

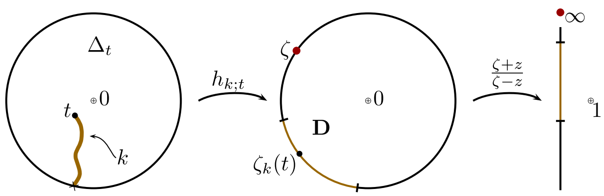
Let $m=1$ or $m=2$ and let $\gamma_1, ..., \gamma_m\colon [0,T]\to\overline{\Omega}\setminus\{0\}$ be Jordan arcs with $\gamma_k(0)\in\partial\mathbb{D}$ and $\Gamma_k:=\gamma_k(0,T]\subset\Omega$. In case $m=2$, we suppose that Γ_1 and Γ_2 are disjoint.

The normalized conformal mapping g_t is defined as before, i.e. g_t maps $\Omega_t:=\Omega\setminus\bigcup_{k=1}^m \gamma_k[0,t]$ onto the circular slit disk D_t.

To simplify the notation, we will also write $\Phi(\xi,z;t)$ instead of $\Phi(\xi,z;D_t)$.



Beside Ω_t, we set $\Delta_k(t):=\mathbb{D}\setminus\gamma_k[0,t]$. Note that $\Delta_k(t)$ is simply connected, whereas Ω_t is n-connected. As before, we denote by $h_{k;t}\colon \Delta_k(t)\to\mathbb{D}$ the unique conformal mapping with the normalization $h_t(0)=0$ and $h_t'(0)>0$.



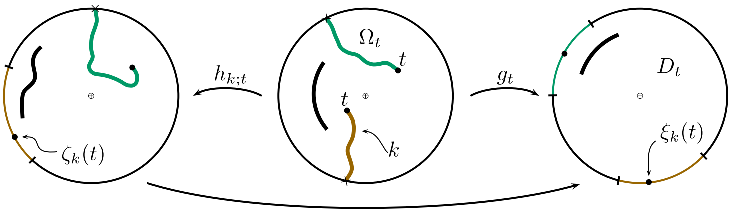
Moreover, we will make use of the driving functions of g_t and $h_{k;t}$ defined by $\xi_k(t):=g_t(\gamma_k(t))$ and $\zeta_k(t):=h_{k;t}(\gamma_k(t))$, respectively, for all $t\in[0,T]$ and all $k=1,...,m$.

Remark 1.11. We note that the driving functions $\xi_k, \zeta_k \colon [0,T] \to \partial\mathbb{D}$ are continuous by Proposition 8 from [2].

In order to give a connection between differentiability of $t \mapsto g_t(z)$ and $t \mapsto h_{k;t}(z)$ we need one further abbreviation. Therefore we set

$$\alpha_k(t) := \left| \frac{d}{dz}(g_t \circ h_{k;t}^{-1})(z)|_{z=\zeta_k(t)} \right|$$

for all $t \in [0,T]$ and all $k = 1, \dots, m$. The derivative is well-defined, as $g_t \circ h_{k,t}^{-1}$ can be extended by the Schwarz refection principle to an analytic function at $z = \zeta_k(t)$.



Note that $\alpha_k(t) \leq 1$ holds for all $t \in [0,T]$ if Ω is simply connected, i.e. if $\Omega = \mathbb{D}$, see Lemma 3.7. Then we find the following theorem.

Theorem 1.12. *Let $t_0 \in [0,T]$ with $\gamma_1(t_0) \neq \gamma_2(t_0)$.*
Let $z_0 \in \Omega_{t_0} \setminus \{0\}$, then the following two conditions are equivalent.
1. *Each function $t \mapsto h_{k;t}(z_0)$ is differentiable at t_0 for every $k = 1, \dots, m$.*
2. *The function $t \mapsto g_t(z)$ is differentiable at t_0 for every $z \in \Omega_{t_0}$.*

If $t \mapsto g_t(z)$ is differentiable at t_0 for every $z \in \Omega_{t_0}$, then

$$(\star) \qquad \dot{g}_{t_0}(z) = g_{t_0}(z) \sum_{k=1}^{m} \lambda_k(t_0) \cdot \Phi\big(\xi_k(t_0), g_{t_0}(z); D_{t_0}\big),$$

where $\lambda_1(t_0), \lambda_2(t_0)$ are uniquely determined non-negative numbers.

If $t \mapsto h_{k;t}(z_0)$ is differentiable at t_0, then $t \mapsto h_{k;t}(z)$ is differentiable at t_0 for every $z \in \Delta_k(t_0)$ and fulfills the following equation

$$(\star\star) \qquad \dot{h}_{k;t_0}(z) = h_{k;t_0}(z) \cdot \mu_k(t_0) \cdot \frac{\zeta_k(t_0) + h_{k;t_0}(z)}{\zeta_k(t_0) - h_{k;t_0}(z)},$$

where $\mu_k(t_0) = \frac{d}{dt} \operatorname{lmr}(h_{k;t})|_{t=t_0} \geq 0$.

Moreover each function $t \mapsto \alpha_k(t)$ is continuous in $[0,T]$ for all $k = 1, \dots, m$ and it holds $\alpha_k(t) > 0$ and $\lambda_k(t_0) = \alpha_k^2(t_0) \cdot \mu_k(t_0)$.

Remark 1.13. The value $\lambda_k(t_0)$ can be given explicitly:
Let $t,\tau\in[0,T]$, set

$$\Omega_k(t,\tau):=\Omega\setminus\left(\gamma_k[0,t]\cup\bigcup_{\substack{j=1\\ j\neq k}}^{m}\gamma_j[0,\tau]\right)$$

and denote by $f_{k;t,\tau}$ the unique normalized conformal mapping from $\Omega_k(t,\tau)$ onto a circular slit disk. Then

$$\lambda_k(t_0)=\lim_{t\to t_0}\frac{\operatorname{lmr}(f_{k;t,t_0})-\operatorname{lmr}(f_{k;t_0,t_0})}{t-t_0},$$

see Lemma 3.1.

Remark 1.14. Note that Theorem 1.12 implies

- Theorem 1.2: consider the case $\Omega=\mathbb{D}$,
- Theorem 1.6: let $m=1$,
- Theorem 1.7 (disjoint case): apply Theorem 1.12 twice; first you pass from the multiply connected case with two slits to equation ($\star\star$), then you pass to the simply connected case with two slits.

Thus, what remains to show are Theorems 1.3, 1.5, and 1.7 for the branch point case and Theorem 1.12.

The rest of this paper is organized as follows: The proof of Theorem 1.12 is given in Section 3 and in Section 4 we prove Theorems 1.3 and 1.5. The proof of Theorem 1.7 for the branch point case is given in Appendix A.

We start with Section 2, where we give three applications of Theorem 1.12.

2. Applications and examples

Theorem 1.12 can be used to prove several results concerning the Loewner equation for multiple slits. In this chapter we use the same notation as in Section 1.2 and we let $m=2$.

If we have no further information about the parametrizations γ_k of the slits Γ_k ($k=1,2$), it is still possible to show that equation ($\star$) holds for almost all $t\in[0,T]$.

First, as the functions $t\mapsto\operatorname{lmr}(h_{k;t})$ are strictly increasing, the derivatives $\mu_1(t),\mu_2(t)$ exist almost everywhere. Thus we immediately get from Theorem A that the functions $t\mapsto h_{k;t}(z)$ are differentiable almost everywhere for all $z\in\Delta_k(T)$ and all $k=1,2$. Together with Theorem 1.12 we find the following corollary, which has been already proved in [2] by using different tools.

Corollary 2.1. (Corollary 5 in [2]) *There exists a null-set $\mathcal{N}$ with respect to the Lebesgue measure such that the functions $t \mapsto g_t(z)$ are differentiable on $[0,T]\setminus\mathcal{N}$ for all $z \in \Omega_T$ and it holds*

$$\dot{g}_t(z) = g_t(z) \sum_{k=1}^{m} \lambda_k(t) \cdot \Phi\big(\xi_k(t), g_t(z); t\big)$$

for all $t \in [0,T]\setminus\mathcal{N}$ and each $z \in \Omega_t$. Furthermore, the functions $\lambda_k(t_0)$ fulfill the condition $\sum_{k=1}^{m} \lambda_k(t_0) = 1$ if the condition $g_t'(0) = c\, e^t$ holds in a neighborhood of t_0 with some constant $c > 0$.

Note that this is true for arbitrary parametrizations of the slits γ_k, i.e. we do not assume any normalization like $g_t'(0) = e^t$.

Next we will demonstrate how Theorem 1.12 can be used to find new parametrizations for Γ_1, Γ_2, in order to get "nice" (Komatu–)Loewner equations, i.e. equations with differentiability everywhere (and not only almost everywhere).

First, we let $L := \operatorname{lmr}(g_T)$ and $L_k := \operatorname{lmr}(h_{k;T})$. Note that $L_k < L$ by the monotonicity of lmr.

Corollary 2.2. *Assume $\gamma_1(0) \neq \gamma_2(0)$. Then there exist parametrizations $\widetilde{\gamma}_1, \widetilde{\gamma}_2 \colon [0,L] \to \overline{\mathbb{D}}$ of the slits Γ_1 and Γ_2 such that the following holds: Denote by $\tilde{g}_s$ the unique normalized conformal mapping from $\widetilde{\Omega}_s := \Omega \setminus (\widetilde{\gamma}_1[0,s] \cup \widetilde{\gamma}_2[0,s])$ onto a circular slit disk $\widetilde{D}_s$ and let $\tilde{\xi}_k(s) := \tilde{g}_s(\widetilde{\gamma}_k(s))$.*

Then the function $s \mapsto \tilde{g}_s$ is continuously differentiable in $[0,L]$ with

$$\dot{\tilde{g}}_s(z) = \tilde{g}_s(z) \sum_{k=1}^{2} \tilde{\lambda}_k(s) \cdot \Phi\big(\tilde{\xi}_k(s), \tilde{g}_s(z), \widetilde{D}_s\big), \quad \textit{for all } s \in [0,L] \tag{4}$$

with continuous functions $\tilde{\xi}_k(s)$, $\tilde{\lambda}_k(s) \geq 0$ and $\tilde{\lambda}_1(s) + \tilde{\lambda}_2(s) = 1$ for all $s \in [0,L]$.

Proof. First of all we assume that each slit Γ_k is parameterized in such a way that $\operatorname{lmr}(h_{k;t})$ is continuously differentiable for all $t \in [0,T]$, e.g. $\operatorname{lmr}(h_{k;t}) = \frac{L_k}{T} \cdot t$. (If not, then we can reparametrize γ_1 and γ_2.)

In the notation of Theorem 1.12, this means that $\mu_k(t) = \frac{L_k}{T}$ for all $t \in [0,T]$.

Then, by Theorem A and Remark 1.11, the trajectories $t \mapsto h_{k;t}(z)$ are continuously differentiable and fulfill equation ($\star\star$) for each $t \in [0,T]$.

By Theorem 1.12, the trajectories $t \mapsto g_t(z)$ fulfill equation ($\star$) for all $t \in [0,T]$. The right side of equation ($\star$) depends continuously on t:

the driving functions are continuous because of Remark 1.11, and Lemma 19 in [2] implies the continuity of the function Φ. The continuity of the weights $t \mapsto \lambda_k(t)$

is an immediate consequence of the relation $\lambda_k(t)=\alpha_k^2(t)\cdot\mu_k(t)$ (see Theorem 1.12) together with the continuity of α_k and μ_k.

Hence, $t\mapsto g_t(z)$ is continuously differentiable in $[0,T]$.

Note that, in general, the weights $t\mapsto\lambda_k(t)$ don't sum up to 1.

In order to get normalized weights, we consider the following increasing homeomorphism $u(t):=\text{lmr}(g_t)=\text{lmr}(\gamma_1[0,t]\cup\gamma_2[0,t])$ that maps $[0,T]$ onto $[0,L]$. It is continuously differentiable with $\dot{u}(t)=\lambda_1(t)+\lambda_2(t)$ for all $t\in[0,T]$. This follows easily by differentiating ($\star$) w.r.t. z at the point $z=0$.

Now we set $\widetilde{\gamma}_k(s):=\gamma_k(u^{-1}(s))$ for all $s\in[0,L]$. Since the function $s\mapsto \text{lmr}(\widetilde{\gamma}_k[0,s])$ is the composition of two continuously differentiable functions it is continuously differentiable as well. Consequently, by using Theorem 1.12 in the same way as before, the trajectories $s\mapsto\tilde{g}_s(z)$ are continuously differentiable and fulfill the stated differential equation for all $s\in[0,L]$.

Finally, as $\text{lmr}(\tilde{g}_s)=s$ for all $s\in[0,L]$, we have $\tilde{\lambda}_1(s)+\tilde{\lambda}_2(s)=1$ for all $s\in[0,L]$. $\square$

Remark 2.3. The proof of Corollary 2.2 shows that there exist "many" parametrizations $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ such that (4) holds and tells us how to construct them. This is based on the fact that we are not restricted to claim $\text{lmr}(h_{k;t})=\frac{L_k}{T}\cdot t$. Instead, we can choose the initial parametrization in such a way that $\text{lmr}(h_{k;t})=u_k(t)$ holds, where $u_k\colon[0,T]\to[0,L_k]$ is an arbitrary continuously differentiable increasing homeomorphism.

In [3] it was shown that one can even choose $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ such that $\tilde{\lambda}_1(t)$ and $\tilde{\lambda}_2(t)$ are constant. Furthermore, this additional condition makes $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ unique.

The next application is a bit more technical, but quite useful, e.g. for constructing certain counterexamples mentioned in the introduction.

Assume that $u_1\colon[0,L]\to[0,L_1]$ is a given increasing homeomorphism. It is easy to see that we can find an increasing homeomorphism $v_1\colon[0,L]\to[0,T]$, such that $\text{lmr}(h_{1,v_1(s)})=u_1(s)$ for all $s\in[0,L]$. Now consider the following question:

Can we find an increasing homeomorphism $v_2\colon[0,L]\to[0,T]$ such that the function $s\mapsto\tilde{g}_s$ from Corollary 2.2 satisfies (4) with $\tilde{\lambda}_1(s)+\tilde{\lambda}_2(s)=1$ for all $s\in[0,L]$?

The following statement gives a partial answer to this question for the simply connected case, i.e. $\Omega=\mathbb{D}$. The proof depends on an inequality for the logarithmic mapping radius (see inequality (7)) that is only known to be true for the simply connected case.

Proposition 2.4. *Assume $\gamma_1(0)\neq\gamma_2(0)$. Let $\Omega=\mathbb{D}$ and $u_1\colon[0,L]\to[0,L_1]$ be an increasing Lipschitz continuous function with a Lipschitz constant $K<1$.*

Let $v_1\colon [0,L]\to[0,T]$ *be the increasing homeomorphism such that* $\mathrm{lmr}(h_{1,v_1(s)})=u_1(s)$ *for all* $s\in[0,L]$. *Then there is a unique increasing homeomorphism* $v_2\colon [0,L]\to[0,T]$ *such that the following holds*:

Denote by $\tilde{g}_s$ *the unique normalized conformal mapping from* $\Omega_s:=\mathbb{D}\setminus((\gamma_1\circ v_1)[0,s]\cup(\gamma_2\circ v_2)[0,s])$ *onto* $\mathbb{D}$. *Then* $\mathrm{lmr}(\tilde{g}_s)=s$ *for all* $s\in[0,L]$.

Moreover, if $s\mapsto u_1(s)$ *is continuously differentiable in* $[0,L]$, *then the function* $s\mapsto\tilde{g}_s(z)$ *is continuously differentiable and satisfies* (4) *for all* $s\in[0,L]$ *and all* $z\in\mathbb{D}\setminus(\Gamma_1\cup\Gamma_2)$ *with* $\tilde{\lambda}_1(s)+\tilde{\lambda}_2(s)=1$ *for all* $s\in[0,L]$.

The proof of this proposition is given in Section 3.

Example 2.5. Let $\Omega=\mathbb{D}$ and Γ_1, Γ_2 be disjoint slits with $L=1$. Then $L_1, L_2<1$. Consequently we find an $\varepsilon>0$ so that $L_1+\varepsilon<1$ as well. Then we define

$$u_1:[0,1]\longrightarrow[0,L_1],\quad s\longmapsto u_1(s):=\begin{cases}(L_1+\varepsilon)s & \text{if } t\in[0,\frac{1}{2}],\\ (L_1-\varepsilon)s+\varepsilon & \text{if } s\in(\frac{1}{2},1].\end{cases}$$

By $v_1:[0,1]\to[0,L_1]$ we denote the homeomorphism such that $\mathrm{lmr}(h_{1;v_1(s)})=u_1(s)$.

u_1 is Lipschitz continuous with Lipschitz constant $K=L_1+\varepsilon<1$. By Proposition 2.4 we find a homeomorphism $v_2\colon [0,1]\to[0,L_2]$ so that $\mathrm{lmr}(\tilde{g}_s)=s$ for all $s\in[0,1]$.

The function $s\mapsto h_{1;v_1(s)}$ is not differentiable at $s=\frac{1}{2}$ by Theorem A as $u_1(s)=\mathrm{lmr}(h_{1,v_1(s)})$ is not differentiable at $s=\frac{1}{2}$.

Thus, by Theorem 1.12, the function $s\mapsto\tilde{g}_s$ is not differentiable at $s=\frac{1}{2}$. However, the function $s\mapsto\mathrm{lmr}(\tilde{g}_s)=s$ is differentiable at $s=\frac{1}{2}$.

Finally, we consider the slightly different setting of two slits with one common starting point. The next example shows that the converse of Theorem 1.5 is not true.

Example 2.6. Let γ_1, $\gamma_2\colon [0,T]\to\overline{\mathbb{D}}$ be parametrizations of two slits satisfying the conditions of Theorem 1.5. Let g_t be defined as in Theorem 1.5.

Furthermore, let $h_{k;t}$ be the unique normalized mapping from $\mathbb{D}\setminus\gamma_k[0,t]$ onto $\mathbb{D}$.

Without restricting generality we may assume $L:=\mathrm{lmr}(g_T)=1$. Moreover, let $L_k:=\mathrm{lmr}(h_{k;T})$. Then $L_k<1$ and we find analogously to Example 2.5 an $\varepsilon>0$ so that $L_1+\varepsilon<1$.

Next, let $u\colon [0,1]\to[0,L_1]$ be defined by

$$s\longmapsto u(s)=\begin{cases}(L_1+\varepsilon)s & \text{if } t\in[0,\frac{1}{2}],\\ (L_1-\varepsilon)s+\varepsilon & \text{if } s\in(\frac{1}{2},1].\end{cases}$$

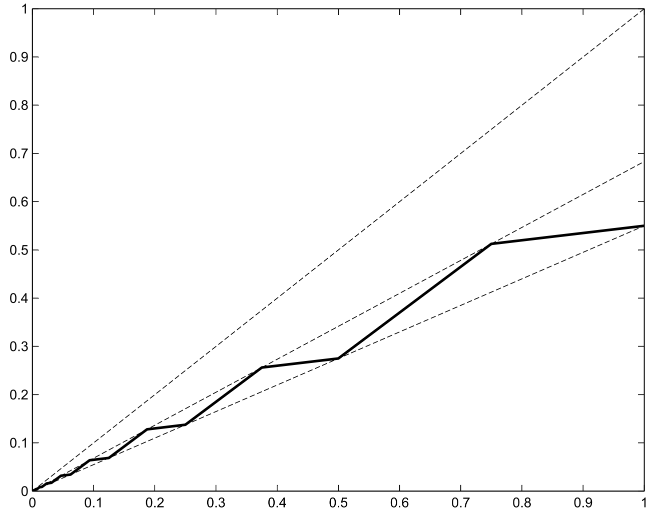


Figure 2. The function u_1 from Example 2.6.

We will use u to construct another increasing homeomorphism $u_1\colon [0,1]\to[0,L_1]$ (see Figure 2):

$$u_1(s) := \begin{cases} \frac{1}{2^n}u(2^n s-1)+\frac{L_1}{2^n} & \text{if } s\in(\frac{1}{2^n},\frac{2}{2^n}] \text{ with } n\in\mathbb{N}, \\ 0 & \text{if } s=0. \end{cases}$$

We have $|u_1(t_2)-u_1(t_1)|\le(L_1+\varepsilon)(t_2-t_1)$ for all $0\le t_1\le t_2\le 1$, so u_1 is strictly increasing and Lipschitz continuous. Moreover we denote by $v_1\colon [0,1]\to[0,T]$ the unique homeomorphism having the property that $\operatorname{lmr}(h_{1;v_1(s)})=u_1(s)$ holds for all $s\in[0,1]$. Now we find a unique homeomorphism $v_2\colon [0,1]\to[0,T]$ such that $\operatorname{lmr}(\tilde{g}_s)=s$ holds for all $s\in[0,1]$, where $\tilde{g}_s$ denotes the unique normalized mapping from $\mathbb{D}\setminus((\gamma_1\circ v_1)[0,s]\cup(\gamma_2\circ v_2)[0,s])$ onto $\mathbb{D}$. This is possible to do using the first part of the proof of Proposition 2.4, as it is applicable to the branch point case as well.

On the one hand, by Theorem 1.7, the function $s\mapsto\tilde{g}_s$ is differentiable at $s=0$. On the other hand, $s\mapsto h_{1;v_1(s)}$ is not differentiable at $s=0$ in accordance with Theorem A, because by construction, $\operatorname{lmr}(h_{1;v_1(s)})=u_1(s)$ and $u_1'(0)$ does not exist.

3. Proof of Theorem 1.12 and Proposition 2.4

As we have mentioned in the introduction, all statements can be easily generalized to the case $m>2$, so we will use a notation indicating this case as well.

First of all, for all $t,\tau\in[0,T]$, we set

$$\Omega_k(t,\tau):=\Omega\setminus\left(\gamma_k[0,t]\cup\bigcup_{\substack{j=1\\ j\neq k}}^{m}\gamma_j[0,\tau]\right)$$

and denote by $f_{k;t,\tau}$ the unique normalized mapping $f_t:\Omega_k(t,\tau)\to D_k(t,\tau)$, where $D_k(t,\tau)$ is a circular slit disk. Consequently we have $g_t=f_{k;t,t}$ and $\Omega_t=\Omega_k(t,t)$ as well.

Next, provided that the limits exist, we define

$$\lambda_k(t_0):=\lim_{t\to t_0}\frac{\operatorname{lmr}(f_{k;t,t_0})-\operatorname{lmr}(f_{k;t_0,t_0})}{t-t_0}\quad\text{and}\quad\mu_k(t_0):=\lim_{t\to t_0}\frac{\operatorname{lmr}(h_{k;t})-\operatorname{lmr}(h_{k;t_0})}{t-t_0}.$$

Finally we set $\xi_k(t,\tau):=f_{k;t,\tau}(\gamma_k(t))$,

$$S_{k;\underline{t},\overline{t},\tau}:=f_{k;\underline{t},\tau}\big(\gamma_k[\underline{t},\overline{t}\,]\big)\subset\mathbb{D}\cup\big\{\xi_k(t,\tau)\big\}\quad\text{and}\quad s_{k;\underline{t},\overline{t},\tau}:=f_{k;\overline{t},\tau}\big(\gamma_k[\underline{t},\overline{t}\,]\big)\subset\partial\mathbb{D},$$

and $\sigma_{k;\underline{t},\overline{t}}:=h_{k;\overline{t}}(\gamma_k[\underline{t},\overline{t}])$ for all $0\le\underline{t}\le\overline{t}\le T$. Next we are going to use results from [2] in order to show that the existence of the above limits $\lambda_k(t_0)$ is equivalent to differentiability of the function $t\mapsto g_t(z)$.

Lemma 3.1. *Let $t_0\in[0,T]$. Then the following three conditions are equivalent*:

1. *Each limit $\lambda_k(t_0)$ exists* $(k=1,\dots,m)$.
2. *The function $t\mapsto g_t(z)$ is differentiable at t_0 for every $z\in\Omega_{t_0}$.*
3. *The function $t\mapsto g_t(z)$ is differentiable at t_0 for every $z\in\Omega_{t_0}$ and fulfills equation ($\star$) for all $z\in\Omega_{t_0}$.*

Proof. First of all note that (1.)⇒(3.) follows immediately from Theorem 2 of [2]. On top of this, (3.)⇒(2.) is trivial, so the only thing we are going to prove is (2.)⇒(1.).

For this, let $t_0\in[0,T]$ and $t>t_0$. The other case $t<t_0$ can be treated in the same way. Since $t\mapsto g_t(z)$ is differentiable at t_0 so is $t\mapsto\log(g_t(z))$ for every $z\in\Omega_{t_0}\setminus\{0\}$. The function $\log(g_t(z))$ is multiple valued, but its derivative is single valued, so the following limit exists and is independent of the branch of the logarithm:

$$\lim_{t\searrow t_0}\frac{1}{t-t_0}\operatorname{Re}\left(\log\frac{g_t(z)}{g_{t_0}(z)}\right)=\lim_{t\searrow t_0}\frac{1}{t-t_0}\ln\left|\frac{g_t(z)}{g_{t_0}(z)}\right|.$$

Next we use Lemma 10 from [2] and the mean value theorem to get

$$\ln\left|\frac{g_t(z)}{g_{t_0}(z)}\right|=\frac{1}{2\pi}\sum_{k=1}^{m}\int_{s_{k;t_0,t,t}}-\ln\left|\left(g_{t_0}\circ g_t^{-1}\right)(\xi)\right|\operatorname{Re}\big(\Phi\big(\xi,g_t(z);t\big)\big)\,|d\xi|$$

$$(5) \qquad = \frac{1}{2\pi}\sum_{k=1}^{m} \mathrm{Re}\big(\Phi\big(\xi_{t,t_0}^{(k)}, g_t(z);t\big)\big) \int_{s_{k;t_0,t,t}} -\ln\big|\big(g_{t_0}\circ g_t^{-1}\big)(\xi)\big|\,|d\xi|.$$

Note that $\Phi(\xi_{t,t_0}^{(k)}, g_t(z);t)$ tends to $\Phi(\xi_k(t_0), g_{t_0}(z);t_0)$ as $t\searrow t_0$ by Lemma 19 from [2]. This is based on the fact $s_{k;t_0,t,t}\ni\xi_{t,t_0}^{(k)}\to\xi_k(t_0)$ as $t\searrow t_0$, see Proposition 8 from [2]. We write

$$\int_{s_{k;t_0,t,t}} -\ln\big|\big(g_{t_0}\circ g_t^{-1}\big)(\xi)\big|\,|d\xi| =: c_k(t,t_0).$$

Note that for each $k\in\{1,\dots,m\}$, $\mathrm{Re}(\Phi(\xi_{t,t_0}^{(k)}, g_t(z);t))$ and $c_k(t,t_0)$ are positive. Moreover, the limit $\lim_{t\searrow t_0}\frac{1}{t-t_0}\ln|g_t(z)/g_{t_0}(z)|$ exists by assumption for any $z\in\Omega_{t_0}$. Summarizing, (5) shows that $\frac{c_k(t,t_0)}{t-t_0}$ is bounded for all $t\in(t_0,T]$. Together with Lemma 19 from [2] we find

$$(6) \qquad \ln\left|\frac{g_t(z)}{g_{t_0}(z)}\right| = \frac{1}{2\pi}\sum_{k=1}^{m}\mathrm{Re}\big(\Phi\big(\xi_k(t_0), g_{t_0}(z);t_0\big)\big)c_k(t,t_0) + o\big(|t-t_0|\big).$$

From the proof of Theorem 2 of [2] we can see that $\lambda_k^+(t_0):=\lim_{t\searrow t_0}\frac{\mathrm{lmr}(f_{k;t,t_0})-\mathrm{lmr}(f_{k;t_0,t_0})}{t-t_0}$ exists if and only if $\lim_{t\searrow t_0}\frac{c_k(t,t_0)}{t-t_0}$ exists. Consequently we are going to prove the existence of the limit $\lim_{t\searrow t_0}\frac{c_k(t,t_0)}{t-t_0}$.

For this purpose we show that we find $z_1,\dots,z_m\in\Omega_{t_0}$ (independently of t) such that the matrix $A:=[a_{j,k}]_{j,k=1}^m$, with $a_{j,k}:=\mathrm{Re}(\Phi(\xi_k(t_0), g_{t_0}(z_j);t_0))$, is invertible. Then, (6) yields

$$\big(c_1(t,t_0),\dots,c_m(t,t_0)\big)^T = \frac{1}{2\pi}A^{-1}\left(\ln\left|\frac{g_t(z)}{g_{t_0}(z_1)}\right|,\dots,\ln\left|\frac{g_t(z)}{g_{t_0}(z_m)}\right|\right)^T + o\big(|t-t_0|\big),$$

and the existence of the limits $\lim_{t\searrow t_0}\frac{c_k(t,t_0)}{t-t_0}$ follows immediately.

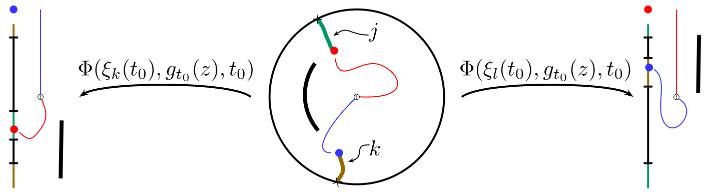
To find $z_1,\dots,z_m$, recall that $\Phi(\xi_k(t_0), g_{t_0}(\gamma_k(t_0));t_0)=\infty$ and $\mathrm{Re}(\Phi(\xi_k(t_0), g_{t_0}(\gamma_j(t_0));t_0))=0$ if $j\neq k$. For $k\in\{1,\dots,m\}$ consider the preimage L_k of the curve $\delta(x):=1+ix$, $x\geq 0$, under the mapping $z\mapsto\Phi(\xi_k(t_0), g_{t_0}(z);t_0)$. Since L_k is a slit in $\Omega(t_0)$ landing at the point $\gamma_k(t_0)$, $\mathrm{Re}(\Phi(\xi_j(t_0), g_{t_0}(z);t_0))\to 0$ as $z\in L_k$ tends to $\partial\Omega(t_0)$ when $j\neq k$, while $\mathrm{Re}(\Phi(\xi_k(t_0), g_{t_0}(z);t_0))=1$ for all $z\in L_k$ by construction.

Thus we can choose $z_k\in\Omega(t_0)$ close enough to $\gamma_k(t_0)$ in order to get

$$\mathrm{Re}\big(\Phi\big(\xi_k(t_0), g_{t_0}(z_k);t_0\big)\big) = 1 \quad\text{and}\quad \mathrm{Re}\big(\Phi\big(\xi_j(t_0), g_{t_0}(z_k);t_0\big)\big) < \frac{1}{m} \text{ for all } j\neq k.$$

Consequently the matrix A is a diagonally dominant matrix, so it is invertible as well. $\square$

The next lemma is a similar statement for the functions $h_{k;t}$. Note, however, that here we only need differentiability of $t\mapsto h_{k;t}(z_0)$ for one fixed $z_0\in\Delta_k(t_0)\setminus\{0\}$.



Lemma 3.2. *Let be $t_0 \in [0,T]$, $z_0 \in \Delta_k(t_0)\setminus\{0\}$ and $k \in \{1,...,m\}$. Then the following three conditions are equivalent*

1. *The limit $\mu_k(t_0)$ exists.*
2. *The function $t \mapsto h_{k;t}(z_0)$ is differentiable at t_0.*
3. *The function $t \mapsto h_{k;t}(z)$ is differentiable at t_0 for every $z \in \Delta_k(t_0)$ and fulfills equation ($\star\star$) for all $z \in \Delta_k(t)$.*

Proof. First of all note that (1.)⇒(3.) follows immediately from Theorem 2 from [2]. On top of this, (3.)⇒(2.) is trivial, so the only thing we need to prove is (2.)⇒(1.).

Let $t > t_0$ and $k \in \{1,...,m\}$. Analogous to the proof of the previous lemma, we find

$$\log\left(\frac{h_{k;t}(z_0)}{h_{k;t_0}(z_0)}\right) = \frac{1}{2\pi}\int_{\sigma_{k;t,t_0}} -\ln\left|\left(h_{k;t_0} \circ h_{k;t}^{-1}\right)(\zeta)\right| \frac{\zeta_k(t)+h_{k;t}(z_0)}{\zeta_k(t)-h_{k;t}(z_0)}\,|d\zeta|$$
$$= \frac{\zeta_k(t_0)+h_{k;t_0}(z_0)}{\zeta_k(t_0)-h_{k;t_0}(z_0)}\,\frac{1}{2\pi}\int_{\sigma_{k;t,t_0}} -\ln\left|\left(h_{k;t_0} \circ h_{k;t}^{-1}\right)(\zeta)\right|\,|d\zeta| + o(|t-t_0|).$$

The other case $t < t_0$ holds in the same way. From the proof of Theorem 2 of [2] we can see that the limit

$$\lim_{t \searrow t_0} \frac{1}{t-t_0}\int_{\sigma_{k;t,t_0}} -\ln\left|\left(h_{k;t_0} \circ h_{k;t}^{-1}\right)(\zeta)\right|\,|d\zeta|$$

exists if and only if $\mu_k^+(t_0)$ exists, where

$$\mu_k^+(t_0) := \lim_{t \searrow t_0} \frac{\operatorname{lmr}(h_{k;t}) - \operatorname{lmr}(h_{k;t_0})}{t-t_0}.$$

Since $t \mapsto h_{k;t}(z_0)$ is differentiable at t_0 the proof is complete □

Remark 3.3. The implication (2.)⇒(3.) in the previous lemma says that differentiability of $t \mapsto h_{k;t}(z)$ at t_0 for just one point $z_0 \in \Delta_k(t_0)\setminus\{0\}$ implies differentiability at t_0 for all $z \in \Delta_k(t_0)$.

We don't know whether the same is true in the case of $m>1$ slits. Note, however, that the proof of Lemma 3.1 shows that there are m points such that differentiability of $t\mapsto g_t(z_1), ..., t\mapsto g_t(z_m)$ at t_0 together implies differentiability of $t\mapsto g_t(z)$ for all $z\in\Omega_{t_0}$.

Before we can proof Theorem 1.12, we need some preliminary lemmas.

Lemma 3.4. *Let $A,B\subset\mathbb{D}$ be bounded domains and assume there exists an $R>0$ so that*

$$A\cap B_R(1)=B\cap B_R(1)=\mathbb{D}\cap B_R(1)$$

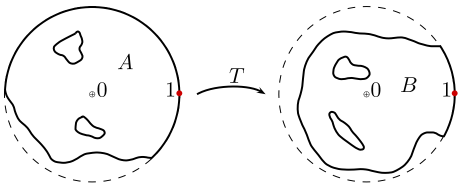
holds, where $B_R(z_0):=\{z\in\mathbb{C}\,|\,|z-z_0|<R\}$. Moreover let $T:A\to B$ be a conformal mapping from A onto B, where $T(1)=1$.

Then $c:=T'(1)>0$ and for any $\delta>0$ there exists $\varepsilon>0$ such that the inequality

$$|z|^{c+\delta}\leq\left|T(z)\right|\leq|z|^{c-\delta}$$

holds for all $z\in A\cap B_\varepsilon(1)$.

Proof. First of all, we can extend the function T to a conformal map in $B_R(1)$ by using the Schwarz reflection principle. As the arc $\partial\mathbb{D}\cap B_R(1)$ is mapped onto an arc of $\partial\mathbb{D}$ and $T(1)=1$, we have $c:=T'(1)>0$.



Now we can choose $\varepsilon\in(0,R)$ small enough such that

$$\left|\frac{T'(z)}{T(z)}-\frac{c}{z}\right|<\delta,\quad\text{for all } z\in B_\varepsilon(1).$$

Next we set $\gamma_\theta(r):=r\cdot e^{i\theta}$ for all $r\in[r_0,1]$ and all $|\theta|<\phi$. Hereby we can choose r_0 close enough to 1 and $\phi>0$ small enough to get $\gamma_\theta(r)\in B_\varepsilon(1)$ for all $r\in[r_0,1]$ and all $\theta\in(-\phi,\phi)$. Moreover, for $r\in[r_0,1]$ and $\theta\in(-\phi,\phi)$, we define

$$h_\theta(r):=\mathrm{Re}\left(\log\frac{T(\gamma_\theta(r))}{(\gamma_\theta(r))^c}\right)=\ln\left|\frac{T(\gamma_\theta(r))}{(\gamma_\theta(r))^c}\right|.$$

Note that there is an analytic branch of the logarithm of $\frac{T(z)}{z^c}$ in $B_\varepsilon(1)$, so we find

$$\begin{aligned}\left|\frac{\partial}{\partial r}h_\theta(r)\right| &= \left|\mathrm{Re}\left(\frac{d}{dz}\log\left(\frac{T(z)}{z^c}\right)\Big|_{z=\gamma_\theta(r)}\cdot\dot{\gamma}_\theta(r)\right)\right| \\ &= \left|\mathrm{Re}\left(\left(\frac{T'(z)}{T(z)}-\frac{c}{z}\right)\Big|_{z=\gamma_\theta(r)}\cdot e^{i\theta}\right)\right| \\ &\le \left|\frac{T'(z)}{T(z)}-\frac{c}{z}\Big|_{z=\gamma_\theta(r)}\right| \le \delta.\end{aligned}$$

Moreover, we have $h_\theta(1)=0$ so we find

$$\ln\bigl(r^\delta\bigr)=\delta\ln(r)\le h_\theta(r)\le -\delta\ln(r)=\ln\bigl(r^{-\delta}\bigr).$$

Finally we get $\ln(|z|^\delta)\le|\frac{T(z)}{z^c}|\le\ln(|z|^{-\delta})$ for all $z\in\{r\cdot e^{i\theta}\,|\,r\in[r_0,1],\ \theta\in(-\phi,\phi)\}$, so the proof is complete. □

Lemma 3.5. *The function* $t\mapsto\alpha_k(t)$ *is continuous and positive for all* $t\in[0,T]$. *Moreover, for* $t_0\in[0,T]$,

$$\left|\left(f_{k;t,\tau}\circ h_{k;t}^{-1}\right)'(a)\right|\longrightarrow\alpha_k(t_0)$$

as $[0,T]^2\times\partial\mathbb{D}\ni(t,\tau,a)\to(t_0,t_0,\zeta_k(t_0))$.

Proof. First of all, $\alpha_k(t)$ is positive, as the mapping $g_t\circ h_{k;t}^{-1}$ can be extended analytically to a conformal map in a small neighborhood around $\zeta(t)$. Consequently the derivative can not vanish.

The continuity of α_k follows from the second statement of the lemma, which we are going to prove below, because

$$\left|\left(f_{k;t,t}\circ h_{k;t}^{-1}\right)'\left(\zeta_k(t)\right)\right|=\alpha_k(t)$$

holds for all $t\in[0,T]$ and because $\partial\mathbb{D}\ni\zeta_k(t)\to\zeta_k(t_0)$ as $t\to t_0$ by Remark 1.11.

Note that we find an $\varepsilon>0$, so that the mapping $H_{k;t,\tau}:=f_{k;t,\tau}\circ h_{k;t}^{-1}$ extends analytically to $B_\varepsilon(\zeta_k(t))$ by the Schwarz reflection principle. Since $\zeta_k(t)\to\zeta_k(t_0)$ as t tends to t_0, we find a small neighborhood U around $\zeta_k(t_0)$, where $H_{k;t,\tau}$ is analytic if t and τ are close enough to t_0. By Proposition 7 from [2], $H_{k;t,\tau}$ converges locally uniformly in $U\cap\mathbb{D}$ to $H_{k;t_0,t_0}$ as $(t,\tau)\to(t_0,t_0)$. Using a normality argument, it is easy to see that $H_{k;t,\tau}$ converges in fact locally uniformly on U to $H_{k;t_0,t_0}$, so we have

$$H_{k;t,\tau}(a)\longrightarrow H_{k;t_0,t_0}\bigl(\zeta_k(t_0)\bigr)\quad\text{as }[0,T]^2\times\partial\mathbb{D}\ni(t,\tau,a)\longrightarrow\bigl(t_0,t_0,\zeta_k(t_0)\bigr).$$

Finally, we find $|H'_{k;t,\tau}(a)|\to|H'_{k;t_0,t_0}(\zeta_k(t_0))|=\alpha_k(t_0)$ as $(t,\tau,a)\to(t_0,t_0,\zeta_k(t_0))$, so the proof is complete. □

Lemma 3.6. *Let be $t_0\in[0,T]$ and $k\in\{1,...,m\}$. Then*

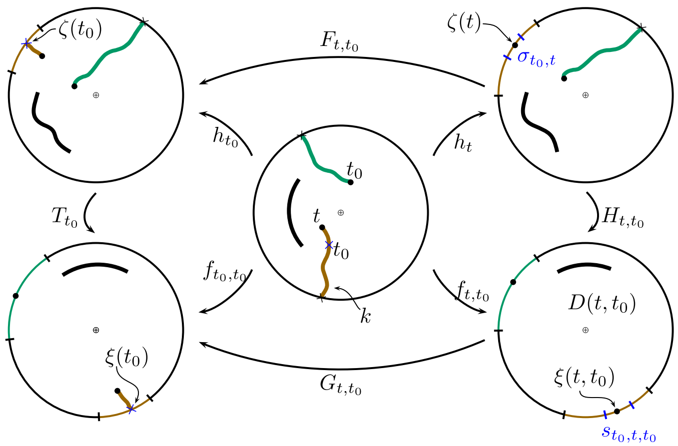
$$\lim_{t\to t_0}\frac{\operatorname{lmr}(f_{k;t,t_0})-\operatorname{lmr}(f_{k;t_0,t_0})}{\operatorname{lmr}(h_{k;t})-\operatorname{lmr}(h_{k;t_0})}=\alpha_k^2(t_0).$$

Consequently the limit $\lambda_k(t_0)$ exists if and only if the limit $\mu_k(t_0)$ exists. Moreover, in this case $\lambda_k(t_0)=\alpha_k^2(t_0)\cdot\mu_k(t_0)$ holds.

Proof. First of all we are going to prove the case $t\searrow t_0$, i.e. we show that

$$\lim_{t\searrow t_0}\frac{\operatorname{lmr}(f_{k;t,t_0})-\operatorname{lmr}(f_{k,t_0,t_0})}{t-t_0}=\alpha_k^2(t_0)\cdot\lim_{t\searrow t_0}\frac{\operatorname{lmr}(h_{k;t})-\operatorname{lmr}(h_{k;t_0})}{t-t_0}.$$

Let be $t_0\in[0,T]$ and $k\in\{1,...,m\}$. Since there is no risk of confusion we omit the index k.



Then we have with $G_{t,t_0}:=f_{t_0,t_0}\circ f_{t,t_0}^{-1}$,

$$\begin{aligned}\operatorname{lmr}(f_{t_0,t_0})-\operatorname{lmr}(f_{t,t_0})&=\log\left(\frac{d}{dz}G_{t,t_0}(z)\Big|_{z=0}\right)=\log\left(\frac{G_{t,t_0}(z)}{z}\right)\Big|_{z=0}\\&=\frac{1}{2\pi i}\int_{\partial D(t,t_0)}\log\left(\frac{G_{t,t_0}(\xi)}{\xi}\right)\frac{d\xi}{\xi}\\&=\frac{1}{2\pi}\int_{\partial D(t,t_0)}\log\left(\frac{G_{t,t_0}(\xi)}{\xi}\right)d\arg\xi\\&=\frac{1}{2\pi}\int_{\partial D(t,t_0)}\ln\left|\frac{G_{t,t_0}(\xi)}{\xi}\right|d\arg\xi,\end{aligned}$$

as $\operatorname{lmr}(f)$ is a real quantity. $|G_{t,t_0}|$ is constant on each concentric slit, so we find

$$\operatorname{lmr}(f_{t_0,t_0})-\operatorname{lmr}(f_{t,t_0})=\frac{1}{2\pi}\int_{\partial\mathbb{D}}\ln\left|\frac{G_{t,t_0}(\xi)}{\xi}\right||d\xi|=\frac{1}{2\pi}\int_{s_{t_0,t,t_0}}\ln\left|G_{t,t_0}(\xi)\right||d\xi|.$$

Next we set $H_{t,t_0}:=f_{t,t_0}\circ h_t^{-1}$, $F_{t,t_0}:=h_{t_0}\circ h_t^{-1}$ and $T_{t_0}:=f_{t_0,t_0}\circ h_{t_0}^{-1}$. Consequently we have by substitution, the mean value theorem and by using the relation $G_{t,t_0}\circ H_{t,t_0}=T_{t_0}\circ F_{t,t_0}$

$$\begin{aligned}\operatorname{lmr}(f_{t_0,t_0})-\operatorname{lmr}(f_{t,t_0})&=\frac{1}{2\pi}\int_{\sigma_{t_0,t}}\left|H'_{t,t_0}(\zeta)\right|\cdot\ln\left|(G_{t,t_0}\circ H_{t,t_0})(\zeta)\right||d\zeta|\\&=\frac{1}{2\pi}\left|H'_{t,t_0}(\zeta_{t,t_0})\right|\int_{\sigma_{t_0,t}}\ln\left|(T_{t_0}\circ F_{t,t_0})(\zeta)\right||d\zeta|\end{aligned}$$

for some $\zeta_{t,t_0}\in\sigma_{t_0,t}$. Since $\zeta_{t,t_0}\to\zeta(t_0)$, we find $|H'_{t,t_0}(\zeta_{t,t_0})|\to\alpha(t_0)$ as $t\searrow t_0$ by Lemma 3.5.

Moreover, the function $\widetilde{T}_{t_0}(z):=\frac{1}{\xi(t_0)}\cdot T_{t_0}(\zeta(t_0)z)$ is a mapping that fulfills the conditions of Lemma 3.4, so we find for every $\delta>0$ an $\varepsilon>0$, so that

$$|z|^{c+\delta}\leq\left|\widetilde{T}_{t_0}(z)\right|\leq|z|^{c-\delta}$$

holds for all $z\in B_\varepsilon(1)\cap\mathbb{D}$, where $c=\widetilde{T}'_{t_0}(1)>0$. Note that $c=|T'_{t_0}(\zeta(t_0))|=\alpha(t_0)$. As a consequence of $|\xi(t_0)|=|\zeta(t_0)|=1$ we get

$$|z|^{c+\delta}\leq\left|T_{t_0}(z)\right|\leq|z|^{c-\delta}$$

for all $z\in B_\varepsilon(\zeta(t_0))$. On top of this, if t is close enough to t_0 we get $F_{t,t_0}(\zeta)\in B_\varepsilon(\zeta(t_0))$ for all $\zeta\in\sigma_{t_0,t}$. Thus we have for all $t\in(t_0,t_0+\rho)$ where $\rho(\delta)>0$ is small

$$\begin{aligned}&\frac{1}{2\pi}\left|H'_{t,t_0}(\zeta_{t,t_0})\right|\left(\alpha(t_0)+\delta\right)\int_{\sigma_{t_0,t}}\ln\left|F_{t,t_0}(\zeta)\right||d\zeta|\\&\qquad\leq\operatorname{lmr}(f_{t_0,t_0})-\operatorname{lmr}(f_{t,t_0})\leq\frac{1}{2\pi}\left|H'_{t,t_0}(\zeta_{t,t_0})\right|\left(\alpha(t_0)-\delta\right)\int_{\sigma_{t_0,t}}\ln\left|F_{t,t_0}(\zeta)\right||d\zeta|.\end{aligned}$$

Moreover in the same way as before we can see that

$$\frac{1}{2\pi}\int_{\sigma_{t_0,t}}\ln\left|F_{t,t_0}(\zeta)\right||d\zeta|=\operatorname{lmr}(h_{t_0})-\operatorname{lmr}(h_t).$$

By combining this with the previous inequality we get for all $t\in(t_0,t_0+\rho)$,

$$\left|H'_{t,t_0}(\zeta_{t,t_0})\right|\left(\alpha(t_0)-\delta\right)\leq\frac{\operatorname{lmr}(f_{t_0,t_0})-\operatorname{lmr}(f_{t,t_0})}{\operatorname{lmr}(h_{t_0})-\operatorname{lmr}(h_t)}\leq\left|H'_{t,t_0}(\zeta_{t,t_0})\right|\left(\alpha(t_0)+\delta\right).$$

As $\delta>0$ is arbitrary, we get in the limit case the existence of $\lambda(t_0)$ if and only if $\mu(t_0)$ exists. Moreover we find

$$\lambda(t_0)=\alpha(t_0)^2\cdot\mu(t_0)$$

as $|H'_{t,t_0}(\zeta_{t,t_0})|$ tends to $\alpha(t_0)$ by Lemma 3.5, so the proof is complete.

The other case $t\nearrow t_0$, i.e.

$$\lim_{t\nearrow t_0}\frac{\operatorname{lmr}(f_{k;t,t_0})-\operatorname{lmr}(f_{k;t_0,t_0})}{t-t_0}=\alpha_k^2(t_0)\cdot\lim_{t\nearrow t_0}\frac{\operatorname{lmr}(h_{k;t})-\operatorname{lmr}(h_{k;t_0})}{t-t_0}$$

follows in the same way. □

Lemma 3.7. *Let Ω be simply connected, i.e. $\Omega=\mathbb{D}$. Then $\alpha_k(t)\le 1$ for all $t\in[0,T]$.*

Proof. First, let $0\le\underline{t}<\overline{t}\le T$, $0\le\underline{\tau}<\overline{\tau}\le T$ and $A:=f_{k;\underline{t},\underline{\tau}}(\gamma_k[\underline{t},\overline{t}])$, $B:=f_{k;\underline{t},\underline{\tau}}(\bigcup_{j\neq k}\gamma_j[\underline{\tau},\overline{\tau}])$. By using the chain rule we get

$$\operatorname{lmr}(A)=\operatorname{lmr}(f_{k;\overline{t},\underline{\tau}})-\operatorname{lmr}(f_{k;\underline{t},\underline{\tau}}),\quad \operatorname{lmr}(B)=\operatorname{lmr}(f_{k;\underline{t},\overline{\tau}})-\operatorname{lmr}(f_{k;\underline{t},\underline{\tau}})\quad\text{and}$$
$$\operatorname{lmr}(A\cup B)=\operatorname{lmr}(f_{k;\overline{t},\overline{\tau}})-\operatorname{lmr}(f_{k;\underline{t},\underline{\tau}}).$$

Furthermore, as Ω is simply connected, we have the following inequality (see [12]):

$$\operatorname{lmr}(A\cup B)\le\operatorname{lmr}(A)+\operatorname{lmr}(B).$$

By combining this inequality with the previous equations we obtain

$$\operatorname{lmr}(f_{k;\overline{t},\overline{\tau}})-\operatorname{lmr}(f_{k;\underline{t},\overline{\tau}})\le\operatorname{lmr}(f_{k;\overline{t},\underline{\tau}})-\operatorname{lmr}(f_{k;\underline{t},\underline{\tau}}).\tag{7}$$

Next we find together with Lemma 3.6

$$\alpha_k^2(t_0)=\lim_{t\to t_0}\frac{\operatorname{lmr}(f_{k;t,t_0})-\operatorname{lmr}(f_{k;t_0,t_0})}{\operatorname{lmr}(h_{k;t})-\operatorname{lmr}(h_{k;t_0})}=\lim_{t\searrow t_0}\frac{\operatorname{lmr}(f_{k;t,t_0})-\operatorname{lmr}(f_{k;t_0,t_0})}{\operatorname{lmr}(f_{k;t,0})-\operatorname{lmr}(f_{k;t_0,0})}\le 1.\qquad\square$$

Proof of Theorem 1.12. This follows immediately from Lemmas 3.1, 3.2, 3.5 and 3.6. □

Proof of Proposition 2.4.

(1) First, we find a unique continuous function $v_2\colon [0,L]\to[0,T]$ so that $\mathrm{lmr}(\tilde{g}_s)=s$ since $\mathrm{lmr}(h_{1,v_1(s)})=u_1(s)\le Ks<s$. Note that the continuity is an immediate consequence of Proposition 7 from [2]. Consequently it remains to prove that v_2 is bijective. First we note that it is clear that $v_2([0,L])=[0,T]$, so it remains to show that v_2 is injective.

Let $0\le s_1<s_2\le L$ and assume $v_2(s_1)=v_2(s_2)$. We denote by $f_{t,\tau}\colon \mathbb{D}\backslash(\gamma_1[0,t]\cup\gamma_2[0,\tau])\to\mathbb{D}$ the normalized Riemann map from $\Omega\backslash(\gamma_1[0,t]\cup\gamma_2[0,\tau])$ onto $\mathbb{D}$. By using (7) with $\underline{t}:=v_1(s_1)$, $\overline{t}:=v_1(s_2)$, $\underline{\tau}:=v_2(s_1)$ and $\overline{\tau}:=v_2(s_2)$ we obtain

$$\begin{aligned} s_2-s_1 &= \mathrm{lmr}(f_{\overline{t},\overline{\tau}})-\mathrm{lmr}(f_{\underline{t},\underline{\tau}}) = \mathrm{lmr}(f_{\overline{t},\underline{\tau}})-\mathrm{lmr}(f_{\underline{t},\underline{\tau}}) \\ &\le \mathrm{lmr}(f_{\overline{t},0})-\mathrm{lmr}(f_{\underline{t},0}) = \mathrm{lmr}(h_{1;\overline{t}})-\mathrm{lmr}(h_{1;\underline{t}}) < s_2-s_1. \end{aligned}$$

This is a contradiction, so v_2 needs to be bijective. Note that this argumentation does not use the fact that $\gamma_1(0)\neq\gamma_2(0)$.

(2) Now we suppose that u_1 is continuously differentiable and prove that (4) holds. First we set $\widetilde{\gamma}_1(s):=(\gamma_1\circ v_1)(s)$, $\widetilde{\gamma}_2(s):=(\gamma_2\circ v_2)(s)$ and denote by $\tilde{f}_{s_1,s_2}\colon\mathbb{D}\backslash(\widetilde{\gamma}_1[0,s_1]\cup\widetilde{\gamma}_2[0,s_2])\to\mathbb{D}$ the normalized Riemann map from $\mathbb{D}\backslash(\widetilde{\gamma}_1[0,s_1]\cup\widetilde{\gamma}_2[0,s_2])$ onto $\mathbb{D}$. Let be $Z=\{0,...,s_N\}$ a partition of the interval $[0,s]$ and

$$S_1(s,Z) := \sum_{l=0}^{N-1} \mathrm{lmr}(\tilde{f}_{s_{l+1},s_l})-\mathrm{lmr}(\tilde{f}_{s_l,s_l}) \quad \text{and}$$

$$S_2(s,Z) := \sum_{l=0}^{N-1} \mathrm{lmr}(\tilde{f}_{s_l,s_{l+1}})-\mathrm{lmr}(\tilde{f}_{s_l,s_l}).$$

Since $\mathrm{lmr}(\tilde{g}_s)=s$ for all $s\in[0,L]$, by Proposition 17 from [2] the limits $c_k(s):=\lim_{|Z|\to0}S_k(s,Z)$ exist and form increasing and Lipschitz continuous functions $s\mapsto c_k(s)$, with $c_1(s)+c_2(s)=s$ for all $s\in[0,L]$. On the one hand, again by Proposition 17 from [2], the limits

$$\tilde{\lambda}_k(s) = \lim_{t\to s}\frac{\mathrm{lmr}(\tilde{f}_{k;t,s})-\mathrm{lmr}(\tilde{f}_{k;s,s})}{t-s}$$

exist and coincide with $\dot{c}_k(s)$ for every point $s\in[0,L]$ at which c_k is differentiable. On the other hand, according to Lemmas 3.5 and 3.6, the continuous differentiability of u_1 implies that $s\mapsto\tilde{\lambda}_1(s)$ is continuous on $[0,L]$. Therefore, c_1 and hence $c_2(s)=s-c_1(s)$ are, in fact, continuously differentiable. It follows that $s\mapsto\tilde{\lambda}_2(s)$ is also continuous and that $\tilde{\lambda}_1(s)+\tilde{\lambda}_2(s)=1$ for all $s\in[0,L]$. Now it remains to apply Theorem 2 from [2] to conclude that $\tilde{g}_s$ satisfies equation (4) for all $s\in[0,L]$. □

4. Proof of Theorems 1.3 and 1.5

In this section we prove Theorems 1.3 and 1.5. We will use a different setting, namely the upper half-plane and the chordal Loewner equation, instead of the radial case in the unit disk. Here, the role of the logarithmic mapping radius is played by the so called half-plane capacity, which has nicer properties for our purpose. First, we describe the chordal Loewner equation and prove the chordal analogs of Theorems 1.3 and 1.5. At the end of this chapter we justify why it makes sense to consider this different setting.

Denote by $\mathbb{H}:=\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)>0\}$ the upper half-plane. A bounded subset $A\subset\mathbb{H}$ is called a *(compact) hull* if $A=\mathbb{H}\cap\overline{A}$ and $\mathbb{H}\setminus A$ is simply connected. By g_A we denote the unique conformal mapping from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ with *hydrodynamic normalization*, i.e.

$$g_A(z)=z+\frac{b}{z}+\mathcal{O}\left(|z|^{-2}\right)\quad\text{for } |z|\to\infty \tag{8}$$

and for some $b\geq 0$. The quantity $\mathrm{hcap}(A):=b$ is called *half-plane capacity* of A. We note four important properties of hcap; see [7], p. 69 and p. 71.

Lemma 4.1.
(a) $\mathrm{hcap}(c\cdot A)=c^2\cdot\mathrm{hcap}(A)$ *for every* $c>0$ *and every hull* A.
(b) *If* A_1, A_2 *are two hulls such that* $A_1\cup A_2$ *is also a hull, then*

$$\mathrm{hcap}(A_1\cup A_2)\leq\mathrm{hcap}(A_1)+\mathrm{hcap}(A_2).$$

This inequality is strict if both hulls are nonempty.

(c) *If* A_1, A_2 *are two hulls such that* $A_1\cup A_2$ *is a hull as well, then* $\mathrm{hcap}(A_1)\geq\mathrm{hcap}(g_{A_2}(A_1))$. *If both hulls are nonempty, then the inequality is strict.*

(d) *If* A_1, A_2 *are hulls with* $A_1\subset A_2$, *then* $\mathrm{hcap}(A_2)-\mathrm{hcap}(A_1)=\mathrm{hcap}(g_{A_1}(A_2\setminus A_1))$.

If $\gamma\colon[0,T]\to\overline{\mathbb{H}}$ is a simple curve, i.e. a continuous, one-to-one function with $\gamma(0)\in\mathbb{R}$ and $\gamma((0,T])\subset\mathbb{H}$, then we call the hull $\Gamma:=\gamma((0,T])$ a *slit*. If the function $t\mapsto b(t):=\mathrm{hcap}(\gamma((0,t]))$ is differentiable at t_0, then the family $g_t:=g_{\gamma(0,t]}$, $0\leq t\leq T$, satisfies the following *chordal Loewner equation* (see [7], Chapter 5):

$$\dot{g}_{t_0}(z)=\frac{\dot{b}(t_0)}{g_{t_0}(z)-U(t_0)}, \tag{9}$$

where $U(t_0)=g_{t_0}(\gamma(t_0))$.

γ is called *half-plane parametrization* of Γ if $\text{hcap}(\gamma(0,t])=t$ for all $t\in[0,T]$.([1])

Furthermore, we will need the following definition:

Let $\varphi\in(0,\pi)$. We say that Γ *approaches* $\mathbb{R}$ *at* $x\in\mathbb{R}$ *in* φ*-direction* if for every $\varepsilon>0$ there is a $t_0>0$ such that $\gamma(0,t_0]$ is contained in the set $\{z\in\mathbb{H}\,|\,\varphi-\varepsilon<\arg(z-x)<\varphi+\varepsilon\}$.

We will need the following lemma about half-plane capacities of straight line segments.

Lemma 4.2. *Let* $b_1,b_2>0$ *and let* Γ_1,Γ_2 *be two line segments starting at* 0 *with angles* $\alpha_1,\alpha_2\in(0,\pi)$, $\alpha_1<\alpha_2$, *and* $\text{hcap}(\Gamma_1)=b_1, \text{hcap}(\Gamma_2)=b_2$. *Then*

$$\text{hcap}(\Gamma_1\cup\Gamma_2)\to b_1+b_2$$

as $(\alpha_1,\alpha_2)\to(0,\pi)$.

Proof. Let $\gamma_j:[0,b_j]\to\Gamma_j$ be the half-plane parametrization of Γ_j, i.e. $\text{hcap}(\gamma_j(0,t])=t$.

We will use a formula which translates the half-plane capacity of an arbitrary hull A into an expected value of a random variable derived from a Brownian motion hitting this hull. Let B_s be a Brownian motion started in $z\in\mathbb{H}\setminus A$. We write $\mathbf{P}^z$ and $\mathbf{E}^z$ for probabilities and expectations derived from B_s. Let τ_A be the smallest time s with $B_s\in\mathbb{R}\cup A$. Then formula (3.6) of Proposition 3.41 in [7] tells us

$$\text{hcap}(A)=\lim_{y\to\infty} y\,\mathbf{E}^{yi}\big[\text{Im}(B_{\tau_A})\big].$$

Let $\varrho=\tau_{\Gamma_1}$ and $\sigma=\tau_{\Gamma_2}$. Then we have (compare with the proof of Proposition 3.42 in [7])

$$\text{hcap}(\Gamma_1)+\text{hcap}(\Gamma_2)-\text{hcap}(\Gamma_1\cup\Gamma_2)$$
$$=\lim_{y\to\infty} y\big(\mathbf{E}^{yi}\big[\text{Im}(B_\sigma);\sigma>\varrho\big]+\mathbf{E}^{yi}\big[\text{Im}(B_\varrho);\sigma<\varrho\big]\big).$$

Here we use the notation $\mathbf{E}^z[X;A]:=\mathbf{E}^z[X\mathbf{1}_A]$, where X is a random variable and $\mathbf{1}_A$ is the indicator function of the event A.

In the following we will estimate the term $\mathbf{E}^{yi}[(B_\sigma);\sigma>\varrho]$, assuming that y is so large that yi is not contained in the union of the two slits.

([1]) Sometimes (e.g. in [7], p. 93), a parametrization γ is called half-plane parametrization if $\text{hcap}(\gamma(0,t])=2t$ for all $t\in[0,T]$. The reason is explained in [7], p. 99.

First we note that $\gamma_j(1)$ and $\mathrm{Im}(\gamma_j(1))$ can be computed explicitly; see Example 3.39 in [7]:

$$\gamma_j(1)=\sqrt{2}\cdot\left(\sqrt{\alpha_j/\pi}\right)^{2\alpha_j/\pi-1}\cdot\left(\sqrt{1-\alpha_j/\pi}\right)^{1-2\alpha_j/\pi}e^{i\alpha_j}\cdot\sqrt{b_j} \tag{10}$$

and consequently

$$\mathrm{Im}\big(\gamma_1(1)\big)=\sin(\alpha_j)\cdot\sqrt{2}\cdot\left(\sqrt{\alpha_j/\pi}\right)^{2\alpha_j/\pi-1}\cdot\left(\sqrt{1-\alpha_j/\pi}\right)^{1-2\alpha_j/\pi}\cdot\sqrt{b_j}.$$

Note that $\mathrm{Im}(\gamma_j(1))\to 0$ and $|\gamma_j(1)|\to\infty$ as $\alpha_j\to 0$ or $\alpha_j\to\pi$.

Let $R>0$ and assume that α_1 is so close to 0 that $\mathrm{Im}(\gamma_1(1))<R$ and

$$\big|\gamma_1(1)\big|>R \tag{$*$}$$

and write

$$\mathbf{E}^{yi}\big[\mathrm{Im}(B_\sigma);\sigma>\varrho\big]=\mathbf{E}^{yi}\big[\mathrm{Im}(B_\sigma);\sigma>\varrho\wedge|B_\varrho|<R\big]+\mathbf{E}^{yi}\big[\mathrm{Im}(B_\sigma);\sigma>\varrho\wedge|B_\varrho|\geq R\big].$$

The first summand: We have $\mathbf{E}^{yi}[\mathrm{Im}(B_\sigma);\sigma>\varrho\wedge|B_\varrho|<R]\leq\mathrm{Im}(\gamma_2(1))\cdot\mathrm{P}\{B_\varrho\in\Gamma_1\cap\{|z|<R\}\}$. Now we use that the limit $\lim_{y\to\infty}y\,\mathbf{P}\{B_\varrho\in\Gamma_1\cap\{|z|<R\}\}$ exists; see [7], p. 74; and that there exists a universal constant c_2 such that

$$\lim_{y\to\infty}y\,\mathbf{P}\big\{B_\varrho\in\Gamma_1\cap\big\{|z|<R\big\}\big\}\leq c_2\,\mathrm{diam}\big(\Gamma_1\cap\{|z|<R\}\big)=c_2\cdot R;$$

see [7], p. 74. Thus we get

$$\lim_{y\to\infty}y\,\mathbf{E}^{yi}\big[\mathrm{Im}(B_\sigma);\sigma>\varrho\wedge|B_\varrho|<R\big]\leq c_2R\cdot\mathrm{Im}\big(\gamma_2(1)\big)\to 0\quad\text{as } (\alpha_1,\alpha_2)\to(0,\pi).$$

The second summand: First we have $\mathbf{E}^{yi}[\mathrm{Im}(B_\sigma);\sigma>\varrho\wedge|B_\varrho|\geq R]\leq\mathrm{Im}(\gamma_2(1))\cdot\mathbf{P}^{yi}\{B_\sigma\in\Gamma_2;\sigma>\varrho\wedge|B_\varrho|\geq R\}$.

A Brownian motion satisfying $\sigma>\varrho\wedge|B_\varrho|\geq R$ will hit Γ_1 at a point Q with $|Q|\geq R$ and afterward it has to hit Γ_2 without hitting the real axis. Call the probability of this event p_Q.

From ($*$) it follows that the Brownian motion hitting Q has to leave the half-disk $\{z\in\mathbb{H}\cup\mathbb{R}\,|\,|z-\mathrm{Re}(Q)|<R\}$ without hitting the real axis; see Figure 3. From Beurling's estimate (Theorem 3.76 in [7]) it follows that $p_Q\leq c_1\cdot\mathrm{Im}(Q)\leq c_1\cdot\mathrm{Im}(\gamma_1(1))$. (2) So we get

$$\mathbf{P}^{yi}\big\{B_\sigma\in\Gamma_2;\sigma>\varrho\wedge|B_\sigma|\geq R\big\}\leq\mathbf{P}^{yi}\{B_\sigma\in\Gamma_2\}\cdot c_1\cdot\mathrm{Im}\big(\gamma_1(1)\big).$$

(2) Note that Theorem 3.76 in [7] gives an estimate on the probability that a Brownian motion started in $\mathbb{D}$ will not have hit a fixed curve, say $[0,1]$, when leaving $\mathbb{D}$ for the first time. The estimate we use can be simply recovered by mapping the half-circle $\mathbb{D}\cap\mathbb{H}$ conformally onto $\mathbb{D}\setminus[0,1]$ by $z\mapsto z^2$.

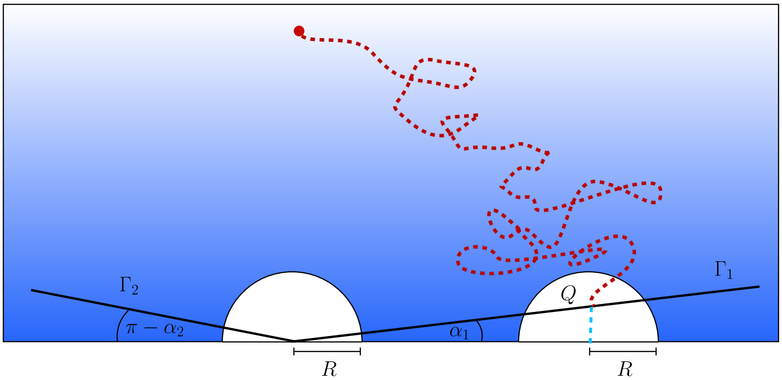


Figure 3. A Brownian motion with $\sigma > \varrho$ and $|B_\varrho| \geq R$.

Again we have $\lim_{y\to\infty} y\,\mathbf{P}^{yi}\{B_\sigma \in \Gamma_2\} \leq c_2 \operatorname{diam}(\Gamma_2) = c_2 \cdot |\gamma_2(1)|$.

Thus, using (10), we have

$$\lim_{y\to\infty} y\,\mathbf{E}^{yi}\big[\operatorname{Im}(B_\sigma); \sigma > \varrho \wedge |B_\varrho| \geq R\big]$$
$$\begin{aligned}
&\leq \operatorname{Im}\big(\gamma_2(1)\big)\cdot c_2 \cdot |\gamma_2(1)| \cdot c_1 \cdot \operatorname{Im}\big(\gamma_1(1)\big)\\
&= c_1 c_2 \operatorname{Im}\big(\gamma_1(1)\big) \sin(\alpha_2) \cdot |\gamma_2(1)|^2\\
&= 2c_1 c_2 b_1 \cdot \operatorname{Im}\big(\gamma_1(1)\big) \cdot \sin(\alpha_2) \cdot (1 - \alpha_2/\pi)^{1-2\alpha_2/\pi} \cdot (\alpha_2/\pi)^{2\alpha_2/\pi - 1}.
\end{aligned}$$

Note that

$$\operatorname{Im}\big(\gamma_1(1)\big) \to 0, \quad \sin(\alpha_2)\cdot(1-\alpha_2/\pi)^{1-2\alpha_2/\pi} \to \pi \quad \text{and} \quad (\alpha_2/\pi)^{2\alpha_2/\pi-1} \to 1$$

and consequently $\lim_{y\to\infty} y\,\mathbf{E}^{yi}[\operatorname{Im}(B_\sigma); \sigma > \varrho \wedge |B_\varrho| \geq R] \to 0$ as $(\alpha_1, \alpha_2) \to (0, \pi)$.

In the same way we obtain $\lim_{y\to\infty} y\,\mathbf{E}^{yi}[\operatorname{Im}(B_\varrho); \sigma < \varrho] \to 0$ as $(\alpha_1, \alpha_2) \to (0, \pi)$ and thus

$$\operatorname{hcap}(\Gamma_1 \cup \Gamma_2) \to \operatorname{hcap}(\Gamma_1) + \operatorname{hcap}(\Gamma_2) \quad \text{as } (\alpha_1, \alpha_2) \to (0, \pi). \qquad \square$$

Let Γ_1, Γ_2 be two slits with parametrizations γ_1 and γ_2. Furthermore, we let $h_1(t) := \operatorname{hcap}(\gamma_1(0,t])$, $h_2(t) := \operatorname{hcap}(\gamma_2(0,t])$ and $c(t) := \operatorname{hcap}(\gamma_1(0,t] \cup \gamma_2(0,t])$.

Theorem 4.3. *Let $b_1, b_2 \geq 0$ and let Γ_1, Γ_2 be two slits with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \{p\} \subset \mathbb{R}$, such that Γ_j approaches p in α_j-direction for $j = 1, 2$, with $0 < \alpha_1 \leq \alpha_2 < \pi$. Assume that $h_1(t)$ and $h_2(t)$ are differentiable for $t = 0$ with $b_1 = \dot{h}_1(0)$, $b_2 = \dot{h}_2(0)$. Then $c(t)$ is differentiable at $t = 0$.*

(i) *If $b_1 = 0$ or $b_2 = 0$, then $\dot{c}(0) = \max\{b_1, b_2\}$.*

If $b_1, b_2 > 0$, *then*

(ii) $\max\{b_1, b_2\} \le \dot{c}(0) < b_1 + b_2$,

(iii) $\dot{c}(0) = \max\{b_1, b_2\}$ *if and only if* $\alpha_1 = \alpha_2$ *and*

(iv) $\dot{c}(0) \to b_1 + b_2$ *as* $(\alpha_1, \alpha_2) \to (0, \pi)$.

Proof. By translation we can assume that $p=0$.

For $t>0$, we define $G_t = (\gamma_1(0,t] \cup \gamma_2(0,t])/\sqrt{t}$. By Lemma 4.1(a) we have

$$c(t)/t = \mathrm{hcap}\big(\gamma_1[0,t]/\sqrt{t} \cup \gamma_1[0,t]/\sqrt{t}\big) = \mathrm{hcap}(G_t).$$

First, we assume that Γ_1 and Γ_2 are straight line segments. Since $\mathrm{hcap}(\gamma_j[0,t]/\sqrt{t}) = h_j(t)/t \to \dot{h}_j(0)$ as $t \to 0$ for $j=1,2$, we conclude that the tip of the line segment $\gamma_j[0,t]/\sqrt{t}$ converges to the tip of the line segment L_j with the same angle and half-plane capacity $\dot{h}_j(0) = b_j = \mathrm{hcap}(L_j)$.

From [9], Lemma 4.10, it follows that $\mathrm{hcap}(G_t) \to \mathrm{hcap}(L_1 \cup L_2)$ as $t \to 0$. Consequently, $c(t)$ is differentiable at $t=0$ with $\dot{c}(0) = \mathrm{hcap}(L_1 \cup L_2)$.

If $\mathrm{hcap}(L_1)=0$ or $\mathrm{hcap}(L_2)=0$, then $\mathrm{hcap}(L_1 \cup L_2) = \max\{\mathrm{hcap}(L_1), \mathrm{hcap}(L_2)\}$. This proves (i).

If, on the other hand, $\mathrm{hcap}(L_1), \mathrm{hcap}(L_2) > 0$, then Lemma 4.1(b) gives

$$\max\big\{\mathrm{hcap}(L_1), \mathrm{hcap}(L_2)\big\} \le \mathrm{hcap}(L_1 \cup L_2) < \mathrm{hcap}(L_1) + \mathrm{hcap}(L_2),$$

hence $\max\{b_1, b_2\} \le \dot{c}(0) < b_1 + b_2$.

We have $\dot{c}(0) = b_j$ if and only if $\mathrm{hcap}(L_j) = \mathrm{hcap}(L_1 \cup L_2)$, i.e. $L_j = L_1 \cup L_2$ which is equivalent to $\alpha_1 = \alpha_2$ and $\mathrm{hcap}(L_j) \ge \mathrm{hcap}(L_{3-j})$.

Since $\mathrm{hcap}(L_1 \cup L_2) \to \mathrm{hcap}(L_1) + \mathrm{hcap}(L_2)$ as $(\alpha_1, \alpha_2) \to (0, \pi)$ by Lemma 4.2, we get $\dot{c}(0) \to b_1 + b_2$ as $(\alpha_1, \alpha_2) \to (0, \pi)$. Thus, we have shown all statements of the theorem for the case of two line segments.

Now we pass on to the general case.

For $j=1,2$ let L_j be the straight line segment starting at 0 with angle α_j and $\mathrm{hcap}(L_j) = b_j$.

Since Γ_j approaches 0 in α_j-direction, we have $\mathbb{H} \setminus (\gamma_j[0,t]/\sqrt{t}) \to \mathbb{H} \setminus L_j$ as $t \to 0$ in the sense of kernel convergence w.r.t. the point ∞.([3])

From this it follows that $\mathbb{H} \setminus G_t \to \mathbb{H} \setminus (L_1 \cup L_2)$ as $t \to 0$ and, by the definition of hcap [see (8)] and the Carathéodory Kernel Convergence Theorem, we obtain

$$\mathrm{hcap}(G_t) \to \mathrm{hcap}(L_1 \cup L_2) \quad \text{as } t \to 0.$$

Hence $c(t)$ is differentiable at $t=0$ with $\dot{c}(0) = \mathrm{hcap}(L_1 \cup L_2)$.

([3]) Here, ∞ is a boundary point of $\mathbb{H}$ on the Riemann sphere. However, in our case, kernel convergence in $\mathbb{H}$ w.r.t. ∞ can be defined by extending the conformal mapping g_A analytically to $\mathbb{C} \setminus \overline{A \cup A^*}$, where A^* stands for the reflection of A w.r.t. the real axis.

Thus, by using the case of two line segments, we immediately get the statements (i), (ii), (iii) and (iv). □

Theorem 4.4. *There exist two slits* Γ_1, Γ_2, *with* $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \{0\}$, *such that* $h_j(t) = t$ *for all* $t \in [0, \operatorname{hcap}(\Gamma_j)]$, *but* $c(t)$ *is not differentiable at* $t=0$.

Proof. Assume that Γ is a slit starting at 0 with half-plane parametrization $\gamma\colon (0,T] \to \mathbb{C}$ having the property $\Gamma \subset \{z \in \mathbb{H} \mid \operatorname{Re}(z) > 0\}$ and assume further that Γ is self-similar in the following sense:

$$1/2 \cdot \Gamma \subset \Gamma.$$

Lemma 4.1(a) implies that $\gamma(0, 1/4^n \cdot T] = 1/2^n \cdot \Gamma$ for every $n \in \mathbb{N}$.

Now let Γ^* be the reflection of Γ with respect to the imaginary axis, i.e. $\Gamma^* := \{-\overline{z} \mid z \in \Gamma\}$. Denote by γ^* the half-plane parametrization of Γ^* and let $K_t = \gamma(0,t] \cup \gamma^*(0,t]$.

Then also K_1 is self-similar, i.e. $1/2 \cdot K_t \subset K_t$ and thus for any $t \in [0,T]$ the half-plane capacity $c(t) := \operatorname{hcap}(K_t)$ of the hull K_t satisfies $c(t/4) = c(t)/4$ and consequently

$$\frac{c(t/4^n)}{t/4^n} = \frac{c(t)}{t}$$

for every $n \in \mathbb{N}$. Hence, if we assume that $c(t)$ is differentiable at $t=0$, then $c(t)$ is linear with $c(t) = \dot{c}(0) \cdot t$.

Below we construct such a self-similar slit Γ having the property that $c(t)$ is not linear, which gives us the desired contradiction.

Let $0 \le \varepsilon < 1/2$ and let A be the curve that connects the points $3/4i + \varepsilon/2$, $i + \varepsilon$, $1/2 + i$, $1/2 + 3/2i$ and $3/2i + \varepsilon$ by straight line segments. Note that A and $1/2 \cdot A$ intersect only at $3/4i + \varepsilon/2$.

Now we define the slit

$$\Gamma := \bigcup_{n=0}^{\infty} 1/2^n \cdot A,$$

see Figure 4. Of course, this slit is self-similar, i.e.

$$1/2 \cdot \Gamma \subset \Gamma.$$

Let Γ^* be the reflection of Γ w.r.t. the imaginary axis. Now let $\gamma, \gamma^*\colon (0,T] \to \mathbb{C}$ be the parametrizations of Γ and Γ^* by half-plane capacity.

For each $t \in (0,T]$ we can define K_t as the smallest hull containing $\gamma(0,t] \cup \gamma^*(0,t]$. Note that $K_t = \gamma(0,t] \cup \gamma^*(0,t]$ for $\varepsilon > 0$. Only for $\varepsilon = 0$, the complement of the union has bounded components. Let $c(t) := \operatorname{hcap}(K_t)$ and let t_2 and t_1 be defined by $\gamma(t_1) = 3/4i + \varepsilon/2$ and $\gamma(t_2) = i + \varepsilon$.

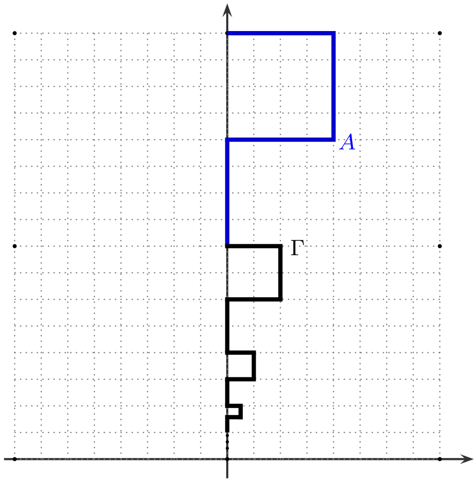


Figure 4. A and Γ for $\varepsilon = 0$.

The quantities $t_2, t_1, c(t_2), c(t_1)$ depend continuously on ε, as the domains $\mathbb{H} \setminus \gamma(0,t]$, $\mathbb{H} \setminus K_t$ depend continuously on ε w.r.t. kernel convergence at ∞ (see the proof of Theorem 4.3).

For $\varepsilon = 0$ we have $K_{t_2} \setminus K_{t_1} = \gamma(t_1, t_2]$ and we obtain

$$t_2 - t_1 \underset{\text{Lemma 4.1(d)}}{=} \text{hcap}\big(g_{\gamma(0,t_1]}(\gamma(t_1,t_2])\big) \underset{\text{Lemma 4.1(c)}}{>} \text{hcap}\big(g_{K_{t_1}}(\gamma(t_1,t_2])\big)$$
$$= c(t_2) - c(t_1).$$

Here, we apply Lemma 4.1(c) for $A_1 = g_{\gamma(0,t_1]}(\gamma(t_1,t_2])$ and $A_2 = g_{\gamma(0,t_1]}(K_{t_1} \setminus \gamma(0,t_1])$. Note that $g_{K_{t_1}} = g_{A_2} \circ g_{\gamma(0,t_1]}$.

Now choose an $\varepsilon > 0$ so small that we still have

$$\frac{c(t_2) - c(t_1)}{t_2 - t_1} < 1. \tag{11}$$

Assume $c(t)$ is differentiable at $t = 0$ in this case. Then c is linear as we have seen before. As $T = \text{hcap}(\Gamma) < c(T) = \dot{c}(0) \cdot \text{hcap}(\Gamma)$, we have $\dot{c}(0) > 1$.

On the other hand, $\dot{c}(0) < 1$ by (11); a contradiction. □

The following lemma gives the connection between the chordal and the radial case that we need for our purpose. The proof is given in Appendix A.

Lemma 4.5. *Let γ_1 and γ_2 be the parametrizations of two disjoint slits in a circular slit disk Ω with $\gamma_1(0) = \gamma_2(0) = 1$. In the following, K_t is either defined by*

(i) $K_t = \gamma_1[0,t]$ *for all* t *or*
(ii) $K_t = \gamma_1[0,t] \cup \gamma_2[0,t]$ *for all* t.

Next, let g_t *be the normalized conformal mapping from* $\Omega \setminus K_t$ *onto a circular slit disk.*

For t *small enough, we can map the hulls into the upper half-plane* $\mathbb{H}$ *by the mapping* $F(z) := -i\log(z)$ *(with* $\log(1)=0$*) and* $A_t := -i\log(K_t)$ *will be a family of increasing* $\mathbb{H}$*-hulls. Then we have:*

$t \mapsto \operatorname{lmr}(g_t)$ *is differentiable at* $t=0$ *if and only if* $t \mapsto \operatorname{hcap}(A_t)$ *is differentiable at* $t=0$*. In this case*

$$\frac{d}{dt}\operatorname{hcap}(A_t)(0) = 2\,\frac{d}{dt}\operatorname{lmr}(g_t)(0).$$

Now we have all means to prove Theorems 1.3 and 1.5.

Proof of Theorem 1.3. In order to get the desired example in the radial case, we take the two slits from Theorem 4.4 and map them, at least locally around 0, into the unit disk by the mapping $z \mapsto e^{iz}$. This gives us two slits Γ_1, Γ_2 in the unit disk with parametrizations γ_1, γ_2. According to Lemma 4.5, case (i), $\gamma_1(t)$ and $\gamma_2(t)$ are Loewner parametrizations at $t=0$.

However, the mapping $t \mapsto g_t$ is not differentiable at $t=0$ because of Lemma 4.5, case (ii), and Theorem 1.7. □

Proof of Theorem 1.5. Theorem 1.5 follows immediately from Theorem 4.3, Lemma 4.5 and Theorem 1.7. □

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Appendix A

Proof of Lemma 4.5. First of all we set $\Omega_t := \Omega \setminus K_t$ and $H_t := \mathbb{H} \setminus A_t$ Then we denote by $h_t : H_t \to \mathbb{H}$ the unique Riemann mapping with hydrodynamic normalization. Moreover we set $s_t := g_t(\partial K_t \cap \partial \Omega_t) \subset \partial \mathbb{D}$ and $\tilde{s}_t := h_t(\partial A_t \cap \partial H_t) \subset \partial \mathbb{H}$. Note that g_t^{-1} and h_t^{-1} can be extended continuously to $\partial \mathbb{D}$ and $\partial \mathbb{H}$ by Theorem 2.1 from [11], so we find

$$\operatorname{lmr}(g_t) = -\frac{1}{2\pi}\int_{s_t} \log\left|g_t^{-1}(\zeta)\right| |d\zeta|,$$

$$\operatorname{hcap}(A_t)=\frac{1}{\pi}\int_{\tilde{s}_t}\operatorname{Im}\big(h_t^{-1}(w)\big)|dw|.$$

A rigorous proof of the first equation can be found in [2], equation ($\star$), p. 12. The second formula can be found, e.g., in [8], equation (2.5).

If t is small enough, K_t will be close to 1, i.e for each $\varepsilon>0$ we find a $t_0>0$ so that $K_t\subset B_\varepsilon(1)$ for all $t\in[0,t_0]$. By Schwarz reflection we see that the function

$$T_t(\zeta):=g_t\big(\exp\big(i\cdot h_t^{-1}(\zeta)\big)\big)$$

can be extended to a conformal mapping in a small neighborhood around $\tilde{s}_k$ for all $t\in[0,t_0]$. Next we get with $h_t^{-1}(\zeta)=-i\log(g_t^{-1}(T_t(\zeta)))$ and by usage of the Mean-value Theorem

$$\operatorname{hcap}(A_t)=\frac{1}{\pi}\int_{\tilde{s}_t}\operatorname{Im}\big(-i\log\big(g_t^{-1}\big(T_t(w)\big)\big)\big)|dw|=-\frac{1}{\pi}\int_{\tilde{s}_t}\log|g_t\big(T_t(w)\big)||dw|$$
$$=-\frac{1}{\pi}\int_{s_t}\log|g_t(\zeta)|\frac{1}{|T_t'(T_t^{-1}(\zeta))|}|d\zeta|=2\frac{1}{|T_t'(\zeta_t)|}\operatorname{lmr}(K_t),$$

where $\zeta_t\in\tilde{s}_t$. Using a normality argument analogous to the proof of Lemma 3.5, $|T_t'(\zeta)|$ tends uniformly to 1 on a small neighborhood around 0 as $t\to0$. Thus $|T_t'(\zeta_t)|\to1$ as t tends to zero. □

Proof of Theorem 1.7 (branch point case).

(a) Let $s_t:=g_t(\gamma_1[0,t]\cup\gamma_2[0,t])$ and $F_t:=h_t\circ g_t^{-1}$. Then equation ($\star$) on p. 12 from [2] gives us

$$\operatorname{lmr}(g_t)=-\frac{1}{2\pi}\int_{s_t}\log\big|g_t^{-1}(\zeta)\big||d\zeta|=-\frac{1}{2\pi}\int_{s_t}\log\big|h_t^{-1}\big(F_t(\zeta)\big)\big||d\zeta|.$$

Next we write $\tilde{s}_t:=h_t(\gamma_1[0,t]\cup\gamma_2[0,t])$. Each F_t can be extended analytically to s_t, so an easy substitution combined with the Mean-value Theorem shows that

$$\operatorname{lmr}(g_t)=-\frac{1}{2\pi}\int_{\tilde{s}_t}\log\big|h_t^{-1}(w)\big|\frac{1}{|F_t'(F_t^{-1}(w))|}|dw|=\frac{1}{|F_t'(\zeta_t)|}\operatorname{lmr}(h_t).$$

Herein $\zeta_t\in s_t$. Finally s_t tends to $\gamma_1(0)$ and F_t can be extended to an analytic function on $B_\varepsilon(\gamma_1(0))$ for all t small enough and a small $\varepsilon>0$. Consequently $F_t'(\zeta_t)\to1$ as F_t tends uniformly to the identical mapping on $B_\varepsilon(\zeta_0)$.

(b) By using the same methods as in Lemma 10 from [2] we get

$$\log\frac{g_t^{-1}(z)}{z}=\frac{1}{2\pi}\int_{s_t}\log\big|g_t^{-1}(\zeta)\big|\Phi(\zeta,z;D_t)|d\zeta|.$$

Substituting $z = g_t(w)$ in the above equality and using the Mean-value Theorem, we get

$$\log \frac{g_0(w)}{g_t(w)} = \frac{1}{2\pi} \Phi\big(\zeta_t, g_t(w), D_t\big) \int_{s_t} \log\big|g_t^{-1}(\zeta)\big| |d\zeta| = -\Phi\big(\zeta_t, g_t(w), D_t\big) \operatorname{lmr}(g_t)$$

with $\zeta_t \in s_t$. Hereby, the continuity of Φ follows from Lemma 19 from [2]. Moreover this lemma gives $\Phi(\zeta_t, g_t(w), D_t) \to \Phi(\gamma_1(0), w, D_0)$ as t tends to 0, so the family $t \mapsto g_t$ is differentiable at 0 iff $t \mapsto \operatorname{lmr}(g_t)$ is differentiable.

Summarized part (a) proves (3.)⇔(4.), part (b) proves (1.)⇔(4.) and part (b) applied to $\Omega = \mathbb{D}$ proves (2.)⇔(3.). □

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