Communicated 12 January 1949 by H. Cramér

Stochastic processes and integral equations

By Ulf Grenander

1. In the following we shall consider stochastic processes, i. e. real functions $x(t, \omega)$ of the two arguments t and ω. t will denote a real parameter in a finite or infinite interval T. ω will be a point in an abstract space Ω, on which is defined a measure of probability in the usual way. For fixed $t \in T$, $x(t, \omega)$ is supposed to be measurable in Ω and the following integrals existing

$$\begin{cases} m(t) = \int_{\Omega} x(t, \omega)\, d P(\omega) = E x(t) \\ r(s, t) = E[x(s) - m(s)][x(t) - m(t)]. \end{cases}$$

Moreover, we suppose that $x(t)$ is continuous in the mean, i. e.

$$E[x(t+h) - x(t)]^2 \to 0$$

for every t as $h \to 0$. Let $T = (a, b)$! Suppose that we know $r(s, t)$ and that $m(t)$ is an unknown constant $= m$, which is to be estimated. Under certain assumptions it can be shown that one is led to consider estimates of the form

$$m^* = \int_a^b x(t) f(t)\, d t$$

where $f(t)$ is quadratically integrable and the integral is taken in the sense of Karhunen (Über lineare Methoden in der Wahrscheinlichkeitsrechnung, Ann Ac Sci Fenn, series A, I, 37, Helsinki 1947). To determine the best estimate of this form we demand that m^* will be unbiased and of minimum variance, i. e.

$$\begin{cases} E m^* = m \\ E[m^* - m]^2 = min. \end{cases}$$

But this implies

$$\begin{cases} \int_a^b f(t)\, d t = 1 \\ I = \int_a^b \int_a^b r(s, t) f(s) f(t)\, d s\, d t = min. \end{cases}$$

As $\int\int r(s,t) f(s) f(t)\, d s\, d t = E\,[\int [x(t) - m(t)]\; f(t)\, d t]^2 \geq 0$, the kernel $r(s,t)$ has positive eigen-values, so that we can apply the theorem of Mercer and obtain

$$r(s,t) = \sum_1^\infty \frac{\varphi_\nu(s)\,\varphi_\nu(t)}{\lambda_\nu}$$

with the characteristic elements λ and φ to the integral equation

$$\varphi(s) = \lambda \int r(s,t)\,\varphi(t)\,d t.$$

Putting $a_\nu = \int \varphi_\nu(t)\,d t$ and $c_\nu = \int f(t)\,\varphi_\nu(t)\,d t$ we get the conditions

$$\sum_1^\infty c_\nu a_\nu = 1, \quad \sum_1^\infty c_\nu^2 < \infty, \quad \sum_1^\infty \frac{c_\nu^2}{\lambda_\nu} = m\,i\,n.$$

In order to get convergent series we must separate several cases. We state only the main result obtained by applying the Schwarz inequality:

The estimates

$$m_n^* = \frac{1}{\sum_1^n a_\nu^2 \lambda_\nu} \sum_1^n a_\nu \lambda_\nu \int_a^b x(t)\,\varphi_\nu(t)\,d t; \qquad n = 1, 2, \ldots$$

satisfy the relations

$$E\,m_n^* = m; \qquad E\,[m_n^* - m]^2 \downarrow \text{G. L. B.}\, I \ as\ n \to \infty.$$

When m_n^* converges in the mean, its limit is unbiased and of minimum variance.

2. There are obvious generalizations of the above. But to treat the general problem of unbiased estimation we must use another method. Ω may be an N-dimensional Euclidean space, corresponding to an N-dimensional stochastic variable, or it may be some appropriate subset of all real functions, corresponding to a stochastic process. On Ω is defined a measure of probability $P(\omega, \theta)$ depending upon a real parameter θ, $a \leq \theta \leq b$. To each θ corresponds the Hilbert space $L_2(\Omega, \theta)$ of quadratically integrable functions with respect to P_θ and with the usual quadratic metric.

Consider the operation

$$T_\theta\, g = \int_\Omega g(\omega)\, d\,P(\omega, \theta); \qquad g \in L_2(\Omega, \theta).$$

Let the following conditions be satisfied:

A $L_2(\Omega, \theta)$ shall be the same for all θ (consisting of the same elements, but usually with different metric) and the topological structure shall be independent of θ.

B T_θ shall transform $L_2(\Omega)$ into $L_2(\theta;\, a, b)$.

C The functionals T_θ shall be of uniformly bounded norm

$$|\,T_\theta\, g\,| \leq K \cdot \|g\|_{\theta = a}; \qquad a \leq \theta \leq b.$$

It is possible to show that for the validity of A—C the following simpler conditions are sufficient. Let

$$I = \{a_i < x(t) < b_i; \quad i = 1, 2, \ldots n\}$$

be an arbitrary finite-dimensional interval and demand that for every I:

a. $P(I, \theta)$ shall be continuous in θ.

b. $P(I, \theta') \leq C \cdot P(I, \theta'')$ for every θ' and θ''.

Under the assumptions A—C it can be proved, using an important theorem of Hilbert concerning completely continuous infinite quadratic forms that there exist two ortho-normal systems $\{\psi_\nu(\theta)\}$ and $\{\varphi_\nu(\omega)\}$ and a sequence of numbers $\{\lambda_\nu\}$ so that

$$\psi_\nu(\theta) = \lambda_\nu \int_\Omega \varphi_\nu(\omega)\, d\, P(\omega, \theta).$$

Now again we must distinguish between various cases. Let us suppose, e. g., that $\{\psi_\nu\}$ is complete! Then the following theorem is easily obtained:

Theorem I. *Under the said conditions it is necessary and sufficient for the existence of an unbiased estimate $\theta^* \in L_2(\Omega)$ of θ that*

$$\sum \lambda_\nu^2 c_\nu^2 < \infty \quad \textit{where} \quad c_\nu = \int_a^b \theta\, \psi_\nu(\theta)\, d\, \theta.$$

This is, of course, connected with the theorem of Picard for usual integral equations.

3. We shall now treat a different problem with similar methods. Let σ be a bounded measure on the real axis, and $x(t)$ a stochastic process with $|r(s, t)| < K$. Then studying the integral equation

$$z(\omega) = \lambda \int_{-\infty}^{\infty} x(t)\, f(t)\, d\, \sigma(t); \quad z \in L_2(x); \quad f \in L_2(\sigma)$$

it is easy to show that the process can be represented as

$$x(t) = \sum_1^\infty z_\nu \frac{\varphi_\nu(t)}{\sqrt{\lambda_\nu}},$$

where both z_ν and φ_ν are obtained as eigen-functions to symmetric integral equations. This is only a simple generalization of a result due to Karhunen. (Zur Spektraltheorie stochastischer Prozesse, Ann Ac Sci Fenn, series A, I, 34, Helsinki 1946.) But for an orthogonal process $Z(\lambda)$ with $|Z(\lambda)|^2 = \sigma(\lambda)$, one can prove, using Parsevals relation and the defining properties of an orthogonal process:

Theorem II.

$$Z(\lambda) = \sum_{1}^{\infty} z_\nu \int_{-\infty}^{\lambda} \varphi_\nu(x)\, d\,\sigma(x)$$

where $\{z_\nu\}$ *is a CON system in* $L_2(Z)$ *and* $\{\varphi_\nu\}$ *is a CON system in* $L_2(\sigma)$. Applying this to the above one gets

Theorem III. *The process can be represented as*

$$x(t) = \int_{-\infty}^{\infty} f(t, \lambda)\, d\,Z(\lambda)$$

where for every t $f(t, \lambda) \in L_2(\sigma)$ *as a function of* λ *and* $Z(\lambda)$ *is an orthogonal process with* $\|Z(\lambda)\|^2 = \sigma(\lambda)$.

Detailed proofs and various further developments of the methods used here will be reserved for a later publication.

Tryckt den 6 maj 1949

Uppsala 1949. Almqvist & Wiksells Boktryckeri AB