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Directional Poincaré inequalities along mixing flows

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Abstract. We provide a refinement of the Poincaré inequality on the torus $\mathbb{T}^d$: there exists a set $\mathcal{B}\subset\mathbb{T}^d$ of directions such that for every $\alpha\in\mathcal{B}$ there is a $c_\alpha>0$ with

$$\|\nabla f\|^{d-1}_{L^2(\mathbb{T}^d)}\|\langle\nabla f,\alpha\rangle\|_{L^2(\mathbb{T}^d)}\geq c_\alpha\|f\|^d_{L^2(\mathbb{T}^d)}\quad\text{for all } f\in H^1(\mathbb{T}^d) \text{ with mean } 0.$$

The derivative $\langle\nabla f,\alpha\rangle$ does not detect any oscillation in directions orthogonal to α, however, for certain α the geodesic flow in direction α is sufficiently mixing to compensate for that defect. On the two-dimensional torus $\mathbb{T}^2$ the inequality holds for $\alpha=(1,\sqrt{2})$ but is not true for $\alpha=(1,e)$. Similar results should hold at a great level of generality on very general domains.

1. Introduction and main result

1.1. Introduction

The classical Poincaré inequality on the torus $\mathbb{T}^d$ states

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}\geq\|f\|_{L^2(\mathbb{T}^d)}$$

for functions $f\in H^1(\mathbb{T}^d)$ with vanishing mean. A natural interpretation is that a function with small derivatives cannot substantially deviate from its mean on a set of large measure. The purpose of this paper is to derive a substantial improvement; we first state the main result.

Theorem 1. (Directional Poincare inequality) *There exists a set $\mathcal{B}\subset\mathbb{T}^d$ such that for every $\alpha\in\mathcal{B}$ there is a $c_\alpha>0$ so that*

$$\|\nabla f\|^{d-1}_{L^2(\mathbb{T}^d)}\|\langle\nabla f,\alpha\rangle\|_{L^2(\mathbb{T}^d)}\geq c_\alpha\|f\|^d_{L^2(\mathbb{T}^d)}$$

for all $f\in H^1(\mathbb{T}^d)$ with mean 0. If $d\geq 2$, then $\mathcal{B}$ is uncountable but Lebesgue-null.

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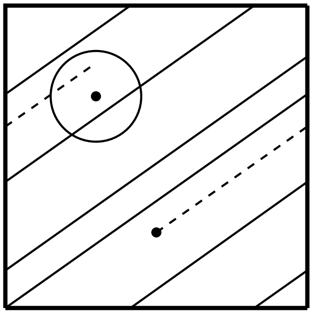

Figure 1. A well-mixing flow transports (*dashed*) every point relatively quickly to a neighborhood of every other point.

The exponents are optimal. The proof is simple and based on elementary properties of Fourier series—we believe it to be of great interest to understand under which conditions comparable inequalities exist on a general Riemannian manifold (M, g) equipped with a suitable vector field.

On the torus, the inequality has strong ties with number theory and can be easily derived at the cost of invoking highly nontrivial results (Schmidt's result on badly approximable numbers, the Khintchine theorem). One remarkable feature is that the inequality holds for $\alpha \in \mathcal{B}$, where $\mathcal{B}$ is a set of Lebesgue measure 0 which shows the inequality to be very delicate (however, as is explained below, slightly weaker statements are very robust). A natural interpretation of the inequality seems to be the following: given two nearby points $x, y \in \mathbb{T}^d$ for which $f(x) \gg f(y)$, the classical Poincaré inequality will detect a large gradient between them. The term $|\langle \nabla f, \alpha \rangle|$ might not detect the large gradient but following the ergodic vector field will relatively quickly lead to a neighborhood of y (see Fig. 1). A priori being in a neighborhood might not imply much because there could be still local oscillations on the scale of the neighborhood, however, since we also invoke a power of $\|\nabla f\|_{L^2(\mathbb{T}^d)}$, this controls the measure of the set on which local oscillations have a strong effect. This heuristic suggests strongly that similar inequalities should hold at a much greater level of generality. We discuss and prove some natural variants in the last section.

1.2. Open problems

It would be of great interest to understand to which extent such inequalities can be true in a more general setup. It is also not clear whether comparable inequalities hold in $L^p(\mathbb{T}^d)$ (our proof heavily uses that $p=2$ but some of the methods might generalize to even p). Generally, for suitable vector fields Y on suitable Riemannian manifolds (M,g) it seems natural to ask whether there exists an inequality of the type

$$\|\nabla f\|_{L^p(M)}^{1-\delta} \big\|\langle \nabla f, Y\rangle\big\|_{L^p(M)}^{\delta} \geq c\|f\|_{L^p(M)}$$

for some $\delta>0$ and all $f\in W^{1,p}(M)$ with mean 0. The parameter δ can be expected to be related to the mixing properties of the flow—it is difficult to predict what the *generic* behavior on a fixed manifold might be (say, for a smooth perturbation of the flat metric on the torus). On $\mathbb{T}^2$ we can rephrase the Khintchine theorem [9] as a statement about generic behavior for the flat metric.

Theorem. (Khintchine, equivalent) *For every $\delta<1/2$, the set of $\alpha\in\mathbb{T}^2$ for which there exists a $c_\alpha>0$ such that*

$$\forall f \in H^1(\mathbb{T}^2) \quad \int_{\mathbb{T}^2} f(x)dx = 0 \quad \Longrightarrow \quad \|\nabla f\|_{L^2(\mathbb{T}^2)}^{1-\delta} \big\|\langle \nabla f, \alpha\rangle\big\|_{L^2(\mathbb{T}^2)}^{\delta} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^2)}$$

has full measure.

This suggests $\delta<1/2$ as a natural threshold that might be achievable by other two-dimensional examples—however, since the inequality is extremely delicate on $\mathbb{T}^2$, the manifolds on which the inequality holds with $\delta=1/2$ might actually be very rare. One would expect new topological effects to appear when considering the sphere $\mathbb{S}^d$ equipped with a nontrivial vector field: the hairy ball theorem dictates that for even d any smooth vector field vanishes somewhere and this will necessitate a change of scaling in the inequality since a function f could be concentrated around the point in which the vector field vanishes. Furthermore, while not every nonvanishing vector field on $\mathbb{S}^3$ has to have a closed orbit (i.e. Seifert's conjecture is false), many of them do—this puts topological restrictions on what directional Poincaré inequalities are possible (since one could set a function to be constant along a periodic orbit and have it decay quickly away from it). However, there should be a variety of admissible inequalities on the flat infinite cylinder $(M,g)=(\mathbb{R}\times\mathbb{T}^{d-1}, \text{can})$ and this could be a natural starting point for future investigations.

2. Proof of the statements

2.1. Outline of the argument

The proof of the classical Poincaré inequality on the torus is a one-line argument if one expands in Fourier series and uses $\hat{f}(0)=\int_{\mathbb{T}^d} f=0$ since

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}^2 = (2\pi)^d \sum_{\substack{k\in\mathbb{Z}^d\\ k\neq 0}} |k|^2|a_k|^2 \geq (2\pi)^d \sum_{\substack{k\in\mathbb{Z}^d\\ k\neq 0}} |a_k|^2 = \|f\|_{L^2(\mathbb{T}^d)}^2.$$

The argument also highlights the underlying convexity of the quadratic form. Our proof will be a direct variation of that result and uses the observation that

$$\begin{aligned}\|\langle \nabla f, \alpha\rangle\|_{L^2(\mathbb{T}^d)}^2 &= \left\| \left\langle \sum_{k\in\mathbb{Z}^d} a_k k e^{ik\cdot x}, \alpha \right\rangle \right\|_{L^2(\mathbb{T}^d)}^2 \\ &= \left\| \sum_{k\in\mathbb{Z}^d} a_k \langle k, \alpha\rangle e^{ik\cdot x} \right\|_{L^2(\mathbb{T}^d)}^2 \\ &= (2\pi)^d \sum_{k\in\mathbb{Z}^d} |a_k|^2 \big|\langle k,\alpha\rangle\big|^2.\end{aligned}$$

This Fourier multiplier is not uniformly bounded away from 0 and will even vanish for certain $k\in\mathbb{Z}^d$ if the entries of α are not linearly independent over $\mathbb{Q}$. If the entries of α are linearly independent over $\mathbb{Q}$, then the Fourier multiplier is always nonnegative but we have no quantitative control on its decay (see below for an example). However, if we denote the Littlewood–Paley projection onto the frequencies satisfying $\{k\in\mathbb{Z}^d : |k|\le N\}$ by $P_{\le N}$, then trivially

$$\|\langle \nabla P_{\le N} f, \alpha\rangle\|_{L^2(\mathbb{T}^d)}^2 \ge \Big(\inf_{\substack{k\in\mathbb{Z}^d\\ |k|\le N}} \big|\langle k,\alpha\rangle\big|^2 \Big) \|P_{\le N} f\|_{L^2(\mathbb{T}^d)}^2.$$

The term in the bracket clearly has great significance in the study of geometry of numbers and has been studied for a long time. It suffices for us to apply the results and use the additional $\|\nabla f\|_{L^2(\mathbb{T}^d)}$ expression to ensure that a fixed proportion of the L^2-mass is contained within a suitable ball of frequency space on which to apply the argument.

2.2. Number theoretical properties

We now discuss subtleties of the inequality in greater detail: it is merely the classical Poincaré inequality for $d=1$. Letting $d=2$ with $\alpha=(1,0)$ yields

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \|\partial_x f\|_{L^2(\mathbb{T}^2)} \ge c \|f\|_{L^2(\mathbb{T}^2)}^2 \quad \text{which is obviously false,}$$

because f might be constant along the x-direction and vary along the y-direction. More generally, the inequality fails for any α with entries linearly dependent over $\mathbb{Q}$ and the functions $\sin(k_1 x + k_2 y)$ for any $k_1, k_2 \in \mathbb{Z}^2$ with $\langle (k_1,k_2), \alpha\rangle = 0$ serve as counterexamples. The next natural example is $\alpha=(\sqrt{2},1)$. Suppose $f\in C^\infty(\mathbb{T})$ and

$$\big\|\big\langle \nabla f, (\sqrt{2},1)\big\rangle\big\|_{L^2(\mathbb{T}^2)} = 0.$$

f is constant along the flow of the vector field $(\sqrt{2},1)$ but every orbit is dense and thus $f\equiv 0$. This is true for any vector with entries that are linearly independent over $\mathbb{Q}$, however, it is not enough to prove the inequality itself: it fails for $(1,e)$ on $\mathbb{T}^2$ despite linear independence. A simple construction for $d=2$ shows that linear independence of the entries of α is not enough: let

$$\alpha=\left(1,\sum_{n=1}^{\infty}\frac{1}{10^{n!}}\right)\sim(1,0.110001...)$$

where the arising number, Liouville's constant, is irrational. If we set

$$f_N(x,y)=\sin\left(10^{N!}\left(\sum_{n=1}^{N}\frac{x}{10^{n!}}-y\right)\right),$$

then

$$\|f_N\|_{L^2(\mathbb{T}^2)}^2=2\pi^2 \quad \text{and} \quad \|\nabla f_N\|_{L^2(\mathbb{T}^2)}\leq 6\cdot 10^{N!}$$

while

$$\big\|\langle\nabla f_N,\alpha\rangle\big\|_{L^2(\mathbb{T}^2)}=\sqrt{2\pi^2}\left(\sum_{n=N+1}^{\infty}\frac{10^{N!}}{10^{n!}}\right)\ll 10^{-2\cdot N!} \quad \text{for } N\geq 3.$$

2.3. An explicit example

The inequalities are not only sharp with respect to exponents, they are actually sharp *on all frequency scales.* This is in stark contrast to classical Poincaré-type inequalities which tend to be sharp for one function (the ground state of the underlying physical system): here, we can exclude all functions having Fourier support in the set $\{\xi:|\xi|\leq N\}$ for arbitrarily large N and still find functions for which the inequality is sharp (up to a constant). We explain this in greater detail for $d=2$ with the admissible direction given by the golden ratio

$$\alpha=\left(1,\frac{1+\sqrt{5}}{2}\right)\in\mathcal{B}.$$

Consider the sequence of functions given by

$$f_n(x,y)=\sin\left(F_{n+1}x-F_ny\right),$$

where F_n is the n-th Fibonacci number. An explicit computation shows that

$$\|\nabla f_n\|_{L^2(\mathbb{T}^2)}\left\|\partial_x f_n+\frac{1+\sqrt{5}}{2}\partial_y f_n\right\|_{L^2(\mathbb{T}^2)}=\sqrt{\frac{F_{n+1}^2}{F_n^2}+1}\left|\frac{F_{n+1}}{F_n}-\frac{1+\sqrt{5}}{2}\right|F_n^2\|f_n\|_{L^2(\mathbb{T}^2)}^2$$

A standard identity for Fibonacci numbers gives that

$$\lim_{n\to\infty}\left|\frac{F_{n+1}}{F_n}-\frac{1+\sqrt{5}}{2}\right|F_n^2=\frac{1}{\sqrt{5}},$$

which implies that

$$\lim_{n\to\infty}\|\nabla f_n\|_{L^2(\mathbb{T}^2)}\left\|\partial_x f_n+\frac{1+\sqrt{5}}{2}\partial_y f_n\right\|_{L^2(\mathbb{T}^2)}\|f_n\|_{L^2(\mathbb{T}^2)}^{-2}=\sqrt{\frac{5+\sqrt{5}}{10}}=\frac{|\alpha|}{\sqrt{5}}.$$

Since these functions f_n have their Fourier transform supported on 4 points in $\mathbb{Z}^2$ and since $F_{n+1}/F_n\to(1+\sqrt{5})/2<2$, we can conclude that every dyadic annulus in Fourier space contains an example for which the inequality is sharp (up to a constant). Put differently, our inequality is close to being attained on every frequency scale. This sequence of f_n has the advantage of simultaneously showing that the following statement is sharp. The proof uses a classical result of Hurwitz.

Proposition. *Let $d=2$ and $\alpha\in\mathbb{T}^2$ be any vector for which*

$$\|\nabla f\|_{L^2(\mathbb{T}^2)}\|\langle\nabla f,\alpha\rangle\|_{L^2(\mathbb{T}^2)}\geq c_\alpha\|f\|_{L^2(\mathbb{T}^2)}^2$$

holds for all $f\in H^1(\mathbb{T}^2)$ with mean 0. Then the constant satisfies

$$c_\alpha\leq\frac{|\alpha|}{\sqrt{5}}.$$

Up to certain transformation, the example above is essentially the only example for which the inequality is tight: normalizing $\alpha=(1,\beta)$, the example shows that the inequality is sharp for $\beta=(1+\sqrt{5})/2$ and there are only countably many other β for which it is sharp (these can be explicity given). For all other numbers the upper bound could be improved to $c_\alpha\leq|\alpha|/\sqrt{8}$. Removing yet another countable set of exceptional directions, we could replace $\sqrt{8}$ by $\sqrt{221}/5$ and the process could be continued (this follows from classical results about the structure of the Markov spectrum, see [2]).

2.4. Badly approximable systems of linear forms

We now introduce the relevant results from number theory. Let $L_1,...,L_\ell:\mathbb{Z}^d\to\mathbb{R}$ be defined as

$$L_1(\mathbf{x})=\alpha_{11}x_1+...\alpha_{1d}x_d=\langle\alpha_1,\mathbf{x}\rangle$$

$$...$$

$$L_\ell(\mathbf{x}) = \alpha_{\ell 1} x_1 + \ldots \alpha_{\ell d} x_d = \langle \alpha_\ell, \mathbf{x} \rangle$$

The relevant question is whether it is possible for all ℓ expressions to be simultaneously close to an integer. Using $\|\cdot\|:\mathbb{R}\to[0,1/2]$ to denote the distance to the closest integer, the pigeonhole principle implies the existence of infinitely many $\mathbf{x}\in\mathbb{Z}^d$ with

$$\max\bigl(\|L_1(\mathbf{x})\|, \ldots, \|L_\ell(\mathbf{x})\|\bigr) \leq \bigl(\max(|x_1|, \ldots, |x_d|)\bigr)^{-\frac{d}{\ell}}.$$

Dirichlet's theorem cannot be improved in the sense that there actually exist badly approximable vectors $\alpha_1, \ldots, \alpha_\ell$ such that for some $c>0$ and all $\mathbf{x}\in\mathbb{Z}^d$

$$\max\bigl(\|L_1(\mathbf{x})\|, \ldots, \|L_\ell(\mathbf{x})\|\bigr) \geq c\bigl(\max(|x_1|, \ldots, |x_d|)\bigr)^{-\frac{d}{\ell}}.$$

The existence of such elements was first shown by Perron [11]. Khintchine [9] has shown that the ℓd-dimensional Lebesgue measure of such tuples $(\alpha_1, \ldots, \alpha_\ell)$ is 0 and Schmidt [13] has proven that their Hausdorff dimension is ℓd.

2.5. Proof of Theorem 1

Proof. We will prove the statement explicitly for the following set $\mathcal{B}$: for any $(d-1)$-dimensional badly approximable vector α_{d-1}, consider the linear form L: $\mathbb{Z}^{d-1}\to\mathbb{R}$ given by

$$L(\mathbf{x}) := \langle \alpha_{d-1}, \mathbf{x} \rangle$$

and the concatenation

$$\alpha = (1, \alpha_{d-1}).$$

Recall that $\|\cdot\|:\mathbb{R}\to[0,1/2]$ denotes the distance to the closest integer and is trivially 1-periodic. Let now $k\in\mathbb{Z}^d$ with $k\neq 0$. If k vanishes on all but the first component, then

$$\bigl|\langle \alpha, k \rangle\bigr| = |k_1| \geq 1.$$

If k does not vanish on all but the first component, then

$$\begin{aligned}\bigl|\langle \alpha, k \rangle\bigr| &= \bigl|k_1 + L\bigl((k_2, \ldots, k_d)\bigr)\bigr| \geq \bigl\|L\bigl((k_2, \ldots, k_d)\bigr)\bigr\| \\ &\geq \frac{c}{\max(|k_2|, \ldots, |k_{d-1}|)^{d-1}} \geq \frac{c}{|k|^{d-1}},\end{aligned}$$

where $c>0$ is some constant which exists because α_{d-1} is badly approximable. Let now

$$f=\sum_{k\in\mathbb{Z}^d} a_k e^{ik\cdot x}\in H^1\left(\mathbb{T}^d\right)$$

and note that $a_0=0$ because f has mean value 0. We have

$$\left\|\langle\nabla f,\alpha\rangle\right\|^2_{L^2(\mathbb{T}^d)}=(2\pi)^d\sum_{k\in\mathbb{Z}^d}|a_k|^2\left|\langle k,\alpha\rangle\right|^2\geq c^2(2\pi)^d\sum_{k\in\mathbb{Z}^d}\frac{|a_k|^2}{|k|^{2d-2}}.$$

It is easy to see that

$$\sum_{|k|\geq 2\frac{\|\nabla f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{L^2(\mathbb{T}^d)}}}|a_k|^2\leq\frac{\|f\|^2_{L^2(\mathbb{T}^d)}}{2}$$

because the opposite inequality would imply that

$$\begin{aligned}\|\nabla f\|^2_{L^2(\mathbb{T}^d)}=\sum_{k\in\mathbb{Z}^d}|k|^2|a_k|^2&\geq\sum_{|k|\geq 2\frac{\|\nabla f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{L^2(\mathbb{T}^d)}}}|k|^2|a_k|^2\\ &\geq 4\frac{\|\nabla f\|^2_{L^2(\mathbb{T}^d)}}{\|f\|^2_{L^2(\mathbb{T}^d)}}\sum_{|k|\geq 2\frac{\|\nabla f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{L^2(\mathbb{T}^d)}}}|a_k|^2\geq 2\|\nabla f\|^2_{L^2(\mathbb{T}^d)},\end{aligned}$$

which is absurd. Altogether, we now have

$$\begin{aligned}\left\|\langle\nabla f,\alpha\rangle\right\|^2_{L^2(\mathbb{T}^d)}\geq c^2(2\pi)^d\sum_{k\in\mathbb{Z}^d}\frac{|a_k|^2}{|k|^{2d-2}}&\geq c^2(2\pi)^d\sum_{\substack{k\in\mathbb{Z}^d\\ |k|\leq 2\frac{\|\nabla f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{L^2(\mathbb{T}^d)}}}}\frac{|a_k|^2}{|k|^{2d-2}}\\ &\geq\frac{c^2(2\pi)^d}{2^{2d-2}}\frac{\|f\|^{2d-2}_{L^2(\mathbb{T}^d)}}{\|\nabla f\|^{2d-2}_{L^2(\mathbb{T}^d)}}\sum_{|k|\leq 2\frac{\|\nabla f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{L^2(\mathbb{T}^d)}}}|a_k|^2\\ &\geq\frac{c^2(2\pi)^d}{2^{2d-2}}\frac{\|f\|^{2d-2}_{L^2(\mathbb{T}^d)}}{\|\nabla f\|^{2d-2}_{L^2(\mathbb{T}^d)}}\frac{\|f\|^2_{L^2(\mathbb{T}^d)}}{2}.\end{aligned}$$

Rearranging gives the result. □

We remark that a classical insight of Liouville allows to give a completely self-contained proof in the most elementary case. If we pick $\alpha=(\sqrt{2},1)$, then the only information required to make the above argument work is the existence of a $c>0$ such that

$$\big|\langle k,\alpha\rangle\big| = |k_1\sqrt{2}+k_2| \geq \frac{c}{|k|} \quad \text{for all } k=(k_1,k_2)\in\mathbb{Z}^2.$$

However, this follows at once with $c=1/3$ from

$$1 \leq \big|2k_1^2-k_2^2\big| = \big|(\sqrt{2}k_1-k_2)(\sqrt{2}k_1+k_2)\big| \leq 3|k|\big|(\sqrt{2}k_1+k_2)\big|.$$

A similar argument works for any $(1,\alpha)$ with α algebraic over $\mathbb{Q}$ (Liouville's theorem). More generally, a classical characterization of badly approximable numbers in one dimension as those numbers with a bounded continued fraction expansion implies that our proof works for

$$\alpha=(1,\beta)\in\mathbb{R}^2$$

if β has a bounded continued fraction expansion. A theorem of Lagrange (see e.g. [8]) implies that this is always the case if β is a quadratic irrational. Moreover, this characterization is sharp on $\mathbb{T}^2$: if β has an unbounded continued fraction expansion, then the inequality is not true for $(1,\beta)$ and the sequence

$$f_n(x)=e^{2\pi i k_n\cdot x}$$

with $k_n=$ (numerator, − denominator) of rational approximations of β coming from the continued fraction expansion will serve as a counterexample. A very interesting special case is Euler's continued fraction formula for e (see, e.g. [7]), which implies that e has an unbounded continued fraction expansion and that the inequality with $\alpha=(1,e)$ fails on $\mathbb{T}^2$.

2.6. Proof of the Proposition

Proof. The direction α has to have both entries different from 0. We use a classical result of Hurwitz [4] which guarantees the existence of infinitely many $k=(k_1,k_2)\in\mathbb{Z}^2$ with

$$\left|\frac{\alpha_1}{\alpha_2}-\frac{k_1}{k_2}\right| \leq \frac{1}{\sqrt{5}k_2^2}.$$

For any such (k_1,k_2), this can be rewritten as

$$|\alpha_1 k_2-\alpha_2 k_1| \leq \frac{|\alpha_2|}{\sqrt{5}|k_2|}.$$

We now consider $f(x)=e^{2\pi i k\cdot x}$. Simple computation yields

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \left\| \langle \nabla f, \alpha \rangle \right\|_{L^2(\mathbb{T}^2)} \leq \frac{1}{\sqrt{5}} \frac{|\alpha_2||k|}{|k_2|} \|f\|^2_{L^2(\mathbb{T}^2)}.$$

However, as $|k|\rightarrow\infty$, we have that $k_1/k_2\rightarrow\alpha_1/\alpha_2$ and thus

$$\frac{|\alpha_2||k|}{|k_2|} = |\alpha_2|\sqrt{\frac{k_1^2}{k_2^2}+1} \longrightarrow |\alpha_2|\sqrt{\frac{\alpha_1^2}{\alpha_2^2}+1} = |\alpha|. \quad \square$$

The constant in the result of Hurwitz is sharp for the golden ratio $\alpha=\alpha_1/\alpha_2=(1+\sqrt{5})/2=\phi$. Moreover, it is known (see e.g. [1]) that for every

$$\alpha \in \mathbb{R}\setminus\mathbb{Q} \quad \text{which is not of the form} \quad \frac{a\phi+b}{c\phi+d} \quad a,b,c,d \in \mathbb{Z} \quad |ad-bc| = 1,$$

the constant $\sqrt{5}$ could be replaced by $\sqrt{8}$. Our example showing the sharpness of the Proposition using Fibonacci numbers was therefore, in some sense, best possible.

2.7. Fractional derivatives

As is obvious from the proof, fine properties of the derivative did not play a prominent role, indeed, the proof really only requires an understanding of how fast the induced Fourier multiplier grows. This allows for various immediate generalizations. We introduce pseudodifferential operators $P(D)$ on $H^s(\mathbb{T}^d)$ via

$$P(D) \sum_{k\in\mathbb{Z}^d} a_k e^{ik\cdot x} := \sum_{k\in\mathbb{Z}^d} a_k P(k) e^{ik\cdot x}.$$

Our proof can always be applied if $|P(k)|\rightarrow\infty$ if $|k|\rightarrow\infty$. One example on $\mathbb{T}^2$ would be that for $\alpha\in\mathcal{B}$ and all $s>0$

$$\left\|\nabla^s f\right\|^{\frac{1}{s}}_{L^2(\mathbb{T}^2)} \left\| \langle \nabla f, \alpha_i \rangle \right\|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^{1+\frac{1}{s}}_{L^2(\mathbb{T}^2)}$$

which is again sharp by the same reasoning as above. The following variant was proposed by Raphy Coifman: if we define

$$D^s \sum_{k\in\mathbb{Z}^d} a_k e^{ik\cdot x} := \sum_{k\in\mathbb{Z}^d} a_k k|k|^{s-1} e^{ik\cdot x},$$

then the d-th derivative along the flow is large

$$\left\| \langle D^d f, \alpha \rangle \right\|_{L^2(\mathbb{T}^d)} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^d)}.$$

2.8. Several ergodic directions

We can also derive a statement for more than one ergodic direction. The same heuristics as above still apply: the main difference is that incorporating a control in more than one ergodic direction poses additional restrictions and requires less global control in the sense that a proportionately smaller power of $\|\nabla f\|_{L^2(\mathbb{T}^d)}$ is necessary.

Theorem 2. *Let $1 \le \ell \le d-1$. Then there exists a set $\mathcal{B}_\ell \in (\mathbb{T}^d)^\ell$ such that for every $(\alpha_1, \alpha_2, ..., \alpha_\ell) \in \mathcal{B}_\ell$ there is a $c_\alpha > 0$ with*

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}^{d-1} \left(\sum_{i=1}^{\ell} \left\| \langle \nabla f, \alpha_i \rangle \right\|_{L^2(\mathbb{T}^d)} \right)^{\ell} \ge c_\alpha \|f\|_{L^2(\mathbb{T}^d)}^{d-1+\ell}$$

for all $f \in H^1(\mathbb{T}^d)$ with mean 0.

Proof. We consider $\ell \le d-1$ vectors $\beta_1, \beta_2, ..., \beta_\ell$ from $\mathbb{T}^{d-1}$ with ℓ associated linear forms $L_i : \mathbb{Z}^{d-1} \to \mathbb{R}$ via $L_i = \langle \beta_i, \mathbf{x} \rangle$ such that they form a system of badly approximable linear forms and set

$$\alpha_1 = (1, \beta_1)$$
$$...$$
$$\alpha_\ell = (1, \beta_\ell)$$

The same reasoning as before (distinguishing between k vanishing outside of the first component or not) implies again for every single $1 \le i \le \ell$

$$\left| \langle \alpha_i, k \rangle \right| = \left| k_1 + L_i\big((k_2, ..., k_d)\big) \right| \ge \left\| L_i\big((k_2, ..., k_d)\big) \right\|$$

from which we derive that whenever k is not concentrated on the first component

$$\sum_{i=1}^{\ell} \left| \langle \alpha_i, k \rangle \right| \ge \max\big(\|L_1(k)\|, ..., \|L_\ell(k)\| \big) \ge c\big(\max(|k_2|, ..., |k_d|) \big)^{-\frac{d-1}{\ell}}.$$

If k is concentrated on the first component, we get a bound of ℓ, which is much larger. For

$$f = \sum_{\substack{k \in \mathbb{Z}^d \\ k \neq 0}} a_k e^{ik \cdot x}$$

a simple computation shows that

$$\sum_{i=1}^{\ell} \left\| \langle \nabla f, \alpha_i \rangle \right\|_{L^2(\mathbb{T}^d)}^2 = (2\pi)^d \sum_{k \in \mathbb{Z}^d} |a_k|^2 \left(\sum_{i=1}^{\ell} \left| \langle k, \alpha_i \rangle \right|^2 \right)$$

$$\geq c^2 \sum_{k \in \mathbb{Z}^d} |a_k|^2 \big(\max(|k_1|, ..., |k_d|)\big)^{-\frac{2(d-1)}{\ell}}$$

$$\geq c^2 \sum_{k \in \mathbb{Z}^d} |a_k|^2 |k|^{-\frac{2(d-1)}{\ell}}.$$

The rest of the argument proceeds as before; finally, we recall that any two norms in finite-dimensional vector spaces are equivalent and thus, up to some absolute constants depending only on ℓ,

$$\left(\sum_{i=1}^{\ell} \left\| \langle \nabla f, \alpha_i \rangle \right\|_{L^2(\mathbb{T}^d)} \right)^{\ell} \sim_{\ell} \left(\sum_{i=1}^{\ell} \left\| \langle \nabla f, \alpha_i \rangle \right\|_{L^2(\mathbb{T}^d)}^2 \right)^{\frac{\ell}{2}}$$

and the result follows. $\square$

2.9. The Hausdorff dimension

As is obvious from the proof, we are not so much interest in the distance to the lattice but care more about the distance to the origin. It seems that there is ongoing research in that direction [3], [5] and [6], which is concerned with establishing bounds on the dimension of the set

$$\max\big(\left|L_1(\mathbf{x})\right|, ..., \left|L_\ell(\mathbf{x})\right|\big) \geq c\big(\max(|x_1|, ..., |x_d|)\big)^{-\frac{d}{\ell}+1},$$

where $|\cdot|$ is the absolute value on $\mathbb{R}$. For any such system of linear forms given by $\alpha_1, ..., \alpha_\ell$ satisfying that inequality, we can improve Theorem 2 with the same proof to

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}^{d-\ell} \left(\sum_{i=1}^{\ell} \left\| \langle \nabla f, \alpha_i \rangle \right\|_{L^2(\mathbb{T}^d)} \right)^{\ell} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^d)}^d$$

for all $f \in H^1(\mathbb{T}^d)$ with mean 0. We also remark the following simple proposition (the essence of which is contained in [5]).

Proposition. *Let $d \geq 2$. $\alpha \in \mathbb{R}^d$ is admissible in Theorem* 1 *if and only if there exists $\lambda \in \mathbb{R}$ such that*

$$\alpha = (\lambda, \lambda\beta)$$

and β is badly approximable linear form in $\mathbb{R}^{d-1}$.

Using the result of Schmidt [13], we see that the Hausdorff dimension of the set of badly approximable vectors in $\mathbb{R}^{d-1}$ is $d-1$ and the construction increases the dimension by 1. The Hausdorff dimension of the set of admissible vectors in Theorem 1 is therefore d. The argument is indirectly contained in the earlier proofs.

2.10. Variants

This subsection is concerned with inequalities of the type

$$\|\nabla f\|_{L^2(\mathbb{T}^2)}^{1-\delta} \left\| \langle \nabla f, \alpha \rangle \right\|_{L^2(\mathbb{T}^2)}^{\delta} \geq c \|f\|_{L^2(\mathbb{T}^2)}$$

for some $0 < \delta \leq 1/2$. The case $\delta = 1/2$ was discussed above and following the same arguments immediately imply that $\delta > 1/2$ is impossible. However, the threshold $\delta = 1/2$ is also sharp.

Theorem. (Khintchine) *For every $\delta < 1/2$, the set of $\alpha \in \mathbb{T}^2$ for which there exists a $c_\alpha > 0$ such that*

$$\|\nabla f\|_{L^2(\mathbb{T}^2)}^{1-\delta} \left\| \langle \nabla f, \alpha \rangle \right\|_{L^2(\mathbb{T}^2)}^{\delta} \geq c \|f\|_{L^2(\mathbb{T}^2)}$$

holds for all $f \in L^2(\mathbb{T}^2)$ with mean 0 *has full measure.*

Another celebrated result in Diophantine approximation is the Thue–Siegel–Roth theorem stating that for every irrational algebraic number α and every $\varepsilon > 0$ we have

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c_\alpha}{q^{2+\varepsilon}}$$

for some $c_\alpha > 0$. This immediately implies, along the same lines as above, that for every vector of the form $\alpha = (1, \beta)$ with β being an irrational algebraic number and every $\varepsilon > 0$

$$\|\nabla f\|_{L^2(\mathbb{T}^2)}^{1/2+\varepsilon} \left\| \langle \nabla f, \alpha \rangle \right\|_{L^2(\mathbb{T}^2)}^{1/2-\varepsilon} \geq c_{\alpha,\varepsilon} \|f\|_{L^2(\mathbb{T}^2)}.$$

Recall that the inequality does *probably* not hold for $\alpha=(1,\pi)$ because π is *probably* not badly approximable. However, there are weaker positive results. A result of Salikhov [12] implies the existence of a $c>0$ such that

$$\left|\pi-\frac{p}{q}\right| \geq \frac{c}{q^8}.$$

Repeating again the same argument as above, we can use this to derive

$$\|\nabla f\|_{L^2(\mathbb{T}^2)}^{7/8} \left\| \left\langle \nabla f,(1,\pi)\right\rangle \right\|_{L^2(\mathbb{T}^2)}^{1/8} \geq c\|f\|_{L^2(\mathbb{T}^2)}.$$

Similarly, a result of Marcovecchio [10] shows

$$\|\nabla f\|_{L^2(\mathbb{T}^2)}^{13/18} \left\| \left\langle \nabla f,(1,\log 2)\right\rangle \right\|_{L^2(\mathbb{T}^2)}^{5/18} \geq c\|f\|_{L^2(\mathbb{T}^2)}$$

and similar results are available for other numbers (i.e. $\pi^2, \log 3, \zeta(3), \ldots$).

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Directional Poincaré inequalities along mixing flows

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