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Modules of systems of measures on polarizable Carnot groups

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Abstract. The paper presents a study of Fuglede's p-module of systems of measures in condensers in polarizable Carnot groups. In particular, we calculate the p-module of measures in spherical ring domains, find the extremal measures, and finally, extend a theorem by Rodin to these groups.

1. Introduction

Let Ω be a bounded domain in a polarizable Carnot group G, a particular case of which is the Euclidean space $\mathbb{R}^n$, $n\geq 2$, and let D_0 and D_1 be two disjoint compacts in the closure $\overline{\Omega}$ of Ω. The triple $(\Omega; D_0, D_1)$ is called a *condenser* in G, and the compacts D_0 and D_1 are called its plates. Two important quantities intimately related to the condenser $(\Omega; D_0, D_1)$ are the module of the family of curves $\Gamma(\Omega; D_0, D_1)$ connecting the compacts D_0 and D_1 in Ω, and the module of the surfaces $\Sigma(\Omega; D_0, D_1)$ separating D_0 and D_1 in Ω.

Let $(X, \mathfrak{M}, m)$ be an abstract measure space with a fixed measure $m\colon \mathfrak{M}\to [0, +\infty]$ defined on the σ-algebra $\mathfrak{M}$ of subsets of X. We denote by $\mathcal{M}$ the system of all measures μ in X, whose domains of definition contain $\mathfrak{M}$, and by E some subset of $\mathcal{M}$. The definition of the p-module of a system of measures given by Fuglede in [13] (see Definition 1, Section 2.1) is a nice and useful generalization of the notion of the p-module of a system of curves or surfaces. The *extremal system of measures* $E_0\subseteq E$ and the *extremal function* in this definition are known explicitly only in few trivial cases although they play an important role. Thereby, it is well known that the module of the family of all locally rectifiable curves Γ in

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a spherical ring domain $\Omega=R_{ab}$ in $\mathbb{R}^n$, $n\geq 2$, connecting two boundary concentric spheres of radii a, b, $0<a<b<\infty$, is equal to the module of the family of radial curves Γ_0 connecting these boundary spheres, and that the family Γ_0 is extremal. In the same spirit, the family of concentric spheres of radii r, $a<r<b$, separating the boundary spheres is extremal for the module of all Lipschitz separating surfaces in R_{ab}, [15], [43], [45], and [47]. The extremal functions are also known.

After a thorough analysis has been developed on Carnot groups, see e.g., [18], [19], and [46], analogous problems can be formulated for these groups. The only locally rectifiable curves on Carnot groups are so-called horizontal curves. In order to define a ring domain on a Carnot group one can use an analogue of the Euclidean norm which is a homogeneous function with respect to the anisotropic dilation δ_s, $s>0$, respecting grading of the corresponding Lie algebra. Unfortunately, the radial curves $\delta_{(\cdot)}$ (considered as functions of s) are not horizontal in general. On the Heisenberg group, Korányi and Reimann [24] found another family of 'radial' curves which are horizontal and orthogonal (in a correct sense) to the spheres defined as the level sets of the homogeneous norm. It was also shown that this family of curves is extremal for the ring domain. The existence of a homogeneous norm and of the corresponding horizontal family of radial curves was proved for some other classes of Carnot groups in [2], which received the name *polarizable Carnot groups*. In Section 3 we show that the family of concentric spheres separating two boundaries of the spherical ring domain is extremal for the p-module of all separating sufficiently smooth countably rectifiable surfaces in this ring domain on polarizable Carnot groups. We also present the extremal function.

Finding the extremal function and the extremal system of measures for the p-module is, in general, a quite difficult task, possible to be completed only in few cases. A result by Rodin [38] provides a method for finding the extremal function that leads to an explicit calculation of the 2-module of an extremal family of curves in the plane. In Section 4, we prove an extension of Rodin's theorem [38] to polarizable Carnot groups.

2. Module of a system of measures and polarizable groups

In this auxiliary section we give some necessary definitions used to obtain the main results in what follows.

2.1. Module of a system of measures

Recall that we consider an abstract measure space $(X,\mathfrak{M},m)$ with a fixed measure $m\colon \mathfrak{M}\to[0,+\infty]$ defined on the σ-algebra $\mathfrak{M}$ of subsets of X. The set

$\mathcal{M}$ is the collection of measures μ in X, whose domains of definition contain $\mathfrak{M}$. With an arbitrary system of measures $E\subset\mathcal{M}$ we associate the class of measurable functions $\rho\colon X\to[0,\infty]$ satisfying the condition

$$\int_X \rho\, d\mu \geq 1, \quad \mu\in E. \tag{1}$$

We call such ρ an *admissible function*, and we write $\rho\wedge\mu$ if (1) holds for a measure μ, and $\rho\wedge E$ if (1) holds for every $\mu\in E$.

Definition 1. For $0<p<\infty$, the p-module $M_p(E)$ of the system of measures E is defined as

$$M_p(E) = \inf_{\rho\wedge E} \int_X \rho^p\, dm,$$

interpreted as $+\infty$ if the set $\{\rho\colon \rho\wedge E\}$ is empty.

If a property holds for a system E, except for some subsystem $E^0\subset E$ with $M_p(E^0)=0$ we say that the property holds for M_p-*almost all* μ.

We call a system of measures E_0, $E_0\subseteq E$ *extremal* for the module $M_p(E)$ if there is a measurable function $\rho_0\colon X\to[0,+\infty]$ such that

$$\int_X \rho_0\, d\mu = 1 \quad \text{and} \quad M_p(E) = M_p(E_0) = \int_X \rho_0^p\, dm,$$

for M_p-almost all $\mu\in E_0$. The function ρ_0 is called the *extremal function*.

Definition 1 is a natural generalization of the concept of the module of a family of curves or surfaces in $\mathbb{R}^n$, $n\geq 2$. Given a family Γ of locally rectifiable curves or a family of Lipschitz surfaces in the space $X=\mathbb{R}^n$, one can regard $\mathfrak{M}$ as the Borel σ-algebra, m as the Lebesgue n-dimensional measure, and μ as the one-dimensional Hausdorff measures on curves or the surface measures on Lipschitz surfaces. The construction for these cases was carefully developed in [13, Chapter 2] and [34]. The module of a system of measures introduced in Definition 1 possesses the properties formulated in the following proposition.

Proposition 2. [13, Chapter 1] *Let $\mathcal{M}$ be a system of measures μ in a measure space $(X,\mathfrak{M},m)$, such that the domains of μ contain $\mathfrak{M}$. We write $\bar{\mu}$ for the completion of μ. The following properties hold*

(1) *$M_p(E)\leq M_p(E')$ if $E\subset E'$ and $E,E'\subset\mathcal{M}$;*
(2) *$M_p(E)\leq\sum_{i=1}^{\infty} M_p(E_i)$ if $E=\bigcup_{i=1}^{\infty} E_i$, and $E_i\subset\mathcal{M}$;*
(3) *If $A\subset X$ and $\overline{m}(A)=0$, then $\bar{\mu}(A)=0$ for M_p-a.a. $\mu\in\mathcal{M}$;*
(4) *If $\rho\in L^p(X,\overline{m})$, then ρ is $\bar{\mu}$-integrable for M_p-a.a. $\mu\in\mathcal{M}$;*

(5) *If* $\|\rho_i-\rho\|_{L^p(X,\overline{m})}\to 0$, *then there is a subsequence* ρ_{i_j}, *such that*

$$\int_X |\rho_{i_j}-\rho|\,d\bar{\mu}\longrightarrow 0 \quad \text{for } M_p\text{-a.e. } \mu\in\mathcal{M};$$

(6) *Let* $E\subset\mathcal{M}$. *Then* $M_p(E)=0$, *if and only if, there exists a non-negative function* $\rho\in L^p(X,m)$, *such that*

$$\int_X \rho\,d\mu=+\infty \quad \text{for every } \mu\in E;$$

(7) *If* $p>1$ *and* $E\subset\mathcal{M}\setminus\{\mu\equiv 0\}$, *then there exists a non-negative function* ρ, *such that*

$$\int_X \rho^p\,dm=M_p(E), \quad \text{and} \quad \int_X \rho\,d\mu\geq 1 \text{ for } M_p\text{-a.e. } \mu\in E;$$

(8) *If* $p\geq 2$, $E_1\subset E_2\subset\ldots$ *are sets of complete measures, and* $E=\bigcup E_i$, *then* $M_p(E)=\lim_{i\to\infty}M_p(E_i)$.

2.2. Polarizable groups

Definition 3. A Carnot group G is a connected simply connected Lie group, whose Lie algebra $\mathfrak{g}$ is nilpotent and possesses a stratification $\mathfrak{g}=\bigoplus_{j=1}^{l}V_j$, where $[V_1,V_j]=V_{j+1}$ for all $j\in\mathbb{N}$ with $V_j=\{0\}$, whenever $j>l$. The positive integer l is called the step of the group.

We assume that the underlying layer V_1 is endowed with an inner product $\langle .\,,.\rangle_0$. Let $X_1,\ldots,X_k$ be an orthonormal basis of V_1 with respect to this inner product. The vector fields $X_1,\ldots,X_k$ are usually called *horizontal*, and a sub-bundle HG of the tangent bundle TG of the group G with the typical fiber $H_gG=\mathrm{span}\{X_1(g),\ldots,X_k(g)\}\subset T_gG$, $g\in G$, is called the *horizontal sub-bundle.* As a consequence, any vector $v\in H_gG$ is also called horizontal. The inner product $\langle .\,,.\rangle_0$ on V_1 defines a left-invariant sub-Riemannian metric on G, which we denote by the same symbol, by means of left translations, $\|v\|_0^2=\langle v,v\rangle_0$ for $v\in H_gG$. Let us use the normal coordinates of the first kind, where an element $g\in G$ is identified with $(x_1,\ldots,x_k,t_{k+1},\ldots,t_m)\in\mathbb{R}^m$ by the formula

$$g=\exp\left(\sum_{i=1}^{k}x_iX_i+\sum_{i=k+1}^{m}t_iT_i\right),$$

where $T_{k+1},...,T_m$ denote a set of vectors extending the horizontal basis $X_1,...,X_k$ to the entire basis of $\mathfrak{g}$. The stratified structure of the Lie algebra naturally defines the dilation δ_s, $s>0$, that can be written as

$$\begin{aligned}\delta_s g &= \delta_s(x_1,...,x_k,t_{k+1},...,t_{k+\dim(V_2)},...,t_{k+\sum_{j=1}^{l-1}\dim(V_j)+1},...,t_m)\\ &= (sx_1,...,sx_k,s^2t_{k+1},...,s^2t_{k+\dim(V_2)},...,s^lt_{k+\sum_{j=1}^{l-1}\dim(V_j)+1},...,s^lt_m),\end{aligned}$$

in the introduced coordinates. As a simply connected nilpotent group, admitting dilations δ_s, G is globally diffeomorphic to $\mathfrak{g}\cong\mathbb{R}^m$, $m=\sum_{i=1}^l \dim V_i$, via the exponential map, see [10, Proposition 1.2]. The number m is the topological dimension of the group. The sub-Riemannian metric induces a distance function d_{cc} on G called the Carnot-Carathéodory distance in a similar way as for the Riemannian metric. The Hausdorff dimension of the metric space (G,d_{cc}) is equal to $Q=\sum_{i=1}^l i\dim V_i$, see [32]. The number Q is called *homogeneous dimension* of G, and it will play an important role in the forthcoming calculations. The Haar measure on G is induced by the exponential map from the Lebesgue measure on $\mathfrak{g}\cong\mathbb{R}^m$. A norm N_G on the group G is called homogeneous if it is given by a homogeneous function of order one with respect to the dilation δ_s: $N_G(\delta_s g)=sN_G(g)$ for all $g\in G$.

The horizontal gradient ∇_0 is the unique horizontal vector field such that

$$\langle\nabla_0 f, v\rangle_0 = v(f), \quad \text{for any section } v \text{ of } HG,\ f\in C^\infty(G).$$

The horizontal gradient is expressed in the orthonormal basis $X_1,...,X_k$ as $\nabla_0 f=(X_1f,...,X_kf)$.

Given a domain $U\subset G$, a function $u\in C^2(U)$ (or, more general, a distribution u) is called p-harmonic if it satisfies the p-sub-Laplacian equation

$$\Delta_{0,p}u := \sum_{i=1}^k X_i\big(\|\nabla_0 u\|_0^{p-2}X_iu\big)=0, \tag{2}$$

in U, and ∞-harmonic if it satisfies the ∞-sub-Laplacian equation

$$\Delta_{0,\infty}u := \frac{1}{2}\langle\nabla_0\|\nabla_0 u\|_0^2, \nabla_0 u\rangle_0 = 0,$$

in U. By a result by Folland [9, Theorem 2.1], there exists a unique fundamental solution u_2 in any Carnot group G of the homogeneous dimension $Q\geq 3$ to the Kohn sub-Laplacian $\Delta_{0,2}$, which is smooth away from zero and homogeneous of degree $2-Q$: $u_2\circ\delta_s=s^{2-Q}u_2$.

Definition 4. (See [2].) We say that a Carnot group G is polarizable if the fundamental solution u_2 of the Kohn sub-Laplacian $\Delta_{0,2}$ has the property that the homogeneous norm $N_G{=}u_2^{\frac{1}{2-Q}}$ associated with u_2 is ∞-harmonic away from zero in G.

Examples of polarizable groups are $\mathbb{R}^m$, n-th Heisenberg group $\mathbb{H}^n$, and H(eisenberg)-type groups introduced by Kaplan [21], which definition will be given in Section 2.2.2. The main result of [2] states that it is possible to carry out a construction of some sort of spherical coordinates in any polarizable group in the same way as it had been done in [24] for the Heisenberg group.

Let G be a polarizable Carnot group, and let N_G be the norm from Definition 4. We denote by $\mathcal{Z}$ the characteristic set of the function N_G,

$$\mathcal{Z} := \{0\} \cup \{g \in G \setminus \{0\} \mid \nabla_0 N_G(g) = 0\}.$$

The radial flow is a solution to the Cauchy initial-value problem in $G\setminus\mathcal{Z}$

$$\begin{cases} \dfrac{\partial}{\partial s}\,\phi(s,g) = \dfrac{N_G(\phi(s,g))}{s}\cdot\dfrac{\nabla_0 N_G(\phi(s,g))}{\|\nabla_0 N_G(\phi(s,g))\|_0^2}, \\ \phi(1,g){=}g. \end{cases} \tag{3}$$

Proposition 5. (*See* [2].) *The flow ϕ satisfies the following properties*:

(i) $N_G(\phi(s,g)){=}sN_G(g)$ *for* $s{>}0$, *and* $g{\in}G\setminus\mathcal{Z}$;

(ii) $\|(\partial\phi/\partial s)\|_0$ *is independent of* s, *i.e.*,

$$\|(\partial\phi/\partial s)\|_0 = \frac{N_G(g)}{\|\nabla_0 N_G(\phi(s,g))\|_0} =: \lambda(g)^{-1},$$

for a non-zero real-valued function λ *on* $G\setminus\mathcal{Z}$;

(iii) $\det D_g\phi(s,g){=}s^Q$ *for* $s{>}0$ *and* $g{\in}G\setminus\mathcal{Z}$, *where* $D_g\phi$ *denotes the differential of the map* $\phi(s,\cdot){:}G\setminus\mathcal{Z}{\to}G\setminus\mathcal{Z}$, *where* $G\setminus\mathcal{Z}$ *is considered as a domain in* $\mathbb{R}^m$.

Definition 6. An absolutely continuous curve $c\colon I{\to}G$ is called horizontal if the vector $\frac{d}{ds}c(s)$ is horizontal, i.e., $\frac{d}{ds}c(s){\in}H_{c(s)}G$ for all $s{\in}I$, when it is defined.

The solution ϕ to the Cauchy problem (3) is a horizontal curve simply because its tangent vector $\frac{\partial\phi(s,g)}{\partial s}$ is proportional to the horizontal gradient of some function, namely, of the homogeneous norm.

We remark that if $c\colon I\to G$ is a horizontal curve whose locus belongs to the level set of the function N_G, then

$$\frac{d}{d\tau}N_G(c(\tau))=\left\langle \nabla_0 N_G, \frac{d\,c(\tau)}{d\tau}\right\rangle_0=0.$$

We say that the horizontal gradient is orthogonal to the level set meaning that it is orthogonal to any horizontal curve (whose locus belongs to the level set) with respect to the inner product $\langle .\,,\,.\rangle_0$. As a consequence, we conclude that the flow ϕ solving the Cauchy problem (3) is orthogonal to the level set of the function N_G, where the orthogonality is understood with respect to the inner product $\langle .\,,\,.\rangle_0$ of the tangent vector to the flow and the tangent vectors to the horizontal curves lying on the level set.

Note that not all Carnot groups are polarizable. It is known that the fundamental solution u to 2-sub-Laplacian always exists, but its norm $N(u)=u^{1/(2-Q)}$ is not necessarily ∞-harmonic. An example of such kind of groups can be the anisotropic H-type groups, see reasoning in [2]. Anisotropic H-type groups were studied, for instance, in [5] and [7]. Now we continue with examples of polarizable groups.

2.2.1. Euclidean space

First of all, notice that $G=\mathbb{R}^k$ is a Carnot group of step 1, where the Lie algebra is given by the space $V_1=\mathrm{span}\{X_1,...,X_k\}$ with $X_j=\frac{\partial}{\partial x_j}$. Since all the commutators of X_j, $j=1,...,k$ vanish, the spaces $V_2=...=V_l=\{0\}$ are empty. The exponential map is a map identifying $\mathbb{R}^k$ with V_1. The Kohn sub-Laplacian $\Delta_{0,2}$ is the usual Laplacian Δ_2, whose fundamental solution $u_2(x)=|x|^{2-k}$ defines the homogeneous norm $N_{\mathbb{R}^k}(x)=|x|$, which is the Euclidean norm of an element $x\in\mathbb{R}^k$. It is trivial to check that $|\cdot|$ is an ∞-harmonic function. The radial flow $\phi(s,x)=sx$, $x\in\mathbb{R}^k$ is a solution to the corresponding Cauchy problem (3).

2.2.2. H-type groups

Definition 7. We say that a Carnot group G is of Heisenberg type (H-type) if its Lie algebra $\mathfrak{g}=V_1\oplus V_2$ of G is 2 step, and if it is endowed with an inner product $\langle .\,,\,.\rangle$, and admits a linear map $J\colon V_2\to\mathrm{End}(V_1)$, compatible with the inner product in the following sense:

(1) V_1 is orthogonal to V_2,
(2) $\langle J_Z U, V\rangle=\langle Z,[U,V]\rangle$ for all $Z\in V_2$ $U,V\in V_1$, and
(3) $J_Z^2=-\|Z\|^2\,\mathrm{Id}_{V_1}$ for all $Z\in V_2$, where $\|Z\|^2=\langle Z,Z\rangle$.

The inner product $\langle .\,,\,.\rangle_0$ on V_1 is the restriction of $\langle .\,,\,.\rangle$ to V_1.

Let G be a group of H-type. Since the exponential map of G is an analytic diffeomorphism, we can define real analytic mappings $u\colon G\to V_1$ and $z\colon G\to V_2$ by $g=\exp\left(u(g)+z(g)\right)$, $g\in G$. The function

$$N_G(g)=\left(\|u(g)\|^4+16\|z(g)\|^2\right)^{1/4} \tag{4}$$

is a homogeneous norm on G. It is well known that N_G is smooth on $G\backslash\{0\}$, see [21, Theorem 2]. It was shown in [2, Proposition 5.6] that H-type groups are polarizable with the norm N_G, defined by (4).

2.2.3. Heisenberg group

A typical example of an H-type group is the Heisenberg group.

Definition 8. The n-dimensional Heisenberg group $\mathbb{H}^n$ is a nilpotent Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, and whose Lie algebra $\mathfrak{h}$ is graded

(1) $\mathfrak{h}=V_1\oplus V_2$, where $\dim(V_1)=2n$, and $\dim(V_2)=1$, and

(2) $\mathfrak{h}$ admits the following commutation relations:

$$[V_1,V_1]=V_2,\quad [V_1,V_2]=[V_2,V_2]=\{0\}.$$

We choose a basis $X_1,\dots X_n,Y_1,\dots Y_n$ of V_1 an element Z of V_2, such that $[X_j,Y_j]=Z$ for $j=1,\dots,n$, and otherwise zero. Define the linear map $J\colon V_1\to V_1$ by $J(X_j)=Y_j$, $J(Y_j)=-X_j$. The inner product $\langle .\,,\,.\rangle_{V_2}$ on V_2 is chosen such that $\langle Z,Z\rangle_{V_2}=1$. The inner product $\langle .\,,\,.\rangle_{V_1}$ is defined by setting

$$\langle U_1,U_2\rangle_{V_1}=\alpha\quad\text{if } [U_1,JU_2]=\alpha Z,\ \alpha\in\mathbb{R}$$

for $U_1,U_2\in V_1$. Now it remains to check that the conditions of Definition 7 hold for the inner product $\langle .\,,\,.\rangle=\langle .\,,\,.\rangle_{V_1}+\langle .\,,\,.\rangle_{V_2}$, and the map J_Z is given by $J_Z=\langle Z,Z\rangle^{1/2}J$.

Using the normal coordinates of the first kind $g=(x,y,t)$, $x,y\in\mathbb{R}^n$, $t\in\mathbb{R}$, we write the homogeneous norm in the form (4).

3. Extremal measures on polarizable groups

3.1. The p-module of $\Gamma(R_{ab};S_a,S_b)$ on polarizable groups

Let G be a polarizable group and let

$$B_r=\{g\in G\mid N_G(g)<r\},\quad S_r=\{g\in G\mid N_G(g)=r\}.$$

We want to present the extremal function and the extremal family of curves for the condenser $(R_{ab}; S_a, S_b)$ in the problem of the p-module.

A result by Korányi and Reimann [24] gives an explicit value of

$$M_p(\Gamma((R_{ab}; S_a, S_b))), \quad \text{in } \mathbb{H}^1,$$

where $p{=}4$ is the homogeneous dimension of $\mathbb{H}^1$. The p-module of the family of curves for the spherical ring domain in H-type groups and in polarizable Carnot groups, in terms of p-capacity is given in [2] and [6]. The value of the p-capacity in the Carnot groups, see Definition 17, and of the p-module of the family of curves connecting S_a and S_b in the spherical ring domain coincide, see [28]. We give here a brief calculation of $M_p(\Gamma(R_{ab}; S_a, S_b))$ in a polarizable Carnot group, for the completeness.

We recall that if $c\colon [a,b]\to G$ is an absolutely continuous curve in a Carnot group G, which is not horizontal for some open interval $I\subset[a,b]$, then it is non-rectifiable. It was shown in [35] that even c is only continuous and rectifiable, the tangent vector $\dot{c}(s)$ exists and is horizontal for almost all $s\in[a,b]$. Thus, we can restrict ourselves to horizontal curves computing the p-module of a family of curves, because the p-module of a family of non-rectifiable curves vanishes [13]. Note also that any system of curves for $0<p<1$ has vanishing p-module [13].

Let ϕ be a solution to the Cauchy problem (3) satisfying the initial data $\phi(1,\xi)=\xi$, $\xi\in S_1$. The existence of the radial flow $\phi(\cdot,\xi)\colon (0,\infty)\to G$ solving (3), allows us to write the integral over G in terms of spherical coordinates. Namely, the following proposition holds.

Proposition 9. [2] *Let G be a polarizable Carnot group. There exists a unique Radon measure dv on $S_1\setminus\mathcal{Z}$, such that the integration formula*

$$\int_G f(g)\, d\mathbf{g} = \int_{S_1\setminus\mathcal{Z}}\int_0^\infty f(\phi(s,\xi))s^{Q-1}\,ds\,dv(\xi) \tag{5}$$

is valid for all $f\in L^1(G)$, where $d\mathbf{g}$ denotes the Haar measure on G.

Observe that the only information one needs is the existence of homogeneous norm N_G satisfying Proposition 5 in order to carry on the construction of spherical coordinates and forthcoming calculation of $M_p(\Gamma((R_{ab}; S_a, S_b)))$ on a polarizable Carnot group. The following theorem is a reformulation of a result from [2].

Theorem 10. *Let G be a polarizable Carnot group of Hausdorff dimension Q with a homogeneous norm N_G associated to Folland's solution to the Kohn sub-Laplacian. Let $\Gamma{=}\Gamma(R_{ab}; S_a, S_b)$ be a family of horizontal locally rectifiable curves* *connecting the boundaries S_a and S_b in R_{ab}. Then,*

$$M_p(\Gamma) = C_{S_1}(p) C_{ab}^{1-p}(p,Q),$$

for $p>1$, where

$$C_{S_1}(p) = \int_{S_1\setminus \mathcal{Z}} \lambda^p(\xi)\, dv(\xi), \quad \text{and} \quad C_{ab}(p,Q) = \int_a^b s^{\frac{1-Q}{p-1}}\, ds.$$

Proof. Let us use the integration in spherical coordinates (5) in order to calculate the module of Γ. For all admissible functions ϖ we have

$$1 \le \int_{\phi(\cdot,\xi)} \varpi = \int_a^b \varpi(\phi(s,\xi))\lambda(\xi)^{-1}\, ds \quad \Longrightarrow \quad \lambda(\xi) \le \int_a^b \varpi(s,\xi)\, ds,$$

where $\lambda(\xi)^{-1} = \|\frac{\partial\phi(s,\xi)}{\partial s}\|_0 = \frac{1}{\|\nabla_0 N_G(\phi(s,\xi))\|_0}$, $\xi\in S_1$. Hölder's inequality implies

$$\lambda(\xi)^p \le \left(\int_a^b (\varpi s^{\frac{Q-1}{p}}) s^{-\frac{Q-1}{p}}\, ds\right)^p \le \left(\int_a^b \varpi^p s^{Q-1}\, ds\right)\left(\int_a^b s^{\frac{1-Q}{p-1}}\, ds\right)^{p-1}.$$

Therefore, $\int_a^b \varpi^p(\phi(s,\xi)) s^{Q-1}\, ds \ge C_{ab}(p,Q)^{1-p}\lambda(\xi)^p$, and

$$\int_{R_{ab}} \varpi^p(g)\, d\mathbf{g} = \int_{S_1\setminus\mathcal{Z}} \int_a^b \varpi^p(\phi(s,\xi)) s^{Q-1}\, ds\, dv(\xi) \ge C_{ab}(p,Q)^{1-p} C_{S_1}(p).$$

If we denote the family of curves generated by the radial flow ϕ by Γ_0, then Γ_0 is a subfamily of Γ. Taking infimum over admissible functions we obtain

$$M_p(\Gamma) \ge M_p(\Gamma_0) \ge C_{ab}(p,Q)^{1-p} C_{S_1}(p). \tag{6}$$

In order to find an upper bound for $M_p(\Gamma)$ we choose the extremal function given by

$$\varpi_0 = \begin{cases} ((\tau+1)C_{ab}(p,Q))^{-1}\, \|\nabla_0(N_G^{\tau+1})\|_0, \ \tau+1 = \frac{p-Q}{p-1}, & p\neq Q, \\ C_{ab}(p,Q)^{-1}\, \|\nabla_0(\log N_G)\|_0, & p=Q. \end{cases} \tag{7}$$

Considering ϖ_0 along the flow $\phi(s,\xi)$ of radial curves we obtain

$$\varpi_0(\phi(s,\xi)) = C_{ab}(p,Q)^{-1}\, s^{\frac{1-Q}{p-1}} \lambda(\xi),$$

where $N_G(\phi(s,\xi)) = s$. Using formula (5), we calculate

$$\int_{R_{ab}} \varpi_0^p \, d\mathbf{g} = \int_{S_1\setminus\mathcal{Z}} \int_a^b \varpi_0^p(\phi(s,\xi)) s^{Q-1} \, ds \, dv(\xi)$$
$$= C_{ab}(p,Q)^{-p} \int_{S_1\setminus\mathcal{Z}} \lambda(\xi)^p \, dv(\xi) \int_a^b s^{\frac{1-Q}{p-1}} \, ds = C_{ab}(p,Q)^{1-p} C_{S_1}(p).$$

The function ϖ_0 is admissible for Γ as it will be shown in Lemma 11. Finally, we have $M_p(\Gamma) \leq \int_{R_{ab}} \varpi_0^p \, d\mathbf{g} = C_{ab}(p,Q)^{1-p} C_{S_1}(p)$. $\square$

Lemma 11. *The function ϖ_0 defined by (7) for all $p>1$ is admissible for the module $M_p(\Gamma)$ of the family of curves connecting S_a and S_b in the spherical ring domain R_{ab}.*

Proof. Let $\gamma\colon [0,l_\gamma] \to G$ be a curve from Γ parametrized by arc-length $\|\dot\gamma\|_0 = 1$ satisfying the boundary conditions $a = N_G(\gamma(0))$ and $b = N_G(\gamma(l_\gamma))$. Then, by the Schwarz inequality, we have $\langle \nabla_0 N_G, \dot\gamma(s)\rangle_0 \leq \|\nabla_0 N_G\|_0$ for almost all $s\in[0,l_\gamma]$. It follows that

$$\int_\gamma \varpi_0 = \int_0^{l_\gamma} \varpi_0(\gamma(s)) \, ds$$
$$\geq \big((\tau+1) C_{ab}(p,Q)\big)^{-1} \int_0^{l_\gamma} (\tau+1) N_G^\tau(\gamma(s)) \langle \nabla_0 N_G(\gamma(s)), \dot\gamma(s)\rangle_0 \, ds$$
$$= \big((\tau+1) C_{ab}(p,Q)\big)^{-1} \int_0^{l_\gamma} \frac{d}{ds} N_G^{\tau+1}(\gamma(s)) \, ds,$$

for $p\neq Q$. Since $N_G^{\tau+1}(\gamma)\colon [0,l_\gamma] \to R_{ab}$ is absolutely continuous, the Fundamental Theorem of Calculus results in

$$\int_\gamma \varpi_0 \geq \left((\tau+1) \int_a^b s^\tau \, ds\right)^{-1} \big(N_G^{\tau+1}(\gamma(l_\gamma)) - N_G^{\tau+1}(\gamma(0))\big) = 1.$$

Similar calculations prove the result for $p=Q$. $\square$

Corollary 12. (*See* [2].) *The family of radial curves Γ_0 satisfying (3) is the extremal family for the module R_{ab} of the spherical ring domain R_{ab} in polarizable Carnot groups. The function ϖ_0 given by (7) is extremal. Moreover, calculating the integral $C_{ab}(p,Q)^{1-p}$, we obtain*

$$M_p(\Gamma) = \begin{cases} C_{S_1}(p) (\frac{|p-Q|}{p-1})^{p-1} |(b^{\frac{p-Q}{p-1}} - a^{\frac{p-Q}{p-1}})|^{1-p}, & p\neq Q, \\ C_{S_1}(p) (\log\frac{b}{a})^{1-Q}, & p=Q. \end{cases}$$

As it was mentioned, the H-type groups are polarizable [2, Proposition 5.6], and the form of the homogeneous norm is given by (4). The value of the constant $C_{S_1}(p)$ for H-type groups was explicitly calculated in [2] and it is given by

$$C_{S_1}(p) = \int_{S_1\setminus\mathcal{Z}} \lambda(\xi)^p \, dv(\xi) = \frac{2\pi^{k+l/2}\,\Gamma(\frac{k+p}{4})}{4^l\Gamma(\frac{k}{2})\Gamma(\frac{k+2l+p}{4})},$$

where $k{=}\dim V_1$, $l{=}\dim V_2$. The details for the Euclidean space can be found in [15], [43], and [47].

3.2. The p-module of a family of separating sets in R_{ab}

Let us define separating sets in a topological sense and describe the associated system of measures. Let D_0, D_1 be disjoint compact sets in the closure $\overline{\Omega}$ of a domain $\Omega{\in}G$. We denote $\Omega^*{=}\Omega{\cup}D_0{\cup}D_1$.

Definition 13. We say that a set σ separates D_0 from D_1 in Ω if

1. $\sigma{\cap}\Omega$ is closed in Ω;
2. There are disjoint sets U_0 and U_1 which are open in $\Omega^*\setminus\sigma$ such that $\Omega^*\setminus\sigma{=}U_0{\cup}U_1$, $D_0{\subset}U_0$ and $D_1{\subset}U_1$.

Let Σ denote the class of all sets that separate D_0 from D_1. With every $\sigma{\subset}\Sigma$ we associate a complete measure μ in the following way. For every Hausdorff H^{Q-1}-measurable set $A{\subset}G$ we define

$$\mu(A) = H^{Q-1}(A{\cap}\sigma{\cap}\Omega),$$

where H^{Q-1} is the $(Q-1)$-dimensional Hausdorff measure taken with respect to the sub-Riemannian metric on the group. Thus the system of measures E associated with the collection Σ is the family of measures μ, defined above. From the properties of the Hausdorff measure, it is clear that the Borel sets in G (here $\sigma{\cap}\Omega$ is closed in Ω, and therefore, is Borel) are μ-measurable. We use the notation $M_q(\Sigma)$ for the q-module of the family of measures E associated with the collection $\Sigma{=}\Sigma(\Omega; D_0, D_1)$ of sets separating D_0 and D_1 in Ω. Thus, according to Definition 1 for any given $1{<}q{<}\infty$,

$$M_q(\Sigma) = \inf_{\rho\wedge\Sigma} \int \rho^q \, d\mathbf{g},$$

where $d\mathbf{g}$ is the Haar measure in G and $\rho\wedge\Sigma$ means that ρ is a non-negative Borel function such that $\int_{\sigma\cap\Omega} \rho \, d\mu{\geq}1$ for every $\sigma{\in}\Sigma$.

Let $\Sigma=\Sigma(R_{ab};S_a,S_b)$ denote the class of sets σ that separate S_a from S_b in $R_{ab}\subset G$ and such that $\sigma\cap R_{ab}$ has locally finite H^{Q-1}-Hausdorff measure. The subset $\Sigma'\subset\Sigma$ of separating sets such that $H^{Q-1}(\sigma'\cap R_{ab})=\infty$, $\sigma'\in\Sigma'$ will have vanishing q-module. The class Σ is not empty, since it contains intrinsic G-regular hypersurfaces, see [11]. Let us specify these measures on spheres S_s in detail. The integration formula (5) implies that the volume element $d\mathbf{g}$ along the flow defined by ϕ can be written as

$$d\mathbf{g}=s^{Q-1}\,ds\,dv(\xi)=s^{Q-1}\left\|\frac{\partial\phi}{\partial s}\right\|_0^{-1}dv(\xi)\left\|\frac{\partial\phi}{\partial s}\right\|_0 ds=s^{Q-1}\lambda(\xi)\,dv(\xi)\lambda(\xi)^{-1}\,ds.$$

Hence, the measure $d\mu_1(\xi)=\lambda(\xi)\,dv(\xi)$, $\xi\in S_1$ is absolutely continuous with respect to the Radon measure $dv(\xi)$ and represents a $(Q-1)$-dimensional element of a surface measure on the unit sphere S_1. The element of the surface measure on the sphere S_s of radius s is given by $d\mu_s=s^{Q-1}\lambda(\xi)\,dv(\xi)$. The part $\lambda(\xi)^{-1}\,ds$ defines the element of length of the curve $\phi(\cdot,\xi)$ and was used in the previous section. In the case $G=\mathbb{R}^n$, we obtain that $\lambda(\xi)\equiv1$, and $d\mu_1=dv$ is the Euclidean surface measure element on the unit sphere.

Theorem 14. *Let G be a polarizable Carnot group of Hausdorff dimension Q with the homogeneous norm N_G associated to Folland's solution to the Kohn sub-Laplacian. Let E be the family of measures associated to the collection $\Sigma=\Sigma(R_{ab};S_a,S_b)$ of sets separating S_a and S_b in R_{ab}. Then for $q>1$, we obtain*

$$M_q(E)=M_q(\Sigma)=K_{ab}(q,Q)K_{S_1}^{1-q}(q),$$

where

$$K_{S_1}(q)=\int_{S_1\setminus\mathcal{Z}}\lambda^{\frac{q}{q-1}}(\xi)\,dv(\xi)\quad\text{and}\quad K_{ab}(q,Q)=\int_a^b s^{(1-q)(Q-1)}\,ds. \tag{8}$$

Proof. Let ρ be an admissible function for the family E. Then, for any sphere S_s, $a<s<b$, we have

$$1\le\left(\int_{S_s\setminus\mathcal{Z}}\rho(\phi(s,\xi))\,d\mu_s\right)^q=\left(\int_{S_1\setminus\mathcal{Z}}s^{Q-1}\rho(\phi(s,\xi))\lambda(\xi)\,dv(\xi)\right)^q$$
$$\le s^{q(Q-1)}\left(\int_{S_1\setminus\mathcal{Z}}\rho^q(\phi(s,\xi))\,dv(\xi)\right)\left(\int_{S_1\setminus\mathcal{Z}}\lambda^{\frac{q}{q-1}}(\xi)\,dv(\xi)\right)^{q-1}.$$

Thus,

$$\int_{S_1\setminus\mathcal{Z}}\rho^q(\phi(s,\xi))\,dv(\xi)\ge s^{-q(Q-1)}\left(\int_{S_1\setminus\mathcal{Z}}\lambda^{\frac{q}{q-1}}(\xi)\,dv(\xi)\right)^{1-q}.$$

Then, we arrive at the inequality

$$\int_{R_{ab}} \rho^q \, d\mathbf{g} = \int_a^b s^{Q-1} \, ds \int_{S_1 \setminus \mathcal{Z}} \rho^q \, dv \geq \int_a^b s^{(1-q)(Q-1)} \left(\int_{S_1 \setminus \mathcal{Z}} \lambda^{\frac{q}{q-1}}(\xi) \, dv(\xi) \right)^{1-q}.$$

Making use of notations (8), we come to a lower bound for the module $M_q(\Sigma)$ of the family of separating sets

$$M_q(\Sigma) \geq M_q(\Sigma_0) \geq K_{ab}(q,Q) K_{S_1}(q)^{1-q},$$

where $\Sigma_0 = \Sigma_0(R_{ab}; S_a, S_b)$ is the family of spheres $S_s = \{g \in G \,|\, N_G(g) = s\}$ for $a<s<b$ which separate the boundaries of the spherical ring domain R_{ab}.

Now we turn to the estimation of $M_q(\Sigma)$ from above. The extremal function in this case is given by the following expression

$$(9) \quad \rho_0(g) = \begin{cases} (\tau+1)^{\frac{1}{1-q}} K_{S_1}^{-1}(q) \|\nabla_0 (N_G^{\tau+1}(g))\|_0^{\frac{1}{q-1}}, \ \tau = (q-1)(1-Q), & q \neq \frac{Q}{Q-1}, \\ K_{S_1}^{-1}(q) \|\nabla_0 (\log N_G(g))\|_0^{\frac{1}{q-1}}, & q = \frac{Q}{Q-1}. \end{cases}$$

Restricting the value of ρ_0 to the sphere $N_G(g) = s$ we conclude that

$$\rho_0(\varphi(s,\xi)) = \rho_0(g) = K_{S_1}^{-1}(q) s^{1-Q} \lambda^{\frac{1}{q-1}}(\xi).$$

Thus,

$$\int_{R_{ab}} \rho_0^q \, d\mathbf{g} = K_{S_1}^{-q}(q) \int_a^b s^{Q-1+q(1-Q)} \, ds \int_{S_1 \setminus \mathcal{Z}} \lambda^{\frac{q}{q-1}}(\xi) \, dv(\xi) = K_{ab}(q,Q) K_{S_1}^{1-q}(q).$$

The function ρ_0 is admissible for the family of separating sets as it will be proved in Subsection 3.3. Finally, taking the infimum over the admissible functions, we obtain

$$M_q(\Sigma) \leq \int_{R_{ab}} \rho_0^q \, d\mathbf{g} = K_{ab}(q,Q) K_{S_1}^{1-q}(q).$$

This finishes the proof. $\square$

Corollary 15. *The family Σ_0 of the spheres $\{S_s, a<s<b\}$ is the extremal family for the module $M_q(\Sigma)$ of sets Σ separating the spheres S_a and S_b in the spherical ring domain R_{ab} on polarizable Carnot groups. The function ρ_0 given by (9) is extremal. In particular, $\int_{S_s} \rho_0 \, d\mu_s = 1$ for any sphere S_s, $a<s<b$.*

Let us observe the following relations revealed by Theorems 10 and 14.

Corollary 16. *For* $\frac{1}{p}+\frac{1}{q}=1$, *Theorems* 10 *and* 14 *imply*
1. $K_{ab}(q,Q)=C_{ab}(p,Q)$,
2. $K_{S_1}(q)=C_{S_1}(p)$,
3. $M_p^{\frac{1}{p}}(\Gamma)M_q^{\frac{1}{q}}(\Sigma)=1$,
4. $\rho_0=C_{ab}^{p-1}(p,Q)C_{S_1}^{-1}(p)\varpi_0^{p-1}$, $p\neq Q$

3.2.1. Relations between $M_p(\Gamma)$, $M_q(\Sigma)$, and the capacity $\mathrm{cap}_p(R_{ab})$.

Before we proceed to show that the function ρ_0 is admissible for the family E, we recall the relations between $M_p(\Gamma)$, $M_q(\Sigma)$, and the capacity $\mathrm{cap}_p(R_{ab})$. We say that a function $u\colon\Omega\to\mathbb{R}$ belongs to the Sobolev space $W^{1,p}(\Omega)$ if $u\in L^p(\Omega)$ and the horizontal gradient $\nabla_0 u$ exists in the sense of distributions and $\|\nabla_0 u\|_0\in L^p(\Omega)$.

Definition 17. Let Ω be a domain in G, and let D_0, D_1 be two disjoint compacts in the closure $\overline{\Omega}$ of Ω. A function $u\in W^{1,p}(\Omega)$ is called admissible for the condenser $(\Omega; D_0, D_1)$, if $u|_{D_0}=0$ and $u|_{D_1}=1$. The value

$$\mathrm{cap}_p(\Omega; D_0, D_1)=\inf\int_\Omega \|\nabla_0 u\|_0^p\, d\mathbf{g},$$

where the infimum is taken over all admissible functions u is called a p-capacity of the condenser $(\Omega; D_0, D_1)$.

The relations between the admissible function u for the p-capacity of a condenser $(\Omega; D_0, D_1)$, and the admissible function ρ for the p-module of a family of curves connecting D_0 and D_1 is as follows. Let ρ be an admissible function for the family of curves connecting D_0 and D_1. Then the function $u(x)=\min\{1,\inf\int_{\beta_x}\rho\}$ is admissible for the p-capacity of the condenser $(\Omega; D_0, D_1)$, where the infimum is taken over all locally rectifiable curves β_x in Ω connecting D_0 and the point $x\in\Omega$. Moreover,

$$|\nabla u|\leq\rho\quad\text{almost everywhere in }\Omega.$$

This immediately implies the inequality

$$\mathrm{cap}_p(\Omega; D_0, D_1)\leq\int_{\mathbb{R}^n}|\nabla u|^p\,dx\leq\int_{\mathbb{R}^n}\rho^p\,dx\leq M_p(\Gamma),$$

by taking infimum on the right-hand side over all admissible functions ρ for the p-module. On the other hand, if u is an admissible $W^{1,p}$-function for the p-capacity of $(\Omega; D_0, D_1)$, then

$$\rho(x)=\begin{cases}|\nabla u(x)|, & x\in\Omega,\\ 0, & x\in\mathbb{R}^n\setminus\Omega\end{cases}$$

is an admissible function for the module $M_p(\Gamma)$ of the family of curves connecting D_0 and D_1, that implies the inequality

$$M_p(\Gamma) \leq \int_{\mathbb{R}^n} \rho^p \, dx = \int_{\mathbb{R}^n} |\nabla u|^p \, dx \leq \mathrm{cap}_p(\Omega; D_0, D_1)$$

upon taking infimum over all admissible functions u for the p-capacity.

Let us also mention a relation between the extremal functions ϖ_0, ρ_0 for the modules $M_p(\Gamma)$ and $M_q(\Sigma)$, and the extremal function u for the p-capacity of the condenser $(R_{ab}; S_a, S_b)$. It is well known that the variational equation for the problem of finding the p-capacity on a polarizable Carnot group G (and particularly in $\mathbb{R}^n$) is the p-sub-Laplacian equation, and the extremal function u for the p-capacity is a solution to the p-sub-Laplace equation in G with prescribed boundary values on D_0 and D_1. It was shown [2], that the function

$$\tilde{u}(g) = \begin{cases} c_p N_G^{\tau+1}, \ \tau+1 = \frac{p-Q}{p-1}, & \text{for } p \neq Q, \\ c_Q \log N_G, & \text{for } p = Q, \end{cases}$$

is a fundamental solution to the p-sub-Laplace equation on G for some appropriate choice of constants, see [6] for an analogous result on H-type groups. One can easily check that

$$u(g) = \frac{N_G^{\tau+1}(g) - a^{\tau+1}}{b^{\tau+1} - a^{\tau+1}}, \quad g \in R_{ab} \tag{10}$$

is extremal for the p-capacity of $(R_{ab}; S_a, S_b)$. In the case $p=Q$ we substitute $N_G^{\tau+1}$ by $\log N_G$.

3.3. Admissibility of ρ_0 for $M_q(\Sigma)$

In this section we will show that the function ρ_0 defined in (9) is admissible for a system E of Hausdorff measures H^{Q-1} associated with a family Σ of sets separating S_a and S_b in R_{ab}. The core idea of the proof is to show that if u is an extremal function for the p-capacity of $(R_{ab}; S_a, S_b)$, then $\|\nabla_0 u\|_0^{p-1}$ is an admissible function for $M_q(\Sigma)$ with $\frac{1}{p} + \frac{1}{q} = 1$. This method goes back to Gehring [15] who proved a similar result for $\mathbb{R}^3$, which was extended for the n-capacities and n-modules by Ziemer [47] in $\mathbb{R}^n$. Later, Shlyk [43] generalized the proof to $\mathbb{R}^n$ for arbitrary values of $p \neq n$. The same result was implicitly presented in [1] for $\mathbb{R}^n$, and in [28] for arbitrary Carnot groups. Here we want to follow the ideas of Gehring [15].

Theorem 18. *Let Σ be a family of sets separating S_a and S_b in R_{ab}, $\sigma\in\Sigma$, and let u be the extremal function for the p-capacity of $(R_{ab}; S_a, S_b)$. Let E be the family of $(Q-1)$-dimensional Hausdorff measures H^{Q-1} associated with Σ. Then the integral $\int_\sigma \|\nabla_0 u(g)\|_0^{p-1}\, dH^{Q-1}(g)$ exists for $M_q(E)$-almost all measures from E and*

$$\int_\sigma \|\nabla_0 u(g)\|_0^{p-1}\, dH^{Q-1}(g) \geq \operatorname{cap}_p(R_{ab}; S_a, S_b), \quad \frac{1}{p}+\frac{1}{q}=1. \tag{11}$$

The proof is preceded by two lemmas. Before we formulate the statement of the first lemma let us describe some constructions which we will use. Let $\sigma\in\Sigma$, and let $\beta>0$ be such that $\beta<\operatorname{dist}(\sigma,\partial R_{ab})$. We denote by $\sigma(\beta)=\{g\in R_{ab}\,|\operatorname{dist}(g,\sigma)<\beta\}$ and by $d(g)=\operatorname{dist}(g,\sigma)$. Here the distance is understood as $\operatorname{dist}(g,\eta)=N_G(\eta^{-1}g)$, $g,\eta\in G$. By construction, the norm $N_G=u_2^{\frac{1}{2-Q}}$ is smooth away from the identity of G due to the smoothness of the solution u_2. This guarantees that the function d is at least Lipschitz in G.

Lemma 19. *Let u be the extremal function for the p-capacity of the condenser $(R_{ab}; S_a, S_b)$, and let $\sigma\in\Sigma$. Then*

$$\int_{\sigma(\beta)} \|\nabla_0 u(g)\|_0^{p-1}\|\nabla_0 d(g)\|_0\, d\mathbf{g} \geq 2\beta \operatorname{cap}_p(R_{ab}; S_a, S_b).$$

Proof. Denote by $\overline{R}_{ab}$ the closure of R_{ab}, and by A^c the complement to A in G. Let F_0 be a component of $\sigma^c\cap\overline{R}_{ab}$ containing S_a, and let F_1 be a component of $\sigma^c\cap\overline{R}_{ab}$ containing S_b. Let

$$E_k=\{g\in\overline{R}_{ab}\,|\,0<\operatorname{dist}(g,F_k^c)<\beta\},\quad k=0,1.$$

Then $E_k\subset F_k$, $E_0\cup E_1\subset\sigma(\beta)$, and it is sufficient to show that

$$\int_{E_k} \|\nabla_0 u(g)\|_0^{p-1}\|\nabla_0 d(g)\|_0\, d\mathbf{g} \geq \beta \operatorname{cap}_p(R_{ab}; S_a, S_b),\quad k=0,1.$$

We focus ourselves only on the case $k=0$. The case $k=1$ is treated analogously. Define

$$v(g)=\min\{\beta,\operatorname{dist}(g,F_0^c)\}=\begin{cases}0, & \text{if } g\in F_0^c=F_1\cup\sigma,\\ \inf_{\eta\in F_0^c}\operatorname{dist}(g,\eta), & \text{if } g\in E_0,\\ \beta, & \text{if } g\in F_0\setminus E_0.\end{cases}$$

The function v is Lipschitz, from the class $L^p(R_{ab})$, and

$$\|\nabla_0 v(g)\|_0 = \begin{cases} \|\nabla_0 d(g)\|_0 > 0, & \text{almost everywhere in} \quad E_0, \\ 0, & \text{in} \quad R_{ab} \setminus E_0. \end{cases}$$

Thus, the function $w = v - \beta u$ is almost everywhere differentiable and belongs to the class $L^p(R_{ab})$. We use w as a test function on R_{ab} and obtain

$$0 = \int_{R_{ab}} \|\nabla_0 u(g)\|_0^{p-2} \langle \nabla_0 u, \nabla_0 w \rangle_0 \, d\mathbf{g} = \int_{R_{ab}} \|\nabla_0 u(g)\|_0^{p-2} \langle \nabla_0 u, \nabla_0 v - \beta \nabla_0 u \rangle_0 \, d\mathbf{g}.$$

This, together with the Cauchy-Schwarz inequality, implies

$$\begin{aligned} \int_{E_0} \|\nabla_0 u(g)\|_0^{p-1} \|\nabla_0 d(g)\|_0 \, d\mathbf{g} &= \int_{R_{ab}} \|\nabla_0 u(g)\|_0^{p-1} \|\nabla_0 v(g)\|_0 \, d\mathbf{g} \\ &\geq \int_{R_{ab}} \|\nabla_0 u(g)\|_0^{p-2} \langle \nabla_0 u(g), \nabla_0 v(g) \rangle_0 \, d\mathbf{g} \\ &= \beta \int_{R_{ab}} \|\nabla_0 u(g)\|_0^{p} \, d\mathbf{g} = \beta \, \mathrm{cap}_p(R_{ab}; S_a, S_b). \qquad \square \end{aligned}$$

If we were pass to the limit in

$$\frac{1}{2\beta} \int_{\sigma(\beta)} \|\nabla_0 u(g)\|_0^{p-1} \|\nabla_0 d(g)\| \, d\mathbf{g} \geq \mathrm{cap}_p(R_{ab}; S_a, S_b)$$

as $\beta \to 0$, we could finish the proof of Theorem 18 at once. In order to show that the limit exists, we consider the sequences of continuous functions $f_r(g)$ converging to $\|\nabla_0 u(g)\|_0^{p-1}$ $\mathbf{g}$-almost everywhere as $r \to 0$ and such that the limit

$$\frac{1}{2\beta} \int_{\sigma(\beta)} f_r(g) \|\nabla_0 d(g)\| \, d\mathbf{g} \to \int_{\sigma} f_r(g) \, dH^{Q-1} \quad \text{as } \beta \to 0$$

exists. We define the integral mean of $\|\nabla_0 u(g)\|_0^{p-1}$ in the ball by

$$f_r(g) = \frac{1}{\mathbf{g}(B(g,r))} \int_{B(g,r)} \|\nabla_0 u(\eta)\|_0^{p-1} \, d\mathbf{g}(\eta). \tag{12}$$

We also recall the co-area formula for the Carnot groups. Let U be a domain in G. Let $f \in L^1(U)$ be a non-negative function, and let v be a real valued Lipschitz function in U, see [18], [22], and [27]. Then

$$\int_U f(g) \|\nabla_0 v(g)\|_0 \, d\mathbf{g}(g) = \int_{-\infty}^{+\infty} \int_{v^{-1}(s)} f(\eta) \, dH^{Q-1}(\eta) \, ds. \tag{13}$$

Lemma 20. *The integral mean (12) satisfies the inequality*

$$\int_\sigma f_r(g)\,dH^{Q-1}\geq \operatorname{cap}_p(R_{ab};S_a,S_b),$$

whenever $r<\operatorname{dist}(\sigma,\partial R_{ab})$.

Proof. If $H^{Q-1}(\sigma)=\infty$, then there is nothing to prove and we can assume that $H^{Q-1}(\sigma)<\infty$. Let β, r be positive numbers such that $\beta+r<\operatorname{dist}(\sigma,\partial R_{ab})$. Let $L_\eta(\sigma)=\eta\sigma$ be the left translation of the set σ by an element $\eta\in G$. Then, changing variables and using Fubini's theorem we come to

$$\begin{aligned}\int_{\sigma(\beta)} & f_r(g)\|\nabla_0 d(g)\|_0\,d\mathbf{g}(g)\\ &=\frac{1}{\mathbf{g}(B(g,r))}\int_{B(0,r)}d\mathbf{g}(\eta)\int_{\sigma(\beta)}\|\nabla_0 u(\eta g)\|_0^{p-1}\|\nabla_0 d(g)\|_0\,d\mathbf{g}(g).\end{aligned}\tag{14}$$

Observe that $d(g)=\operatorname{dist}(g,\sigma)=\operatorname{dist}(\eta g,\eta\sigma)=\operatorname{dist}(\psi,\eta\sigma)$, with $\psi=\eta g$. Then, making change of variables $\psi=\eta g$, we write the last integral in the form

$$\begin{aligned}\frac{1}{\mathbf{g}(B(g,r))}&\int_{B(0,r)}d\mathbf{g}(\eta)\int_{\eta\sigma(\beta)}\|\nabla_0 u(\psi)\|_0^{p-1}\|\nabla_0\operatorname{dist}(\psi,\eta\sigma)\|_0\,d\mathbf{g}(\psi)\\ &=\frac{1}{\mathbf{g}(B(g,r))}\int_{B(0,r)}d\mathbf{g}(\eta)\int_{(\eta\sigma)(\beta)}\|\nabla_0 u(\psi)\|_0^{p-1}\|\nabla_0 d(\psi)\|_0\,d\mathbf{g}(\psi)\\ &\geq 2\beta\operatorname{cap}_p(R_{ab};S_a,S_b),\end{aligned}$$

where the last inequality follows from Lemma 19. Moreover, applying the co-area formula (13), we obtain

$$\int_{\sigma(\beta)}f_r(g)\|\nabla_0 d(g)\|_0\,d\mathbf{g}(g)=\int_0^\beta\int_{d^{-1}(s)}f_r(\zeta)dH^{Q-1}(\zeta)\,ds.$$

Let $F(s)$ denote the interior integral in the right-hand side,

$$F(s)=\int_{d^{-1}(s)}f_r(\zeta)\,dH^{Q-1}(\zeta),\quad \zeta\in d^{-1}(s).$$

Since the manifold structure of the group G coincides with $\mathbb{R}^m$ we can define polyhedral sets. If $\sigma\in\Sigma$ is the boundary ∂P of a polyhedral set P in G, then it is obvious that

$$F(s)=\int_{d^{-1}(s)}f_r(\zeta)\,dH^{Q-1}(\zeta)\to F(0)=2\int_{\partial P}f_r(\zeta)\,dH^{Q-1}(\zeta),\quad \text{as } s\to 0$$

since the function f_r is continuous on G. Moreover

$$\begin{aligned}\operatorname{cap}_p(R_{ab}; S_a, S_b) &\le \lim_{\beta\to 0} \frac{1}{2\beta} \int_{\partial P(\beta)} f_r(g) \|\nabla_0 d(g)\|_0 \, d\mathbf{g}(g) \\ &= \lim_{\beta\to 0} \frac{1}{2\beta} \int_0^\beta F(s)\, ds = \int_{\partial P} f_r(\zeta)\, H^{Q-1}(\zeta). \end{aligned} \tag{15}$$

Let now $\sigma\in\Sigma$ be an arbitrary set. Since $H^{Q-1}(\sigma)<\infty$ it is of finite G-perimeter, see [12], and can be approximated by polyhedral sets, see [33]. Let $U=G\setminus\sigma$ containing S_b. The set U is of finite G-perimeter and can be approximated by polyhedral sets $\{P_i\}$. There is an index i_0 such that $S_b\subset P_i$ and $S_a\subset\overline{R}_{ab}\setminus U$ for all $i\ge i_0$. Let $\partial^* U$ denote the reduced boundary of U, then the inclusions $\partial^*U\subset\partial U\subset\sigma$ imply

$$\begin{aligned}\int_\sigma f_r(\zeta)\, dH^{Q-1}(\zeta) \ge \int_{\partial^* U} f_r(\zeta)\, dH^{Q-1}(\zeta) = \lim_{i\to\infty} \int_{\partial P_i} f_r(\zeta)\, dH^{Q-1}(\zeta) \\ \ge \operatorname{cap}_p(R_{ab}; S_a, S_b)\end{aligned}$$

by (15), that finishes the proof. $\square$

Proof of Theorem 18. Let $\sigma\in\Sigma$. We can assume that for any $r<r_0<\operatorname{dist}(\partial F_0, \partial R_{ab})$, the support of f_r belongs to R_{ab}. Then,

$$f_r \to \|\nabla_0 u\|_0^{p-1} \quad \mathbf{g}\text{-almost everywhere as } r\to 0,$$

and $\int_{R_{ab}} f_r^q\, d\mathbf{g} \le \int_{R_{ab}} \|\nabla_0 u\|_0^p\, d\mathbf{g}<\infty$. The Lebesgue dominated convergence theorem implies that f_r converges to $\|\nabla_0 u\|_0^{p-1}$ in $L^q(R_{ab})$ as $r\to 0$. Therefore, there is a subsequence (that we will denote by the same symbol) f_r, such that

$$\int_\sigma \big| f_r - \|\nabla_0 u\|_0^{p-1} \big|\, dH^{Q-1} \to 0 \quad \text{as } r\to 0,$$

for $M_q(E)$-almost all measures $\mu\in E$ by Theorem 2, item (5). Thus, the integral $\int_\sigma \|\nabla_0 u\|_0^{p-1}\, dH^{Q-1}$ exists, and moreover, inequality (11) holds. $\square$

Corollary 21. *The function ρ_0 defined in (9) is admissible for the module $M_q(\Sigma)$ of the family of sets separating S_a and S_b in R_{ab}.*

Proof. As it was mentioned, the function (10) is extremal for $\operatorname{cap}_p(R_{ab}; S_a, S_b)$ and $\varpi_0=\|\nabla_0 u\|_0$ by (7). Therefore,

$$\operatorname{cap}_p(R_{ab}; S_a, S_b) = \int_{R_{ab}} \|\nabla_0 u\|_0^p\, d\mathbf{g} = \int_{R_{ab}} \varpi_0^p\, d\mathbf{g} = M_p(\Gamma),$$

where $\Gamma = \Gamma(R_{ab}; S_a, S_b)$ is the family of all locally rectifiable curves connecting S_a and S_b. Moreover, $\rho_0 = C_{ab}^{p-1}(p,Q) C_{S_1}^{-1}(p) \|\nabla_0 u\|_0^{p-1}$ as (9) shows. Thus, for any $\sigma \in \Sigma$, we obtain

$$\begin{aligned}\int_\sigma \rho_0 \, dH^{Q-1} &= C_{ab}^{p-1}(p,Q) C_{S_1}^{-1}(p) \int_\sigma \|\nabla_0 u\|_0^{p-1} \, dH^{Q-1} \\ &\geq C_{ab}^{p-1}(p,Q) C_{S_1}^{-1}(p) \ \operatorname{cap}_p(R_{ab}, S_a, S_b) \\ &\geq C_{ab}^{p-1}(p,Q) C_{S_1}^{-1}(p) \ M_p(\Gamma) = 1. \quad \square\end{aligned}$$

3.3.1. Historical overview

Löwner introduced 3-capacity in $\mathbb{R}^3$ in the late 50', see [26], and showed that $\operatorname{cap}_3(\Omega; D_0, D_1) > 0$. Gehring [15] proved that the Löwner 3-capacity (or the conformal capacity) for a ring domain in $\mathbb{R}^3$, coincides with the module $M_3(\Gamma)$ of a family of curves, which was calculated by Väisälä earlier in [45], and that it is also equal to the module $M_{3/2}(\Sigma)^{-2}$ of the family of compact piecewise smooth surfaces Σ separating D_0 and D_1 in a bounded domain $\Omega \subset \mathbb{R}^3$. The latter notion was used by Šabat [41] in his study of quasiconformal maps in $\mathbb{R}^3$. The restriction to smooth surfaces was relaxed in [25] provided that the admissible functions behave sufficiently nice. Later in 1966-68, Ziemer showed that the module $M_n(\Gamma)$ of a family of curves connecting D_0 and D_1 in a bounded domain $\Omega \subset \mathbb{R}^n$ is in the following relation with the module $M_{\frac{n}{n-1}}(\Sigma)$ of the family of sets separating D_0 and D_1 [47]:

$$(16) \qquad \big(M_n(\Gamma)\big)^{\frac{1}{n}} \big(M_{\frac{n}{n-1}}(E)\big)^{\frac{n-1}{n}} = 1.$$

In order to relax the conditions on admissible functions, the method of symmetrization in [14] and surface-theoretical approximation theorems permit to consider general separating sets, see [8]. Shlyk showed in [43], that for a rather general condenser $(\Omega; D_0, D_1)$ in $\mathbb{R}^n$, the equality (16) can be extended as follows

$$(17) \qquad \big(M_p(\Gamma)\big)^{\frac{1}{p}} \big(M_q(\Sigma)\big)^{\frac{1}{q}} = 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

For further interesting generalizations for modules in $\mathbb{R}^n$ see [1]. Some extensions to the Carnot groups can be found in [29]–[31], and for arbitrary metric measure spaces, for instance, in [42].

Ziemer proved [48], [49], that the capacity $\operatorname{cap}_p(\Omega; D_0, D_1)$ coincides with the module $M_p(\Gamma(\Omega; D_0, D_1))$ of the family of curves connecting D_0 and D_1 in Ω, where the domain $\Omega \subset \mathbb{R}^n$ is assumed to be bounded. Hesse [20] extended his result to unbounded domains. In particular, he showed that the set of the admissible functions for the p-module of a family of curves can be restricted from non-negative Borel measurable functions in $\mathbb{R}^n$ to lower semicontinuous L_p-functions in $\mathbb{R}^n$, which are continuous in Ω, provided that $D_0 \cup D_1 \subset \Omega$. Shlyk [44] generalized the result of Hesse from a connected open set (domain) Ω to an arbitrary open set in $\overline{\mathbb{R}}^n$.

4. Rodin's theorem for polarizable Carnot group

Let G be a polarizable group of topological dimension m, of homogeneous dimension Q, and let $\phi(s,\xi)$ be a solution to the Cauchy problem (3), where $\xi \in S_1$. Then the curves $\phi(s,\xi)$, $s \in [a,b]$ connect the spheres $S_a = \phi(a, S_1)$ and $S_b = \phi(b, S_1)$, and we denote the spherical ring domain by R_{ab} and the family of curves by Γ_0.

Theorem 22. *Let $f\colon G \to G$ be a C^1-smooth orientation preserving contact map, and let $c_\xi(s) = f(\phi(s,\xi))$. Set $1/p + 1/q = 1$, $p, q > 1$, and*

$$\ell(\xi) = \int_a^b \left(\frac{\|\dot{c}_\xi(t)\|_0}{J_f s^{Q-1}} \right)^q J_f s^{Q-1}\, ds, \quad \xi \in S_1.$$

Then

$$\rho_0(y) = \frac{1}{\ell(\xi)} \left(\frac{\|\dot{c}_\xi\|_0}{J_f s^{Q-1}} \right)^{\frac{1}{p-1}} \circ f^{-1}, \quad \phi(s,\xi) \in R_{ab},$$

$y = f(\phi(s,\xi)) \in R'_{ab} = f(R_{ab})$, is the extremal function for the p-module $M_p(f(\Gamma_0))$, which is calculated to be $M_p(f(\Gamma_0)) = \int_{R'_{ab}} \rho_0^p\, dy = \int_{S_1 \setminus \mathcal{Z}} \ell^{1-p}(\xi)\, dv(\xi)$.

Proof. Let us first observe that $\int_{c_\xi} \rho_0 = 1$. Indeed,

$$\begin{aligned}\int_{c_\xi} \rho_0 &= \int_a^b (\rho_0 \circ f)\|\dot{c}_\xi\|_0\, ds = \frac{1}{\ell} \int_a^b \left(\frac{\|\dot{c}_\xi\|_0}{J_f s^{Q-1}} \right)^{\frac{1}{p-1}} \|\dot{c}_\xi\|_0\, ds \\ &= \frac{1}{\ell} \int_a^b \left(\frac{\|\dot{c}_\xi\|_0}{J_f s^{Q-1}} \right)^q J_f s^{Q-1}\, ds = 1,\end{aligned}$$

for all $\xi \in S_1 \setminus \mathcal{Z}$. Therefore, ρ_0 is admissible for $f(\Gamma_0)$ and

$$M_p(f(\Gamma_0)) \le \int_{R'_{ab}} \rho_0^p\, d\mathbf{g}. \tag{18}$$

On the other hand, for any ρ admissible for $f(\Gamma_0)$ we have $\int_{c_\xi} \rho \ge 1$, and therefore,

$$\int_{c_\xi} (\rho - \rho_0) \ge 0.$$

This implies that

$$\frac{1}{\ell^{p-1}(\xi)}\int_a^b [(\rho-\rho_0)\circ f\,]\,\|\dot{c}_\xi\|_0\,ds\geq 0.$$

Then

$$\int_{S_1\setminus\mathcal{Z}}\int_a^b \big((\rho-\rho_0)\rho_0^{p-1}\circ f\big)J_f s^{Q-1}\,ds\,dv(\xi)\geq 0.$$

Equivalently,

$$\int_{R'_{ab}}\rho\rho_0^{p-1}\,d\mathbf{g}\geq\int_{R'_{ab}}\rho_0^p\,d\mathbf{g}.$$

The Hölder inequality yields

$$\left(\int_{R'_{ab}}\rho^p\,d\mathbf{g}\right)^{1/p}\left(\int_{R'_{ab}}\rho_0^{(p-1)q}\,d\mathbf{g}\right)^{1/q}\geq\int_{R'_{ab}}\rho\rho_0^{p-1}\,d\mathbf{g}\geq\int_{R'_{ab}}\rho_0^p\,d\mathbf{g},$$

or since $(p-1)q=p$,

$$\int_{R'_{ab}}\rho^p\,d\mathbf{g}\geq\int_{R'_{ab}}\rho_0^p\,d\mathbf{g}.$$

Taking the infimum in the above inequality over all admissible ρ we conclude that

$$M_p(f(\Gamma_0))\geq\int_{R'_{ab}}\rho_0^p\,d\mathbf{g}. \tag{19}$$

Comparing (18) and (19) we see that the function ρ_0 is extremal for the module $M_p(f(\Gamma_0))$. Now we can calculate the p-module as

$$\begin{aligned}M_p(f(\Gamma_0))&=\int_{R'_{ab}}\rho_0^p\,d\mathbf{g}=\int_{S^1\setminus\mathcal{Z}}\int_a^b[\rho_0^p\circ f]\,J_f s^{Q-1}\,ds\,dv(\xi)\\&=\int_{S_1\setminus\mathcal{Z}}\int_a^b\frac{1}{\ell^p(\xi)}\left(\frac{\|\dot{c}_\xi\|_0}{J_f s^{Q-1}}\right)^{\frac{p}{p-1}}J_f s^{Q-1}\,ds\,dv(\xi)\\&=\int_{S_1\setminus\mathcal{Z}}\ell^{1-p}(\xi)\,dv(\xi).\qquad\square\end{aligned}$$

Remark 23. Let $\mathbf{u}\colon D\to G$, $\theta\mapsto\xi=u(\theta)$, be a parametrization of the unit sphere S_1 on a rectangular $D\subset\mathbb{R}^{m-1}$ and let $J_{\mathbf{u}}$ be the Jacobian. Then by making use of the change of variables we obtain

$$M_p(f(\Gamma_0))=\int_{S_1\setminus\mathcal{Z}}\ell^{1-p}(\xi)\,dv(\xi)=\int_D\ell^{1-p}(u(\theta))J_{\mathbf{u}}\,d\theta.$$

The classical Rodin's theorem [38, Theorem 14] corresponds to the case $p=n=2$ and a rectangular domain Ω in the plane. Rodin's theorem provides an estimate from below of the module of a family of curves by the module of its subfamily, which can be used in a several ways. For example, it was used by Rodin and Warschawski to characterize the boundary behavior of conformal maps, see, e.g., [39] and [40]. On the other hand, such estimates were used in $\mathbb{R}^2$ to obtain conditions under which μ-homeomorphisms of the half-plane possess homeomorphic extension to the boundary, see [3], [16], and [17].

Rodin's theorem for $G=\mathbb{R}^n$ was studied at [4] and it was proved for a wider class of condensers. In particular,

- $\Omega=[a,b]\times D$ is a cylinder in $\mathbb{R}^n$, where D is a compact set in $\mathbb{R}^{n-1}$, $D_0=\{a\}\times D$, $D_1=\{b\}\times D$, and $\mathbf{u}\equiv\mathrm{id}$.
- A spherical ring domain $\Omega=R_{ab}$ in $\mathbb{R}^n$ bounded by the concentric spheres S_a and S_b of radii a and b respectively. Then $S_1=\mathbf{u}(D)$ is the unit sphere in $\mathbb{R}^n$, $S_{ab}\ni s\xi=s\mathbf{u}(\theta)$, $\xi\in S_1$.
- A conical cylinder $\Omega=\{(\beta tx,t): x\in D\subset\mathbb{R}^{n-1},\ t\in[a,b],\ \beta>0\}$.

Various possibilities of the regularity assumptions for the homeomorphism f were also considered in [4]. For instance, f can be assumed to be a $W^{1,p}$-Sobolev homeomorphism, $p\geq 1$, and such that the Jacobian $J_f>0$ almost everywhere in Ω. Another option is to consider quasiconformal homeomorphisms f, which are $W^{1,n}$-Sobolev and of bounded distortion, see details in [4].

Further, we present a couple of examples.

Example 24. Let us start with a spherical ring $R_{1r}\subset\mathbb{R}^n$ bounded by the unit sphere S_1 and a sphere S_r of radius r, $1<r<\pi$. Recall that R_{1r} is given by the spherical transformation $\mathbf{u}\colon (\theta_1,\dots,\theta_{n-1},s)\to(\xi,s)=(x_1,\dots,x_n)$, where $\theta_1\in[0,2\pi)$, $\theta_k\in[0,\pi]$, $k=2,3,\dots,n-1$, $s\in[1,r]$, and

$$
\begin{aligned}
x_1 &= s\sin\theta_1\sin\theta_2\dots\sin\theta_{n-1},\\
x_2 &= s\cos\theta_1\sin\theta_2\dots\sin\theta_{n-1},\\
x_3 &= s\cos\theta_2\sin\theta_3\dots\sin\theta_{n-1},\\
&\ \ \vdots\\
x_n &= s\cos\theta_{n-1}.
\end{aligned}
$$

The Jacobian $J_{\mathbf{u}}$ of the spherical transformation $\mathbf{u}$ is

$$J_{\mathbf{u}}(\theta_1,\dots,\theta_{n-1},s)=s^{n-1}\omega=s^{n-1}\sin\theta_2\sin^2\theta_3\dots\sin^{n-2}\theta_{n-1}.$$

Let us define the twisting map $f\colon R_{1r}\to R_{1r}$ by the shear transform

$$\mathbf{u}^{-1}\circ f\circ\mathbf{u}\colon(\theta_1,\theta_2,...,\theta_{n-1},s)\to((\theta_1+s-1),\theta_2,...,\theta_{n-1},s),$$

i.e., the boundary sphere S_1 remains unchanged while the spheres S_s rotate by an angle $s-1$, $s\in(1,r]$ by the element of $\mathrm{SO}(n)$

$$\begin{pmatrix} \cos(s-1) & \sin(s-1) & 0 & 0 & \dots & 0\\ -\sin(s-1) & \cos(s-1) & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ & & & \ddots & & \\ \dots & \dots & \dots & & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Observe that $J_f=J_{\mathbf{u}^{-1}\circ f\circ\mathbf{u}}=1$, and the radial intervals from Γ_0 are mapped onto the curves $c(s)$ with $|\dot c|=\sqrt{1+s^2}$. Theorem 22 implies that

$$\ell(\xi)=\int_1^r\left(1+s^2\right)^{q/2}s^{(n-1)(1-q)}\,ds.$$

As we see the value of the integral does not depend on $\xi\in S_1$ and we denote it by K. The p-module of $f(\Gamma_0)$ is

$$M_p(f(\Gamma_0))=\int_{S_1}\ell^{1-p}\,dH^{n-1}=K^{1-p}\int_{S_1}dH^{n-1}=K^{1-p}\omega(S_1).$$

Since $f(\Gamma_0)$ is a subfamily of the family $\Gamma=\Gamma(S_{ab};S_a,S_b)$, then $M_p(f(\Gamma_0))\leq M_p(\Gamma)$. Taking into account that

$$M_p(\Gamma)=\begin{cases}(\frac{p-1}{|p-n|})^{1-p}|b^{\frac{p-n}{p-1}}-a^{\frac{p-n}{p-1}}|^{1-p}\omega(S_1), & \text{for } p\neq n,\\ (\log b/a)^{1-n}\omega(S_1), & \text{for } p=n,\end{cases}$$

we obtain correct inequalities

$$\int_1^r\left(1+t^2\right)^{\frac{p}{2(p-1)}}t^{\frac{n-1}{1-p}}\,dt\geq\frac{p-1}{|p-n|}\left(r^{\frac{p-n}{p-1}}-1\right),\quad p>1,\ p\neq n,$$

and

$$\int_1^r\frac{\left(1+t^2\right)^{\frac{n}{2(n-1)}}}{t}\,dt\geq\log r,\quad p=n\geq 2.$$

Example 25. In this example we consider the spherical ring domain $R_{1b}=\{(x_1,x_2,t)\colon 1\leq N_{\mathbb{H}}(x_1,x_2,t)\leq b\}$ in the Heisenberg group $\mathbb{H}=\mathbb{H}^1$ with respect to the homogeneous norm $N_{\mathbb{H}}$. For $(x_1,x_2,t)\in S_{ab}$ we write $(x_1,x_2,t)=\phi(\xi,s)$, $\xi\in S_1$, where $\xi=\mathbf{u}(\theta,\alpha)$ and $\theta\in[0,2\pi)$, $\alpha\in(-\pi/2,\pi/2)$. Thus the spherical coordinates are given by

$$x_1=s\sqrt{\cos\alpha}\cos\theta,\quad x_2=s\sqrt{\cos\alpha}\sin\theta,\quad \text{and}\quad t=s^2\sin\alpha, \tag{20}$$

$\theta\in[0,2\pi)$, $\alpha\in(-\pi/2,\pi/2)$, $s\in[1,b]$. The horizontal vector fields in the polar form are

$$X_1=\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial t}=\sqrt{\cos\alpha}\left(\cos(\theta-\alpha)\frac{\partial}{\partial s}+\frac{2}{s}\sin(\theta-\alpha)\frac{\partial}{\partial\alpha}-\frac{\sin\theta}{s\cos\alpha}\frac{\partial}{\partial\theta}\right),$$
$$X_2=\frac{\partial}{\partial x_2}-2x_1\frac{\partial}{\partial t}=\sqrt{\cos\alpha}\left(\sin(\theta-\alpha)\frac{\partial}{\partial s}-\frac{2}{s}\cos(\theta-\alpha)\frac{\partial}{\partial\alpha}+\frac{\cos\theta}{s\cos\alpha}\frac{\partial}{\partial\theta}\right).$$

The horizontal norm of the horizontal gradient of a smooth function $f(\theta,\alpha,s)$ is calculated as $\|\nabla_0 f\|_0=(X_1^2(f)+X_2^2(f))^{1/2}$. The 3-dimensional Hausdorff measure element on the sphere $S_s\setminus\mathcal{Z}$, written in polar coordinates is $d\mu_s=s^3\sqrt{\cos\alpha}\,d\alpha d\theta$. In particular, the area of the unit sphere S_1 is calculated as

$$\operatorname{Area}(S_1)=4\sqrt{2\pi}\;\Gamma^2\left(\frac{3}{4}\right).$$

The radial flow $\phi(s,\xi)=\phi(s,\theta,\alpha)$ in $\mathbb{H}$ orthogonal to the sphere $S_1\setminus\mathcal{Z}$ is a solution to the initial-value problem (3) in the particular case of the homogeneous Heisenberg norm $N_{\mathbb{H}}$, given by

$$x_1(s)=s\sqrt{\cos\alpha}\cos\left(\theta-\tan\alpha\,\log s\right),$$
$$x_2(s)=s\sqrt{\cos\alpha}\sin\left(\theta-\tan\alpha\,\log s\right),$$
$$t(s)=s^2\sin\alpha,$$

where $1\leq s\leq b$, and $\theta\in[0,2\pi)$, $\alpha\in(-\pi/2,\pi/2)$ are fixed. The horizontal norm of $\dot\phi(s,\theta,\alpha)=\frac{\partial}{\partial s}\phi(s,\theta,\alpha)$ is $\|\dot\phi(s,\theta,\alpha)\|_0=\cos^{-1/2}\alpha$.

An analogue to Theorem 22 and Remark 23 can be formulated for the spherical ring domain in $\mathbb{H}$ as follows. Let Γ_0 denote the family of curves $\phi_{\theta\alpha}(\cdot)\colon[1,b]\to\mathbb{H}$ given by radial flow $\phi(s,\theta,\alpha)$ for every fixed $\alpha\in(-\pi/2,\pi/2)$, $\theta\in[0,2\pi)$. In order to preserve the horizontal nature of the families of curves we require from a smooth map $f\colon\mathbb{H}\to\mathbb{H}$ to be the contact map, that is a map whose differential preserves the horizontal planes $\operatorname{span}\{X_1(g),X_2(g)\}$ for all $g\in\mathbb{H}$, see, for instance, [23].

Proposition 26. *Let $f\colon \mathbb{H}\to\mathbb{H}$ be a C^1-smooth orientation preserving contact map, and let $c_{\theta\alpha}(s)=f(\phi(s,\theta,\alpha))$. Set $1/p+1/q=1$, $p,q>1$, and*

$$\ell(\theta,\alpha)=\int_1^b\left(\frac{\|\dot c_{\theta\alpha}\|_0}{J_f s^3\sqrt{\cos\alpha}}\right)^q J_f s^3\sqrt{\cos\alpha}\,dr,\quad \alpha\in[-\pi/2,\pi/2],\ \theta\in[0,2\pi).$$

Then

$$\rho_0(y)=\frac{1}{\ell(\theta,\alpha)}\left(\frac{\|\dot c_{\theta\alpha}\|_0}{J_f s^3\sqrt{\cos\alpha}}\right)^{\frac{1}{p-1}}\circ f^{-1},\quad \alpha\in(-\pi/2,\pi/2),\ \theta\in[0,2\pi),$$

$y=f(x_1,x_2,t)=f(\phi(\theta,\alpha,s))\in R'_{1b}=f(R_{1b})$, is the extremal function for the p-module $M_p(f(\Gamma_0))$ which is calculated as

$$M_p(f(\Gamma_0))=\int_{R'_{1b}}\rho_0^p\,dy=\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\ell^{1-p}(\theta,\alpha)\,d\theta\,d\alpha.$$

Let us now calculate the p-module of $f(\Gamma_0)$, where f is a contact C^1-smooth orientation preserving map. If we try to create a twisting map similarly to Example 24 in the spherical coordinates (20) written as

$$\mathbf{u}^{-1}\circ f\circ\mathbf{u}\colon(\theta,\alpha,s)\to(\theta+\omega(s),\alpha,s),\quad \omega(1)=0,$$

i.e, the boundary sphere S_1 remains unchanged while the spheres S_s rotate to the angle $\omega(s)$, $s\in(1,b]$, then the condition of horizontality for the curves $f(\phi_{\theta\alpha}(\cdot))$ is quite rigid, which leads us to $\omega(s)\equiv 0$. Let us try to modify the twisting map by

$$\mathbf{u}^{-1}\circ f\circ\mathbf{u}\colon(\theta,\alpha,s)\to(\theta+\tan\alpha\log s+\omega_1(s),\alpha+\omega_2(s),s),\quad \omega_1(1)=\omega_2(1)=0.$$

Then the image $c_{\theta\alpha}(s)=f(\phi_{\theta\alpha}(s))$ is written in coordinates as

$$\begin{aligned}
x_1(s)&=s\sqrt{|\cos(\alpha+\omega_2(s))|}\cos(\theta+\omega_1(s)),\\
x_2(s)&=s\sqrt{|\cos(\alpha+\omega_2(s))|}\sin(\theta+\omega_1(s)),\\
t(s)&=s^2\sin(\alpha+\omega_2(s)).
\end{aligned}$$

The horizontality condition $\dot t=2(\dot x_1x_2-\dot x_2x_1)$ is equivalent to

$$\dot\omega_1=-\frac{1}{2}\dot\omega_2-\frac{1}{s}\tan(\alpha+\omega_2).$$

For example,

$$\omega_2(s)=s-1\quad\text{and}\quad \omega_1(s)=\frac{1-s}{2}-\int_1^s\frac{\tan(\alpha+\tau-1)}{\tau}\,d\tau.$$

Then $J_f = 1$ and

$$\|\dot{c}_{\theta\alpha}(s)\|_0 = \frac{1}{2}\sqrt{\frac{4+s^2}{\cos(\alpha+s-1)}}.$$

In Proposition 26 we calculate

$$\ell(\theta,\alpha) = \ell(\alpha) = \int_1^b \left(\frac{4+s^2}{4\cos(\alpha+s-1)}\right)^{q/2} s^{3-q} (\cos\alpha)^{\frac{1}{2}(1-q)}\, ds.$$

and the p-module of $f(\Gamma_0)$ is

$$M_p(f(\Gamma_0)) = 2\pi \int_{-\pi/2}^{\pi/2} \ell^{1-p}(\alpha)\, d\alpha.$$

Unfortunately, the integral inequality which follows from the monotonicity of the module $M_p(f(\Gamma_0)) \le M_p(\Gamma_0) = M_p(\Gamma)$ is quite difficult, and there is very little hope to obtain simple inequalities as in Example 24.

Let us mention here works [36] and [37], where the author used the techniques of the module of a family of surfaces to solve an extremal problem in some class of contact quasiconformal maps between ring domains in the Heisenberg group.

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Modules of systems of measures on polarizable Carnot groups

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