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On certain asymptotic solutions of Riccati's differential equation

By HERBERT OLSSON

1. Discussing Schrödinger's wave equation G. WENTZEL[1] has shown a method, the so-called W B K-method (Wentzel, Brillouin, Kramers) for calculating solutions of Riccati's differential equation. Later on R. E. LANGER[2] discussing asymptotic solutions of ordinary differentilal equations of the second order expresses certain critical remarks concerning the W B K-method and establishes that the procedure is of only formal and minor significance.

As the W B K-method forms a very simple procedure for calculating solutions in the case that the analysis converges it seems to be of some interest to prove that if the series thus calculated diverges, on certain conditions they can represent asymptotic solutions and thus contrary to LANGER's assumption the W B K-method in certain cases can be used even when the series calculated in this way diverges.

2. Consider the differential equation

$$y'' - (\varrho^2 f^2(x) + g(x))\, y = 0 \tag{1}$$

where ϱ is a parameter independent of x, capable of assuming indefinitely great real values and suppose that $f^2(x)$ and $g(x)$ are real and that $f^2(x)$ is bounded from zero when

$$a \leq x \leq b.$$

If the value

$$z(x, \varrho) = \frac{y'(x, \varrho)}{\varrho \cdot y(x, \varrho)}$$

should approach a fixed limit when $\varrho \to \infty$ it is necessary that $y(x, \varrho)$ does not vanish if ϱ is large enough. For assume that, when $\varrho > \varrho_0$, it should be found such values of ϱ, say ϱ_1, so that for some x_1 of the interval $a - b$ an integral $y(x, \varrho)$ and $y'(x, \varrho)$ vanish at the same time, we obtain

$$z = \lim_{x \to x_1} \frac{y'(x, \varrho_1)}{\varrho_1 y(x, \varrho_1)} = \lim_{x \to x_1} \frac{y''(x, \varrho_1)}{\varrho_1 y'(x, \varrho_1)}$$

[1] Eine Verallgemeinerung der Quantenbedingungen für die Zwecke der Wellenmechanik, Zeitschrift der Physik 38 (1926) p. 518.

[2] The asymptotic solutions of ordinary differential equations of the second order, American Mathematical Society 2. 40. 1934 p. 545 spec. p. 551.

and if

$$\frac{y'}{\varrho_1 y} \to a$$

when

$$\varrho_1 \to \infty$$

we obtain that

$$\frac{y''}{\varrho_1 y'} = \left(f^2(x) + \frac{g(x)}{\varrho_1^2}\right) \cdot \frac{\varrho_1 y}{y'} \to \frac{1}{a} f^2(x)$$

and thus

$$a = \pm f(x)$$

and thus z cannot approach a fixed limit when $\varrho \to \infty$ for all values of x of the interval $a - b$.

Thus when $\varrho > \varrho_0$, z must be regular and finite and we have to consider solutions of the equation (1) which, when

$$\varrho > \varrho_0$$

where ϱ_0 is large enough, are positive all over the interval $a - b$.

By the condition

$$y(a, \varrho) = y_0 \quad \text{and} \quad y'(a, \varrho) = y_0' \tag{2}$$

we suppose that a positive integral $y(x, \varrho)$ is defined and that $\eta(x, \varrho_1)$ defines an integral of (1) where

$$\varrho_1 > \varrho > \varrho_0$$

which satisfies the same conditions (2) as $y(x, \varrho)$. Hence we have

$$\eta(x, \varrho_1) > y(x, \varrho)$$

when $x > a$ or because $y(x, \varrho)$ is a positive integral the solution $\eta(x, \varrho_1)$ cannot vanish over the interval $a - b$ for any values $\varrho_1 > \varrho$.

Thus considering such solutions of (1) which do not vanish on $a - b$ we know that the following possibilities can exist

$$\lim_{\varrho \to \infty} \frac{y'(x, \varrho)}{\varrho\, y(x, \varrho)} = \pm f(x). \tag{3}$$

3. The differential equation (1) transforms by[1]

$$y = e^{\varrho \int_{a_1}^{x} u(x, \varrho)\, dx} \tag{4}$$

into Riccati's equation

$$\varrho u' + \varrho^2 u^2 - \varrho^2 f^2 - g = 0 \tag{5}$$

and thus if $u(x, \varrho)$ is a solution of (5) we know from above that the following possibilities can exist

[1] Where a_1 is any constant.

$$1)\ 2)\ \lim_{\varrho \to \infty} u(x, \varrho) = \pm f(x)$$

$$3)\ \lim_{\varrho \to \infty} u(x, \varrho) \quad \text{cannot approach a fixed limit}$$

$$4)\ \lim_{\varrho \to \infty} u(x, \varrho) = \infty.$$

Supposing now that

$$u(x, \varrho) = u_0(x) + z_0(x, \varrho)$$

the function z_0 satisfies Riccati's equation

$$\varrho z_0^1 + \varrho^2 z_0^2 + 2\varrho^2 z_0 u_0 + \varrho u_0^1 - g = 0$$

by choosing $u_0(x)$ so that

$$u_0(x) = \pm f(x).$$

Proceeding in this way $u(x, \varrho)$ can be written in the form

$$u(x, \varrho) = \sum_{\nu=0}^{n} u_\nu(x) \cdot \frac{1}{\varrho^\nu} + z_n(x, \varrho) \tag{6}$$

where z_n is a function of x and ϱ and the coefficients $u_\nu(x)$ can be calculated in succession to

$$u_1 = -\frac{u_0^1}{2u_0}, \qquad u_2 = -\frac{u_1^1 + u_1^2 - g}{2u_0}$$

and

$$u_{\nu+1} = -\frac{1}{2u_0}\left(u_\nu^1 + \sum_{p=1}^{\nu} u_p \cdot u_{\nu+1-p}\right)$$

if $\nu \geq 2$, which calculation is identical with the procedure of the W B K-method.

The function z_n then satisfies Riccati's equation

$$\varrho z_n^1 + \varrho^2 z_n^2 + 2\varrho^2 z_n \sum_{\nu=0}^{n} u_\nu \frac{1}{\varrho_\nu} + \frac{u_{n-1}^2}{\varrho^{n-1}} + \frac{u_n^2}{\varrho^{n+1}} + \cdots + \frac{u_n^1}{\varrho^{n-1}} = 0 \tag{7}$$

and for z_n the following possibilities can exist

$$1)\ \lim_{\varrho \to \infty} z_n(x, \varrho) = 0$$

$$2)\ \lim_{\varrho \to \infty} z_n(x, \varrho) \quad \text{cannot approach a fixed limit}$$

$$3)\ \lim_{\varrho \to \infty} z_n(x, \varrho) = \infty.$$

4. Now let

$$\varrho^{n-1} z_n = v_n \tag{8}$$

the equation (7) then can be written

$$v_n^1 + \frac{v_n^2}{\varrho^{n-2}} + 2\varrho v_n \sum_{\nu=0}^{n} u_\nu \frac{1}{\varrho^\nu} + \frac{u_{n-1}^2}{\varrho} + \frac{u_n^2}{\varrho^3} + \cdots + \frac{u_n^1}{\varrho^2} = 0. \tag{9}$$

In equation (9) we write

$$\sum_{\nu=1}^{n} u_\nu \frac{1}{\varrho^\nu} + \frac{v_n}{2\varrho^{n-1}} = \varepsilon_n(x, \varrho)$$

and

$$\frac{u_{n-1}^2}{\varrho} + \frac{u_n^2}{\varrho^3} + \cdots\cdots\cdots + \frac{u_n^1}{\varrho} = \delta_n(x, \varrho).$$

Now considering the first possibility. For every fixed integral value of $n \geq 1$ and any x of the interval $a - b$ we have

$$\lim_{\varrho \to \infty} \varepsilon_n(x, \varrho) = 0$$

and

$$\lim_{\varrho \to \infty} \delta_n(x, \varrho) = 0$$

and thus equation (9) can be written

$$v_n^1 + 2\varrho v_n (u_0 + \varepsilon_n(x, \varrho)) + \delta_n(x, \varrho) = 0. \tag{9 a}$$

Writing

$$v^1 + 2\varrho v (u_0 + \varepsilon_n(x, \varrho)) = 0 \tag{10}$$

we know that the equations (9 a) and (10) at every point x of the interval $a - b$ for every fixed value n have solutions $v_n(x, \varrho)$ and $v(x, \varrho)$ and if these solutions satisfy the same initial conditions we have

$$|v(x, \varrho) - v_n(x, \varrho)| < \frac{\varepsilon}{2}$$

where ε is any positive number provided that ϱ is large enough.

From equation (10) we have

$$v(x, \varrho) = c(\varrho)\, e^{-2\varrho \int_a^x (u_0 + \varepsilon_n(x, \varrho))\, dx}$$

where

$$|c(\varrho)| < k \varrho^{n-1}$$

and if the integral

$$I_1 = \int_a^x u_0(x)\, dx$$

converges, and there is no restriction in the supposition that both the integral I_1 and ϱ are positive, we also have

$$|v(x, \varrho)| < \frac{\varepsilon}{2}$$

and thus

$$|v_n(x, \varrho)| < \varepsilon$$

and we can write

$$z_n(x, \varrho) = \frac{1}{\varrho^{n-1}} \varepsilon_n^1(x, \varrho)$$

where

$$\lim_{\varrho \to \infty} \varepsilon_n^1(x, \varrho) = 0.$$

5. By using the W B K-method we thus have shown the following theorem: *If $f^2(x)$ is bounded from zero on $a - b$ and the integral*

$$I_1 = \int_a^x f(x)\, dx$$

converges then the equation (5) *has asymptotic solutions which have the forms*

$$u(x, \varrho) = \pm f(x) + \frac{u_1(x)}{\varrho} + \frac{u_2(x)}{\varrho^2} + \cdots\cdots + \frac{u_n(x) + \varepsilon_n(x, \varrho)}{\varrho^n}$$

over the interval $a - b$.

6. Now writing

$$u(x, \varrho) = \pm f(x) + \frac{u_1(x)}{\varrho} + \frac{u_2(x) + \varepsilon_2(x, \varrho)}{\varrho^2}$$

from equation (4) we have

$$y(x, \varrho) = |f(x)|^{-\frac{1}{2}} e^{\pm \varrho \int_{a_1}^{x} f(x)\, dx} \cdot e^{\frac{1}{\varrho} \int_{a_1}^{x} (u_2(x) + \varepsilon_2(x, \varrho))\, dx}$$

If

$$I_2 = \int_a^x u_2(x)\, dx$$

converges, that is if the integral

$$I_2 = \int_a^x \left(\frac{g}{f} + \frac{f''}{2 f^2} - \frac{3 f'^2}{4 f^3} \right) dx$$

converges when $a \leq x \leq b$ we thus have the known theorem:[1]

If $f^2(x)$ is bounded from zero on $a - b$ and the integrals I_1 and I_2 converge then the equation (1) *has asymptotic solutions*

$$y(x) = |f(x)|^{-\frac{1}{2}} e^{\pm \varrho \int_{a}^{x} f(x)\, dx} \left(1 + \frac{a_1(x)}{\varrho} + \frac{a_2(x)}{\varrho^2} + \cdots + \frac{a_n(x) + \varepsilon_n(x, \varrho)}{\varrho^n} \right)$$

over the interval $a - b$.

[1] Loc. cit. LANGER p. 550.